Computer Science Department of The University of Auckland
CITR at Tamaki Campus (http://www.citr.auckland.ac.nz)

CITR-TR-65 September 2000

A New Approximation Scheme for Digital Objects and Curve Length Estimations

T. Asano[1], Y. Kawamura[1], R Klette[2], K. Obokata[1]

Abstract

The paper introduces a new approximation scheme for planar digital curves. This scheme defines an approximating sausage 'around' the given digital curve, and calculates a minimum-length polygon in this approximating sausage. The length of this polygon is taken as an estimator for the length of the curve being the (unknown) preimage of the given digital curve. Assuming finer and finer grid resolution it is shown that this estimator converges to the true perimeter of an r-compact polygonal convex bounded set. This theorem provides theoretical evidence for practical convergence of the proposed method towards a 'correct' estimation of the length of a curve.

---

[1] School of Information Science, JAIST Asahidai, Tatsunokuchi, 923-1292 Japan
[2] Centre for Image Technology and Robotics, The University of Auckland, Tamaki Campus, Auckland, New Zealand

A New Approximation Scheme for Digital Objects and Curve Length Estimations

T. Asano[1], *Y. Kawamura*[1], *R. Klette*[2], *and K. Obokata*[1]

[1]School of Information Science, JAIST
Asahidai, Tatsunokuchi, 923-1292 Japan
{*t-asano, kawamura, obokata*}*@jaist.ac.jp*
[2]CITR, University of Auckland, Tamaki Campus, Building 731
Auckland, New Zealand
r.klette@auckland.ac.nz

Abstract

The paper introduces a new approximation scheme for planar digital curves. This scheme defines an approximating sausage 'around' the given digital curve, and calculates a minimum-length polygon in this approximating sausage. The length of this polygon is taken as an estimator for the length of the curve being the (unknown) preimage of the given digital curve. Assuming finer and finer grid resolution it is shown that this estimator converges to the true perimeter of an r-compact polygonal convex bounded set. This theorem provides theoretical evidence for practical convergence of the proposed method towards a 'correct' estimation of the length of a curve.

Keywords: Digital geometry, digital curves, multigrid convergence, length estimation.

1 Introduction and Preliminary Definitions

Curve length estimation in image analysis may be based on methods ensuring convergence of estimated values towards the true length assuming an increase in grid resolution. For example, the *digital straight segment approximation method* (DSS method), see [2, 7], and the *minimum length polygon approximation method* assuming one-dimensional grid continua as boundary sequences (GC-MLP method), see [8], are methods for which there are convergence theorems when specific convex sets are assumed to be the given input data, see [5, 6, 9]. This paper studies the convergence properties of a new minimum length polygon approximation method based on so-called *approximation sausages* (AS-MLP method).

Motivations for studying this new technique are as follows: the resulting DSS approximation polygon depends upon starting point and the orientation of the boundary scan, it is not uniquely defined, but it may be calculated for any given digital object. The resulting GC-MLP approximation polygon is uniquely defined, but it assumes a one-dimensional grid continua as an input polygon which is only possible if the given digital object does not have cavities of width 1 or 2. The new method leads to a uniquely defined polygon, and it may be calculated for any given digital object. Furthermore, it will be of interest

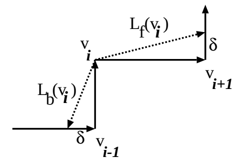

Figure 1: Definition of the forward and backward approximating segments associated with a vertex v_i.

to compare experimental data for this new technique with others. However, we will not report on experiments in this paper.

Let r be the *grid resolution* defined as being the number of grid points per unit. We consider *r-grid points* $g^r_{i,j} = (i/r, j/r)$ in the Euclidean plane, for integers i, j. Any r-grid point is assumed to be the center point of an *r-square* with *r-edges* of length $1/r$ parallel to the coordinate axes, and *r-vertices*.

The digitization model for our new approximation method is just the same as that considered in case of the DSS method, see [3, 4, 5]. That is, let S be a set in the Euclidean plane, called *real preimage*. The set $C_r(S)$ is the union of all those r-squares whose center point $g^r_{i,j}$ is in S. The boundary $\partial C_r(S)$ is the *r-frontier* of S. Note that $\partial C_r(S)$ may consists of several non-connected curves even in the case of a bounded convex set S. A set S is *r-compact* iff there is a number $r_S > 0$ such that $\partial C_r(S)$ is just one (connected) curve, for any $r \geq r_0$. This definition of r-compactness has been introduced in [5] in the context of showing multigrid convergence of the DSS method.

As a main result in this paper we show that the length of the approximating polygonal curve generated by the new scheme converges to the real perimeter of a given curve when it is convex.

2 Approximation Scheme

Given a connected region S in the Euclidean plane and a grid resolution r, the r-frontier of S is uniquely determined. We consider r-compact sets S, and grid resolutions $r \geq r_S$ for such a set, i.e. $\partial C_r(S)$ is just one (connected) curve. In such a case the r-frontier of S can be represented in the form $P = (v_0, v_1, \ldots, v_{n-1})$ in which the vertices are clockwise ordered so that the interior of S lies to the right of the boundary. Note that all arithmetics on vertex indices is modulo n.

Let δ be a real number between 0 and $1/(2r)$. For each vertex of P we define forward and backward shifts: The *forward shift* $f(v_i)$ of v_i is the point on the edge (v_i, v_{i+1}) at the distance δ from v_i. The *backward shift* $b(v_i)$ is that on the edge (v_{i-1}, v_i) at the distance δ from v_i.

For example, in the approximation scheme as detailed below we will replace an edge (v_i, v_{i+1}) by a line segment $(v_i, f(v_{i+1}))$ interconnecting v_i and the forward shift of v_{i+1}, which is referred to as the *forward approximating segment* and denoted by $L_f(v_i)$. The *backward approximating segment* $(v_i, b(v_{i-1}))$ is defined similarly and denoted by $L_b(v_i)$. Refer to Fig. 1 for illustration. Now we have three sets of edges, original edges of the r-frontier, forward and backward approximating segments. Based on these edges we define a connected region $A_r(S)$, which is homeomorphic to the annulus, as follows:

Given a polygonal circuit P describing an r-frontier in clockwise orientation. By reversing P we obtain a polygonal circuit Q in counterclockwise order. In the initialization step of our approximation procedure we consider P and Q as the *external* and *internal* bounding polygons of a polygon P_B homeomorphic to the annulus. It follows that this initial polygon P_B has area contents zero, and as a set of points it coincides with $\partial C_r(S)$.

Now we 'move' the external polygon P 'away' from $C_r(S)$, and the internal polygon Q 'into' $C_r(S)$ as specified below. This process will expand P_B step by step into a final polygon which contains $\partial C_r(S)$, and where the Hausdorff distance between P and Q becomes non-zero. For this purpose, we add forward and backward approximating segments to P and Q in order to increase the area contents of the polygon P_B.

To be precise, for any forward or backward approximating segment $L_f(v_i)$ or $L_b(v_i)$ we first remove the part lying in the interior of the current polygon P_B and updating the polygon P_B by adding the remaining part of the segment as a new boundary edge. The direction of the edge is determined so that the interior of P_B lies to the right of it.

Definition 2.1 *The resulting polygon P_B is referred to as the* approximating sausage *of the r-frontier and denoted by $A_r(S)$.*

The width of such an approximating sausage depends on the value of δ. It is easy to see that as far as the value of δ is at most half of the grid size, i.e., less or equal $1/(2r)$, the approximating sausage $A_r(S)$ is well defined, that is, it has no self-intersection. It is also immediately clear from the definition that the Hausdorff distance from the r-frontier $\partial C_r(S)$ to the boundary of the sausage $A_r(S)$ is at most $\delta < 1/(2r)$.

We are ready to define the final step in our AS-MLP approximation scheme for estimating the length of a digital curve. Our method is similar to that of the GC-MLP as introduced in [8].

Definition 2.2 *Assume a region S having a connected r-frontier. An* AS-MLP curve *for approximating the boundary of S is defined as being a shortest closed curve $\gamma_r(S)$ lying entirely in the interior of the approximating sausage $A_r(S)$, and encircling the internal boundary of $A_r(S)$.*

It follows that such an AS-MLP curve $\gamma_r(S)$ is uniquely defined, and that it is a polygonal curve defined by finitely many straight segments. Note that this curve depends upon the choice of the approximation constant δ. An example of such a shortest closed curve $\gamma_r(S)$ is given in Fig. 2.

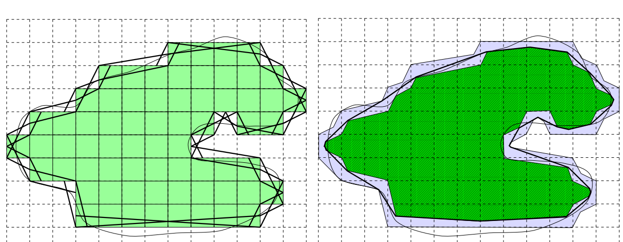

Figure 2: Construction of approximating sausage and approximation by shortest internal path.

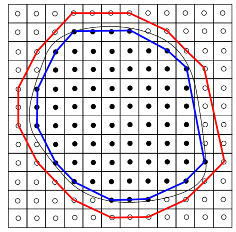

Figure 3: Interior r-grid points (filled circles) and exterior points (empty circles) with the convex hulls CH_{in} of a set of interior points and CH_{out} of a set of exterior points adjacent to interior ones.

3 Convergence Theorem

In this section we prove the main result of this paper about the multigrid convergence of the AS-MLP curve based length estimation of the perimeter of a given set S.

Theorem 3.1 *The length of the approximating polygonal curve $\gamma_r(S)$ converges to the perimeter of a given region S if S is a r-compact polygonal convex bounded set.*

We begin a proof of this theorem with an investigation of geometric properties of the r-frontier of a convex polygonal region S.

We first classify r-grid points into interior and exterior ones depending on whether they are located inside of the region S or not. Then, CH_{in} is defined to be the convex hull of the set of all interior r-grid points. CH_{out} is the convex hull of the set of those exterior r-grid points adjacent horizontally or vertically to interior ones. See Fig. 3 for illustration.

Lemma 3.2 *The difference between the lengths of CH_{in} and CH_{out} is exactly $4\sqrt{2}/r$.*

Now, we are ready to prove the following lemma which will be of crucial importance for proving the convergence theorem.

Lemma 3.3 *Given an r-compact polygonal convex bounded set S, the approximating polygonal curve $\gamma_r(S)$ is contained in the region bounded by CH_{in} and CH_{out}.*

Proof We first note that any convex vertex of the r-frontier is located between CH_{in} and CH_{out}. Any such vertex is characterized by an interior r-grid point p and two exterior r-grid points horizontally and vertically adjacent to p. Then, the polygonal edge of CH_{out} cannot come closer to CH_{in} beyond the edge between the two exterior points, which passes through the convex corner vertex. Thus, such a convex corner vertex is always located between CH_{in} and CH_{out}. This guarantees that the approximating polygonal curve $\gamma_r(S)$ cannot go beyond CH_{out}.

On the other hand, a concave vertex can be inside CH_{in}. However, such a concave corner is excluded from the approximating sausage $A_r(S)$. Suppose that the horizontal r-edge incident to a concave corner vertex p is intersected by a polygon edge of CH_{in} and hence

p is located inside CH_{in}. Then, by the definition of $A_r(S)$, the approximating segment associated with the polygon edge is parallel to the edge. Since $\gamma_r(S)$ is a shortest closed circuit in $A_r(S)$, it never comes into the concave part by crossing the approximating segment. Therefore, the shortest path cannot cross any polygonal edge of CH_{in}. QED

Let CH be the convex hull of the set of vertices of the approximating polygonal curve $\gamma_(S)$. The convex hull CH is also bounded by CH_{in} and CH_{out}. Obviously, the vertices of CH are all intersections of approximating segments. Furthermore, exterior intersections do not contribute to CH, where external (internal, resp.) intersections are those on the external (internal, resp.) boundary of the approximating sausage. Therefore, we can evaluate the perimeter of CH. How far an internal intersection from the boundary of CH_{in}? the longer an approximating segment, the closer its associated intersection to the inner hull CH_{in}. Thus, it is farthest at a corner defined by two unit edges. Thus, the maximum distance from CH_{in} to CH is bounded by $\sqrt{2}/6$, which implies that the perimeter of CH is bounded by $\sqrt{2}\pi/3$.

Lemma 3.4 *Let CH be the convex hull of all internal intersection defined above. Then, the approximating polygonal curve $\gamma_r(S)$ lies between the two convex hulls CH_{in} and CH. The maximum gap between CH_{in} and CH is bounded by $\sqrt{2}/6$, and for their perimeter we have*

$$Perimeter(CH) \leq Perimeter(CH_{in}) + 4\sqrt{2}/r. \tag{1}$$

So, if the approximating polygonal curve $\gamma_r(S)$ is convex, then we are done. Unfortunately, it is not always convex. In the remaining part of this section we evaluate the largest possible difference on lengths between $\gamma_r(S)$ and CH.

Lemma 3.5 *The approximating polygonal curve $\gamma_r(S)$ is concave when two consecutive long edges of lengths d_{i-1} and d_i with intervening unit edge satisfy $d_i > 3d_{i-1}+1$.*

By careful calculation we find that the difference δ for each concave part depicted in Fig. 4 where $d_i = q, d_{i-1} = p$ is expressed by

$$\begin{aligned}
\delta &= P1P2 + P2P3 - P1P3 \\
&= \sqrt{\left(\frac{q}{r(4q-1)} + \frac{p}{r(4p-1)}\right)^2 + \left(\frac{2q-1}{r(4q-1)} - \frac{2p}{r(4p-1)}\right)^2} \\
&\quad + \sqrt{\left(\frac{2q(2q-1)}{r(4q-1)} - \frac{q}{r(4q-1)}\right)^2 + \left(-\frac{2q-1}{r(4q-1)} - \frac{2q-1}{r(4q-1)}\right)^2} \\
&\quad - \sqrt{\left(\frac{2q(2q-1)}{r(4q-1)} + \frac{p}{r(4p-1)}\right)^2 + \left(-\frac{2q-1}{r(4q-1)} - \frac{2p}{r(4p-1)}\right)^2}
\end{aligned}$$

It is verified that δ is monotonically decreasing with p while monotonically increasing for q. Then, we have a bound $\delta < 0.0234/r$. Also we can see that such concave parts can happen at most 8 times.

Theorem 3.6 *S is a bounded, convex polygonal set. Then, there exists a grid resolution r_0 such that for all$r \geq r_0$ it holds that any AS-MLP approximation of the r-frontier $\partial C_r(S)$ is connected polygon with perimeter l_r and*

$$|Perimeter(S) - l_r| \leq (4\sqrt{2} + 8 * 0.0234)/r = 5.844/r. \tag{2}$$

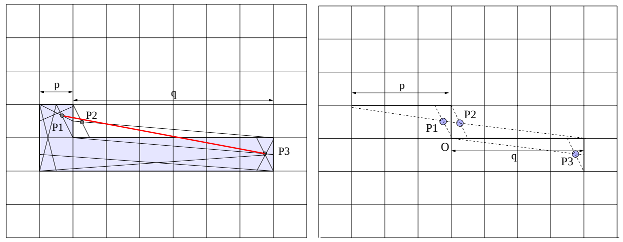

Figure 4: A concave part.

4 Conclusions

We have presented a new scheme for approximating a digital curve and compared its performance with the existing methods, DSS and MLP. We proved that the length of the approximating polygonal curve generated by the scheme converges to the real perimeter of a given curve when it is convex. The error bound for our scheme is about $5.844/r$, which is between DSS and MLP. The new method has been tested for several examples of digitized curves following [4]. However, experimental results will be reported in another paper due to space limitations in this publication.

References

[1] T. Buelow, R. Klette: Rubber band algorithm for estimating the length of digitized space-curves. Accepted for ICPR'2000, Barcelona, September 2000.

[2] A. Hübler, R. Klette, K. Voss: Determination of the convex hull of a finite set of planar points within linear time. *EIK* **17** (1981) 121–139.

[3] R. Klette, V. Kovalevsky, B. Yip: On the length estimation of digital curves. Proc. *Vision Geometry VIII*, SPIE Volume **3811**, Denver, July (1999) 117–129.

[4] R. Klette and Ben Yip: The length of digital curves. *Machine GRAPHICS & VISION* **9** (2000) to appear.

[5] R. Klette, J. Žunić: Multigrid Convergence of Calculated Features in Image Analysis. Accepted for *Journal of Mathematical Imaging and Vision.*

[6] V. Kovalevsky, S. Fuchs: Theoretical and experimental analysis of the accuracy of perimeter estimates. in: *Robust Computer Vision* (W. Förstner, S. Ruwiedel, eds.), Wichmann, Karlsruhe (1992) 218–242.

[7] A. Rosenfeld: Digital straight line segments. *IEEE Trans. Comp.* **C-28** (1974) 1264–1269.

[8] F. Sloboda, B. Zaťko, P. Ferianc: Minimum perimeter polygon and its application. in: *Theoretical Foundations of Computer Visi/-on* (R. Klette, W.G. Kropatsch, eds.), Mathematical Research **69**, Akademie Verlag, Berlin (1992) 59-70.

[9] F. Sloboda, B. Zaťko, J. Stoer: On approximation of planar one-dimensional continua. in: *Advances in Digital and Computational Geometry*, (R. Klette, A. Rosenfeld and F. Sloboda, eds.) Springer, Singapore (1998) 113–160.