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Graph Laplacians and Fourier Transforms on Boolean

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Graph Laplacians and Fourier Transforms on Boolean Domains *

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Abstract

We identify the eigenvalue problem for the graph Laplacian on a Hamming graph as an analog to the problem of modes of vibration on a continuum, and show that the eigenvectors of the graph Laplacian matrix are the basis functions for the Fourier transform on the Boolean domain.

Keywords: Graph Laplacian, Eigenvalue problem, Fourier transform, Boolean domain.

1 Introduction

A recent tutorial paper [1] described how the Fourier transform on a Boolean domain was re-invented by Kahn et al. [2] to show the relation between the influence factors of variables on a Boolean function and the coefficients of its Fourier transform. Since that paper Fourier transforms on Boolean domains have been applied in such areas as the sampling theorem, approximation of Boolean functions, noise reduction in Boolean functions, complexity problems, etc. Another important application of general Fourier transforms (not necessarily of Boolean type) is their wide use in image processing [3].

In this paper we exhibit the Fourier transform kernel as a solution of the eigenvalue problem for the graph Laplacian on an n-dimensional Hamming graph. We arrived at this result by pursuing the analogy between the geometric and graph-theoretic pictures. As pointed out by Brooks [4], one of the reasons for passing back and forth between the geometric and graph-theoretic pictures is that a problem which appears difficult from one point of view may be relatively easy, or even already solved, from the other point of view. Another reason is that attitudes towards various results may differ markedly in the two areas, and comparing them may be an important source of insight.

Fourier transforms and eigenvalue problems are both applied in nature: we have mentioned some applications of Fourier transforms on Boolean domains, and there are numerous applications of them on real domains; on the other hand, eigenvalues and eigenvectors of Hamming graphs have applications to coding theory [5,6], and, of course, eigenfunctions derived from Laplacians on real domains are used in many practical problems.

This paper is organized as follows: In Section 2 Hamming graphs and hypercubes are identified as graph-theoretic and geometric aspects of the same object; in Section 3 the eigenvalue problem for the Laplacian is described for each aspect; and in Section 4 it is shown that the eigenvectors for the graph Laplacian form the kernel of the Fourier transform on the Boolean domain.

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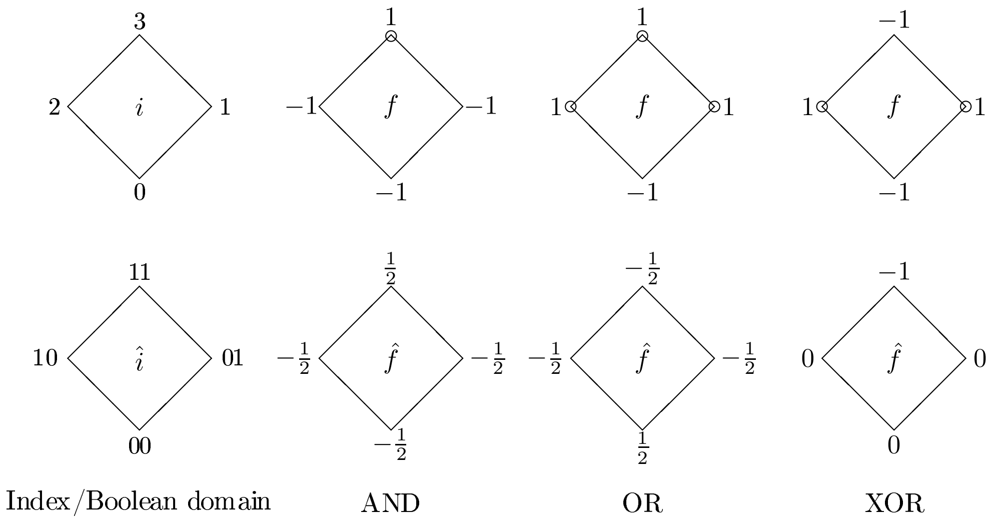

Figure 1: The first column shows the 2D Hamming graph with vertices indexed by i and the corresponding Boolean domain indexed by the binary number $\hat{i}$; the other three columns show the Fourier transform pairs, f and $\hat{f}$ for the Boolean functions AND, OR and XOR of two variables.

2 Hamming Graphs and Hypercubes

A binary Hamming graph is a graph corresponding to the vertices and edges of a unit hypercube. If $\{\epsilon_i\}$ denotes the standard orthonormal basis of R^n, the n-dimensional unit cube is $\gamma_n = \{\Sigma_{i=0}^{n-1}\xi_i\epsilon_i | 0 \leq \xi_i \leq 1\}$. A vertex of a Hamming graph is indexed by an integer i corresponding to its coordinate $\hat{i}$ regarded as a binary number. So a Hamming graph consists of a vertex set $V = \{0, 1, \cdots, 2^n - 1\}$ with the edge set $E = \{\{i,j\} | i,j \in V, (\hat{i} \oplus \hat{j}) = 1\}$, where $(\hat{i} \oplus \hat{j}) = \sum_{k=0}^{n-1} \hat{i}_k \oplus \hat{j}_k$ (bitwise exclusive-or followed by a sideway add). Note that the Boolean domain $\{0,1\}^n$ can also be indexed by a bit-string corresponding to an integer in binary, and there is a one to one correspondence between the Hamming graph vertices and the Boolean domain. A Hamming graph can be used to visualize a Boolean function $f : \{0,1\}^n \rightarrow \{-1, 1\}$ by assigning a value to each vertex or putting circles on vertices with value 1 as shown in the upper right three figures of Fig. 1. According to the "dictionary" between graphs and manifolds [4] the Hamming graph and the hypercube are corresponding entities in the two spaces, and we expect that whatever happens in one space also happens (in full or in part) in the other. In the next section we consider the eigenvalue problem in both spaces.

3 The Laplacian Eigenvalue Problem

Consider the vibration problem for a hypercubic medium, for simplicity the 2D case of the vibration of a square membrane. Under appropriate conditions the transverse displacement $U(x, y, t)$ from the equilibrium position of the membrane satisfies the wave equation

$$\Delta U = \frac{1}{c^2}\frac{\partial^2 U}{\partial t}, \tag{1}$$

where c is the wave velocity. The solution is a triple Fourier series [7,8] of which the spatial factors of each term are solutions of the eigenvalue problem

$$\Delta u = \lambda u, \tag{2}$$

and the shape of the membrane at $t = 0$ is given by the Fourier series

$$u(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin(m\pi x) \sin(n\pi y). \tag{3}$$

The functions $\sin(m\pi x)$ and $\sin(n\pi y)$ are eigenfunctions, with eigenvalues m and n, of Eq. (2). The eigenvalues are chosen to satisfy the boundary condition. The graph-theoretic counterparts of Eqs. (2) and (3) are the eigenvalue problem for the graph Laplacian on the Hamming graph and the Fourier expansion of Boolean functions. In the next section we consider how these are related.

4 Graph Laplacians and Fourier Transforms on Boolean Domains

The *Laplacian* of a graph G, denoted by $L(G)$, is defined to be $D - A$, where $A = (a_{ij})$ is the adjacency matrix of G (i.e., $a_{ij} = 1$ if $\{i, j\}$ is an edge of G and 0 otherwise) and D is the diagonal matrix with $(d_{ii}) = d_i$, the degree of the i-th node [9]. For the n-dimensional Hamming graph H, a_{ij} is 1 if $(\hat{\imath} \oplus \hat{\jmath}) = 1$ and 0 otherwise, and $d_i = n$ for $0 \leq i \leq 2^n - 1$, that is, the Hamming graph is an equi-degree or regular graph. We denote the Laplacian matrix by $L(H) = (h_{ij})$. The eigenvalue problem for the Hamming graph Laplacian is

$$L(H)u = \lambda u, \tag{4}$$

where λ is the eigenvalue and u is the eigenvector.

On the other hand the Fourier transform of a Boolean function is defined as follows [1]. The i-th Fourier coefficient of a Boolean function f is

$$\hat{f}(\hat{\imath}) = \frac{1}{2^n} \sum_{j=0}^{2^n-1} f(\hat{\jmath}) \chi_{\hat{\imath}}(\hat{\jmath}), \tag{5}$$

and f can be uniquely represented as

$$f(\hat{\jmath}) = \sum_{i=0}^{2^n-1} \hat{f}(\hat{\imath}) \chi_{\hat{\imath}}(\hat{\jmath}), \tag{6}$$

where $\chi_{\hat{\imath}}(\hat{\jmath}) = (-1)^{(\hat{\imath} \cdot \hat{\jmath})}$ is the basis function for the Fourier transform with $(\hat{\imath} \cdot \hat{\jmath}) = \sum_{k=0}^{n-1} \hat{\imath}_k \cdot \hat{\jmath}_k$. (It is also the kernel of the 1D Hadamard transformation [3].) Some examples of Fourier expansions of Boolean functions are shown in Fig. 1 as pairs f and $\hat{f}$ for the two-variable functions AND, OR and XOR. As an example, Table 1 shows the kernel $\chi_{\hat{\imath}}(\hat{\jmath})$ for $n = 3$, and the upper left 4×4 matrix is the kernel for $n = 2$, needed to find the Fourier transforms $\hat{f}$ in Fig. 1.

It is apparent that Eqs. (4) and (6) correspond to Eqs. (2) and (3), respectively, and by analogy with the continuum we expect the basis functions for the Fourier transform to be the eigenvectors of the graph Laplacian. This is the content of the following theorem.

Theorem 1 *The eigenvectors of the graph Laplacian for a binary Hamming graph are precisely the basis functions of the Fourier transform on the corresponding Boolean domain (and hence also the basis functions of the 1-dimensional Hadamard transform). Explicitly,* $U = (u_{ij})$*, where* $u_{ij} = (-1)^{(\hat{\imath} \cdot \hat{\jmath})}$ *is a matrix whose columns are eigenvectors of* $L(H)$*, that is* $L(H) \cdot U = U \cdot C$ *where* C *is a diagonal matrix of eigenvalues.*

Table 1: The Fourier transform kernel $\chi_{\hat{i}}(\hat{j}) = (-1)^{(\hat{i}\cdot\hat{j})}$ (shown in row i and column j) for $n = 3$. It is also the kernel of the 1D Hadamard transformation for eight points.

	0	1	2	3	4	5	6	7
0	+1	+1	+1	+1	+1	+1	+1	+1
1	+1	−1	+1	−1	+1	−1	+1	−1
2	+1	+1	−1	−1	+1	+1	−1	−1
3	+1	−1	−1	+1	+1	−1	−1	+1
4	+1	+1	+1	+1	−1	−1	−1	−1
5	+1	−1	+1	−1	−1	+1	−1	+1
6	+1	+1	−1	−1	−1	−1	+1	+1
7	+1	−1	−1	+1	−1	+1	+1	−1

Proof: The (i, j)-element $(HU)_{ij}$ of $L(H) \cdot U$ is

$$(HU)_{ij} = \sum_{k=0}^{2^n-1} h_{ik} \cdot u_{kj} = \sum_k (d_{ik} - a_{ik}) \cdot (-1)^{(\hat{k}\cdot\hat{j})}. \tag{7}$$

For d_{ik}, only the term with $k = i$ contributes to the sum, and has value n, while for a_{ik}, there are n different k's which contribute to to the sum and Eq. (7) becomes

$$(HU)_{ij} = n \cdot (-1)^{(\hat{i}\cdot\hat{j})} - \sum_{k\in K} (-1)^{(\hat{k}\cdot\hat{j})}, \tag{8}$$

where $K = \{k|(\hat{k} \oplus \hat{i}) = 1\}$ with cardinality $|K| = n$. If we put $(-1)^{(\hat{k}\cdot\hat{j})} = a_k \cdot (-1)^{(\hat{i}\cdot\hat{j})}$ Eq. (8) can be written

$$(HU)_{ij} = (n - \sum_{k=0}^{n-1} a_k) \cdot (-1)^{(\hat{i}\cdot\hat{j})}. \tag{9}$$

Let the bit in which $\hat{i}$ and $\hat{k}$ differ be the p-th bit. Then $a_k = -1$ if the p-th bit of $\hat{j}$ is 1 and $a_k = 1$ otherwise. The sum $\sum_k a_k$ is n, when all a_k's are 1, and a change of a_k from 1 to -1 subtracts 2 from this sum. Since the number of 1's in $\hat{j}$ is $(\hat{j})$, the sum is therefore $\sum_k a_k = n - 2 \cdot (\hat{j})$. Thus Eq. (9) becomes

$$(HU)_{ij} = \{n - (n - 2 \cdot (\hat{j}))\} \cdot (-1)^{(\hat{i}\cdot\hat{j})} = 2 \cdot (\hat{j}) \cdot (-1)^{(\hat{i}\cdot\hat{j})} = 2 \cdot (\hat{j}) \cdot u_{ij}, \tag{10}$$

which proves the theorem. ■

Eq. (10) shows that the diagonal matrix of eigenvalues is

$$(c_{jj}) = 2 \cdot (\hat{j}). \tag{11}$$

This agrees with the known eigenvalues and multiplicities of Hamming graph adjacency matrices. The eigenvalues θ_j and their multiplicities f_j of a general Hamming graph matrix (for q symbols instead of 2) are given in [5] as

$$\begin{aligned} \theta_j &= q(d - j) - d \\ f_j &= \tbinom{d}{j}(q - 1)^j, \end{aligned} \tag{12}$$

where d is the dimension of the Hamming code, q is the number of symbols, and j runs from 0 to d. Putting $q = 2$ and $d = n$ and subtracting θ_j from n, the eigenvalues θ'_j and their multiplicities f'_j of Hamming graph Laplacian matrix are therefore seen to be

$$
\begin{aligned}
\theta'_j &= 2j \\
f'_j &= \tbinom{n}{j}.
\end{aligned} \tag{13}
$$

The distinct eigenvalues are $\{2 \cdot (\hat{j}) | 0 \leq j \leq 2^n - 1\} = \{2j | 0 \leq j \leq n\}$ and their multiplicities are $|\{i | (\hat{i}) = j, i \in \{0, 1, \cdots, 2^n - 1\}| = \binom{n}{j}$.

Equation (11) is striking, and there must be some unknown reason underlying it.

As an example, here is Eq. (10) for $n = 3$. (The matrix U on the right of the product is as in Table 1.)

$$
\begin{aligned}
&\begin{bmatrix}
3 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\
-1 & 3 & 0 & -1 & 0 & -1 & 0 & 0 \\
-1 & 0 & 3 & -1 & 0 & 0 & -1 & 0 \\
0 & -1 & -1 & 3 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 3 & -1 & -1 & 0 \\
0 & -1 & 0 & 0 & -1 & 3 & 0 & -1 \\
0 & 0 & -1 & 0 & -1 & 0 & 3 & -1 \\
0 & 0 & 0 & -1 & 0 & -1 & -1 & 3
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\
1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\
1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\
1 & -1 & -1 & 1 & -1 & 1 & 1 & -1
\end{bmatrix} \\
&= \begin{bmatrix}
0 & 2 & 2 & 4 & 2 & 4 & 4 & 6 \\
0 & -2 & 2 & -4 & 2 & -4 & 4 & -6 \\
0 & 2 & -2 & -4 & 2 & 4 & -4 & -6 \\
0 & -2 & -2 & 4 & 2 & -4 & -4 & 6 \\
0 & 2 & 2 & 4 & -2 & -4 & -4 & -6 \\
0 & -2 & 2 & -4 & -2 & 4 & -4 & 6 \\
0 & 2 & -2 & -4 & -2 & -4 & 4 & 6 \\
0 & -2 & -2 & 4 & -2 & 4 & 4 & -6
\end{bmatrix}
\end{aligned} \tag{14}
$$

5 Concluding Remarks

We have shown that the kernel of the Fourier transform on a Boolean domain is a matrix whose columns are eigenvectors of the graph Laplacian for the corresponding Hamming graph. The diagram below illustrates the concepts discussed in this paper and how they are related. Not all are new. The object of this paper has been to draw attention to the missing arrow (shown thick) which has so far been unnoticed.

$$
\begin{tikzcd}
\text{Graph Laplacian eigenvalue problem} \arrow[r, leftrightarrow] \arrow[d, very thick] & \text{Real Laplacian eigenvalue problem} \arrow[d, "\text{Square boundary}"] \\
\text{Fourier expansion of Boolean function} \arrow[r, leftrightarrow] & \text{Fourier expansion of real function}
\end{tikzcd}
$$

We note that, for the continuum, different shapes of boundary lead to different sets of eigenfunctions, while for the graph-theoretic analog there is a unique set of eigenvectors, presumably because there is no freedom to shape the boundary. One graph-theoretic significance of our new arrow can be found in coding theory: the Hamming graph is distance regular and distance transitive (see [5] and [6]) and the entire structure of this graph and its related codes can be determined from the related Bose-Mesner algebra, which is an algebra with two fixed bases, one of which is the basis formed by the eigenvectors of the graph Laplacian. This basis can be used to study special subalgegras of the Bose-Mesner algebra related to the so-called fusion schemes [10].

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