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Analysis of Finite Difference Algorithms for Linear Shape from Shading

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Abstract

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Abstract. This paper presents and analyzes four explicit, two implicit and four semi-implicit finite difference algorithms for the linear shape from shading problem. Comparisons of accuracy, solvability, stability and convergence of these schemes indicate that the weighted semi-implicit scheme and the box scheme are better than the other ones because they can be calculated more easily, they are more accurate, faster in convergence and unconditionally stable.

Keywords: shape from shading, PDEs, finite difference schemes, stability, convergence

1 Introduction

The basic problem in shape from shading is to recover surface values $Z(x, y)$ of an object surface from its variation in brightness. The surface function Z is assumed to be defined in image coordinates (x, y). It is identical to the *depth* of surface points visualized in an image, i.e., to the Euclidean distance between image plane and surface point.

We assume parallel illumination of a Lambertian surface. The illumination is characterized by an orientation $(p_s, q_s, -1)$ and its intensity E_0. For the surface we assume that it may be modelled by a function $Z(x, y)$, and $\rho(x, y)$, with $0 \leq \rho(x, y) \leq 1$, denotes the *albedo* (i.e. the reflectance constant) at point (x, y). The surface function $Z(x, y)$ satisfies the following *image irradiance equation*

$$\frac{1 + p_s p + q_s q}{\sqrt{1 + p_s^2 + q_s^2}\sqrt{1 + p^2 + q^2}} = \rho(x, y) \cdot E_0 \cdot E(x, y) \tag{1}$$

over a compact image domain Ω, where $(p, q) = (p(x, y), q(x, y))$ is the surface gradient with $p = \partial Z/\partial x$ and $q = \partial Z/\partial y$ at point $(x, y) \in \Omega$, and $E(x, y)$ is the image brightness at this point formed by an orthographic (parallel) projection of reflected light onto the xy-image plane. Throughout this paper we assume that $E_0 \cdot \rho(x, y) = 1$, for all points $(x, y) \in \Omega$, i.e. we assume a constant albedo for all projected surface points. This approach is called *albedo-dependent shape recovery*, see Klette et al. [5]. The above nonlinear, first-order partial differential equation has been studied with a variety of different techniques (see, e.g., Horn

[1,3]; Horn and Brooks [2]; Tsai and Shah [13]; Lee and Kuo [9]; Kimmel and Bruckstein [4]). The traditional approaches employ regularization techniques. However, Oliensis [10] discovered that in general, shape from shading must be assumed to be ill-posed, and regularization techniques should be used with caution. Furthermore, Zhang et al. [15] pointed out that all shape from shading algorithms produce generally poor results. Therefore, new shape from shading methods should be developed to provide more accurate, and realistic results. Pentland [11] proposed a method based on the linearity of the reflectance map in the surface gradient (p, q), which greatly simplifies the shape from shading problem. This leads to the following *linear image irradiance equation:*

$$\frac{1 + p_s p + q_s q}{\sqrt{1 + p_s^2 + q_s^2}} = E(x, y) \ . \tag{2}$$

As an example, such a special case arises e.g. in recovering the shape of parts of the lunar surface ("Maria of the moon"). Defining

$$F(x, y) = E(x, y)\sqrt{1 + p_s^2 + q_s^2} - 1 \ ,$$

we can rewrite (2) as

$$p_s p + q_s q = F(x, y) \ ,$$

that is

$$p_s \frac{\partial Z}{\partial x}(x, y) + q_s \frac{\partial Z}{\partial y}(x, y) = F(x, y) \ . \tag{3}$$

Horn [1] first proposed a method for recovery of shapes described by (3). For the sufficient conditions assuring the well-posedness of the problem (3) we refer the reader to Kozera [6]. Kozera and Klette [7,8] presented four algorithms based on explicit finite difference methods. Ulich [14] also discussed two explicit and one implicit finite difference algorithm for (3). So far it has not yet been studied which finite difference algorithms for (3) are better in practical use. The method used for the proof of stability and convergence are relatively complicated.

In this paper, we consider (3) over a rectangle domain

$$\Omega = \{(x, y) \in \mathbb{R}^2 : \quad 0 \leq x \leq a, \quad 0 \leq y \leq b\}$$

with the following initial condition

$$Z(x, 0) = \phi(x), 0 \leq x \leq a \tag{4}$$

and boundary conditions

$$Z(0, y) = \psi_0(y), \quad 0 \leq y \leq b \tag{5}$$

$$Z(a, y) = \psi_1(y), \quad 0 \leq y \leq b \tag{6}$$

where the given functions $\phi(x)$, $\psi_0(y)$ and $\psi_1(y)$ satisfy

$$\phi \in C([0, a]) \cap C^2((0, a)), \quad \psi_0, \psi_1 \in C([0, b]) \cap C^2((0, b))$$

and $\phi(0) = \psi_0(0), \phi(a) = \psi_1(a)$, and $(p_s, q_s) \neq (0,0)$. Throughout this paper we assume that the Cauchy's problem (3)–(6) is *well-posed* over a rectangle Ω, that is, there exists a unique solution $Z(x,y)$ to the corresponding partial differential equation satisfying the boundary conditions ((5) and/or (6)) and depending continuously on the given initial condition (4), and we also suppose that the solution $Z(x,y)$ is sufficiently smooth, at least $Z(x,y) \in C^2(\bar{\Omega})$, see Kozera [6].

The organization of the the rest of the paper is as follows. In Section 2 we present ten different discretizations of equation (3): four explicit, two implicit and four semi-implicit schemes. The initial condition $Z(x,0)$ is used for all these methods, but different boundary conditions $Z(0,y)$ and/or $Z(a,y)$ are required. In Section 3 we discuss the accuracy, solvability, consistency, stability and convergence of these methods. The conclusions are given in Section 4.

2 Finite Difference Algorithms

Suppose that the rectangular domain Ω is divided into small grids by parallel lines $x = x_i$ $(i = 0,1,\ldots,M)$ and $y = y_j$ $(j = 0,1,\ldots,N)$, where $x_i = ih$, $y_j = jk$ and $Mh = a$, $Nk = b$, M and N are integers, h is the grid constant in x-direction (i.e. distance between neighboring grid lines) and k is the grid constant in y-direction. For convenience, we shall denote by $Z(i,j)$ the value $Z(x_i, y_j)$ of solution $Z(x,y)$ on the grid point (x_i, y_j).

2.1 Explicit Schemes

Forward-Forward (FF) Scheme: Approximating $\partial Z/\partial x$ and $\partial Z/\partial y$ with the forward difference quotient gives the following discretization for (3):

$$p_s \frac{Z(i+1,j) - Z(i,j)}{h} + q_s \frac{Z(i,j+1) - Z(i,j)}{k} + O(h+k) = F(i,j) \ ,$$

where $O(h+k) = -hZ_{xx}(x_i, \theta_1)/2 + kZ_{yy}(x_j, \theta_2)/2$, $x_i \leq \theta_1 \leq x_{i+1}$, $y_{j-1} \leq \theta_2 \leq y_j$. Denoting by $Z_{i,j}$ an approximation of $Z(i,j)$ and then dropping the truncation error $O(h+k)$ gives

$$p_s \frac{Z_{i+1,j} - Z_{i,j}}{h} + q_s \frac{Z_{i,j+1} - Z_{i,j}}{k} = F_{i,j}.$$

A rearrangement of this equation then yields the two-level explicit scheme

$$Z_{i,j+1} = (1+c)Z_{i,j} - cZ_{i+1,j} + \frac{k}{q_s} F_{i,j}, \qquad (7)$$
$$i = 0,1,\ldots,M-1; j = 0,1,\ldots,N-1,$$

where the corresponding finite difference initial conditions $Z_{i,0} (i = 0,1\ldots,M)$ and boundary conditions $Z_{M,j} (j = 0,1,\ldots,N)$ are given, $c = \frac{p_s k}{q_s h}$, $q_s \neq 0$.
Remark 1: The FF scheme is classified as *explicit* because the value of $Z_{i,j+1}$ at the $(j+1)$th level is calculated directly from known values of $Z_{i,j}$ and $Z_{i+1,j}$ at

the previous jth level. It is a two-level scheme because values of Z at only two levels of j are involved in the scheme.
Remark 2: As mentioned above, the truncation error of the FF scheme is in the order of $O(h+k)$.
Remark 3: Given a linear shape from shading problem (3), the FF scheme with the above boundary condition recovers the unknown shape over a domain of influence which coincides with the entire Ω, that is,

$$D_{FF_M} = \Omega.$$

But if we only give the following boundary conditions $Z_{0,j} (j = 0, 1, \ldots, N)$, then the domain of influence of the FF scheme is as follows

$$D_{FF_0} = \{(x, y) \in \mathbb{R}^2 : \ 0 \le x \le a, \quad 0 \le y \le (-b/a)(x - a)\}.$$

The same scheme with different boundary conditions coincides with different domains of influence. Therefore, boundary conditions are very important to the finite difference algorithms for the linear shape from shading (3).

Backward-Forward (BF) Scheme: This is a modification of the FF finite difference scheme, in which the forward difference approximation for $\partial Z/\partial x$ is replaced by the backward different quotient. If we use the above techniques, we can get the following two-level explicit scheme

$$\begin{aligned} Z_{i,j+1} = cZ_{i-1,j} + (1-c)Z_{i,j} + \frac{k}{q_s} F_{i,j}, \\ i = 1, \ldots, M; j = 0, \ldots, N-1, \end{aligned} \tag{8}$$

where initial conditions $Z_{i,0} (i = 0, 1, \ldots, M)$ and boundary conditions $Z_{0,j} (j = 0, 1, \ldots, N)$ are given, $c = \frac{p_s k}{q_s h}$, $q_s \neq 0$.
Remark 1: The truncation error of the BF scheme is $O(h+k)$.
Remark 2: The domain of influence of the BF scheme with the above boundary condition, D_{BF_0}, is entire Ω. But if the corresponding finite difference boundary conditions are given by $Z_{M,j} (j = 0, 1, \ldots, N)$, then the domain of influence is

$$D_{BF_M} = \{(x, y) \in \mathbb{R}^2 : \ 0 \le x \le a, \quad 0 \le y \le (b/a)x\}.$$

Lax-Friedrichs (LF) Scheme: Another explicit scheme for solving the linear shape from shading problem is the Lax-Friedrichs scheme. Approximating $\partial Z/\partial x$ with the central difference quotient and $\partial Z/\partial y$ with the forward difference approximation, and then replacing $Z_{i,j}$ by its average at the $(i+1)$th and $(i-1)$th levels.

$$p_s \frac{Z_{i+1,j} - Z_{i-1,j}}{2h} + q_s \frac{Z_{i,j+1} - \frac{1}{2}(Z_{i+1,j} + Z_{i-1,j})}{k} = F_{i,j},$$

or, equivalently

$$Z_{i,j+1} = \frac{1-c}{2} Z_{i+1,j} + \frac{1+c}{2} Z_{i-1,j} + \frac{k}{q_s} F_{i,j}, \tag{9}$$
$$i = 1, \ldots, M-1; j = 0, 1, \ldots, N-1,$$

where initial conditions $Z_{i,0}(i = 0, 1, \ldots, M)$, boundary conditions $Z_{0,j}$ and $Z_{M,j}(j = 0, 1, \ldots, N)$ are given, $c = \frac{p_s k}{q_s h}$, $q_s \neq 0$. It holds that the truncation error of the LF scheme is $O\left(h^2 + k + h^2/k\right)$, and the domain of influence of the LF scheme, D_{LF}, is entire Ω.

Leapfrog Scheme: Approximating both $\partial Z/\partial x$ and $\partial Z/\partial y$ with the central difference quotient yields

$$p_s \frac{Z_{i+1,j} - Z_{i-1,j}}{2h} + q_s \frac{Z_{i,j+1} - Z_{i,j-1}}{2k} = F_{i,j}.$$

This leads to the following three-level explicit scheme

$$Z_{i,j+1} = Z_{i,j-1} + c(Z_{i-1,j} - Z_{i+1,j}) + \frac{2k}{q_s} F_{i,j}, \tag{10}$$
$$i = 1, \ldots, M-1; j = 1, \ldots, N-1,$$

where $Z_{i,0}(i = 0, \ldots, M)$, $Z_{0,j}$ and $Z_{M,j}(j = 0, \ldots, N)$ are given, $c = \frac{p_s k}{q_s h}$, $q_s \neq 0$.
Remark 1: The truncation error of the leapfrog scheme is $O(h + k)$, and the domain of influence of the leapfrog scheme, D_{Leap}, is entire Ω.
Remark 2: The leapfrog scheme is a multistep scheme because $Z_{i,j+1}$ at the $(j+1)$th level is calculated using values at the two previous levels: $Z_{i-1,j}$ and $Z_{i+1,j}$ at the jth level and $Z_{i,j-1}$ at the $(j-1)$th level. To start the computations of the leapfrog scheme we must specify the values of $Z_{i,0}$ and $Z_{i,1}$ for all i, usually $Z_{i,1}$ can be calculated by another scheme, e.g., the FF scheme, the BF scheme or others.

2.2 Implicit Schemes

Each of the methods described previously is explicit. At the new $(j+1)$th level, the finite difference scheme contains only the one unknown value $Z_{i,j+1}$, which is calculated explicitly from values of Z known at previous jth levels. These algorithms are easy to program and require few computations to determine the values of Z at each new $(j+1)$th level. Unfortunately, the accuracy of the explicit schemes is usually low since the truncation errors are only $O(h+k)$, and we will see that the stability of the explicit schemes require much smaller step size.

Central-Backward (CB) Scheme: As before, approximating $\partial Z/\partial x$ with the central difference quotient and $\partial Z/\partial y$ with the backward difference quotient, we get

$$p_s \frac{Z_{i+1,j+1} - Z_{i-1,j+1}}{2h} + q_s \frac{Z_{i,j+1} - Z_{i,j}}{k} = F_{i,j}.$$

This equation can be rearranged to yield the following two-level implicit scheme.

$$-\frac{c}{2}Z_{i-1,j+1}+Z_{i,j+1}+\frac{c}{2}Z_{i+1,j+1}=Z_{i,j}+\frac{k}{q_s}F_{i,j}, \qquad (11)$$
$$i=1,\ldots,M-1;\, j=0,\ldots,N-1,$$

where the initial conditions $Z_{i,0}(i=0,\ldots,M)$, the boundary conditions $Z_{0,j}$ and $Z_{M,j}(j=0,\ldots,N)$ are given, $c=\frac{p_s k}{q_s h}$, $q_s \neq 0$.

Remark 1: The truncation error of the CB scheme is $O\left(h^2+k\right)$, the domain of influence of the scheme, D_{CB}, is entire Ω.

Remark 2: The CB scheme is implicit, since there are three unknown values of Z at the same $(j+1)$th level. Assuming values are known at the jth level we may substitute $i=1$ to $M-1$ in (11) to obtain the following set of tridiagonal linear algebraic equations with unknowns $Z_{i,j+1}, i=1,\ldots,M-1$:

$$\begin{bmatrix} 1 & \frac{c}{2} & & & \\ -\frac{c}{2} & 1 & \frac{c}{2} & & \\ \ddots & \ddots & \ddots & & \\ & -\frac{c}{2} & 1 & \frac{c}{2} \\ & & -\frac{c}{2} & 1 \end{bmatrix} \begin{bmatrix} Z_{1,j+1} \\ Z_{2,j+1} \\ \vdots \\ Z_{M-2,j+1} \\ Z_{M-1,j+1} \end{bmatrix} = \begin{bmatrix} Z_{1,j} \\ Z_{2,j} \\ \vdots \\ Z_{M-2,j} \\ Z_{M-1,j} \end{bmatrix} + \qquad (12)$$
$$\frac{k}{q_s}\begin{bmatrix} F_{1,j} \\ F_{2,j} \\ \vdots \\ F_{M-2,j} \\ F_{M-1,j} \end{bmatrix} + \frac{c}{2}\begin{bmatrix} Z_{0,j+1} \\ 0 \\ \vdots \\ 0 \\ -Z_{M,j+1} \end{bmatrix}, \; j=0,1,\ldots,N-1.$$

The computations of the CB scheme will take much more time because it requires to solve a linear algebraic systems at each j level.

Crank-Nicolson (CN) Scheme: Another implicit finite difference algorithm used to solve the linear shape from shading problem is the CN scheme:

$$p_s\frac{1}{2}\left[\frac{Z_{i+1,j}-Z_{i-1,j}}{2h}+\frac{Z_{i+1,j+1}-Z_{i-1,j+1}}{2h}\right]+q_s\frac{Z_{i,j+1}-Z_{i,j}}{k}=F_{i,j}.$$

Rearrangement of this equation then yields a two-level CN scheme

$$-\frac{c}{4}Z_{i-1,j+1}+Z_{i,j+1}+\frac{c}{4}Z_{i+1,j+1}=\frac{c}{4}Z_{i-1,j}+Z_{i,j}-\frac{c}{4}Z_{i+1,j}+\frac{k}{q_s}F_{i,j}\;, \qquad (13)$$
$$i=1,\ldots,M-1;\, j=0,1,\ldots,N-1,$$

where $Z_{i,0}(i=0,1,\ldots,M), Z_{0,j}$ and $Z_{M,j}(j=0,\ldots,N)$ are given, $c=\frac{p_s k}{q_s h}$, $q_s \neq 0$.

Remark 1: The truncation error of the CN scheme is $O\left(h^2+k^2\right)$. The domain of influence of the scheme, D_{CN}, is entire Ω.

Remark 2: At each j level, the CN scheme also requires to solve a tridiagonal systems of linear equations with the following coefficients matrix:

$$\begin{bmatrix} 1 & \frac{c}{4} & & & \\ -\frac{c}{4} & 1 & \frac{c}{4} & & \\ \ddots & \ddots & \ddots & & \\ & -\frac{c}{4} & 1 & \frac{c}{4} & \\ & & -\frac{c}{4} & 1 & \end{bmatrix}$$

2.3 Semi-implicit Schemes

An explicit finite difference scheme for (3) contains only one unknown value of Z at each j level. The unknown value is calculated directly from the known values of Z at the previous levels. Therefore, explicit schemes are easy to be computed. The disadvantage of explicit schemes is that their accuracy is lower since the order of their truncation errors is usually lower. For an implicit scheme, there are three unknown values of Z at each j level. Implicit schemes are more accurate than explicit schemes since the order of the truncation errors of implicit schemes is higher than that of explicit schemes. However, the computation of implicit schemes takes much more time than that of explicit schemes because implicit schemes require to solve a linear algebraic systems for each j. In order to overcome the drawbacks and take the advantages of explicit and implicit schemes, we consider the following semi-implicit schemes which contain two unknown values of Z at each j level.

Forward-Backward (FB) Scheme: Approximating $\partial Z/\partial x$ with the forward difference quotient and $\partial Z/\partial y$ with the backward difference quotient gives

$$p_s \frac{Z_{i+1,j} - Z_{i,j}}{h} + q_s \frac{Z_{i,j} - Z_{i,j-1}}{k} = F_{i,j},$$

or, equivalently

$$Z_{i+1,j} = (1-d)Z_{i,j} + dZ_{i,j-1} + \frac{h}{p_s} F_{i,j}, \qquad (14)$$
$$i = 0, 1, \ldots, M-1; j = 1, 2, \ldots, N,$$

where $Z_{i,0} (i = 0, \ldots, M)$ and $Z_{0,j} (j = 0, \ldots, N)$ are given, $p_s \neq 0, d = 1/c$.
Remark 1: The truncation error of the FB scheme is $O(h + k)$. The domain of influence of the scheme, D_{FB_0}, is entire Ω.
Remark 2: The FB scheme involves two unknown values of Z at the jth level, but it can be calculated in the order $i = 1$ to $M - 1$, since $Z_{i,j}$ is known when $Z_{i+1,j}$ is computed.
Remark 3: Semi-implicit schemes have the advantages of both explicit and implicit schemes, such as computationally simple, very stable, not need in solving of a set of linear algebraic equations, etc.

Backward-Backward (BB) Scheme: Approximating both $\partial Z/\partial x$ and $\partial Z/\partial y$ with the backward difference scheme yields

$$Z_{i,j} = \frac{1}{1+c} Z_{i,j-1} + \frac{c}{1+c} Z_{i-1,j} + \frac{k}{q_s(1+c)} F_{i,j}, \quad (15)$$
$$i = 1, \ldots, M; j = 1, \ldots, N,$$

where $Z_{i,0}(i = 0, 1, \ldots, M)$ and $Z_{0,j}(j = 0, 1, \ldots, N)$ are given, $c = \frac{p_s k}{q_s h}$, $c \neq -1$, $q_s \neq 0$. The truncation error of the BB scheme is $O(h + k)$. The domain of influence of the BB scheme, D_{BB_0}, is entire Ω.

Weighted Semi-implicit (WS) Scheme: The Weighted semi-implicit scheme for (1) is:

$$p_s \frac{1}{2} \left[\frac{Z_{i,j+1} - Z_{i-1,j+1}}{h} + \frac{Z_{i+1,j} - Z_{i,j}}{h} \right] + q_s \frac{Z_{i,j+1} - Z_{i,j}}{k} = F_{i,j}.$$

Rearranging gives the two-level semi-implicit scheme

$$Z_{i,j+1} = Z_{i,j} + \frac{c}{2+c}(Z_{i-1,j+1} - Z_{i+1,j}) + \frac{k}{q_s(2+c)} F_{i,j}, \quad (16)$$
$$i = 1, \ldots, M-1; j = 0, 1, \ldots, N-1,$$

where $Z_{i,0}(i = 0, 1, \ldots, M), Z_{0,j}$ and $Z_{M,j}(j = 0, \ldots, N)$ are given, $c = \frac{p_s k}{q_s h}$, $c \neq -2$, $q_s \neq 0$. The truncation error of the WS scheme is $O\left(h + k^2\right)$. The domain of influence of the WS scheme, D_{WS}, is entire Ω.

The WS scheme also involves two unknown values of Z at the same $(j+1)$th level. However, commencing with a given boundary value $Z_{0,j}$ and $Z_{M,j}$, the values of $Z_{i,j+1}$ can be computed in the order $i = 1$ to $M-1$, since $Z_{i-1,j+1}$ is known at each application of (16).

Box Scheme: The box scheme for solving (3) is as follows:

$$\frac{p_s}{2} \left[\frac{Z_{i+1,j+1} - Z_{i,j+1}}{h} + \frac{Z_{i+1,j} - Z_{i,j}}{h} \right] +$$
$$\frac{q_s}{2} \left[\frac{Z_{i,j+1} - Z_{i,j}}{k} + \frac{Z_{i+1,j+1} - Z_{i+1,j}}{k} \right] = F_{i,j}.$$

Rearrangement of this equation then yields

$$Z_{i+1,j+1} = Z_{i,j} + \frac{1-c}{1+c}(Z_{i+1,j} - Z_{i,j+1}) + \frac{2k}{q_s(1+c)} F_{i,j}, \quad (17)$$
$$i = 0, 1, \ldots, M-1; j = 0, 1, \ldots, N-1,$$

where $Z_{i,0}(i = 0, 1, \ldots, M)$ and $Z_{0,j}(j = 0, 1, \ldots, N)$ are given, $c = \frac{p_s k}{q_s h}$, $c \neq -1$, $q_s \neq 0$. The truncation error of the box scheme is $O\left(h^2 + k^2\right)$. The domain of influence of the box scheme, D_{box_0}, is entire Ω.

3 Analysis of Finite Difference Algorithms

Given the above list of schemes we are naturally led to the question of which of them are useful and which are not. In this section we firstly determine which schemes have solutions that approximate solutions of the shape from shading problem (3). Later on we determine which schemes are more accurate than others and also investigate the efficiency of the various schemes.

3.1 Consistency

Definition 1. *A finite difference scheme is said to be* consistent *with a partial differential equation iff as the grid constants tend to zero, the difference scheme becomes in the limit the same as the partial differential equation at each point in the solution domain.*

Theorem 1. *All the above finite difference schemes are consistent with* (3)*, the LF scheme is consistent if k/h is constant.*

Proof. As mentioned in Section 2, the truncation errors of the FF, BF, leapfrog, FB and BB schemes are $O(h+k)$; the truncation error of the CB is $O\left(h^2+k\right)$, WS is $O\left(h+k^2\right)$, CN and box are $O\left(h^2+k^2\right)$. It is obvious to see that the limit of all the truncation errors is zero as $h, k\rightarrow 0$. Therefore, all the above finite difference schemes except the LF scheme are consistent with (3). On the other hand, the truncation error of the LF scheme, $O\left(h^2+k+h^2/k\right)$, tends to zero as $h, k\rightarrow 0$ if k/h is constant. So the LF scheme is also consistent with (3). □

3.2 Solvability

Theorem 2. *Let $c=\frac{p_s k}{q_s h}$ be a fixed constant, $q_s \neq 0$. Then,*
(a) *all the above explicit schemes are solvable;*
(b) *the CB scheme is solvable if $|c|<1$, the CN scheme is solvable if $|c|<2$;*
(c) *the FB scheme is solvable if $p_s \neq 0$, the BB and box schemes are solvable if $c \neq -1$, the WS scheme is solvable if $c \neq -2$.*

Proof. At first we prove (a) and (b). From equations (7)–(10), it is obvious that all the explicit schemes are solvable.

For the CB scheme, we must solve a set of tridiagonal linear algebraic systems (12) at each j level. Therefore, the necessary conditions under which the CB scheme is solvable is that the coefficients matrix of the systems

$$\begin{bmatrix} 1 & \frac{c}{2} & & & \\ -\frac{c}{2} & 1 & \frac{c}{2} & & \\ \ddots & \ddots & \ddots & & \\ & -\frac{c}{2} & 1 & \frac{c}{2} \\ & & -\frac{c}{2} & 1 \end{bmatrix}$$

is strict diagonally dominant. From the definition of a strict diagonally dominant matrix, we have

$$|-c/2| + |c/2| < 1, \quad \text{i.e.,} \quad |c| < 1.$$

The solvability of the CN scheme can be proved analogously.

Secondly we prove (c). The FB scheme involves two unknown values of $Z_{i+1,j}$ and $Z_{i,j}$ at the jth level, but $Z_{i+1,j}$ can be calculated in the order $i = 1$ to $M-1$, since $Z_{i,j}$ is known when the initial values $Z_{i,0}(i = 0, \ldots, M)$ and boundary conditions $Z_{0,j}(j = 0, \ldots, N)$ are given. That is, the FB scheme is solvable. The solvability of the BB scheme can be proved analogously. About the WS scheme, it also involves two unknown values $Z_{i,j+1}$ and $Z_{i-1,j+1}$ at the $(j+1)$th level. However, commencing with a given boundary value $Z_{0,j}$ and $Z_{M,j}(j = 0, \ldots, N)$, the values of $Z_{i,j+1}$ can be computed in the order $i = 1$ to $M-1$, since $Z_{i-1,j+1}$ is known at each application of the WS scheme. Using the same method, we can see that the box scheme is also solvable under the given conditions. □

3.3 Stability

Definition 2. *A finite difference scheme is said to be* stable *iff the difference between the numerical solution and the exact solution of the difference scheme does not increase as the number of rows of calculation at successive j levels in the solution domain is increased.*

In order to obtain the stability of all the schemes in the paper, we need the following lemma.

Lemma 1. (Von Neumann criterion of stability) [12]: *Given a finite difference scheme with constant coefficients*

$$L_h Z_{i,j} = F_{i,j}, \tag{18}$$

where L_h is a finite difference operator. Let $Z_{i,j} = g^j e^{I\theta i}$ with $I^2 = -1, \theta = 2\pi l h, l = \pm 1$ (g is called amplification factor*). If we substitute this value $Z_{i,j}$ into the homogenous finite difference scheme associated with* (18)*, and eliminating the common factor, we obtain an expression for g. Then the finite difference scheme* (18) *is stable if and only if there is a constant $K > 0$ (independent of θ, h and k) such that*

$$|g| \leq 1 + Kk \tag{19}$$

for all θ. If k/h is constant, the stability condition (19) *can be replaced by*

$$|g| \leq 1.$$

Theorem 3. *Let $c = \frac{p_s k}{q_s h}$ be a fixed constant, $q_s \neq 0$. Then,*
(a) *the FF scheme is stable if and only if $-1 \leq c \leq 0$;*
(b) *the BF scheme is stable if and only if $0 \leq c \leq 1$;*
(c) *the LF and the leapfrog schemes are stable if and only if $|c| \leq 1$;*
(d) *the CB and the CN schemes are unconditionally stable;*

(e) *the FB scheme is stable if and only if* $d \leq 1$*, where* $d = 1/c$*;*
(f) *the BB scheme is stable if and only if* $c \geq 0$ *or* $c < -1$*;*
(g) *the WS and the box schemes are unconditionally stable.*

Proof. (a) Replacing $Z_{i,j}$ in the FF scheme (7) by $g^j e^{I\theta i}$ for each value of i and j, we have that

$$g^{j+1}e^{I\theta i} = (1+c)g^j e^{I\theta i} - cg^j e^{I\theta(i+1)} ,$$

which gives the amplification factor as

$$\begin{aligned} g &= 1 + c - ce^{I\theta} \\ &= 1 + c\,(1-\cos\theta) - Ic\sin\theta \\ &= 1 + 2c\sin^2\frac{\theta}{2} - I\cdot 2c\sin\frac{\theta}{2}\cos\frac{\theta}{2} , \end{aligned}$$

and

$$\begin{aligned} |g|^2 &= \left(1 + 2c\sin^2\frac{\theta}{2}\right)^2 + \left(2c\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2 \\ &= 1 + 4c(1+c)\sin^2\frac{\theta}{2} . \end{aligned}$$

We see that $|g|$ is less than or equal to 1 if and only if $-1 \leq c \leq 0$. By Lemma 1, the FF scheme is stable iff $-1 \leq c \leq 0$.

(b) The amplification factor for the BF scheme is given by

$$\begin{aligned} g &= 1 - c + ce^{-I\theta} \\ &= 1 - c\,(1-\cos\theta) - Ic\sin\theta . \end{aligned}$$

The magnitude of g is

$$|g|^2 = 1 - 4c(1-c)\sin^2\frac{\theta}{2} .$$

We see that $|g|$ is bounded by 1, that is, the scheme is stable iff $0 \leq c \leq 1$.

(c) For the LF scheme the amplification factor is

$$\begin{aligned} g &= \frac{1-c}{2}e^{I\theta} + \frac{1+c}{2}e^{-I\theta} \\ &= \cos\theta - Ic\sin\theta \end{aligned}$$

and

$$|g|^2 = 1 - (1-c^2)\sin^2\theta .$$

We see that $|g| \leq 1$, that is , the LF scheme is stable if and only if $|c| \leq 1$. On the other hand, the amplification factor of the leapfrog scheme satisfies

$$g^2 + (I.2c\sin\theta)g - 1 = 0 .$$

Solving the quadratic equation gives

$$g_{1,2} = -Ic\sin\theta \pm \sqrt{1-(c\sin\theta)^2}\ .$$

If $|c| \leq 1$, then the quantity under the square root sign is real, and

$$|g_1|^2 = |g_2|^2 = (c\sin\theta)^2 + 1 - (c\sin\theta)^2 = 1$$

for all θ. If $|c| > 1$, then for some θ we have $c\sin\theta > 1$ when the quantity under the square root sign is negative and

$$g_{1,2} = -I\left(c\sin\theta \pm \sqrt{(c\sin\theta)^2 - 1}\right)\ .$$

At least one of these values g_1, g_2 has a modulus greater than 1. The leapfrog scheme is, therefore, stable iff $|c| \leq 1$.

(d) The amplification factor for the CB scheme is $g = 1/(1 + Icsin\theta)$. We have $|g| \leq 1$ for all θ; it is said to be unconditionally stable. It is not difficult to find out that the amplification factor of the CN scheme is

$$g = \frac{1 - I\frac{c}{2}\sin\theta}{1 + I\frac{c}{2}\sin\theta}\ ,$$

so

$$|g|^2 = \frac{1 + \left(\frac{c}{2}\sin\theta\right)^2}{1 + \left(\frac{c}{2}\sin\theta\right)^2} = 1$$

for any value of θ, that is, the CN scheme is also unconditionally stable.

(e) The FB scheme has the amplification factor

$$g = \frac{d}{d - (1-\cos\theta) + I.\sin\theta} = \frac{d}{d - 2\sin^2\frac{\theta}{2} + I.2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\ .$$

Therefore

$$|g|^2 = \frac{d^2}{d^2 + 4(1-d)\sin^2\frac{\theta}{2}}\ .$$

We see that $|g| \leq 1$, that is, the FB scheme is stable iff $d \leq 1$.

(f) The amplification factor for the BB scheme is given by

$$g = \frac{1}{1 + c(1-\cos\theta) + Ic\sin\theta} = \frac{1}{1 + 2c\sin^2\frac{\theta}{2} + I.2c\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\ .$$

The magnitude of g is

$$|g| = \frac{1}{1 + 4c(1+c)\sin^2\frac{\theta}{2}}\ .$$

It is easy to find out that $|g| \leq 1$ iff $c \geq 0$ or $c \leq -1$. Combining this condition with the solvability of the scheme, we have that the BB scheme is stable iff $c \geq 0$ or $c < -1$.

(g) The amplification factor of the WS scheme is

$$g = \frac{2 + c - ce^{I\theta}}{2 + c - ce^{-I\theta}} = \frac{(2 + c(1 - \cos\theta)) - Ic\sin\theta}{(2 + c(1 - \cos\theta)) + Ic\sin\theta} .$$

It follows that $|g| = 1$ for all values of c and θ, that is, the WS scheme is unconditionally stable.

For the box scheme the amplification factor is given by

$$g = \frac{1 + c + (1 - c)e^{I\theta}}{1 - c + (1 + c)e^{I\theta}} = \frac{((1 + c) + (1 - c)\cos\theta) + I(1 - c)\sin\theta}{((1 - c) + (1 + c)\cos\theta) + I(1 + c)\sin\theta} .$$

It follows that $|g| = 1$ for all c and θ, that is, the box scheme is also unconditionally stable. □

Remark: Comparing our results and proof methods with Kozera and Klette [7, 8] and Ulich [14], the ranges of the stability conditions for the FB and BB schemes are wider, and the proof methods for all the schemes described in this paper may be considered to be simpler.

3.4 Convergence

Definition 3. *A solution to a finite difference scheme which approximates a given partial differential equation is said to be* convergent *iff at each grid-point in the solution domain, the solution of the difference scheme approaches the solution of the corresponding partial differential equation as the grid constants tend to zero.*

Such a convergence to the true value, as the grid constants tend to zero, is also called *multigrid convergence* in digital geometry and image analysis. To obtain the convergence of all considered schemes, the following lemma is needed.

Lemma 2. *Given a finite difference scheme for a well-posed initial boundary value problem of a partial differential equation. If the scheme is consistent and stable, then it is convergent.*

Applying Lemma 2, Theorem 2 and Theorem 3 we get the following convergence theorem for the schemes discussed above.

Theorem 4. *Let* $c = \frac{p_s k}{q_s h}$ *be a fixed constant,* $q_s \neq 0$. *Then,*
(a) *the FF scheme is convergent if* $-1 \leq c \leq 0$*;*
(b) *the BF scheme is convergent if* $0 \leq c \leq 1$*;*
(c) *the LF and the leapfrog schemes are convergent if* $|c| \leq 1$*;*
(d) *the CB and the CN schemes are convergent for all* $c \in \mathbb{R}$*;*
(e) *the FB scheme is convergent if* $d \leq 1$*, where* $d = 1/c$*;*
(f) *the BB scheme is convergent if* $c \geq 0$ *or* $c < -1$*;*
(g) *the WS scheme is convergent if* $c \neq -2$*; the box scheme is convergent if* $c \neq -1$.

4 Conclusions

Four explicit, two implicit and four semi-implicit finite difference algorithms are analyzed for linear shape from shading problem. The analysis of different algorithms is achieved by comparing their domains of influence, truncation errors, and consistency, solvability, stability and convergence of each scheme (see Table 1). Finally, we conclude this paper by itemizing a few main results.

- The semi-implicit finite difference algorithms for linear shape from shading are discussed in this paper for the first time. The comparison of accuracy, solvability, stability and convergence of each scheme indicates that the WS and the box schemes are more useful.
- All schemes presented in this paper are supplemented by a full domain of influence, truncation error, consistency, solvability, stability and convergence analysis, see Table 1.
- The domain of influence of each scheme in this paper is entire Ω.
- In comparison with the results obtained by Kozera and Klette [7, 8] and Ulich [14], the range of the stability and convergence of the FB and BB schemes is identified as being larger.

Scheme	Influence Domain	Truncation Error	Solvability	Stability/Convergence
FF	$D_{FF_M} = \Omega$	$O(h+k)$	$q_s \neq 0$	$-1 \leq c \leq 0$
BF	$D_{BF_0} = \Omega$	$O(h+k)$	$q_s \neq 0$	$0 \leq c \leq 1$
LF	$D_{LF} = \Omega$	$O(h^2 + k + h^2/k)$	$q_s \neq 0$	$\|c\| \leq 1$
Leap	$D_{leap} = \Omega$	$O(h+k)$	$q_s \neq 0$	$\|c\| \leq 1$
CB	$D_{CB} = \Omega$	$O(h^2+k)$	$q_s \neq 0, \|c\| < 1$	for all c
CN	$D_{CN} = \Omega$	$O(h^2+k^2)$	$q_s \neq 0, \|c\| < 2$	for all c
FB	$D_{FB_0} = \Omega$	$O(h+k)$	$p_s \neq 0$	$d \leq 1$
BB	$D_{BB_0} = \Omega$	$O(h+k)$	$q_s \neq 0, c \neq -1$	$c \geq 0/c < -1$
WS	$D_{WS} = \Omega$	$O(h+k^2)$	$q_s \neq 0, c \neq -2$	for all c
box	$D_{box_0} = \Omega$	$O(h^2+k^2)$	$q_s \neq 0, c \neq -1$	for all c

Table 1. Domain of influence, truncation error, solvability, stability and convergence

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