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Interactions between Number Theory and Image Analysis

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Abstract

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ABSTRACT

The conceptual design of many procedures used in image analysis starts with models which assume as an input sets in Euclidean space which we regard as real objects. However, the application finally requires that the Euclidean (real) objects have to be modelled by digital sets, i.e. they are approximated by their corresponding digitizations. Also "continuous" operations (for example integrations or differentiations) are replaced by "discrete" counterparts (for example summations or differences) by assuming that such an replacement has only a minor impact on the accuracy or efficiency of the implemented procedure.

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Until now have been only minor impacts of image analysis on developments in number theory, by defining new problems, or by specifying ways how existing results may be discussed in the context of image analysis. There might be a more fruitful exchange between both disciplines in the future.

Keywords: Digital images, image analysis, number theory.

1. INTRODUCTION

Number theory is that field in mathematics which studies the properties of natural numbers. "Mathematics is the queen of sciences - and number theory is the queen of mathematics," according to Carl Friedrich Gauss.[35] Image analysis deals with objects defined on *grid points* having natural numbers as its coordinates. What would be more obvious than assuming that there should be a very strong relationship between both disciplines?

Image analysis often benefits from results based on number theoretical studies. This may be relevant to combinatorial procedures (e.g. number of linear partitions of rectangular grid arrays, simple constructions of digital convex n-gons), upper bounds for required memory allocations (e.g. maximum number of vertices of a convex polygon contained in a rectangular array of grid points), or to accuracy estimates for geometric feature calculations (e.g. multigrid convergence of digital moments towards the real moments).

Some grid point problems in number theory do have an extensive history such as the relationship between the estimated size (area, contents) of a planar region by the number of grid points contained in this region. This problem is, of course, of direct relevance for image analysis.

Let $r(n)$ be the number of different representations of the natural number n by a sum of two squares of integers. Let $r(0) = 1$. For example, $r(1) = 4$ because of $1^2 + 0^2 = 0^2 + 1^2 = (-1)^2 + 0^2 = 0^2 + (-1)^2$, $r(2) = 4$, $r(3) = 0$ etc. Gauss * characterized this function as follows,

$$R(x) = \sum_{0 \le n \le x} r(n) = \sum_{i^2+j^2 \le x} 1 = \pi x + \mathcal{O}\left(\sqrt{x}\right) \ , \tag{1}$$

saying that the size of $r(n)$ has an averaged order of π. This theorem allows a geometric interpretation: $R(x)$ is the number of grid points in or on a circle with midpoint $(0,0)$ and radius $\sqrt{x}$. The theorem by Gauss states that the number of grid points in or on a circle approximates the contents of a circle up to an error in the order of the radius, what is the same as in the order of the perimeter. The function $r(n)$ is an example of a *number-theoretic function*, i.e. defined on all natural numbers and having real or complex values. Another example of a number-theoretic function is the *Euler function* $\varphi(n)$ which counts the number of positive integers less than or equal n which are relatively prime to n. For example, $\varphi(1) = 1$, $\varphi(2) = 1$, $\varphi(3) = 2$, etc. F. Mertens has shown that[22]

$$\sum_{1 \le n \le x} \varphi(n) \ = \ \frac{3}{\pi^2} \cdot x^2 \ + \ \mathcal{O}(x \cdot \log x) \ , \tag{2}$$

which is a very useful theorem for analyzing the maximum number of vertices of a convex grid polygon contained in an $n \times n$-grid.[32] These two examples indicate already that image analysis may benefit from knowledge in number theory.

A further example of a number theoretic function of thinkable relevance for image analysis is the divisor function

$$\sigma_k(n) = \sum_{t|n} t^k \ ,$$

where $\omega(n) = \sigma_0(n)$ specifies the number of divisors of n, and $\sigma(n) = \sigma_1(n)$ is the sum of all divisors. The divisor function may be estimated by[22]

$$\sum_{n \le x} \sigma_k(n) = \frac{\varsigma(k+1)}{k+1} x^{k+1} + \begin{cases} \mathcal{O}\left(x^k\right), & \text{for } k > 1 \\ \mathcal{O}(x \log x), & \text{for } k = 1 \\ \mathcal{O}(x), & \text{for } 0 < k < 1 \end{cases}$$

using Riemann's zeta function

$$\varsigma(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \ , \quad \text{for} \quad s > 1 \ .$$

The divisor function is actually used in [34] for calculating the maximum number of vertices of a convex grid polygon contained in an $n \times n$-grid.

If $(m,n) = 1$ means that integers m and n are relatively prime, and $\omega(n)$ denotes the number of distinct divisors of n as defined above, then it holds that

$$\sum_{\substack{(m,n)=1 \\ a \le n \le b}} 1 = (b-a) \cdot \frac{\varphi(n)}{n} + \mathcal{O}(2^{\omega(n)})$$

where $\varphi(n)$ is the Euler function.

Number theory is a very developed field in mathematics, and grid points are one of the main subjects in this field since H. Minkowski.[25] However, in general it seems that image analysis researchers prefer to use "classical analysis tools" which are (usually) weaker in obtaining "sharp" bounds than "number theoretical tools" are.

The authors believe that a good example for demonstrating the strength of number theoretic arguments is our paper [19] where it is shown that any improvement in Huxley's result[11] (a recent theorem in number theory) leads

*Gauss has not used the $\mathcal{O}$-notation.

to a sharper estimation of errors if real moments are approximated by corresponding discrete moments. This result is stronger than the related previous ones published in the image analysis literature.

This paper has been written to show that there are already many interactions in place between number theory and image analysis which go far beyond the "classical" area estimation problem mentioned above. This paper reviews related work in this field of number-theoretical studies related to image analysis, and, frankly speaking, with a special emphasis on work done by the authors.

2. DIGITAL STRAIGHT LINE SEGMENTS

A digital curve or arc is normally considered to be an ordered sequence of pixels (or grid points) which may result from some digitizing procedure applied to a real curve or arc. Of particular interest has been the characterization of digital straight line segments.[4,6–10,26,27,33,37]

The paper [21] considers segments of lines $y = \alpha \cdot x + \beta$ in the first octant, defined by $0 \leq x \leq m$, $0 \leq \alpha \leq 1$, $0 \leq \beta < 1$ and the following digitizations of such line segments,

$$L = \{(x_i, y_i) \mid x_i = i, y_i = \lfloor \alpha \cdot i + \beta \rfloor,\ i = 0, 1, 2, \dots m\} .$$

Obviously these line digitizations are defined to be the set of those grid points which are on or below the line, but which have an 8-neighbor † which is above the line. In [21] it is shown that the number of such digital line segments is equal to

$$\frac{1}{\pi^2} \cdot m^3 \ + \ \mathcal{O}\left(m^2 \cdot \log m\right) . \tag{3}$$

The first grid point in such a digital line segment is $(0, 0)$, and the digital line segment consists of exactly $m + 1$ grid points. There is an obvious "one – to – one" correspondence between the set of digital straight line segments starting at $(0, 0)$ and the set of linear partitions of an $m \times m$ orthogonal grid, where a *linear partition* of a set S is defined to be any partition of S into sets X and $S \setminus X$ by a line l such that the sets X and $S \setminus X$ belong to different halfplanes defined by line l. Of course, any digital straight line segment consisting of $m + 1$ points and beginning at $(0, 0)$ defines exactly one linear partition, but there are also further linear partitions of the $m \times m$ grid which do not correspond to such specific digital straight lines starting at $(0, 0)$.

The number of linear partitions of an $m \times n$ orthogonal grid is considered in [1]. There it has been shown that the number of such partitions is equal to

$$\frac{3}{\pi^2} \cdot m^2 \cdot n^2 \ + \ \mathcal{O}(m^2 \cdot n \cdot \log n) \ + \ \mathcal{O}(m \cdot n^2 \cdot \log\log n)$$

where $m \leq n$ is assumed.

This result can be understood as the "capacity" of a digital picture of size $m \times n$ with respect to straight lines, i.e., it shows what is the number of straight lines which can be discriminated by digitizations on an $m \times n$ orthogonal grid.

Both asymptotic formulas, on the number of straight line segments and on the number of linear partitions, can be derived by using well known formulas for average values of number-theoretical functions and Riemann-Stieltjes integration, as made available in textbooks such as [3,22,28,29].

Digital curves are normally encoded by a directional code specifying code numbers 0,1,...,7 for all the possible steps to an 8-neighbor of a given grid point, where these numbers follow a clockwise or counter-clockwise order. The study of the "periodical pattern" of directional code sequences of digital straight lines is one of the oldest problems in image analysis,[8] and in number theory it has roots dating back to the Bernoulli brothers.[4] A complete coverage of the history and a solution based on continued fractions is given by K. Voss in [34].

†Standard definition in the image processing literature: for a grid point (x, y) it holds that an *8-neighbor* is any grid points (i, j) with $max\{x - i, y - j\} = 1$.

Freeman formulated his conjecture about a specific appearance pattern of directional codes in his paper in 1971 as follows[9]: there are only two consecutive codes (modulo 8) in the sequence, at least one of them is always isolated (i.e. the same number does not follow immediately again), and the occurrence of these isolated orientations is somehow "uniformly distributed". Rosenfeld[26] has specified this further by showing that the run-length of the non-isolated orientations satisfies these three conditions recursively.

Dorst[6] proposed spirographs as a tool to study periodic behavior. Rothstein and Weiman[27] discussed Farey sequences for characterizing the periodic pattern of directional codes of straight lines passing through the origin. The nth Farey sequence consists of all rational numbers a_0/a_1, with $0 \leq a_0 \leq a_1$ and a_0 and a_1 are relatively prime, sorted in increasing order. For example, for $n = 5$ we have the sequence

$$\frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}$$

In [27] it has been discussed that the set of digital line sequences of length n, passing trough the origin, is in one-one correspondence with the nth Farey sequence. This is actually already a proof for Equ. (3).

If the slope of the straight line is assumed to be a rational number a_0/a_1, with integers $a_0 > a_1 > 1$, then it can be represented as a (finite) continued fraction,[22,29]

$$\frac{a_0}{a_1} = [q_1, q_2, ..., q_n] = \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{\ddots \cfrac{1}{q_{n-1} + \frac{1}{q_n}}}}} \quad ,$$

with integer coefficients $q_i > 0$, for $1 \leq i \leq n$. The Euclidean algorithm may be used to derive such continued fractions:

$$\begin{aligned} \frac{a_0}{a_1} &= q_1 + \frac{a_2}{a_1} \quad \text{with} \quad 0 < \frac{a_2}{a_1} < 1 \ , \\ \frac{a_1}{a_2} &= q_2 + \frac{a_3}{a_2} \quad \text{with} \quad 0 < \frac{a_3}{a_2} < 1 \ , \\ &\cdots\cdots\cdots\cdots , \\ \frac{a_{n-2}}{a_{n-1}} &= q_{n-1} + \frac{a_n}{a_{n-1}} \quad \text{with} \quad 0 < \frac{a_n}{a_{n-1}} < 1 \ , \\ \frac{a_{n-1}}{a_n} &= q_n \quad \text{with} \quad a_{n+1} = 0 \ . \end{aligned}$$

Continued fractions have been used in [4,33] to characterize directional code sequences of digitized straight lines. Related results in number theory[13] have been of use in these studies. We review the related definitions and results as given by K. Voss in [34].

We consider straight lines passing (w.l.o.g.) through the origin and having a rational slope a/b, with $(a, b) = 1$. The *characteristic triangle* of such a line is given by the vertices $(0, 0), (a, 0), (0, b)$. The *concatenation* $T_1 \otimes T_2$ of two characteristic triangles T_1 and T_2 is a characteristic triangle defined by the slope a/b with $a = \frac{1}{c}(a_1 + a_2)$, and $b = \frac{1}{c}(b_1 + b_2)$, and those integer c such that $(a, b) = 1$.

The numerical value of a continued fraction can also be expressed in the form of multiples of q_n,

$$\frac{a_0}{a_1} = [q_1, q_2, ..., q_n] = \frac{\alpha_n q_n + \beta_n}{\gamma_n q_n + \delta_n} \ ,$$

where $\alpha_n, \beta_n, \gamma_n, \delta_n$ are defined by the coefficients $q_1, q_2, ..., q_{n-1}$. For $n = 1$ it is $\alpha_1\delta_1 - \beta_1\gamma_1 = -1$. For $n \geq 1$ it holds

$$[q_1, q_2, ..., q_{n-1}, q_{n+1}] = \frac{\alpha_{n+1} q_{n+1} + \beta_{n+1}}{\gamma_{n+1} q_{n+1} + \delta_{n+1}} = \frac{\alpha_n (q_n q_{n+1} + 1) + \beta_n q_{n+1}}{\gamma_n (q_n q_{n+1} + 1) + \delta_n q_{n+1}} \ ,$$

and thus

$$\frac{\alpha_{n+1} q_{n+1} + \beta_{n+1}}{\gamma_{n+1} q_{n+1} + \delta_{n+1}} = \frac{(\alpha_n q_n + \beta_n) q_{n+1} + \alpha_n}{(\gamma_n q_n + \delta_n) q_{n+1} + \gamma_n} \ .$$

This means that the slope $\frac{a_0}{a_1} = [q_1, q_2, ..., q_n]$ of a characteristic triangle can be expressed as

$$\frac{(\alpha_{n-1}q_{n-1} + \beta_{n-1})\, q_n + \alpha_{n-1}}{(\gamma_{n-1}q_{n-1} + \delta_{n-1})\, q_n + \gamma_{n-1}} = \frac{(\alpha_{n-1}q_{n-1} + \beta_{n-1})\,(q_n - 1) + \alpha_{n-1}\,(q_{n-1} + 1) + \beta_{n-1}}{(\gamma_{n-1}q_{n-1} + \delta_{n-1})\,(q_n - 1) + \gamma_{n-1}\,(q_{n-1} + 1) + \delta_{n-1}} \ .$$

Therefore, the characteristic triangle defined by slope a_0/a_1 is equal to the result of a repeated concatenation $\otimes$ of one characteristic triangle with slope $[q_1, q_2, ..., q_{n-1} + 1]$ and $q_n - 1$ triangles with slope $[q_1, q_2, ..., q_{n-1}]$, which may be expressed by the formula

$$[q_1, q_2, ..., q_n] = [q_1, q_2, ..., q_{n-1} + 1] \otimes (q_n - 1) \cdot [q_1, q_2, ..., q_{n-1}] \ .$$

This allows to prove Freeman's conjecture and Rosenfeld's refined hypothesis as follows:

$$\begin{aligned} [q_1, q_2, ..., q_n] \quad = \quad & (q_{n-1} \cdot [q_1, q_2, ..., q_{n-2}] \otimes [q_1, q_2, ..., q_{n-2} + 1]) \\ & \otimes (q_n - 1) \cdot ((q_{n-1} - 1) \cdot [q_1, q_2, ..., q_{n-2}] \otimes [q_1, q_2, ..., q_{n-2} + 1]) \ . \end{aligned}$$

The isolated code number is $[q_1, q_2, ..., q_{n-2} + 1]$, and $[q_1, q_2, ..., q_{n-2}]$ is the other ("non-isolated") code number at this level of representing a straight line, passing through the origin and having rational slope. The run lengths q_{n-1} and $q_{n-1} - 1$ differ by 1.

3. GEOMETRIC MOMENTS

Moments and function of moments have been extensively used in the area of image analysis, see, for example, [18,23,31,38]. For a given planar region S, the (p, q)-moment of S is defined by

$$m_{p,q} = \iint\limits_S x^p y^q \, dx dy \ .$$

Of course, in practice, instead of real $m_{p,q}(S)$ moments we can use only their "discrete version", so-called *discrete moments*. The discrete (p, q)-moment of a region S is defined to be

$$\mu_{p,q}(S) = \sum_{(i,j) \in D(S)} i^p \cdot j^q = \sum_{\substack{i,j \text{ are integers} \\ (i,j) \in S}} i^p \cdot j^q$$

where $D(S)$ denotes a set of grid points, a digitalization of S. We assume that $D(S)$ is the *Gauss-digitization* of S consisting of all points in S having integer coordinates.

Since a region S can be represented on digital pictures of different resolutions characterized by parameter r showing the number of pixels per unit, it is convenient to consider a dilation $r \cdot S$ of S by factor $r > 0$, defined by

$$r \cdot S = \{(r \cdot x, r \cdot y) \mid (x, y) \in S\}.$$

In general we expect that

$$m_{p,q}(S) \ \approx \ \frac{1}{r^{p+q+2}} \cdot \mu_{p,q}(r \cdot S) \ ,$$

i.e. that this formula defines a "good approximation" of the real moment $m_{p,q}(S)$ calculated from its digitization on a grid of a given resolution r, i.e., calculated from its discrete moment $\mu_{p,q}(r \cdot S)$. Of course, we expect that an increase in grid resolution leads to a decrease in the error in the above approximation. It turns out that the mentioned error in this approximation can be estimated efficiently by knowing the order of a function $f(r)$ denoting the following difference,

$$f(r) \ = \ |m_{0,0}(r \cdot S) - \mu_{0,0}(r \cdot S)| \ = \ |r^2 \cdot m_{0,0}(S) - \mu_{0,0}(r \cdot S)| \ .$$

A classical result is that $f(r) = \mathcal{O}(r)$ if there are (nearly) no assumptions about the boundary of S. In [19] it is shown that $f(r) = \mathcal{O}(r)$ implies the following upper bound for the error in the estimation of real moments of an arbitrary order:

$$|m_{p,q}(S) - \frac{1}{r^{p+q+2}} \cdot \mu_{p,p}(r \cdot S)| = \mathcal{O}\left(\frac{1}{r}\right) .$$

But a recent result due to Huxley, enables us to give a sharper error estimate in the case when the boundary of a convex set S belongs to the C^3 class and does not contain straight line segments, i.e. has positive curvature at all boundary points. Huxley proved in [11]

THEOREM 3.1. *If S is a convex set in the Euclidean plane with a C^3 boundary and positive curvature at every point of the boundary, then it holds that*

$$|r^2 \cdot m_{0,0}(S) - \mu_{0,0}(r \cdot S)| = \mathcal{O}\left(r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}}\right) .$$

The previous result has the following consequence as proved in [19]:

$$|m_{p,q}(S) - \frac{1}{r^{p+q+2}} \cdot \mu_{p,q}(r \cdot S)| = \mathcal{O}\left(\frac{(\log r)^{\frac{47}{22}}}{r^{\frac{15}{11}}}\right)$$

for such a set S as specified in Huxley's theorem.

We consider the Huxley result as being a very strong mathematical result. It is not known yet whether it specifies the best possible upper bound or not. But any improvement of this result defining a new function $f_1(x)$ having a smaller order of magnitude than

$$\mathcal{O}\left(r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}}\right)$$

would "automatically" imply an improvement

$$|m_{p,q}(S) - \frac{1}{r^{p+q+2}} \cdot \mu_{p,q}(r \cdot S)| = \mathcal{O}\left(\frac{f_1(r)}{r^2}\right)$$

in the studied error estimation due to the proof technique developed in [19]. See also the experiments reported in [20] which support the hypothesis that an improvement of Huxley's result is possible. Such experimental studies in image analysis might be of more general importance for number theoretical studies in the future.

We mention a "standard approach" of expressing functions such as $f(r)$: if $f_1(r)$ is going to be expressed in the form $f_1(r) = r^\alpha$, then, due to the example of $y(x) = \sqrt{x}$ (note: this curve passes $\lfloor\sqrt{r}\rfloor$ grid points while $x \in (0, r]$) it already follows that $\alpha < 1/2$ is impossible. So at the moment, there is a gap between $r^{1/2}$ and $r^{7/11+\varepsilon}$. This implies: if real moments of a convex set with C^3 boundary and positive curvature at all of the boundary points are estimated based on a digitization of this set with resolution r then the precision $\mathcal{O}(r^{-15/11+\varepsilon})$ is preserved, while the best (precision) which can be expected is lower bounded by $c \cdot r^{-3/2}$, where c is some positive constant.

4. ZERNIKE MOMENTS

In the previous section it was stated that Theorem 3.1 gives the best known upper bound for the difference

$$|m_{0,0}(r \cdot S) - \mu_{0,0}(r \cdot S)|$$

even in the case when S is a circle in a general position. But there is a new (unpublished yet) result, also due to Huxley,[12] which is related to circles whose center is placed at the origin (equivalently, into an arbitrary integer point).

THEOREM 4.1. *Let C be the unit circle given by $x^2 + y^2 \leq 1$. Then*

$$G(r) = \frac{1}{r^2} \cdot |m_{0,0}(r \cdot C) - \mu_{0,0}(r \cdot C)| \ = \ |\pi - \frac{1}{r^2} \cdot \mu_{0,0}(r \cdot C)| \ = \ \mathcal{O}\left(r^{-\frac{285}{208}}\right) .$$

This result has a direct application to the improvement of error estimations (particularly to the improvement of the so called "geometric error" estimations) in approximating Zernike Moments based on digitized data, see [24].

The use of Zernike moments[36] in image analysis was pioneered by Teague.[30] Since then, the Zernike moments have been frequently utilized for a number of image processing and computer vision tasks.

In order to define the Zernike moments, let us introduce a set of complex orthogonal functions with complete orthogonal basis over the class of square integrable functions defined over the unit disk. Such a set of complex orthogonal functions is called a set of *Zernike functions.* The (p,q)-order Zernike function is defined as follows,

$$V_{p,q}(x,y) = R_{p,q}(\rho) \cdot e^{j \cdot q \cdot \arctan(y/x)} ,$$

where $j = \sqrt{-1}$, $\rho = \sqrt{x^2+y^2} \leq 1$, $|q| \leq p$, $p - |q|$ is an even number, $R_{p,q}(\rho)$ is a polynomial in ρ of degree $p \geq 0$ containing no power of ρ lower than $|q|$.

The orthogonality constraint for the set of functions $\{V_{p,q}(x,y)\}$ has the special form

$$\iint_D V^*_{p,q}(x,y) \cdot V_{p',q'}(x,y) dx dy = \frac{\pi}{p+1} \cdot \beta_{p,p'} \beta_{q,q'}$$

where $\beta_{p,p'} = 1$ if $p = p'$ and 0 otherwise.

The completeness and orthogonality of the set $\{V_{p,q}(x,y)\}$ allow us to represent any square integrable function $f(x,y)$ defined on the unit disk in the way of the following series:

$$f(x,y) = \sum_{p=0}^{\infty} \sum_{q=-p}^{p} \frac{p+1}{\pi} \cdot A_{p,q} \cdot V_{p,q}(x,y) , \quad \text{where} \quad p - |q| \text{ is an even number} . \tag{4}$$

$A_{p,q}$ is the Zernike moment of order p with repetition q:

$$A_{p,q} = \iint_D f(x,y) \cdot V^*_{p,q}(x,y) dx dy.$$

The important feature of the Zernike moments is their rotational invariance. If $f(x,y)$ is rotated by an angle α then it is shown that the Zernike moments $A'_{p,q}$ of the rotated image is given by

$$A'_{p,q} = A_{p,q} \cdot e^{-j \cdot q \cdot \alpha}$$

where $j = \sqrt{-1}$, i.e., the magnitudes of the Zernike moments can be used as rotational invariant features of functions f. The function f may be an image defined on a rectangle which is assumed to be contained in the unit disk.

Of course, the above is valid as long as one uses an integrable "image" function f defined on a region D in the Euclidean plane. In practice, the Zernike moments have to be computed from sampled data. A discrete version of $A_{p,q}$ over a grid point set $\{(x_i, y_j) : 1 \leq i, j \leq n\}$ is given as follows:

$$\overline{A}_{p,q} = \sum_{\substack{i,j \\ x_i^2 + y_j^2 \leq 1}} h_{p,q}(x_i, y_j) \cdot f(x_i, y_j)$$

where $h_{p,q}(x_i, y_j)$ represents an integration of the function $V^*_{p,q}(x,y)$ over a grid square having grid point (x_i, y_j) as its center point.

A detailed analysis of the difference

$$\overline{A}_{p,q} - A_{p,q} = E^n_{p,q} + E^g_{p,q}$$

is given in [24]. The above difference (i.e., the error in the calculation of the Zernike moments from discrete data) is divided into two parts, a so-called "numerical error" denoted by $E^n_{p,q}$, and a "geometric error" denoted by $E^g_{p,q}$.

In the same paper [24] the following estimations are given,

$$|E^n_{p,q}| \le \sqrt{\frac{4 \cdot \pi \cdot \max\limits_{x,y} f(x,y) \cdot V(f)}{p+1}} \cdot \frac{1}{2r}$$

and

$$|E^g_{p,q}| \le \sqrt{\frac{\pi}{p+1}} \cdot \max_{x,y} f(x,y) \cdot \sqrt{G(2 \cdot r)}$$

where $V(f)$ is the total variation of $f(x, y)$. In the previous expression the geometric error is dominant obviously. In [24] the result $G(r) = \mathcal{O}(r^{-15/11})$ from [15] is used, and consequently $|\overline{A}_{p,q} - A_{p,q}|$ is upper bounded by $\mathcal{O}(r^{-15/22})$.

Huxley's recent result [12] improves this error bound to

$$\mathcal{O}\left(r^{-285/416}\right)$$

Moreover, replacing $A_{p,q}$ with $\overline{A}_{p,q}$ into Equ. 4 the authors of [24] gave the following reconstruction formula:

$$\overline{f}_T(x,y) = \sum_{p=0}^{T} \sum_{q=-p}^{p} \frac{p+1}{\pi} \cdot \overline{A}_{p,q} \cdot V_{p,q}(x,y) \qquad \text{where} \quad p - |q| \quad \text{is an even integer.}$$

T is the truncation parameter informing us how many moments are taken into account.

If $Error(\overline{f}_T)$ is defined as a natural performance measure in the following way:

$$Error(\overline{f}_T) = \int\!\!\int_D |\overline{f}_T - f(x,y)|^2 dxdy \ ,$$

it is shown in [24] that

$$Error(\overline{f}_T) = \mathcal{O}\left(\frac{T^2}{r^{15/11}}\right) + \mathcal{O}\left(\frac{1}{T}\right)$$

In order to minimize the above upper bound the authors suggest a choice of T as T^* in the form

$$T^* = c \cdot r^{5/11} \qquad \text{where c is some positive constant}$$

which would imply that $\mathcal{O}(r^{-5/11})$ is an upper bound for the accuracy of the proposed reconstruction method.

By applying the Theorem 4.1 following the same approach we obtain

$$Error(\overline{f}_T) = \mathcal{O}\left(\frac{T}{r^{285/208}}\right) + \mathcal{O}\left(\frac{1}{T}\right) .$$

The above implies that a choice of a smaller number of the used Zernike moments T^* given in the form

$$T^* = c \cdot r^{95/208}$$

leads to an upper error bound of $\mathcal{O}\left(r^{-95/208}\right)$.

5. DIGITAL CONVEX POLYGONS

Convex polygons are of fundamental interest in image analysis (calculations of convex hulls, segmentation of complex shapes into convex polygons etc.). Since in practice we deal with digital pictures of such sets it is straightforward to consider extrema problems related to such convex sets which can be represented on a digital (binary) picture of a given size, let say $m \times m$. Especially it might be of interest

- to estimate the maximum number $\mathcal{N}(m)$ of vertices of such a convex polygon, and
- to give the estimation of an sufficiently large number $\mathcal{B}(m)$ of bits for the encoding (representation) of all digital convex polygons with respect to the size m of a given squared orthogonal grid.

Also, let $\mathcal{C}(m)$ be the number of convex digital polygons contained in a square $m \times m$ grid, for integer $m > 0$. – The first question has been studied in many papers, see, for example [2,32,34], and a complete answer can be found in [2]. Precisely, there it has been shown that the maximum number $\mathcal{N}(m)$ of vertices of a digital convex polygon contained in an $m \times m$ orthogonal grid is equal to

$$\mathcal{N}(m) \ = \ \frac{12}{\sqrt[3]{4 \cdot \pi^2}} \cdot \sqrt[3]{m^2} + \mathcal{O}\left(\sqrt[3]{m} \cdot \log m\right) \ .$$

This result states that the minimal number $\mathcal{B}(m)$ of bits sufficient for coding of digital convex polygon from an $m \times m$ grid, has an order of magnitude larger or equal to $m^{\frac{2}{3}}$. Precisely, since $\mathcal{C}(m) \geq 2^{\mathcal{N}(m)}$ holds trivially, we have

$$\mathcal{B}(m) = \log \mathcal{C}(m) \geq \log \left(2^{\mathcal{N}(m)}\right) = \mathcal{N}(m) = \mathcal{O}(m^{\frac{2}{3}}) \ .$$

On the other hand, a simple coding scheme where any vertex is coded particularly, gives $\mathcal{O}\left(m^{\frac{2}{3}} \cdot \log m\right)$ as an upper bound for $\mathcal{B}(m)$, because any vertex can be coded by $\mathcal{O}(\log m)$ bits. So there is a gap between $m^{2/3}$ and $m^{2/3} \cdot \log m$ for estimating $\mathcal{B}(m)$. The problem is solved in [14], i.e., it is shown that $\mathcal{B}(m) = \mathcal{O}(m^2/3)$. We give a sketch of the proof.

First, it is shown that $\mathcal{B}(m)$ has the same order of magnitude as

$$\log \left(\sum_{ind_1} \prod_{1 \leq i \leq m} \binom{n_i + \varphi(i) - 1}{n_i} \right) ,$$

where the last sum is taken over all ordered m–tuples $(n_1, n_2, \ldots, n_m)$ in the set

$$ind_1 = \{(n_1, n_2, \ldots, n_m) : 1 \cdot n_1 + 2 \cdot n_2 + \ldots + m \cdot n_m \leq m \ \ \wedge \ \ n_1 \geq 0 \ \wedge \ n_2 \geq 0 \ \wedge \ \ldots \ \wedge \ n_m \geq 0\} \ .$$

Further, by considering the number-theoretical function

$$p(m) = \sum_{ind_2} 1 \ ,$$

with

$$ind_2 = \{(n_1, n_2, \ldots, n_m) : 1 \cdot n_1 + 2 \cdot n_2 + \ldots + m \cdot n_m = m \ \wedge \ n_1 \geq 0 \ \wedge \ n_2 \geq 0 \ \wedge \ \ldots \ \wedge \ n_m \geq 0\} \ ,$$

which is equal to the number of unrestricted partitions of a natural number m, and by using a result in number theory, describing the asymptotic behavior of $p(m)$, (see, for example [5), p.79],

$$p(n) \approx \frac{1}{4 \cdot \sqrt{3} \cdot n} \cdot e^{\pi \cdot \sqrt{\frac{2 \cdot n}{3}}}$$

it remains to prove whether the maximum summand of the form

$$\log \left(\prod_{ind_3} \binom{n_i + \varphi(i) - 1}{n_i} \right) \ ,$$

with

$$ind_3 = \{n_i : 1 \leq i \leq m \ \ \wedge \ \ 1 \cdot n_1 + 2 \cdot n_2 + \ldots + m \cdot n_m = m \ \ \wedge \ \ n_1 \geq 0 \ \wedge \ \ldots \ \wedge \ n_m \geq 0\} \ ,$$

has $m^{\frac{2}{3}}$ as the order of magnitude. The last is also proved in [14] implying $\mathcal{B}(m) = \mathcal{O}\left(m^{\frac{2}{3}}\right)$.

We conclude this section by observing that an efficient procedure for the encoding of digital convex polygons contained in an $m\times$-grid by using $\mathcal{O}\left(m^{\frac{2}{3}}\right)$ bits is not described yet.

6. CONCLUSIONS

This paper might contribute to a discussion about interrelationships between image analysis on one hand, providing tools for experimental studies on orthogonal grids, and defining problems for such grids, and number theory, providing proof techniques and basic theorems, and probably interesting new challenges, on the other hand. The paper shows that there are already some interesting interactions in place, and it is felt that there is more potential for the benefit of both disciplines.

The paper also shows that interesting interrelations between image analysis and number theory are motivated by specific analysis tasks in image analysis. Grid points having natural numbers as coordinates are not sufficient to identify strong interactions. For example, H. Minkowski[25] proved the following theorem:

THEOREM 6.1. *Assume a convex set in the n-dimensional Euclidean space $\mathcal{E}^n$ with the origin as its centroid, and a contents greater than or equal 2^n. Then this set contains at least one more grid point (due to symmetry reasons even at least two more grid points) in its interior or on its boundary. If the contents of this set is greater than 2^n then there are besides the origin at least two more grid points in the interior of this convex set.*

This is without any doubt a very fundamental theorem in number theory. However, the authors are not aware of any interesting application of this theorem in the context of image analysis.

Also, there are very interesting results in number theory about the number of grid points on curves $ax^2 + bxy + cy^2 + dx + cy + f = 0$ of second order, having integer coefficients $a, ..., f$, see, for example [22]. In case of the parable it holds that:

THEOREM 6.2. *There are either non or infinitely many grid points on a parable having integer coefficients.*

In case of an ellipse $x^2 + Dy^2 = p$, $D \neq p$, it holds that:

THEOREM 6.3. *If p is a prime number, then there are either non or, if $D > 1$, exactly four, or, if $D = 1$, exactly eight grid points on this ellipse.*

In case of the circle $x^2 + y^2 = n$, $n > 1$, it holds that:

THEOREM 6.4. *There are grid points on this circle if and only if any prime factor of n with a representation $4m + 3$ has an even exponent in the canonic partition of n.*

Again, the authors are not aware of applications of such theorems in the context of image analysis.

On the other hand, due to the fact that algorithms in image analysis are running on $n \times n$ arrays, or on digital sets characterized by features such as diameter or width, having natural numbers n as its values, it is straightforward that the output behavior of such algorithms can be described by a number-theoretic function, defined on all values of n, $n = 0, 1, 2, ...$ Features of interest might be extrema, accuracy, or computing time. The discussed examples of straight line segments, moments, or convex hulls are just examples in this wide field of algorithms or mappings of digital objects into digital objects or features. For example, the study of three-dimensional objects (such as digitized planes[17]) adds not only another dimension to the field of possible problems to be studied.

Of course, such questions often also lead to combinatorial problems, i.e. a mathematical discipline "close" to number theory but which is not a subdiscipline of number theory. Many interactions may be stated between image analysis and combinatorics, see, e.g. work reported in [16,34], and a discussion of such interactions between combinatorics and image analysis might be worth a separate paper.

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