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Digital Geometry – The Birth of a New Discipline

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Abstract

Digital geometry emerged with the rise of computer technologies in the second half of the 20th century as an application-oriented field influenced by the new possibilities in computer graphics and digital image analysis. Digital geometry is a subdiscipline of discrete geometry. Problems in image analysis are defined with respect to Euclidean geometry, and direct links to this geometric discipline are normally established for motivation or comparison. This directs digital geometry toward being a digitized Euclidean geometry.

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Chapter 1

DIGITAL GEOMETRY — THE BIRTH OF A NEW DISCIPLINE

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1. INTRODUCTION

Digital geometry emerged with the rise of computer technologies in the second half of the 20th century as an application-oriented field, influenced by the new possibilities in computer graphics and digital image analysis. It has its mathematical roots in graph theory and discrete topology; it deals with sets of grid points, which are also studied in the geometry of numbers [78, 85], or with cell complexes, which have also been studied in topology since its beginning [2, 76, 86, 87]. Studies of gridding techniques, such as those by Gauss, Dirichlet or Jordan [52], may also be cited as historic context. Digitizations on regular grids are also important for numerical approaches in other fields such as computational fluid dynamics or computational physics [139]. This chapter will deal only with digital geometry in the context of image analysis.

1.1 DISCIPLINES IN GEOMETRY

The history of geometrical knowledge dates back about 4000 years, to societies in Mesopotamia and Egypt. The word *geometry* has been in use for more than 2500 years since the time of classical antiquity. The field emerged as a special discipline in mathematics due to practical needs, e.g. in earth measurement. Distance measurements, or calculations of areas and volumes, are elements of the earliest developments in mathematics. Of course, only simple two- or three-dimensional objects such as polygons, cubes, prisms or cylinders were studied in those days. For example, the calculation of the volume of a frustum of a pyramid was already known in ancient Egypt, as well as definitions of angles, or decompositions of polygons into triangles or rectangles. The law of Pythagoras was known in ancient Mesopotamia, and the use of similarity rules for triangles was common in applications. Archimedes calculated

π to three significant figures using regular 96-gons for inner and outer approximations of the circle.

Needs in applications and science led to the emergence of a broad diversity of geometries: *Euclidean* (Thales of Miletus, Hippocrates of Chios, the secret society of the Pythagoreans, Euclid, Archimedes), *analytical* (Descartes, also known as Cartesius), *perspective* (Alberti, da Vinci, della Francesca, Dürer), *projective* (Desargues, Pascal), *descriptive* (Monge), *non-Euclidean*, such as *elliptical* and *hyperbolical* (Lobachevsky, Bólyai, Riemann), or *combinatorial* (Helly, Borsuk, Erdős).

The Norwegian mathematician S. Lie and the German mathematician F. Klein specified a classification system for all geometries known at their time: *Geometric properties of spatial objects are those which are invariant with respect to a specified group of transformations.* This classification scheme is famous as the 1872 *Erlangener Programm* of F. Klein. Let $\mathbf{B}$ be a manifold and $\mathbf{G}$ a group of transformations defined on this manifold. The theory of invariants with respect to $\mathbf{B}$ and $\mathbf{G}$ defines a geometry, where a non-empty family of objects or figures $\mathcal{F} \subseteq 2^{\mathbf{B}}$ is specified as *objects of interest.*

For example, $\mathbf{B}$ may be the three-dimensional Euclidean space $\mathcal{E}^3 = [\mathbf{R}^3, d]$, defined by the manifold $\mathbf{R}^3$ of triples of real numbers and the Euclidean metric d, $\mathcal{F}$ the family of all bounded polyhedral sets, and $\mathbf{G}$ the group of (i) all similarity transforms, (ii) all affine transforms, or (iii) all projective transforms into the Euclidean plane. These are three different geometries of polyhedra in the sense of the Erlangener Programm.

The study of invariants requires that elements of $\mathbf{B}$ can be understood in some relation to each other, i.e. the base set $\mathbf{B}$ needs to have a *structure* defined by a metric, or a topology (i.e. a system of open and closed sets), or at least a system of neighborhoods $U(x) \subseteq \mathbf{B}$ for all of its elements $x \in \mathbf{B}$. A *neighborhood relation* $y \in U(x)$ is reflexive, symmetric, and non-transitive. An element y is a *proper neighbor* of x iff $y \in U(x)$ and $x \neq y$. The proper-neighborhood relation is irreflexive, symmetric, and non-transitive.

A *geometric discipline* is not only defined by a meaningful combination of a structured set $\mathbf{B}$, a group $\mathbf{G}$ and a family $\mathcal{F}$. There is also the social aspect of *significance* in science and technology.

1.2 DIGITAL GEOMETRY

Digital geometry is a very lively research area in which hundreds of journal papers have been published so far. The book chapter by A. Rosenfeld [49], the chapter by K. Voss in [114], and the digital geometry chapter in [101], as well as the books [21, 38, 74, 79, 82, 113, 131], define *digital geometry* as a theory of n-dimensional *digital spaces* (cellular or grid point spaces) oriented toward the understanding of geometric subjects. In short, digital geometry is characterized by a regular grid or cell structure of its base set $\mathbf{B}$ which allows a well-defined mapping of all of its elements into $\mathbf{Z}^n$, for some value of $n \geq 1$.

Grid point model. We assume an orthogonal grid[1] with grid constant $0 < \vartheta \leq 1$ in n-dimensional Euclidean space $\mathcal{E}^n$, $n \geq 1$, i.e. ϑ is the uniform spacing between grid points parallel to each of the coordinate axes. Furthermore, let $r \geq 1$ be the *grid resolution*, defined as the number of grid points per unit length, i.e. any grid edge is of length $\vartheta = 1/r$. In the case of the Euclidean plane we consider *r-grid points* $g^r_{i,j} = (\vartheta \cdot i, \vartheta \cdot j)$, for integers $i, j \in \mathbf{Z}$ and $\vartheta = 1/r$. For $r = 1$ we simply speak about *grid points* (i, j) in the Euclidean plane $\mathcal{E}^2$.

Assume an *integer metric* (i.e. a metric having values $r \cdot i$, i integer only) on r-grid points, and consider all r-grid points within distance $1/r$ as being proper neighbors of a given r-grid point. This approach results in a one-one correspondence between neighborhood definitions and *graph metrics* on the infinite orthogonal grid, i.e. special integer metrics which are uniquely specified by repeated applications of the definition of points at distance $1/r$. The proper-neighborhood relation specifies an undirected graph on the given set of grid points. Distances in this *neighborhood graph* are defined by the graph metric used.

Cell complex model. Any r-grid point $g^r_{i,j,k} \in \mathcal{E}^3$, with i, j, k integers, is assumed to be the center point of an *r-grid cube* with *r-faces* parallel to the coordinate planes, with *r-edges* of length $1/r$, and *r-vertices*. Assume r to be constant. *Cells* are either cubes, faces, edges or vertices. The intersection of two cells is either empty or a joint *side* of both cells. We consider a non-empty finite set K of cells such that for any cell in K, any side of this cell is also in K. Such a set K is a special finite *Euclidean complex* [89]. Let $dim(a)$ denote the dimension of a cell a, which is 0 for vertices, 1 for edges, 2 for faces and 3 for cubes. Then $[K, \subset, dim]$ is also a *cell complex* with properties such as

(i) the relation $\subset$ of proper inclusion is transitive on K,

(ii) dim is monotone on K with respect to $\subset$, and

(iii) for any pair of cells $a, b \in K$ with $a \subset b$ and $dim(a) + 1 < dim(b)$ there exists a cell $c \in K$ with $a \subset c \subset b$.

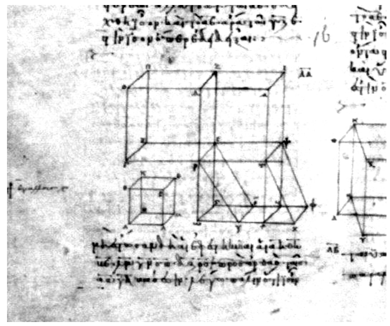

Figure 1.1 A figure from Euclid's "Elements", Book XI Propositions 31–33 on the volumes of parallelepiped solids. Today we could call these 'face-connected r-cubes'. (For a larger, color image see `http://www.ibiblio.org/expo/vatican.exhibit/exhibit/d-mathematics/Math_extra.html`.)

Cell b *bounds* cell a iff $b \subset a$, and b is a *proper side* of a in this case. Two cells a and b are *incident* iff cell a bounds b or cell b bounds a. Two r-faces are *edge-neighbors* iff their intersection contains an r-edge, and they are *proper edge neighbors* if they are also not identical. Further neighborhood relations can be defined in an analogous way; see Figure 1.1. Different cell complex models might be used in special situations; see [38, 70, 87, 120, 134] and the historic review in [63].

Geometric studies of digital images assume digital spaces, i.e. regular grids or cell complexes, as base structures $\mathbf{B}$. The orthogonal grid with $r = 1$, and Euclidean complexes, may be considered to be default models of the base set $\mathbf{B}$ of digital geometry. The continuity or topology definition in Euclidean geometry based on the Euclidean metric needs to be replaced by a connectivity definition or topology definition on $2^{\mathbf{B}}$ to allow specifications of 'regions of interest'. Graph metrics on regular grids specify neighborhoods (which allow connectivity definitions), and cell complexes allow the introduction of a topology of open and closed sets [1, 87, 89, 125]. Regular grid or cell structures do not resemble Euclidean geometry or topology with respect to nondenumerable bounded sets; there is no possibility of smaller and smaller $\varepsilon > 0$, of continuous convergence, or of infinitely refined partitions in digital geometry.

However, the problems in image analysis are defined with respect to Euclidean geometry, and direct links to this geometric discipline are normally established for motivation or comparison. This directs digital geometry toward being a *digitized Euclidean geometry*. Furthermore, increasing need for accuracy in high-resolution digital imaging results in demands for mathematically provable or experimentally evaluated behaviors of algorithms to ensure that further investments in technology (e.g. more accurate digitization or higher image resolution) will actually result in improved data. Interestingly, the history of digital geometry followed these technological demands: the study of small or low-resolution images supported (or still allowed) a graph-theoretical preference, but high-resolution images led away from such points of view.

Eukleides (also known as Euclid) formulated in 13 books an *axiomatic system* of the geometry today known as Euclidean geometry. There were still many open problems related to the axiomatic foundations of geometries at the end of the 19th century. Poincaré, Pasch, Hilbert and others devoted their time to these problems. Hilbert proposed an axiomatic system in 1899. Axiomatic foundations of geometric theories, which have to be complete and consistent, have been studied in logic and geometry, and the field of geometry is one of the best axiomatized disciplines in mathematics [36]. Digital geometry is by now a well-established subfield of geometry, and studies involving transformation groups (in the sense of the Erlangener Programm of F. Klein) and axiomatic foundations are just as important as in any other geometry. The habilitation [41], which unfortunately remained unpublished except for some small notes such as [42], contains valuable contributions toward the definition and analysis of the transformation groups $\mathbf{G}$ of digital geometry and the axiomatic foundations of digital geometry. See also the essay [119] on axiomatic foundations of convexity and linearity. Today, digital geometry is a discipline which provides definitions and basic rules for thousands of running applications worldwide, especially in the context of digital image processing, image analysis, or computer graphics.

1.3 DISCRETE GEOMETRY

Digital geometry is a subdiscipline of *discrete geometry*, which is a theory of invariants in the sense of Klein's Erlangener Programm, with a set of objects $\mathcal{F}$ and a transformation group $\mathbf{G}$ on a manifold $\mathbf{B}$, where at least one of the three constituents $\mathbf{B}$, $\mathbf{G}$, or $\mathcal{F}$ of such a theory is finite or discrete [12, 29, 37].

A family of sets $\mathcal{A} \subseteq 2^{\mathbf{B}}$ is *discrete* in $\mathbf{B}$, where $\mathbf{B}$ is a set equipped with a neighborhood definition, iff for any element $x \in \mathbf{B}$ there exists a neighborhood $U(x) \subseteq \mathbf{B}$ which has a non-empty intersection only with a finite number of sets in $\mathcal{A}$. For example, a family of cells of a Euclidean cell complex is discrete in $\mathbf{R}^n$.

A set $A \subseteq \mathbf{B}$ is discrete in $\mathbf{B}$ iff for any $x \in \mathbf{B}$ there exists $U(x) \subseteq \mathbf{B}$ such that $U(x) \cap A$ is finite. Any set of regular grid points is discrete in $\mathbf{R}^n$.

Packing problems, tessellations, polyhedral geometry, Euclidean cell complexes, geometry on graphs, and finite geometries are examples of subdisciplines of discrete geometry other than digital geometry. Some of them may be considered to be generalizations of digital geometry, if a more abstract base set $\mathbf{B}$ allows a regular grid or a cell complex to act as its specialized interpretation (i.e. a model in the sense of logic). The conferences on 'Discrete Geometry for Computer Imagery', of which [14] was the first outside France, are actually focused on digital geometry.[2]

1.4 CORE PROBLEMS AND STRUCTURE OF THE CHAPTER

This chapter will not provide many technical details. In general we assume that the reader is familiar with the literature on digital geometry and its basic definitions. The intention of this chapter is more to summarize historic developments in the field, to contribute to the problem of how to define 'digital geometry', and to propose some future directions of study. Also, we cannot cite all the important work in the field; we give only a few references. The reader is referred to [102] for an extensive list of more than 800 related publications, or to [94] for 374 references just on digital and computational convexity and straight lines.

There are (at least) three *core tasks* in digital geometry: **(i)** the definition of curves (in 2D and 3D) and of curve length such that the concept is consistent with accurate estimation of the length of digitized Jordan curves (i.e. rectifiable one-dimensional manifolds) in Euclidean spaces; **(ii)** the definition of surfaces in 3D and of surface area such that the concept allows precise estimation of the surface area of digitized Jordan surfaces (i.e. measurable two-dimensional manifolds) in Euclidean spaces; and **(iii)** the definition of differential properties (e.g. curvature) of digitized one- or two-dimensional manifolds in accordance with the corresponding properties of Euclidean sets prior to digitization. In all three cases the task also includes the design of efficient algorithms. Problem **(i)** has been basically solved for polygonal approximations. Solutions to problem **(ii)** exist, but they are either theoretical results without algorithmic implementation, or experimental algorithms without theoretical foundation. There are some first steps toward solutions to problem **(iii)**.

The structure of this chapter is as follows: We start with a review of three classic papers which stand at the beginning of the new discipline; then we review 'traditional digital geometry', which combines studies in digital topology, graph theory for digital spaces, and digital geometry. We propose a more focused definition of digitized Euclidean geometry, which is the subject of the next section. We review developments toward the solution of problems **(i)** and **(ii)** to illustrate the new discipline at least in two subject areas. Finally, a few problems are listed for further study.

2. THREE CLASSIC PAPERS BY A. ROSENFELD AND J. L. PFALTZ

We start our journey through the origins of digital geometry by reviewing three classic papers which, in addition to their specific content, were of general methodological importance. These papers [83, 106, 107] introduced *digital pictures*, i.e. arrays of grid points, and problems about arbitrarily shaped sets of grid points, as a new subject into the scientific literature. Connectedness of grid points was introduced, as well as first approaches to understanding the shape of a set of grid points. Earlier work such as [15, 33, 77, 121] was focused on computer description of line drawings for transmission, plotting or graphics, which has also been very important for the establishment of digital geometry.

2.1 CONNECTEDNESS AND DISTANCE TRANSFORMS

The paper "Sequential operations in digital picture processing" [106] cites, e.g., [24, 61, 126] as preceding work on digital picture processing, and starts with a discussion of *local image transforms.* Its Section 3 introduced *connected components.* Without explicitly using the terms '8-neighborhood' or '4-neighborhood', the paper deals with these neighborhoods. Figure 1.2a (a set containing two grid points) is 8-connected, as is its complementary set. Figure 1.2b is both 4- and 8-connected, and its complement is neither 4- nor 8-connected. Figure 1.2c is not 4- or 8-connected but its complement is both 4- and 8-connected, and

> "the 'paradox' of Figure 1.2d can be (expressed) as follows: If the 'curve' of shaded points is connected ('gapless'), it does not disconnect its interior from its exterior; if it is totally disconnected it *does* disconnect them. This is of course not a mathematical paradox, but it is unsat-

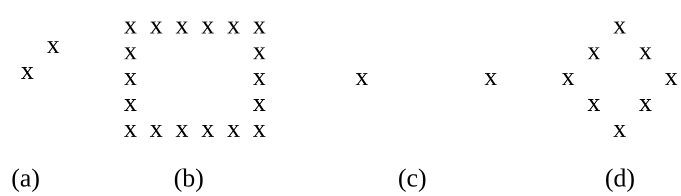

Figure 1.2 Examples of connected and nonconnected sets [106].

> isfying intuitively; nevertheless, connectivity is still a useful concept. It should be noted that if a digitized picture is defined as an array of hexagonal, rather than square, elements, the paradox disappears".[3]

The first case assumes 8-connectedness for 'curve' and 'background', and the second case assumes 4-connectedness for both. The first case occurs if a non-planar neighborhood graph (here: grid points as vertices with horizontal, vertical and diagonal edges) is used. An early discussion about different options for defining 'separating connected curves' is given in [69], and a systematic treatment of the problem under the assumption of a planar neighborhood graph can be found in [131].

In [96, 97] it was shown that 4-curves separate 8-holes and 8-curves separate 4-holes. This led to the suggestion of using these two neighborhood definitions on the same non-planar graph structure, one of them for 'objects' and the other one for 'background'. The drawback is that objects and background then have different topologies. This concept is used in binary image analysis. It definitely fails for multi-level images where different neighborhood definitions for all the different levels are impossible. Cell complexes or planar neighborhood graphs provide sound models for these cases; see, e.g., the discussions in [56, 70, 131].

The paper [106] also specifies an algorithm for sequential *labeling of connected components*, and computation of *distance transforms* based on Blum's and Kotelly's earlier work on visual form perception. The *medial axis* in [10, 84] is approximated by midpoints of 8-circles having an 8-radius which is a local maximum; the resulting, in general disconnected set of labeled midpoints is the *distance transformation* of a picture subset.[4] The paper concludes with applications of the concepts that were introduced: number of connected components, connectivity graph analysis, proximity of picture subsets, elongation of a subset, and shape description based on partitions into squares (i.e. 8-circles).

The material on distance transforms in this paper is actually one of the historic sources of *digital topology*. It initiated a very broad literature on *shape simplification*: shrinking (15 references in [102]), distance transforms (56 references), medial axes (92 references), thinning in 2D (156 references), in 3D (13 references), and in gray level images (29 references). Thus a total of 361 references were devoted to simplification of shape; this is close to 50% of all of the digital geometry references in [102]! Topology preservation [93, 138] is a key aspect of this area of shape simplification; it is not further dealt with in this chapter.

Encodings of picture subsets are compared in the paper "Computer representation of planar regions by their skeletons" [83]. This paper gives a *comparative performance analysis* for three different boundary representations and skeletons (i.e. region encoding using centers and radii of maximal 4-neighborhoods).

2.2 METRICS

Different *integer metrics* are defined and compared in [107] for the planar orthogonal and hexagonal grids. This paper defines and studies two graph metrics d_1 and d_2 corresponding to the 4- and 8-neighborhood (this was the first paper where both were discussed explicitly), as well as a hexagonal d_3 and an octagonal d_4 integer metric. It states that neither the nearest integer to, nor the greatest integer not exceeding, the Euclidean distance, is a metric:

> "For example, let $(i,j) = (1,1), (h,k) = (-1,-1)$; then $\sqrt{(i-h)^2 + (j-k)^2} = \sqrt{8}$, the nearest integer to which is 3, but $\sqrt{i^2 + j^2} = \sqrt{h^2 + k^2} = \sqrt{2}$, the nearest integer to which is 1, so triangularity is violated. Similarly, 'greatest integer not exceeding Euclidean distance' is not a distance function, as shown by the example $(i,j) = (2,3), (h,k) = (-1,-1)$."

However, for every metric d, taking the least integer greater than or equal to the distance $d(\mathbf{p}, \mathbf{q})$ also specifies a metric. The paper discusses an algorithm for the octagonal metric d_4 using dilation by alternating 4- and 8-neighborhoods as structuring elements. It is shown that such octagons are the best possible approximations of Euclidean circles if we are limited to the use of isotropic local operations as specified in Propositions 8 and 9 of that paper. The paper concludes with several proposed applications of integer metrics defined on regular grids. The approximation to Euclidean distance was a central subject in this paper.[5] This early work coincides with the point of view that digital geometry is directed toward digitized Euclidean geometry.

These three papers were of importance in the historic process of establishing a theory of computer-represented pictures: they contain notions and results which were quickly accepted by others as 'basic tools' of the field. See, e.g., [92] for early work in Belgium, [19] for France, the chapter by K. Voss on digital geometry in [114] for Germany, [7] for Italy, [137] for Japan, [13] for Sweden, and [25] for The Netherlands. The name 'digital picture' was pushed aside by 'digital image' in subsequent years.

3. TRADITIONAL DIGITAL GEOMETRY

In [101] we read: "By *digital geometry* we mean the mathematical study of geometrical properties of digital picture subsets." We chose this book chapter to characterize *traditional digital geometry* by briefly listing its basic definitions and a few of the main results in the field.

3.1 SUBJECT LIST ANNO 1979

The book chapter [101] provided a first comprehensive review of the field. It contained the following *basic definitions*: Digital picture (as a binary mapping on a rectangular orthogonal grid), point (grid point with integer coordinates), d-neighbors and d-adjacency for $d \in \{4, 8\}$ (with its follow-ups: d-path, d-connected set, and d-component), background d-component, d-hole, and *simply d-connected* set (i.e. 4-connected with no 8-holes, or 8-connected with no 4-holes). These definitions proved to be sufficient for basic studies in traditional digital geometry, and this discipline became very important for designing algorithms in image analysis:

(1) Segmentation of pictures: The discussion in this 1979 chapter is at first focused on a digital Jordan curve theorem based on the notion of an arc. An *arc* is characterized as a connected set of points, where all but two points have exactly two neighbors in the set, and those two have exactly one. Due to the intuitive desire to consider connection and separation as opposite concepts it is suggested to use opposite (4- or 8-) types of connectedness for a picture subset and its complementary set, the background.[6] This concept allows us to show that the *adjacency graph* of any subset of a digital picture is a tree (Theorem 2.6.1). A *curve* is defined to be a connected set of points in which all points have exactly two neighbors in the set. Theorem 2.4.4 states that any curve has exactly one hole. This is a *digital Jordan curve theorem.*

(2) Simplification of picture subsets: The next main subject in this book chapter is *simple points*, which were called 'deletable points' in earlier work [97, 99]. Theorem 2.5.5 says that a set is an arc iff it is simply connected and has exactly two simple points. Deletable or simple points became very important in research on simplification of shape [124].

(3) Measurements of picture subsets: The book chapter then focuses on *area* (defined to be the number of points in the given set), *perimeter* (two different definitions were suggested, both based on the length of 4-paths), and *genus*. The *shape factor* is defined and discussed for these area and perimeter definitions. A local definition of the genus follows which resembles [79].

(4) Graph metrics for distances in pictures: Integer metrics are introduced named the *city block metric* for a 4-neighborhood and the *chessboard metric* for an 8-neighborhood graph. The *intrinsic diameter* (i.e. the maximum distance in the set between any two points of the set) of a grid point set is compared to the number of steps of a border following (BF) algorithm. A *geodesic* is a shortest path between its endpoints. Characterizations of 4- and 8-geodesics are possible using the notion of *chord length* (= distance between endpoints): an arc is a geodesic iff its arc length is equal to its chord length (Proposition 2.9.3). Properties of *near-geodesic arcs* again illustrate that digital geometry is different from Euclidean geometry. See also [100].

The chapter concludes with a few hints about further topics in digital geometry (homotopy, dimension, elongatedness, straightness, convexity, connected component tracking and counting, adjacency tree construction, and border finding and following), on which papers had already been published before 1979.

3.2 THREE INTERRELATED AREAS

The chapter summarized the first 15 years in the field, which afterwards split into several interrelated areas: *digital topology*, *digital geometry*, and *graph theory or combinatorics for digital spaces*, where a theory of neighborhood graphs in $\mathcal{E}^n$, for $n \geq 1$, can also be seen as a study of combinatorial or topological problems about one-dimensional complexes [131] in a general theory of *cell complexes*.

Digital topology. A first definition of this discipline is Section 1 in [97], which also discusses the concepts of open and closed sets. Notions such as connectedness, digital Jordan curve theorem, genus (see recent work on Euler numbers in [47]), and simple points are related to topology, and are required for specification of the picture subsets to be studied in digital geometry.

Graph theory and combinatorics. Arcs and curves are introduced in [97] as special graph-theoretical objects because grid points and their neighborhoods specify undirected neighborhood graphs. They are the subject of [131]. The proposed perimeter and distance definitions are graph-theoretical concepts [11], and they should not be confused with approximations of the Euclidean perimeter and distance. Graph metrics provide a tool for specifying distance operations on sets of grid points. The intrinsic diameter of a connected picture subset never exceeds half of the border length of that set [99]. Results on convexity [58], centers of picture subsets [57], and geodesics in graphs [135] are further examples using graph theory for regular grids, cell complexes, or even more general graph structures, and many of the results in [131] are of a graph-theoretical character as well. Discrete combinatorial surfaces [30] specify boundaries of (simplicial) cell complexes in local neighborhoods. For combinatorial or number-theoretical problems see, e.g., [66, 132].

Digital geometry. Digital geometry was dominated at an early stage by topological and graph-theoretical concepts. However, [105] redefined the main focus of the field: "*Digital geometry* is the study of geometric properties of sets of lattice points produced by digitizing regions or curves in the plane." Compared to the 1979 book chapter, new concepts in digital geometry include

(5) *digitization*, by mapping sets from the Euclidean plane into grid points;

(6) *digital convexity* (images of convex Euclidean sets, or defined by graph-theoretical approaches using the city block or chessboard distance); and

(7) *digital straightness* (images of Euclidean straight line segments using the grid-intersection digitization)

and there has also been a continuing interest in thinning (see item (2) above).

The digitization of curves or boundaries is dealt with in many publications, see, e.g., [15, 16, 24, 33, 34, 35, 98, 111]. Digitization models are found in [62, 74, 95, 113].

For references on digital convexity we cite [20, 40, 58, 64, 91, 94, 115, 116]. Digital straight segments (DSSs), and approximations of digital boundaries by sequences of maximum-length DSSs, have been studied, e.g., in [15, 25, 33, 35, 43, 98, 133].

Digital straight lines and digital planes are two of the most important basic objects in digital geometry, and they will be dealt with in two sections later on. The intention of staying 'as close as possible' to Euclidean geometry now defines digital geometry as a 'digitized version' of Euclidean geometry. Of course, this excludes the study of image geometries related to fisheye cameras, or digitized non-Euclidean geometries. The regular structure of the base set $\mathbf{B}$ (see our definition above) is crucial for digital geometry. The family of figures $\mathcal{F}$ contains all finite subsets of $2^{\mathbf{B}}$. We do not discuss the group $\mathbf{G}$ of transformations in this chapter; it is discussed in [41].

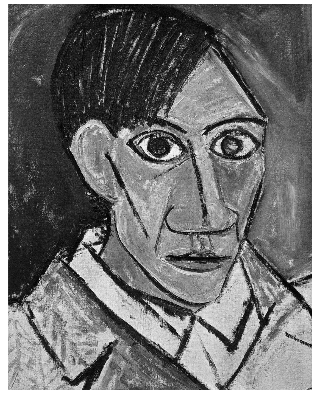

Figure 1.3 Pablo Ruiz Picasso, Selfportrait. Paris. Summer/1907. Oil on canvas. 56 x 46 cm. Národni Gallery, Prague (by courtesy of Enrique Mallen; see color image on `http://www.tamu.edu/mocl/picasso/`).

4. DIGITIZED EUCLIDEAN GEOMETRY

Geometric image analysis approaches are normally motivated by concepts in Euclidean geometry. A straightforward approach is: approximate picture subsets in 2D by *polygons* (Archimedes, 250 BC; Freeman [33]; Montanari [80]; Rosenfeld [98]) or in 3D by *polyhedrons* (della Francesca, 1450; Dürer, 1510; Jordan [52]; Picasso, 1907 (see Figure 1.3); Keppel [54]; Keskes and Faugeras [55]; Morgenthaler and Rosenfeld [81]; Kim and Rosenfeld [59, 60]; Faugeras et al. [28]; Andres et al. [6]; Françon and Papier [31]) and use Euclidean geometry from that moment on for any further object analysis or manipulation step. A theoretical motivation is given by the fact that Jordan curves and surfaces can be approximated by polygonal curves or polyhedral surfaces up to any desired accuracy. This means that if we consider grid resolution as a potentially improvable parameter, then polygonal or polyhedral approximations appear to converge to the original preimage of the given object.

However, this approach may not always be the best (e.g. in the case of curvature estimation for digital curves or surfaces) because it involves an inherent loss of detail, and it generates 'non-smooth' approximations. Smooth boundary approximations in some class $C^{(m)}$ are often required (e.g. a train cannot drive on a track built by following a polygonal curve approximation) and led to the development of splines.

4.1 THREE BASIC DIGITIZATION MODELS

Gridding techniques were used in mathematics before digital image processing came into existence. In the case of the area definition, the fact that the number of grid points in a region may be used as an estimator has been known in number theory since the time of Gauss and Dirichlet [73]. Both of them knew that the number of grid points inside a planar convex curve γ estimates the area of the set bounded by the curve within an order of $\mathcal{O}(l)$, where l is the length of γ.

Definition 1 *For a set S in the Euclidean plane its* Gauss digitization *$G_r(S)$ is defined to be the set of all r-grid points contained in S, i.e.*

$$G_r(S) = \{g^r_{i,j} \ : \ g^r_{i,j} = (i/r, j/r) \in S\} \ .$$

When $r = 1$ the Gauss digitization is denoted by $G(S)$.

Consider all r-grid points as centers of isothetic squares with edge length $1/r$. The set $\mathbf{G}_r(S)$ is the union of all those squares having their center points in $G_r(S)$.

If the given set is actually a curve γ in the plane then the grid-intersection scheme for chain coding [15, 33] is the standard in digital geometry. Of course, this scheme can be adapted to r-grid points for any value of $r > 0$, and the resulting sequence of r-grid points is the *grid intersection digitization* $I_r(\gamma)$, which can be characterized by a start point and a chain code (i.e. a sequence of directional codes). Such code sequences have been studied since the historic papers [35, 98]. For curves in higher-dimensional Euclidean spaces it is appropriate [50, 51] to use the Jordan digitization model $J^+_r(\gamma)$ as defined below.

The important problem of *volume estimation* was studied in [52] based on gridding techniques. Any grid point (i, j, k) in the Euclidean space $\mathcal{E}^3$ is assumed to be the center point of a cube with faces parallel to the coordinate planes and with edges of length 1. The boundary is part of this cube (i.e. it is a closed set). Let S be a set contained in the union of finitely many such cubes. Dilate the set S with respect to an arbitrary point $p \in \mathcal{E}^3$ in the ratio $r : 1$. This transforms S into S^p_r. Let $l^p_r(S)$ be the number of cubes completely contained in the interior of S^p_r, and let $u^p_r(S)$ be the number of cubes having a non-empty intersection with S^p_r. In [52] it is shown that $r^{-3} \cdot l^p_r(S)$ and $r^{-3} \cdot u^p_r(S)$ always converge to limit values $L(S)$ and $U(S)$, respectively, for $r \to \infty$, independently of the chosen point p. Jordan called $L(S)$ the *inner volume* and $U(S)$ the *outer volume* of set S, or the *volume* $vol(S)$ of S if $L(S) = U(S)$. The volume definition based on gridding techniques was further studied, e.g., in [78, 112].

As in the cases of the Gauss and grid-intersection digitizations, we again restrict ourselves to the two-dimensional case in the following definition, i.e. to r-squares, r-edges and r-vertices. Generalizations to higher dimensions are straightforward.

Definition 2 *For a set S in the Euclidean plane its* Jordan digitizations *$J_r^-(S)$ and $J_r^+(S)$ are defined as follows: the set $J_r^-(S)$ (also called the* inner digitization*) contains all r-squares completely contained in the interior of set S, and the set $J_r^+(S)$ (also called the* outer digitization*) contains all r-squares having a non-empty intersection with set S.*

The unions of all cells contained in $J_r^-(S)$ or $J_r^+(S)$ are isothetic polygons $\mathbf{J}_r^-(S)$ or $\mathbf{J}_r^+(S)$, respectively. The Hausdorff-Chebyshev distance[7] between the polygonal boundaries $\partial\mathbf{J}_r^-(S)$ and $\partial\mathbf{J}_r^+(S)$ is greater than or equal to $1/r$ for any non-empty closed set S, and

$$\mathbf{J}_r^-(S) \subset S \subseteq \mathbf{J}_r^+(S)$$

in this case. The Gauss and Jordan digitizations have been used in gridding studies in mathematics (geometry of numbers, number theory, analysis).

The *dilation* of a set $S \subset \mathcal{E}^2$ by a factor $r \geq 1$ is defined to be

$$r \cdot S \ = \ \{(r \cdot x,\ r \cdot y) \ : \ (x, y) \in S\}\ .$$

Following [52], this is a dilation with respect to the origin $(0, 0)$, and other points in $\mathcal{E}^2$ could be chosen to be the fixpoint as well. Sometimes it may be more appropriate to consider sets of the form $r \cdot S$ (the approach preferred, e.g., by Jordan and Minkowski) digitized in the orthogonal grid with unit grid length, instead of sets S digitized in r-grids with grid length $1/r$. The study of $r \to \infty$ corresponds to the increase in grid resolution, and this may be either a study of repeatedly dilated sets $r \cdot S$ in the grid with unit grid length, or of a given set S in repeatedly refined grids. This is a general *duality principle for multigrid studies* [67].

The Gauss, grid-intersection and Jordan digitizations are three basic models for analyzing Euclidean entities in regular grid or cell structures. In multigrid studies there is a choice between refined grid resolutions, i.e. grid constants $1/r$, and $r = 1$ but dilated sets $r \cdot S$, with $r \to \infty$ in both cases.

4.2 MULTIGRID CONVERGENCE

Estimation of the *length of a Jordan curve* is another example where gridding methods have been used, where the aim is minimum error of

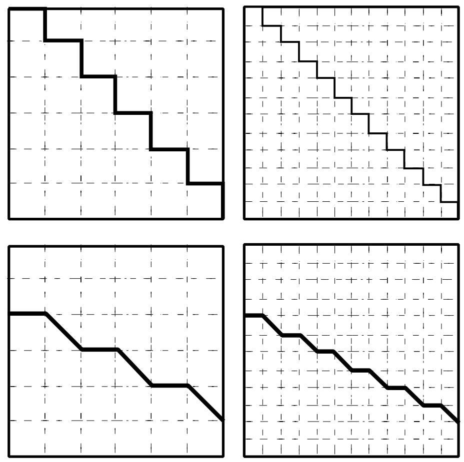

Figure 1.4 Lengths of 4-paths: The length of the staircase, which may be considered to be a digital image of a diagonal line segment, remains constant (twice the side length of the square) no matter what grid resolution is used. Lengths of 8-paths: A digitization of a 22.5° straight line produces the same effect of constant length, where diagonal steps are weighted by $\sqrt{2}$.

the length estimate compared to the true length. The multigrid 'non-convergence to true value' behavior of the lengths of (graph-theoretical!) 4- or 8-paths is illustrated in Figure 1.4.

The problems of measuring the length of a curve or the perimeter of a 2D set, or the area of a closed or open surface of a 3D set based on gridding techniques have an extensive history in digital geometry; see, e.g., [118] for the curve length problem and [53] for the surface area problem. Area estimation in 2D and volume estimation in 3D have been solved with respect to practical needs. Gauss digitization allows accurate estimation by just counting the number of r-grid points in the digitized sets. The area definition in [101] complies with this.

Minkowski [78] proposed solutions to the curve length and surface area problems in pioneering work on morphological operations. However, these proposals (Minkowski addition, continuous reduction of structuring elements, etc.) seem not to be relevant to efficient calculation of these quantitative properties.

A general way to compare results obtained for picture subsets with the true quantities defined by the corresponding operation on the preimage in Euclidean space has been formalized in [113]. Such comparisons are of importance in justifying geometric transformations [49] and operations or property estimations in digital geometry, and their study deserves more attention. The following definition [53] specifies a measure for the speed of convergence toward the true quantity.

Definition 3 *Let $\mathcal{F}$ be a family of sets S, and $dig_r(S)$ a digital image of set S, defined by a digitization mapping dig_r. Assume that a quantitative property P, such as area, perimeter, or a moment, is defined for all sets in family $\mathcal{F}$. An estimator E_P is* multigrid convergent *for this family $\mathcal{F}$ and this digitization model dig_r iff there is a grid resolution $r_S > 0$ for any set $S \in \mathcal{F}$ such that the estimator value $E_P(dig_r(S))$ is defined for any grid resolution $r \geq r_S$, and*

$$|E_P(dig_r(S)) \ - \ P(S)| < f(r)$$

for a function f defined for real numbers, having positive real values only, and converging toward 0 if $r \to \infty$. The function f specifies the convergence speed.

Similar criteria are in common use in numerical mathematics. In general, algorithms or methods can be judged according to a diversity of criteria, such as methodological complexity of underlying theory, expected computational cost of implementation, or run-time behavior and storage requirements of the implemented algorithm. Accuracy is an important criterion as well, and this can be modeled as convergence toward the true value for grid-based calculations. Such concepts support performance evaluation at a theoretical or practical level, as demonstrated (e.g.) in [48] for the definition and calculation of skeletons.

The result of Gauss and Dirichlet on *area estimation* can be described as follows:

Theorem 1 (ca. 1820) *For the family of planar convex sets, the number of r-grid points contained in a set approximates the true area with at least linear convergence speed, i.e. $f(r) = r^{-1}$.*

Today we know that the convergence speed of this estimator is actually at least $r^{-1.3636}$ [46] for planar, bounded, 3-smooth (continuous 3rd derivatives at boundary points, with only a finite number of discontinuities) convex sets, and it cannot be better than $r^{-1.5}$, which is a trivial lower bound.

Theorem 2 (1990) *For the family of planar, bounded, 3-smooth convex sets, the number of r-grid points contained in a set approximates the true area with a convergence speed of $f(r) = r^{\alpha}$, for $-1.5 \leq \alpha < -1.3636$.*

Closing the gap is an open problem which is a famous subject in number theory [73], and is closely related to digital geometry [66].

Figure 1.4 illustrates the fact that set-theoretical convergence in the sense of the Hausdorff-Chebyshev metric is not sufficient for convergence of quantitative properties (the length, in this case). However, the definition of multigrid convergence can also be modified in such a way as to cover set-theoretical convergence with respect to a chosen metric. For example, study of the *convergence* of different proposed digital skeletons *toward a specified medial axis* of a set (in Euclidean space) could be an interesting topic.

In the next two sections we briefly discuss a few aspects of the core problems **(i)** and **(ii)** of digital geometry using the above definitions of digitization and multigrid convergence.

5. APPROXIMATION OF CURVES

Boundaries of digitized planar sets, or digitized curves, can be approximated by sequences of maximal-length digital straight line segments. A *digital straight line* $I_r(\gamma)$ is an 8-curve of r-grid points resulting from the grid-intersection digitization of a the straight line γ in the Euclidean plane, excluding the straight lines $y = x + i/2$, where i is an integer. A digital straight line is *rational* if it can be generated by a γ that has a rational slope. A *digital straight line segment* (DSS) is a finite 8-connected subsequence of a digital straight line.

The boundary of an 8-connected grid point set can be approximated by a polygon, using such a DSS procedure to segment the boundary into a sequence of maximal-length DSSs. The resulting polygon depends on the starting point and the orientation of the scan. Besides this DSS-based approach to the approximation of digital curves by polygons, there are other possible approaches using minimum-length polygons; see [118].

Descriptive tools for boundaries of planar regions do not have to follow this classical approach of polygonal approximation; see, e.g., [75, 122].[8] However, a curve calculated by an approximation algorithm should provide an estimate of the length of the digital curve, or the perimeter of the digital region, which is multigrid convergent to the length of the given Jordan curve prior to digitization.

5.1 A CASE STUDY: DSS CHARACTERIZATION

Any rational digital straight line is periodic, i.e. for its chain code

$$C(\gamma) = (c_i)_{i \in \mathbf{Z}}$$

there exists an $n \in \mathcal{N}$ such that $c_j = c_{j+n}$, for all $j \in \mathbf{Z}$. In the case $r = 1$, for any pair of different grid points $\mathbf{p}, \mathbf{q} \in \mathbf{Z}^2$ there exists an infinite and enumerable set of digital straight lines $F(\gamma)$ containing these two grid points generated by a nondenumerable set of straight lines γ. An initial formulation of necessary conditions for chain codes of digital straight lines is given in [35]:

> "To summarize, we thus have the following three specific properties which all chains of straight lines must possess [33]:
>
> (F1) at most two types of elements can be present, and these can differ only by unity, modulo eight;
>
> (F2) one of the two element values always occurs singly;
>
> (F3) successive occurrences of the element occurring singly are as uniformly spaced as possible."

These properties (actually listed as (1), (2) and (3) in the historic source) were illustrated by examples and based on heuristic insights. The imprecise criterion (F3) is not suitable for a formal proof [82].

The interesting history of DSS characterizations is described in [41]; we follow this unpublished thesis in this subsection. [17] proposed an algorithm for chain code generation of rational digital straight lines based on the periodicity of these code sequences. The generated sequences 'satisfy' criteria (F1), (F2) and (F3). [98] provided a first 'near-exact' characterization of digital straight lines. Let us call an 8-arc *irreducible* iff its set of grid points does not remain 8-connected after removing a point which is not an end point. Then this characterization may be formulated as follows:

Definition 4 *A set $G \subseteq \mathbf{Z}^2$ satisfies the* chord property *iff for any two different points* $\mathbf{p}$ *and* $\mathbf{q}$ *of G, and any point* $\mathbf{r}$ *on the (real) line segment between* $\mathbf{p}$ *and* $\mathbf{q}$*, there exists a (grid) point* $\mathbf{t} \in G$ *such that* $max(|x_{\mathbf{r}} - x_{\mathbf{t}}|, |y_{\mathbf{r}} - y_{\mathbf{t}}|) < 1$.

Theorem 3 (1974) *A finite irreducible 8-arc is a DSS iff it satisfies the chord property.*

There are infinitely many irreducible 8-curves that satisfy the chord property without being digital straight lines. The above theorem was used in [98] to derive the following necessary conditions for the chain code of a DSS:

> (R1) "The runs have at most two directions, differing by 45°, and for one of these directions, the run length must be 1.
>
> (R2) The runs can have only two lengths, which are consecutive integers.
>
> (R3) One of the runs can occur only once at a time.

(R4) ..., for the run length that occurs in runs, these runs can themselves have only two lengths, which are consecutive integers; and so on."

These properties (actually listed as 1), 2), 3) and 4) in the historic source) still do not allow formulation of sufficient conditions for the characterization of a DSS, but they specify (F3) by a recursive argument on run lengths.

For an alternative proof of Theorem 3 see [90]. The property of *evenness* (i.e. "on a DSS the digital slope must be the same everywhere"), as discussed in [44], is equivalent to the chord property.

It was later [7] proved that point sequences generated by the Brons algorithm possess the chord property, and [108] that the formal language of DSS chain codes is context-sensitive. This allowed linear-bounded and cellular automata to be specified for the recognition of DSS chain codes using 'string rewriting rules', but no time complexity discussion was provided for these algorithms.

Hübler et al. [43] defined the criteria (F1–F3) in a precise way following the recursion idea in (R1–R4). To prepare for this definition, we first introduce the following concepts:

Let $S = (s_i)_{i \in I}$ be a finite or infinite sequence of numbers, for an index set $I \subseteq \mathbf{Z}$. A number k is *singular in* S iff it appears in S, and for all $i \in I$, if $s_i = k$ then $s_{i-1} \neq k$ and $s_{i+1} \neq k$, if $i-1$ or $i+1$ is in I. A number k is *nonsingular in* S iff it appears in S and is not singular in S. A sequence S is *reducible* iff it contains no singular number at all, or any subsequence of its nonsingular numbers is of finite length. Assume S to be reducible, and let $R(S)$ be the length of S if S is finite and contains no singular number at all, or the length of the sequence of numbers that results from S by replacing all subsequences of nonsingular numbers in S, which are between two singular numbers in S, by their run lengths, and by deleting all other numbers in S. A recursive application of this *reduction operation* R produces a sequence of number sequences: $S_0 = S$, and $S_{n+1} = R(S_n)$, for all or just a finite sequence of $n \in \mathcal{N}$. We can now give the definition as used in [43]:

Definition 5 *A chain code C of an unbounded 8-curve satisfies the* DSL property *iff $C_0 = C$ is a reducible sequence; $C_{n+1} = R(C_n)$, for $n \in \mathcal{N}$; and any sequence C_n, $n \geq 0$, satisfies the following two conditions:*

(L1) There are at most two different numbers a and b in C_n, and if there are two, then $|a-b| = 1$ (in the case of C_0: modulo 8).

(L2) If there are two different numbers in C_n, at least one of them is singular in C_n.

Following this definition for the case of unbounded chain codes of digital straight lines, it was possible in [43] to derive a definition of a DSS property that allowed the formulation of a necessary and sufficient condition for DSS chain codes. Possible finite sequences of nonsingular numbers at both ends of a sequence require special attention. Let $l(S)$ and $r(S)$ denote the run lengths of nonsingular numbers to the left of the first singular number, or to the right of the last singular number, respectively, for a finite number sequence S.

Definition 6 *A finite chain code C satisfies the* DSS property *iff $C_0 = C$ satisfies conditions (L1) and (L2), and any nonempty sequence $C_n = R(C_{n-1})$, for $n \geq 1$, satisfies (L1) and (L2) and the following two conditions:*

(S1) If C_n contains only one number c, or two different numbers c and $c+1$,then $l(C_{n-1}) \leq c+1$ and $r(C_{n-1}) \leq c+1$.

(S2) If C_n contains two different numbers c and $c+1$, and c is nonsingular in C_n, then if $l(C_{n-1}) = c+1$, C_n starts with c, and if $r(C_{n-1}) = c+1$, C_n ends with c.

A linear-time algorithm for DSS recognition based on this DSS property was described in [43]. A different, but also linear-time DSS algorithm using the same DSS characterization was described in [136]. This paper also contains a proof (with a few small and repairable errors) that this algorithm recognizes just the chain codes of finite, irreducible 8-curves that have the chord property. This concluded the process of specification of Freeman's informal constraints (F1-F3), providing an important set of constraints for the design of efficient DSS recognition procedures.

Theorem 4 (1982) *An unbounded 8-curve is a digital straight line iff its chain code satisfies the DSL property. A finite 8-curve is a digital straight segment iff its chain code satisfies the DSS property.*

Wu's proof of the equivalence of the chord property and the DSS property, for irreducible finite 8-curves, is based on number theory and consists of many case discussions. A much simpler proof based on continued fractions was published (in the same year) in two independent papers [18, 130]. [88] proposed another general way of defining a DSS, not using an algorithmic approach, which allowed a generalization to n-dimensional hyperplanes and introduced *arithmetic geometry*. This will be briefly discussed later on in the context of digital planes.

There are interesting relations between digital straight lines and digital convexity; see, e.g. [20, 60, 94, 103].

By now there are many publications on (efficient) DSS recognition algorithms. The more difficult problem of decomposing an arc into a sequence of DSSs, which includes DSS recognition as a subproblem, is solved in, e.g. [23, 26, 43, 71].

5.2 MULTIGRID CONVERGENCE OF ESTIMATED CURVE LENGTH

The following multigrid convergence theorem addresses the DSS method, and it provides not only an asymptotic $\mathcal{O}(1/r)$ upper bound but even an explicit specification of the asymptotic constant. The value of r_0 depends on the given set, and $\varepsilon_{DSS}(r) \geq 0$ is an algorithm-dependent approximation threshold specifying the maximum Hausdorff-Chebyshev distance (generalizing the Euclidean distance between points to a distance between sets of points) between the r-frontier $\partial\mathbf{G}_r(S)$ and the constructed (not uniquely specified) DSS approximation polygon. Its 'classical' value is the grid constant, as specified in the chord property in [98].

Theorem 5 (2000) *Let S be a convex polygonal set in the Euclidean plane having an 8-connected set $G_r(S)$ for almost all values of r. Then there exists a grid resolution r_0 such that for all $r \geq r_0$, any DSS approximation of the r-frontier $\partial\mathbf{G}_r(S)$ is a connected polygon with perimeter p_r satisfying the inequality*

$$|Perimeter(S) - p_r| \leq \tfrac{2\pi}{r}\left(\varepsilon_{DSS}(r) + \frac{1}{\sqrt{2}}\right) .$$

This theorem and its proof can be found in [67]. The proof is based to a large extent on material given in [68].

The approach in [118] is based on Jordan digitization of sets S in the Euclidean plane. The difference set $\mathbf{O}_r(S) \setminus \mathbf{I}_r(S)$ can be transformed into a subset such that the Hausdorff-Chebyshev distance between its inner and outer boundary is exactly $1/r$, i.e. the grid constant. The perimeter of S can be estimated by the length of a minimum-length polygon (MLP) contained in this subset, and circumscribing the inner boundary of this subset, which is homeomorphic to an annulus. The subset can be described by a sequence of r-squares, where any r-square has exactly two edge-neighboring r-squares in the sequence. Such a sequence is called a *one-dimensional grid continuum* (1D-GC). Such 1D-GCs are treated in the theory of 2D cell complexes in the plane. This specifies an alternative approach (GC-MLP in short) to the approximation of digital curves; it has been experimentally compared with the DSS method in [65].

For the case of GC-MLP approximations there are several convergence theorems in [118], showing that the perimeter of the GC-MLP approximation is a convergent estimator of the perimeter of a bounded, convex, smooth or polygonal set in the Euclidean plane. The following theorem is basically a quotation from [118]; it specifies the asymptotic constant for GC-MLP perimeter estimates.

Theorem 6 (1998) *Let γ be a (closed) convex curve in the Euclidean plane which is contained in a 1D-GC of r-squares, for $r \geq 1$. Then the GC-MLP approximation of this 1D-GC is a connected polygonal curve with length l_r satisfying the inequality*

$$l_r \leq length(\gamma) < l_r + \frac{8}{r} \,.$$

These two theorems are examples of theoretical results about curve length estimators. Assuming $\varepsilon_{DSS}(r) = 1/r$, it follows that the upper error bound for DSS approximations is characterized by[9]

$$\frac{2\pi}{r^2} + \frac{2\pi}{r \cdot \sqrt{2}} \approx \frac{4.5}{r} \quad \text{if} \quad r \gg 1$$

and the upper error bound for GC-MLP approximation is characterized by $8/r$. This ratio of about 1:2 is also the result of the experimental studies reported in [65]. Both theorems are stated for convex sets. However, note that finite unions of convex sets can be used to describe complex sets; see, e.g. [9, 67].

6. APPROXIMATION OF SURFACES

A '3D object' can be modeled by a regular solid, which is defined to be a simply-connected compact set having a well-defined surface area [53]. Algorithms for multigrid-convergent surface area estimation are still a research topic. Obviously, increasing the grid resolution in a digitization of a regular solid in the form of a 3D cell complex [8], and measuring the area of the resulting isothetic surface, does not result in convergence to the true value.

Polyhedrization is a common goal of segmenting the surface of a digitized regular solid, normally given in the form of the boundary points of a 3D grid point set (e.g. using 3D Gauss digitization) or in the form of a two-dimensional grid continuum (2D-GC) defined by the difference between the inner and outer Jordan digitizations. We briefly discuss one possible method of defining a *digital plane segment* (DPS), and add some comments on multigrid-convergent measurement of surface area.

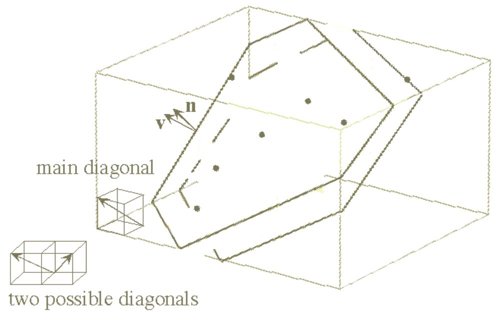

Figure 1.5 Illustration of the main diagonal of a DPS.

6.1 DPS CHARACTERIZATION

We generalize from the definition of a DSS in the plane; however, we now consider 4-paths (e.g. results of digitizations of straight lines using the outer Jordan digitization [118], or boundary sequences of cellular representations of digital sets [71]) instead of the previously discussed 8-paths. A finite 4-path is a DSS in a unit grid iff its main diagonal width is less than $\sqrt{2}$ [3, 71]. In the 3D case we begin by defining a main diagonal. A grid cube has eight directed diagonals. The *main diagonal* of a Euclidean plane is the directed diagonal that has the largest dot product (inner product) with the normal to the plane. Note that in general there may be more than one main diagonal for a Euclidean plane; if so, we can choose any of them as the main diagonal. The distance between two parallel Euclidean planes in the main diagonal direction is called the *main diagonal distance* between the two planes.

Now consider a finite set of faces of r-cubes in 3D space. A Euclidean plane is called a *supporting plane* of this set if it is incident with at least three non-collinear vertices of the set of faces, and all the faces of the set are in only one of the (closed) halfspaces defined by the plane. Note that any non-empty finite set of faces has at least one supporting plane. Any supporting plane defines a *tangential plane*, which is the nearest parallel plane to the supporting plane such that all faces of the given set are within the closed slice defined by the supporting and tangential planes. Note that a tangential plane may be a supporting plane as well. Figure 1.5 gives a rough sketch of such a set of faces, where **n** denotes the normal to the two parallel planes, and **v** is the main diagonal.

Definition 7 *A finite, edge-connected set of faces in 3D space is a* digital planar segment *(DPS) iff it has a supporting plane such that the main diagonal distance between this plane and its corresponding tangential plane is less than* $\sqrt{3}/r$.

Such a supporting plane is called *effective* for the given set of grid faces. Let $\mathbf{v}$ be a vector in a main diagonal direction with a length of $\sqrt{3}/r$, let $\mathbf{n}$ be the normal vector to a pair of parallel planes, and let $d = \mathbf{n} \cdot \mathbf{p}_0$ be the equation of one of these planes. According to our definition of a DPS, all the vertices $\mathbf{p}$ of the faces of a DPS must satisfy the following inequality:

$$0 \leq \mathbf{n} \cdot \mathbf{p} - d < \mathbf{n} \cdot \mathbf{v}$$

Let $\mathbf{n} = (a, b, c)$. Then this inequality becomes

$$0 \leq ax + by + cz - d < |a| + |b| + |c| \ ,$$

i.e. our DPS definition is equivalent to that of a finite, edge-connected subset of faces in a *standard plane* [32]. A *simply-connected DPS* is such that the union of its faces is topologically equivalent (in Euclidean space) to the unit disk.

[88] introduced *arithmetic geometry* which allows characterizations of hyperplanes in n-dimensional spaces. [4, 5] proposed a general definition that linked planes and topology, introducing $|a| + |b| + |c|$ thick planes. For a generalization to n dimensions see [6]. These planes were further specified and used in [32].

DPS recognition is one of the basic problems in digital geometry. It can be stated as follows: Given n vertices $\{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n\}$, does there exist a DPS such that each vertex satisfies the inequality system given above, i.e.

$$0 \leq \mathbf{n} \cdot \mathbf{p}_i - d < \mathbf{n} \cdot \mathbf{v}, \quad i = 1, \ldots, n \ .$$

[128] suggests a method of turning this into a linear inequality system, by eliminating the unknown d as follows:

$$\mathbf{n} \cdot \mathbf{p}_i - \mathbf{n} \cdot \mathbf{p}_j < \mathbf{n} \cdot \mathbf{v}, \quad i, j = 1, \ldots, n \ ,$$

This system of n^2 inequalities can be solved in various ways. [32, 128] use a Fourier elimination algorithm. However, this algorithm is not time-efficient even for very small cell complexes; an incremental algorithm, possibly based on updating lists of effective supporting planes, may be faster.

In fact, in [128] a more advanced elimination technique than Fourier-Motzkin was proposed to eliminate unknowns from systems of inequalities. This technique eliminates all variables at once, whereas the Fourier-Motzkin technique eliminates one variable at a time. Eliminating all variables at once leads to an $O(n^4)$ algorithm for recognizing a DPS, which is faster than the algorithm sketched in [32]. [128] included results for hyperplanes of arbitrary dimension.

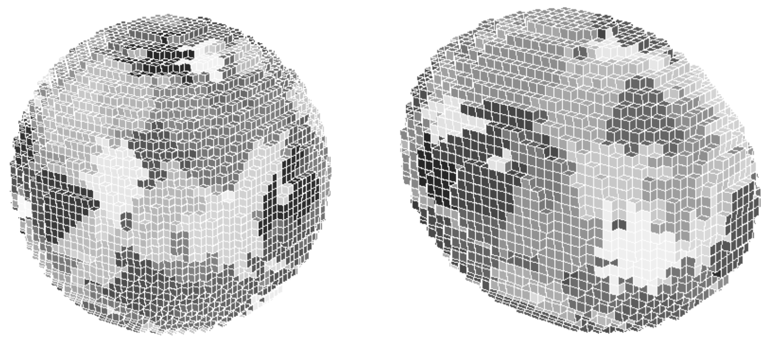

Figure 1.6 Agglomeration of faces of a sphere and an ellipsoid into DPSs.

An evenness property for rectangular plane segments is proven in [128], and the chord property has been extended to hyperplanes of arbitrary dimension in [127]. As in the DSS recognition problem for one-dimensional manifolds, the chord property supports the design of recognition algorithms.

Note that two different definitions are actually used to define digital planes, depending on what kind of connectivity relation is required:

$$0 \leq ax + by + cz - d < |a| + |b| + |c|$$

or

$$0 \leq ax + by + cz - d < max(|a|, |b|, |c|) .$$

The second definition was used in [128], but the results obtained there for the elimination technique are equally valid for the first definition.

6.2 MULTIGRID CONVERGENCE OF ESTIMATED SURFACE AREA

Any DPS algorithm can be used for segmenting a surface of a 3D cell complex into maximal-size DPSs. However, the starting point and the search strategy during the process of 'growing' a DPS are critical for the behavior of such an algorithm. Also, after obtaining maximal-size DPSs, it is not straightforward to derive a polyhedron from the resulting segmentation of the surface.

Analytical surface area calculation of an ellipsoid, with all three semi-axes a, b, c allowed to be different, is a complicated task. If two semi-axes coincide, i.e. in the case of an ellipsoid of revolution, the surface area can be analytically specified in terms of standard functions. The surface area formula in the general case is based on standard elliptic integrals.

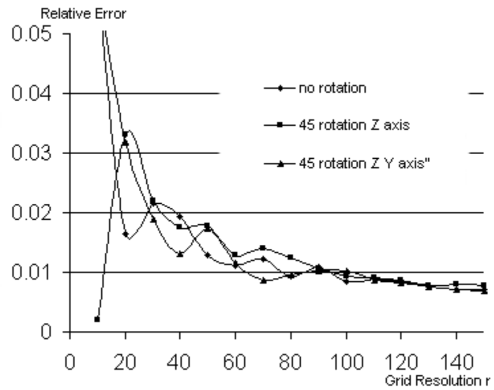

Figure 1.7 Relative errors in surface area estimation for an ellipsoid in three orientations for increasing grid resolution. Figure 1.6 illustrates resolution $r = 40$.

Example 2 in [53], reporting recent work by Tee, specifies an analytical method of computing the surface area of a general ellipsoid. This area can be used in experimental studies as *ground truth* to evaluate the performance of DPS algorithms in surface area estimation. Figure 1.7 shows the error in the estimated value relative the true value for an ellipsoid in three different orientations, using a search depth of 10 in region growing (breadth-first search). In general we find that these DPS-based estimates are 'better' than those based on the convex hull or on marching-cube algorithms. The convex hull and marching-cube methods lead to relative errors of 3.22% and 10.80% for $r = 100$, respectively, while the DPS error is less than 0.8%. The DPS method shows a good tendency to converge, but theoretical work needs to be done to prove this.

On the other hand, there is progress on proving multigrid-convergent behavior of surface area estimation (but without algorithmic implementation so far). [22, 117] introduce the *relative convex hull* $CH_Q(P)$ of a polyhedral solid P which is completely contained in the interior of another polyhedral solid Q. If the convex hull $CH(P)$ is contained in Q then $CH_Q(P) = CH(P)$; otherwise $CH_Q(P) \subseteq Q$ is a 'shrunk version' of the convex hull. To be precise, let $\overline{\mathbf{pq}}$ be the (real) straight line segment from point $\mathbf{p}$ to point $\mathbf{q}$ in $\mathcal{E}^3$, and introduce the following definition[117]:

Definition 8 *A set* $A \subseteq Q \subseteq \mathbf{R}^3$ *is* Q-convex *iff for all* $\mathbf{p}, \mathbf{q} \in A$ *such that* $\overline{\mathbf{pq}} \subseteq Q$ *we have* $\overline{\mathbf{pq}} \subseteq A$*. Let* $P \subseteq Q$*. The* relative convex hull

$CH_Q(P)$ of P with respect to Q is the intersection of all Q-convex sets containing P.

For a set $S \subseteq \mathbf{R}^3$ we defined the inner and outer Jordan digitizations $\mathbf{J}_r^-(S)$ and $\mathbf{J}_r^+(S)$ for grid resolution $r \geq 1$. If S is a regular solid with a defined surface area, let $s(S)$ be its surface area in the Minkowski sense [78].

Theorem 7 (2000) *Let $S \subset \mathbf{R}^3$ be a compact set bounded by a smooth closed Jordan surface ϑS. Then*

$$\lim_{r \to \infty} s\left(CH_{\mathbf{J}_r^+(S)}\left(\mathbf{J}_r^-(S)\right)\right) = s(S) \ .$$

This theorem, from [117], specifies a method of multigrid convergence which requires research on algorithmic implementation, theoretical and experimental convergence speed, and performance evaluation in comparison with other methods such as the DPS segmentation method sketched above.

7. CONCLUSIONS

Only a few concepts from the history of digital geometry could be mentioned here. For example, the core problem of curvature estimation is not discussed at all in this chapter. However, the material presented here demonstrates that there is an increasing interest in deriving *accurate measurements* or *precise shape descriptions* from high-resolution images. Euclidean geometry specifies the ground truth, the correct length or the correct curvature prior to digitization. The concept of multigrid convergence may provide a general methodology for evaluating and comparing different approaches. For example, many skeleton techniques have been suggested; and a convergence analysis (toward a specified medial axis of the Euclidean set) might be a way to compare them.

The measurement of quantitative properties is certainly a main topic in digital geometry. [72] is one of the early publications in this area, and [65] is one of the more recent ones, both focusing on length estimates. Probability-theoretical aspects of digitization errors [129, 133] have only been studied for a few elementary figures and simple geometric problems; further studies should provide answers to open problems such as those listed in [67]. Quantitative properties can be defined at different levels of complexity; see, for example, the degrees of adjacency and surroundedness [104], or distances or similarities between sets of grid points [45], and such properties may deserve more attention.

There are still no solutions to the core problems **(ii)** and **(iii)** with respect to multigrid convergence which combine convergence theorems and algorithmic implementations. Non-polygonal approximations should be studied in connection with core problem **(i)**. For all three problems, it is important to determine what convergence speed $f(r)$ is actually possible. The curve approximation problem **(i)** has been solved only by polygonal approximations that have linear convergence speed $f(r) = c/r$, for different constants c.

Due to the complexity of 2D or 3D shape, digital geometry will continue to be a lively research area related to geometric problems in image analysis.

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This chapter is written for a good friend who has had a major impact on the academic careers of many colleagues working in computer vision today, and the author is happy to be one of these. The author also acknowledges comments and suggestions by (in alphabetical order) Eric Andres, Ulrich Eckhardt, Angel R. Frances, Georgy Gimel'farb, Concettina Guerra, Sergei Matveev, Kensaku Mori, Jean-Pierre Reveillès, Christian Ronse, Azriel Rosenfeld, Radim Sara, Michael Smyth, Garry Tee, Junichiro Toriwaki, Peter Veelaert, and Klaus Voss. The surface DPS approximation and related experiments are a joint project with Hao Jie Sun and will be published in a separate full-length paper.

Notes

1. Hexagonal and trigonal grids are the other two options for regular grids in the plane. Regular grids in dimensions $n \geq 1$ (including toroidal spaces), called *uniform grids*, are studied in [131].

2. The Latin 'digitus' means 'finger' and the actual meaning of the French 'digital' is 'relative to a finger'. Thus 'géométrie digitale' or 'topologie digitale' in specialized papers would refer (for a defender of the French language) to the geometry of fingers, or to geometry (or topology) that you can do with fingers. Of course, French researchers are not opposed to using 'digital geometry' when they write in English, but being used to 'discrete' in their French writings, they may be tempted to use it in English text as well.

3. A mathematical paradox (antinomy) is characterized by a deduction of a contradiction within one theory. Deduction of statements in digital topology which do not resemble statements in Euclidean topology illustrates only a 'syntactic difference'.

4. This representation by local maxima of 8-radii was called a 'skeleton' in [106] and a few subsequent papers. However, the word 'skeleton' was used in [39] for the result of connectivity-preserving thinning, and this became its meaning later on. The 'skeletons' resulting from thinning require additional constraints [13, 27, 123].

5. The distance transformation may be calculated using the Euclidean metric. A sequential algorithm which calculated exactly the minimum Euclidean distance to the border of a planar digital set is given in [109]. The three-dimensional case is dealt with in [110].

6. See our comments earlier on this.

7. Generated by the d_∞ metric, which specifies the chessboard distance on r-grid points.

8. [75] received the award 'Best paper in the journal Pattern Recognition' at IAPR 2000 in Barcelona.

9. Let $f(r) = 2\pi/r^2 + 2\pi/r \cdot \sqrt{2}$. Then it follows that $f(r) \to \pi\sqrt{2}$ as $r \to \infty$. For example, we have $f(20) = 4.76$.

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