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A Wavelet-based Algorithm for Height from Gradients

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Keywords: height from gradients, wavelet transform, connection coefficients, discretization.

1 Introduction

Shading-based 3D shape recovery techniques, e.g. shape from shading (SFS), photometric stereo method (PSM), normally provide gradient values (vector field) for a discrete set of visible points on object surfaces. These gradient values have to be integrated to achieve relative height or depth values. See Fig. 1 for an illustration of such a mapping of a vector field into a depth map. However, no much work was done so far in integration techniques for the gradient vector field.

Essentially there are two main classes of integration techniques for discrete gradient vector field: *local integration techniques* and *global integration techniques* (for a review, see Klette and Schlüns [8]). Suppose that a surface $Z(x,y)$ is defined over a region Ω which is either the real plane R^2 or a bounded subset of this plane, and that the gradient values of this surface at discrete points $(x,y) \in \Omega$

$$p(x,y) = \frac{\partial Z(x,y)}{\partial x} = Z_x, \quad q(x,y) = \frac{\partial Z(x,y)}{\partial y} = Z_y \tag{1}$$

are only available as input data, for instance, in the form of a *needle diagram*(see the left picture in Fig 1). Local integration methods [2], [5], [12] are based on the following curve integrals:

$$Z(x,y) = Z(x_0,y_0) + \int_\gamma p(x,y)dx + q(x,y)dy. \tag{2}$$

where γ is an arbitrarily specified integration path from (x_0,y_0) to $(x,y) \in \Omega$. Starting with initial height values, the methods propagate height values according to a local approximation rule (e.g., based on the 4-neighborhood) using the given gradient data. Such a calculation of relative height values can be repeated by using different scan algorithms. Finally, resulting height values can be determined by averaging operations. Generally, local integration methods are easy to implement and do not explicitly implement any assumption of the integrability condition. However, initial height values have to be provided. The locality of the computations strongly depends on data accuracy, and the propagation of errors may occur due to the propagation of height increments along paths. Therefore, local integration techniques perform badly when the data are noisy.

Integration of discrete gradient vector fields is thought to be an optimization problem in global integration techniques [4], [6], [7]. That is, the problem of finding Z from p and q can be solved by minimizing the following functional (cost function):

$$E = \int\int_\Omega [(Z_x - p)^2 + (Z_y - q)^2]dxdy. \tag{3}$$

Comparing with the local methods, the *Frankot-Chellappa algorithm* [4] is more robust against noise and leads to considerably better results for the task of

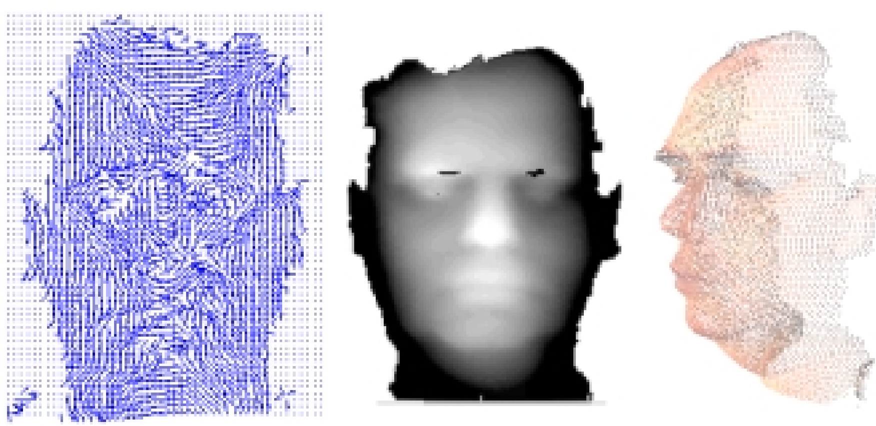

Fig. 1. The left picture shows a needle map representation of surface normals of a human face calculated based on photometric stereo. The middle image shows a depth map obtained from the normals using the global integration method by Frankot-Chellappa. The right image visualizes the recovered 3D shape of the human face.

calculating height from gradients (see Klette at al.[9]). Figure 1 shows a result of the global method. Nevertheless, the height values obtained in the algorithm may suffer from high frequency oscillations and this method needs slightly more computing time.

Wavelets theory has proved to be a powerful tool in various applications such as numerical analysis, pattern recognition, signal and image processing. Using wavelet decomposition it is possible to detect singularities, irregular structure and transient phenomena exhibit by a function. This paper presents a new integration technique for discrete gradient fields. The tensor product of the third-order Daubechies' scaling functions is used to span the solution space. The surface height is described as a linear combination of a set of the scaling basis functions. This method efficiently discretize the cost function associated with the height from gradients problem. After discretization, the height from gradients problem becomes a discrete minimization problem rather than discretized PDE's. To solve the minimization problem, perturbation method is used. The surface height is finally decided after finding the weight coefficients.

The rest of this paper is organized as follows. In the next section, the basic concepts of the wavelet transform and the relevant properties of Daubechies wavelet will be briefly addressed. Then,in Section 3, the proposed wavelet-based algorithm for height from gradients will be described. In this short note the pertinent results are presented only. Finally, a conclusion is given in Section 4.

2 Daubechies wavelet basis and connection coefficients

In this section, we will briefly describe the basic idea of *wavelet transform*. Wavelets are mathematical functions that cut up data into different frequency components, and then study each component with a resolution matched to its scale. They have advantages over traditional Fourier methods in analyzing physical situations where the signal contains discontinuities and sharp spikes. They provide the methods for representing a set complex phenomena in a simpler, more compact, and thus more efficient manner.

Let $\phi(x)$ and $\psi(x)$ are the Daubechies *scaling function* and *wavelet*, respectively. They both are implicitly defined by the following two-scale equation [3]

$$\phi(x) = \sum_{k \in Z} a_k \phi(2x-k), \quad \psi(x) = \sum_{k \in Z} (-1)^k a_{1-k} \phi(2x-k),$$

where a_k are called the Daubechies wavelet filter coefficients. Denote by $L^2(R)$ the space of square integrable functions on the real line. Let V_j be the closure of the function subspace spanned by $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k),\ j,k \in Z$, and suppose that W_j, the orthogonal complementary of V_j in V_{j+1}, be the closure of the function subspace generated by $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k),\ j,k \in Z$. Then the function subspaces V_j and W_j have the following properties: $V_j \subseteq V_{j+1}$, for all $j \in Z$;

$$\bigcap_{j \in Z} V_j = \{0\}; \quad \bigcup_{j \in Z} V_j = L^2(R); \quad V_{j+1} = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_j,$$

where $\oplus$ denotes the orthogonal direct sum. On each fixed scale j, the scaling functions $\{\phi_{j,k}(x), k \in Z\}$ form an orthonormal basis of V_j and the wavelets $\{\psi_{j,k}(x), k \in Z\}$ form an orthonormal basis of W_j. The set of subspaces V_j is called a *multiresolution analysis* of $L^2(R)$.

Let J be a positive integer. A function $f(x) \in V_J$ can be represented by the wavelet series

$$f(x) = \sum_{k \in Z} c_{J,k} \phi_{J,k}(x),$$

where the expansion coefficients $c_{J,k}$ are specified by $c_{J,k} = \int f(x)\phi_{J,k}(x)dx$. Since $V_J = V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_{J-1}$, $f(x)$ can be alternatively represented by

$$f(x) = \sum_{k \in Z} c_{0,k} \phi_{0,k}(x) + \sum_{j=0}^{J-1} \sum_{k \in Z} d_{j,k} \psi_{j,k}(x).$$

The wavelet series expansion coefficients $c_{0,k}$ and $d_{j,k}$ can be computed via the decomposition algorithm [3].

The *connection coefficients* [1], [10] play an important role in representing the relation between the scaling function and differential operators. Throughout this paper, we assume that the scaling function $\phi(x)$ has N *vanishing moments*. For $k \in Z$, we define that

$$\begin{aligned} \Gamma_k^0 &= \int \phi(x)\phi(x-k)dx, \\ \Gamma_k^1 &= \int \phi^{(x)}(x)\phi(x-k)dx, \\ \Gamma_k^2 &= \int \phi^{(x)}(x)\phi^{(x)}(x-k)dx. \end{aligned}$$

Then we have the following properties: $\Gamma_0^1 = 0$; for the scaling function $\phi(x)$ which has N vanishing moments, $\Gamma_k^1 = \Gamma_k^2 = 0, k \notin [-2N+2, 2N-2]$; and

$$\Gamma_k^0 = \begin{cases} 1, & k = 0, \\ 0, & \text{otherwise.} \end{cases}$$

The connection coefficients for Daubechies' wavelet with $N = 3$ vanishing moments are shown in the following table [11]:

3 Wavelet-based height from gradients

In this section, we will derive a new wavelet-based algorithm for solving the height from gradients. First of all, we assume that the size of the domain of the surface $Z(x, y)$ is $M \times M$, and the surface $Z(x, y)$ is represented by a linear combination of a set of the third-order Daubechies scaling basis functions in the following format:

$$Z(x, y) = \sum_{m=0}^{M-1} \sum_{m=0}^{M-1} z_{m,n} \phi_{m,n}(x, y), \tag{4}$$

Γ_k^1	Γ_k^2
Γ_{-4}^1 =0.00034246575342	Γ_{-4}^2 = -0.00535714285714
Γ_{-3}^1 =0.01461187214612	Γ_{-3}^2 = -0.11428571428571
Γ_{-2}^1 =-0.14520547945206	Γ_{-2}^2 = 0.87619047619052
Γ_{-1}^1 =0.74520547945206	Γ_{-1}^2 = -3.39047619047638
Γ_{0}^1 =0.0	Γ_{0}^2 = 5.26785714285743
Γ_{1}^1 =-0.74520547945206	Γ_{1}^2 = -3.39047619047638
Γ_{2}^1 =0.14520547945206	Γ_{2}^2 = 0.87619047619052
Γ_{3}^1 =-0.01461187214612	Γ_{3}^2 = -0.11428571428571
Γ_{4}^1 =-0.00034246575342	Γ_{4}^2 = -0.00535714285714

Table 1. Connection Coefficients with $N = 3$

where $z_{m,n}$ are the weight coefficients, $\phi_{m,n}(x,y)$ are the tensor product of the third-order Daubechies scaling functions, i.e., $\phi_{m,n}(x,y) = \phi(x-m)\phi(y-n)$. For the known gradient values $p(x,y)$ and $q(x,y)$, we assume that

$$p(x,y) = \sum_{m=0}^{M-1}\sum_{m=0}^{M-1} p_{m,n}\phi_{m,n}(x,y), \tag{5}$$

$$q(x,y) = \sum_{m=0}^{M-1}\sum_{m=0}^{M-1} q_{m,n}\phi_{m,n}(x,y), \tag{6}$$

where the weight coefficients $p_{m,n}$ and $q_{m,n}$ can be determined by

$$p_{m,n} = \int\int p(x,y)\phi_{m,n}(x,y)dxdy, \quad q_{m,n} = \int\int q(x,y)\phi_{m,n}(x,y)dxdy.$$

Substituting (4), (5) and (6) into (3), we have

$$\begin{aligned} E = \int\int \Bigg[& \left(\sum_{m,n=0}^{M-1} z_{m,n}\phi_{m,n}^{(x)}(x,y) - \sum_{m,n=0}^{M-1} p_{m,n}\phi_{m,n}(x,y) \right)^2 \\ + & \left(\sum_{m,n=0}^{M-1} z_{m,n}\phi_{m,n}^{(y)}(x,y) - \sum_{m,n=0}^{M-1} q_{m,n}\phi_{m,n}(x,y) \right)^2 \Bigg] dxdy \\ = & \ E_1 \ + E_2, \end{aligned} \tag{7}$$

where $\phi_{m,n}^{(x)}(x,y) = \partial\phi_{m,n}(x,y)/\partial x$ and $\phi_{m,n}^{(y)}(x,y) = \partial\phi_{m,n}(x,y)/\partial y$.

In order to derive the iterative scheme for Z, let $\Delta z_{i,j}$ represent the updating amounts of $z_{i,j}$ in the iterative equation, $z'_{i,j}$ be the value after update. Then $z'_{i,j} = z_{i,j} + \Delta z_{i,j}$. Substituting $z'_{i,j}$ into E_1, E_1 will be changed by an amount ΔE_1, that is,

$$E'_1 = E_1 + \Delta E_1$$

$$
\begin{aligned}
&= \int\int \left[\left(\sum_{m,n=0}^{M-1} z_{m,n} \phi_{m,n}^{(x)}(x,y) - \sum_{m,n=0}^{M-1} p_{m,n} \phi_{m,n}(x,y) \right) \right. \\
&\quad \left. + \Delta z_{i,j} \phi_{i,j}^{(x)}(x,y) \right]^2 dxdy \\
&= E_1 + 2\Delta z_{i,j} \sum_{m,n=0}^{M-1} z_{m,n} \Gamma_{i-m}^2 \Gamma_{j-n}^0 \\
&\quad - 2\Delta z_{i,j} \sum_{m,n=0}^{M-1} p_{m,n} \Gamma_{i-m}^1 \Gamma_{j-n}^0 + \Delta z_{i,j}^2 \Gamma_0^2 .
\end{aligned} \tag{8}
$$

Using the same derivation, we have

$$
\begin{aligned}
E_2' &= E_2 + \Delta E_2 \\
&= E_2 + 2\Delta z_{i,j} \sum_{m,n=0}^{M-1} z_{m,n} \Gamma_{i-m}^0 \Gamma_{j-n}^2 \\
&\quad - 2\Delta z_{i,j} \sum_{m,n=0}^{M-1} q_{m,n} \Gamma_{i-m}^0 \Gamma_{j-n}^1 + \Delta z_{i,j}^2 \Gamma_0^2 .
\end{aligned} \tag{9}
$$

Substituting (8) and (9) into (7), it is shown that

$$
\begin{aligned}
\Delta E &= \Delta E_1 + \Delta E_2 \\
&= 2\Delta z_{i,j} \sum_{m,n=0}^{M-1} z_{m,n} \left(\Gamma_{i-m}^2 \Gamma_{j-n}^0 + \Gamma_{i-m}^0 \Gamma_{j-n}^2 \right) \\
&\quad - 2\Delta z_{i,j} \sum_{m,n=0}^{M-1} p_{m,n} \Gamma_{i-m}^1 \Gamma_{j-n}^0 \\
&\quad - 2\Delta z_{i,j} \sum_{m,n=0}^{M-1} q_{m,n} \Gamma_{i-m}^0 \Gamma_{j-n}^1 + 2\Delta z_{i,j}^2 \Gamma_0^2 .
\end{aligned}
$$

In order to make the cost function decrease as fast as possible, ΔE must be maximized. From $\partial \Delta E / \partial \Delta z_{i,j} = 0$, we have

$$
\Delta z_{i,j} = \frac{1}{2\Gamma_0^2} \sum_{k=-2N+2}^{2N-2} \left[\left(p_{i-k,j} + q_{i,j-k} \right) \Gamma_k^1 - \left(z_{i-k,j} + z_{i,j-k} \right) \Gamma_k^2 \right]
$$

¿From the above results, the iterative equation can be represented as follows:

$$
z_{i,j}^{[t+1]} = z_{i,j}^{[t]} + \Delta z_{i,j}, \qquad t = 0, 1, \dots \tag{10}
$$

The initial values are zero.

4 Conclusions

In this paper, we presented a new iterative algorithm for solving the height from gradients problem. Wavelet transform is a generalization of Fourier transform, and a power tool for efficiently representing images. Therefore, the proposed method takes the advantages of wavelet transform. By applying the wavelet transform, the objective function associated with the original height from gradients problem is converted into the wavelet-based format. In the new iterative algorithm, the step size can be easily determined by maximizing the decrease of the objective function. we only presented the pertinent results in this short note. In the future the new algorithm should be studied in combining and comparing it with existing height from gradients techniques.

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