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Surface Area Estimation for Digitized Regular Solids

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Abstract

Regularly gridded data in Euclidean 3-space are assumed to be digitizations of regular solids with respect to a chosen grid resolution. Gauss and Jordan introduced different digitization schemes, and the Gauss center point scheme is used in this paper. The surface area of regular solids can be expressed finitely in terms of standard functions for specific sets only, but it is well defined by triangulations for any regular solid. We consider surface approximations of regularly gridded data characterized to be polyhedrizations of boundaries of these data. The surface area of such a polyhedron is well defined, and it is parameterized by the chosen grid resolution. A surface area measurement technique is multigrid convergent for a class of regular solids iff it holds that for any set in this class the surface areas of approximating polyhedra of the digitized regular solid converge towards the surface area of the regular solid if the grid resolution goes to infinity. Multigrid convergent volume measurements have been studied in mathematics for more than one hundred years, and surface area measurements had been discussed for smooth surfaces. The problem of multigrid convergent surface area measurement came with the advent of computer-based image analysis. The paper proposes a classification scheme of local and global polyhedrization approaches which allows us to classify different surface area measurement techniques with respect to the underlying polyhedrization scheme. It is shown that a local polyhedrization technique such as marching cubes is not multigrid convergent towards the true value even for elementary convex regular solids such as cubes, spheres or cylinders. The paper summarizes work on global polyhedrization techniques with experimental results pointing towards correct multigrid convergence. The class of general ellipsoids is suggested to be a test set for such multigrid convergence studies.

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ABSTRACT

Regularly gridded data in Euclidean 3-space are assumed to be digitizations of regular solids with respect to a chosen grid resolution. Gauss and Jordan introduced different digitization schemes, and the Gauss center point scheme is used in this paper. The surface area of regular solids can be expressed finitely in terms of standard functions for specific sets only, but it is well defined by triangulations for any regular solid. We consider surface approximations of regularly gridded data characterized to be polyhedrizations of boundaries of these data. The surface area of such a polyhedron is well defined, and it is parameterized by the chosen grid resolution. A surface area measurement technique is *multigrid convergent* for a class of regular solids iff it holds that for any set in this class the surface areas of approximating polyhedra of the digitized regular solid converge towards the surface area of the regular solid if the grid resolution goes to infinity. Multigrid convergent volume measurements have been studied in mathematics for more than one hundred years, and surface area measurements had been discussed for smooth surfaces. The problem of multigrid convergent surface area measurement came with the advent of computer-based image analysis. The paper proposes a classification scheme of local and global polyhedrization approaches which allows us to classify different surface area measurement techniques with respect to the underlying polyhedrization scheme. It is shown that a local polyhedrization technique such as marching cubes is not multigrid convergent towards the true value even for elementary convex regular solids such as cubes, spheres or cylinders. The paper summarizes work on global polyhedrization techniques with experimental results pointing towards correct multigrid convergence. The class of general ellipsoids is suggested to be a test set for such multigrid convergence studies.

Keywords: Surface area, regular solid, ellipsoid, digitization, isosurface, polyhedrization.

1. INTRODUCTION

The study of regularly gridded data in Euclidean 3-space $\mathcal{E}^3 = (\mathcal{R}^3, d_2)$ is closely related to the general problem of the topology of surfaces in 3-space. The definition of a surface (a special 2-manifold) and of its surface area in $\mathcal{E}^3$ provide an important component of theoretical fundamentals for gridding techniques.

C. Jordan[12] and L.E.J. Brouwer[3] studied the segmentation of an n-space by an $(n-1)$-dimensional manifold. A *region* is a connected open subset of a topological space. The general *Jordan-Brouwer segmentation theorem* states that a compact subset C of the Euclidean space $\mathcal{E}^n = [\mathcal{R}^n, d_2]$ cuts $\mathcal{R}^n$ into two regions if C is a topological image of the $(n-1)$-sphere. For example, a circular curve is a 1-sphere. A compact set C *cuts* $\mathcal{R}^n$ if $\mathcal{R}^n \setminus C$ possesses at least two components. Set A is a *topological image* of set B if there exists a one-to-one mapping (a *homeomorphism*) between both sets which is continuous in both directions.

The Jordan-Brouwer segmentation theorem is known as *Jordan curve theorem* for $n = 2$, and as *Jordan surface theorem* for $n = 3$. This paper deals with topological images of the 2-sphere in 3-space which possess measurable surface areas. These sets are assumed to be surfaces of simply-connected solids as studied in image analysis applications such as volume analysis (MRI, confocal microscopy, CT etc) in medical, industrial or scientific imaging.

A topological image of a 2-sphere possesses a measurable surface area if it allows a decomposition into a finite number of measurable smooth surface patches. The surface area of a measurable smooth surface patch is defined via integration, and three (equivalent) formulae are cited in Section 2. A *regular surface* is a topological image of a 2-sphere which allows a segmentation into a finite number of measurable smooth surface patches, see [16]. A regular surface cuts $\mathcal{R}^3$ into a bounded and into an unbounded region. The topological closure of this bounded region (i.e. the union of this region and its regular surface) defines a *regular solid* in $\mathcal{R}^3$. Note that such a regular solid is simply-connected.

In image analysis, measurements are only possible when they are based on captured regularly gridded data of a given regular solid. We assume that these discrete data are defined by an appropriate digitization scheme and a chosen grid resolution. The task consists in estimating the surface area of the given regular solid based on these available regularly gridded data only such that this estimate converges towards the surface area of the given regular solid if the grid resolution goes to infinity.

The paper is organized as follows: Section 2 specifies sets in $\mathcal{E}^3$ being candidates for multigrid surface area studies. These regular solids are compact, simply connected and do have a measurable surface area, i.e. a rectifiable surface. Example 1 in Section 2 cites an historic result from 1890: multigrid surface area calculation may fail to be convergent even if the correct surface is triangulated with increasing grid resolution! Section 3 defines the multigrid digitization approach and explains the problem of multigrid convergence. Section 4 specifies an approach how to classify polyhedrization techniques into local or global ones, allowing to classify surface area measurement strategies based on such polyhedrizations into local or global ones as well. Section 5 mentions some of the "popular" local techniques and shows by experimental evaluation that these techniques fail to be multigrid convergent with respect to the surface area. Section 6 briefly discusses proposals for global techniques. They seem to be more difficult with respect to implementation, but promising to offer multigrid convergent surface area calculations.

2. REGULAR SOLIDS

This section defines regular solids * to be models of "3D objects". The surface area is well defined for regular solids, and this value is later on used as "ground truth" to evaluate the performance of surface area algorithms.

2.1. Smooth Surface Patches

We start with the definition of a general surface model for simply-connected compact sets. A surface is defined to be a finite union of faces or surface patches, and each patch is defined by a graph of a function.

Definition 2.1. *A smooth surface patch in Euclidean 3-space is a set of points* $\mathbf{F} \subseteq \mathcal{R}^3$ *in parametric form* $\mathbf{F} = \mathbf{F_B}(\varphi, \psi, \chi)$ *where* $\mathbf{B} \subseteq \mathcal{R}^2$ *is a simply-connected compact set and* φ, ψ, *and* χ *are three functions differentiable for all positions* (u, v) *in* $\mathbf{B}$, *such that*

$$\mathbf{F_B}(\varphi, \psi, \chi) = \{(x, y, z) : \ x = \varphi(u, v) \wedge y = \psi(u, v) \wedge z = \chi(u, v) \wedge (u, v) \in \mathbf{B}\},$$

and for which it is assumed that each point in $\mathbf{F_B}(\varphi, \psi, \chi)$ *is defined by exactly one point* $(u, v) \in \mathbf{B}$, *and that the rank of the matrix of the first derivatives*

$$\begin{pmatrix} \varphi_u & \psi_u & \chi_u \\ \varphi_v & \psi_v & \chi_v \end{pmatrix}$$

is equal to 2 for all positions $(u, v) \in \mathbf{B}$.

*Regular solids had been called *Jordan sets* in [16] recognizing the role of C. Jordan in introducing curves and surface patches as parameterized sets. However, many mathematicians, see, e.g., [9,23,26], contributed to the definition of surfaces and surface area of sets in 3-space, and a more generic notation is used therefore in this paper.

From this definition it follows that at each position $\mathbf{b} = (u, v)$ at least one of the three subdeterminants

$$A = \begin{vmatrix} \psi_u & \chi_u \\ \psi_v & \chi_v \end{vmatrix}, \quad B = \begin{vmatrix} \chi_u & \varphi_u \\ \chi_v & \varphi_v \end{vmatrix}, \quad \text{and} \quad C = \begin{vmatrix} \varphi_u & \psi_u \\ \varphi_v & \psi_v \end{vmatrix}$$

is not equal to zero. These subdeterminants can be used to define the *area* $J_{area}(\mathbf{F})$ of a smooth surface patch $\mathbf{F}$ as follows

$$J_{area}(\mathbf{F}) = \int_{\mathbf{B}} \sqrt{A^2 + B^2 + C^2}\, \mathrm{d}\mathbf{b} \tag{1}$$

assuming that the surface patch $\mathbf{F}$ is measurable (see Definition 2.3 and Theorem 2.4 below).

For example assume any simply-connected planar compact surface patch $\mathbf{F}$ incident with a plane $\mathbf{P}$. Then $\mathbf{B}$ may be chosen parallel to plane $\mathbf{P}$, with $\varphi(u, v) = const$, $\psi(u, v) = u$, and $\chi(u, v) = v$. The rank of the corresponding matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is two and the equation

$$J_{area}(\mathbf{F}) = \int_{\mathbf{B}} \mathrm{d}\mathbf{b} = J_{area}(\mathbf{B})$$

reduces the 3D measurement problem to a 2D measurement problem. It follows that any simply-connected planar compact set $\mathbf{F}$ is a smooth surface patch according to the definition above, and it is measurable if its 2D projection is measurable.

Assume a set $\mathbf{A}$ incident with a 2-manifold, i.e. a special two-dimensional topological subspace of $\mathcal{E}^n$. The *2D interior* $\mathbf{A}^\circ$ of such a set $\mathbf{A}$ is defined with respect to this topological space.

The measurability definition in [19] of smooth surface patches $\mathbf{F} = \mathbf{F_B}(\varphi, \psi, \chi)$ is based on a triangulation of a bounded superset $\mathbf{B}_1$ of $\mathbf{B}$ satisfying $\mathbf{B} \subseteq \mathbf{B}_1^\circ$, where the first order derivatives of the functions φ, ψ, χ exist and they are continuous in $\mathbf{B}_1^\circ$. Formally this means that $\varphi, \psi, \chi \in C^1(\mathbf{B}_1^\circ)$.

In No. 108 of [19] it is shown that the angles α of such triangulations (of the base sets of the resulting polyhedral faces) have to satisfy the constraint $\alpha < 2\pi/3$. This was independently shown by *O. Hölder*[9] in 1882, *G. Peano*[23] in 1890, and *H. A. Schwarz*[26] in 1890) to avoid inaccurate surface value calculations for curved surfaces. We cite an example as discussed in [19,26].

Example 1. *Assume that the lateral area $\mathbf{L}$ of a straight circular cylinder of radius ρ and of height h is cut by $(k-1)$ planes, $k \geq 2$, which are parallel to the base circles and which segment the cylinder into k congruent parts. Furthermore assume a regular n-gon, $n \geq 3$, in every cross section including both base circles, see Fig. 1 for $k = 4$ and $n = 6$. The axis of the cylinder and any vertex of such an n-gon defines a halfplane, which bisects an edge of the n-gon in the neighboring cross section or base circle. Now we connect for two neighboring n-gons each edge in one n-gon with those vertex of the other n-gon closest to this edge. This results into a triangulation $\mathbf{T}_{k,n}$ (i.e. a specific polyhedrization) of the lateral area $\mathbf{L}$ of the cylinder into $2kn$ congruent triangles having a surface area equal to*

$$J_{area}(\mathbf{T}_{k,n}) \;=\; 2\pi\rho \cdot \frac{\sin(\pi/n)}{\pi/n} \sqrt{\frac{1}{4}\pi^4\rho^2 \left(\frac{\sin(\pi/2n)}{\pi/2n}\right)^4 \left(\frac{k}{n^2}\right)^2 + h^2}\,.$$

If k and n go to infinity then the length of the edges of the triangular faces of $\mathbf{T}_{k,n}$ converges to zero. However, the surface area of $\mathbf{T}_{k,n}$ does not necessarily converge towards the surface area $J_{area}(\mathbf{L}) = 2\pi\rho h$ of the lateral area! This is only true if k and n go to infinity such that k/n^2 converges to zero. If k/n^2 converges to $g > 0$ then $J_{area}(\mathbf{T}_{k,n})$ converges to

$$2\pi\rho \cdot \frac{\sin(\pi/n)}{\pi/n} \sqrt{\frac{1}{4}\pi^4\rho^2 g^2 + h^2}\,.$$

It may even happen that k/n^2 goes to infinity, e.g. $k = n^3$, and then it follows that $J_{area}(\mathbf{T}_{k,n})$ goes to infinity as well!

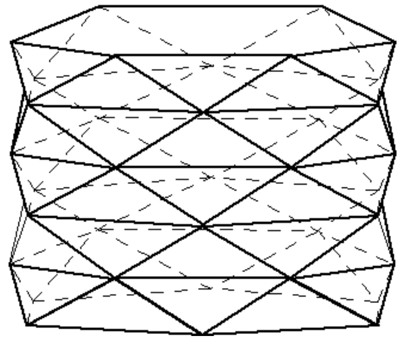

Figure 1. Triangulation of the lateral area of a straight circular cylinder[19,26] defined by $(k-1)$ cutting planes and regular n-gons in cutting planes and base circles, defining a regular grid on the lateral area of the cylinder. If the increase in k is at least $\Omega(n^2)$ then the total surface area of the triangulation does not converge towards the surface area of the lateral area!

Definition 2.2. *Let* $\mathbf{B}_1 \subseteq \mathcal{R}^2$ *be a simply-connected compact set with* $\mathbf{B} \subseteq \mathbf{B}_1^\circ$ *and assume an angle* ω *with* $0 < \omega < \pi/3$. *Then any network* Z *of triangles completely covering* $\mathbf{B}_1$ *and satisfying the following two properties,*

(i) all angles of triangles in Z *are less or equal to* $\pi - \omega$, *and*

(ii) for all triangles in Z *having a non-empty intersection with* $\mathbf{B}$ *it holds that all three vertices are in* $\mathbf{B}_1$,

is called a triangular subdivision Z *of* $\mathbf{B}_1$ *with respect to* $\mathbf{B}$.

Let $\mathbf{F} = \mathbf{F_B}(\varphi, \psi, \chi)$ be a smooth surface patch. Each triangular subdivision Z of $\mathbf{B}_1$ with respect to $\mathbf{B}$ defines a polyhedral approximation $\mathbf{F}(Z)$ of the given smooth surface patch $\mathbf{F} = \mathbf{F_B}(\varphi, \psi, \chi)$ by an orthogonal projection of the vertices † of triangulation Z onto the smooth surface patch $\mathbf{F}$, where the neighborhood relation between vertices in Z is propagated into a neighborhood relation between these resulting vertices on the smooth surface patch $\mathbf{F}$. The surface area $J_{area}(\mathbf{F}(Z))$ is defined to be the sum of all areas of triangular faces of the polyhedral approximation $\mathbf{F}(Z)$.

Definition 2.3. *Let* $\mathbf{F} = \mathbf{F_B}(\varphi, \psi, \chi)$ *be a smooth surface patch. Assume that there exists a simply-connected compact set* $\mathbf{B}_1 \subseteq \mathcal{R}^2$ *such that* $\mathbf{B} \subseteq \mathbf{B}_1^\circ$, *the functions* φ, ψ, χ *are in* $C^{(1)}(\mathbf{B}_1^\circ)$, *and there exists a sequence* Z_1, Z_2, Z_3, ... *of triangular subdivisions of* $\mathbf{B}_1$ *with respect to* $\mathbf{B}$ *such that* $a_t \to 0$ *where* a_t *denotes the maximum length of any side of any triangle in the subdivision* Z_t. *This sequence defines an infinite sequence of polyhedral approximations*

$$\mathbf{F}(Z_1), \quad \mathbf{F}(Z_2), \quad \mathbf{F}(Z_3), \quad \ldots$$

having well-defined surface areas

$$J_{area}(\mathbf{F}(Z_1)), \quad J_{area}(\mathbf{F}(Z_2)), \quad J_{area}(\mathbf{F}(Z_3)), \quad \ldots$$

The smooth surface patch $\mathbf{F}$ *is* measurable *if it has a bounded surface area*

$$J_{area}(\mathbf{F}) = \sup_t J_{area}(\mathbf{F}(Z_t)) \ .$$

The following theorem is a historic result[19] about smooth surface patches, and it allows numerical calculations of surface areas in cases where the parameterization of involved smooth surface patches is known.

†The resulting vertices of this polyhedral approximation are **on** the given smooth surface patch $\mathbf{F}$, and not vertices "close" to the smooth surface patch as in case of digitizations. If the smooth surface patch $\mathbf{F}$ is "unknown", as it is in image analysis applications, then we also have no "access" to such a polyhedral approximation $\mathbf{F}(Z)$.

Theorem 2.4. *For a measurable smooth surface patch* $\mathbf{F} = \mathbf{F_B}(\varphi, \psi, \chi)$,

$$J_{area}(\mathbf{F}) = \int\limits_{\mathbf{B}} \sqrt{A^2 + B^2 + C^2}\, \mathrm{d}\mathbf{b}$$

independent of the chosen parameterization $\mathbf{B}$, φ, ψ, *where* A, B *and* C *are the subdeterminants as defined above.*

Consider a surface patch $\mathbf{F}$ with rectangular Cartesian coordinates x, y, z given by an equation $z = f(x, y)$, and (x, y) is limited to be within a closed, bounded and measurable set $\mathbf{B} \subset \mathcal{R}^2$, and the first-order partial derivatives of f exist and are continuous within a set $\mathbf{B}_1$ satisfying $\mathbf{B} \subseteq \mathbf{B}_1^\circ$. Then the content of the surface patch $\mathbf{F} = \{(x, y, f(x, y)) : (x, y) \in \mathbf{B}\}$ is equal to

$$J_{area}(\mathbf{F}) = \int\limits_{\mathbf{B}} \sqrt{1 + (f_x(x, y))^2 + (f_y(x, y))^2}\, \mathrm{d}\mathbf{b}\,. \tag{2}$$

This follows from Theorem 2.4. Furthermore, often the values

$$E = \varphi_u^2 + \psi_u^2 + \chi_u^2\,, \quad F = \varphi_u\varphi_v + \psi_u\psi_v + \chi_u\chi_v\,, \quad G = \varphi_v^2 + \psi_v^2 + \chi_v^2$$

are used in the theory of curved surfaces, and due to the Lagrange identity it holds that $A^2 + B^2 + C^2 = EG - F^2$, and thus

$$J_{area}(\mathbf{F}) = \int\limits_{\mathbf{B}} \sqrt{EG - F^2}\, \mathrm{d}\mathbf{b}\,. \tag{3}$$

Theorem 2.4 points out that the area of a surface patch is only dependent upon its geometric structure, and Equs. (1), (2), and (3) specify three different formal ways of calculating such an area.

2.2. Regular Surfaces and Surface Area

A single smooth surface patch cannot form a surface of a non-trivial (i.e. having a non-zero volume) set in 3-space. Because of the assumed property that each point in $\mathbf{F_B}(\varphi, \psi, \chi)$ is defined by exactly one point $(u, v) \in \mathbf{B}$ it follows that at least two smooth surface patches are necessary to obtain a closed surface of such a non-trivial set. Furthermore, the assumed C^1 property of functions φ, ψ, and χ allows no discontinuities within a single Jordan face, as it appears at edges of polyhedra.

Definition 2.5. *A* regular solid *is a simply-connected compact set* $\Theta \subset \mathcal{R}^3$ *where its boundary is a union of a finite number of measurable smooth surface patches* $\mathbf{F}_1$, $\mathbf{F}_2$, ..., $\mathbf{F}_n$ *having pairwise disjoint 2D interiors. A* regular surface $\mathbf{S} = S(\Theta)$ *is the boundary* $\partial\Theta$ *of such a regular solid* Θ*, i.e.*

$$\mathbf{S} = \mathbf{F}_1 \cup \mathbf{F}_2 \cup \ldots \cup \mathbf{F}_n,$$

and the surface area *of* Θ *or* $\mathbf{S}$ *is defined as*

$$J_{area}(\mathbf{S}) = J_{area}(\mathbf{F}_1) + J_{area}(\mathbf{F}_2) + \ldots + J_{area}(\mathbf{F}_n)\,.$$

The region $\Theta^\circ = \Theta - \partial\Theta$ is the 3D *interior* of this regular solid Θ. Note that a regular solid is always homeomorphic to the 3-ball, and a regular surface is always homeomorphic to the 2-sphere, i.e. the surface of the 3-ball.

Each simple polyhedron is a regular solid, and curved sets in $\mathcal{E}^3$ may be classified to be regular solids as well. A *smooth regular solid* has a surface which possesses a uniquely defined tangent plane at each of its surface points. Note that this is not necessarily the case for a union of measurable smooth surface patches. The following (well-known) Jordan surface theorem (a conclusion of the general Brouwer-Jordan segmentation theorem) holds for smooth, and also for non-smooth regular surfaces.

Theorem 2.6. *Any surface* $\mathbf{S}$ *of a regular solid cuts* $\mathcal{R}^3$ *into two regions* $\mathbf{I}$, *and* $\mathbf{E} = \mathcal{R}^3 - (\mathbf{S} \cup \mathbf{I})$ *with* $\partial\mathbf{I} = \partial\mathbf{E} = \mathbf{S}$.

The region $\mathbf{E} = E(\mathbf{S})$ is the 3D *exterior* of the regular solid $\mathbf{S} \cup \mathbf{I}$. "Going from $\mathbf{I}$ to $\mathbf{E}$ " means that we have "to leave $\mathbf{I}$ " by passing through its boundary $\partial\mathbf{I} = \mathbf{S}$, i.e. any curve starting in $\mathbf{I}$ and ending in $\mathbf{E}$ intersects the given surface **S** at least once. A regular surface specifies a separation in Euclidean 3-space, as a Jordan curve does in Euclidean 2-space.

EXAMPLE 2. *The surface area of a general ellipsoid in 3-space with semi–axes a, b, c can be expressed finitely in terms of incomplete elliptic integrals.*[22] *This formula is accurate if the ellipsoid is not "nearly spheric". The representation of the surface area may be analytically specified if two radii coincide, i.e. in case of an ellipsoid of revolution. We use a numeric solution which allows accurate calculations independent upon the parameters of the ellipsoid. We explain the theoretical fundamentals of this general program* ‡ *which allows to calculate surface areas of arbitrary ellipsoids. This program has been used in our experiments for providing the true value which will be reported later in this paper.*

Consider an ellipsoid $\mathbf{E}_{a,b,c}$ centered at the coordinate origin in 3-space, with rectangular Cartesian coordinate axes along the semi–axes a, b, c,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\,.$$

For the case in which two axes are equal $b=c$, the surface is generated by rotation around the x–axis of the half–ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}\;=\;1$$

with $y \geq 0$. The surface area $J_{area}(\mathbf{E}_{a,b,b})$ is equal to

$$4\pi ab\int_0^1\sqrt{1-qu^2}\;\mathrm{du}$$

where $u=x/a$ and $q=1-b^2/a^2$. Therefore,

$$J_{area}(\mathbf{E}_{a,b,b})\;=\;\begin{cases} 2\pi b\left(a\times\frac{\arcsin\sqrt{q}}{\sqrt{q}}+b\right) & \text{if}\quad q>0\\ 2\pi b(a+b) & \text{if}\quad q=0\\ 2\pi b\left(a\times\frac{\operatorname{arcsinh}\sqrt{-q}}{\sqrt{-q}}+b\right) & \text{if}\quad q<0.\end{cases}$$

The use of the truncated power series

$$J_{area}(\mathbf{E}_{a,b,b})\;=\;2\pi b\left(a\left[1+\tfrac{1}{6}q+\tfrac{3}{40}q^2+\tfrac{5}{112}q^3\right]+b\right)$$

is suggested for $|q|\ll 1$.

A program for calculating the surface area of a general ellipsoid $\mathbf{E}_{a,b,c}$ may follow Equ. (2) assuming equations $z=f(x,y)$. W.l.o.g. the coordinate axes can be named so that $a\geq b\geq c$. On that surface,

$$\frac{\partial f}{\partial x}\;=\;\frac{-c^2x}{a^2z},\qquad \frac{\partial f}{\partial y}\;=\;\frac{-c^2y}{b^2z}$$

Consider the octant for which x, y, z are all non–negative. Then the surface area for that octant is

$$\int_0^a\int_0^{b\sqrt{1-x^2/a^2}}\sqrt{1+\frac{c^4x^2}{a^4z^2}+\frac{c^4y^2}{b^4z^2}}\;\mathrm{dy\;dx}$$

$$=\;\int_0^a\int_0^{b\sqrt{1-x^2/a^2}}\sqrt{\frac{\frac{z^2}{c^2}+\frac{c^2}{a^2}\frac{x^2}{a^2}+\frac{c^2}{b^2}\frac{y^2}{b^2}}{z^2/c^2}}\;\mathrm{dy\;dx}$$

$$=\;\int_0^a\int_0^{b\sqrt{1-x^2/a^2}}\sqrt{\frac{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{c^2}{a^2}\frac{x^2}{a^2}+\frac{c^2}{b^2}\frac{y^2}{b^2}}{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\;\mathrm{dy\;dx}\,.$$

‡Program and its theoretical derivation, as reported in this example, by Garry Tee (Auckland).

The integral with respect to y is

$$\int_0^{b\sqrt{1-x^2/a^2}} \sqrt{\frac{1-\left(1-\frac{c^2}{a^2}\right)\frac{x^2}{a^2}-\left(1-\frac{c^2}{b^2}\right)\frac{y^2}{b^2}}{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}\ \mathrm{dy}$$

$$= \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{\frac{1-\left(1-\frac{c^2}{a^2}\right)\frac{x^2}{a^2}}{1-\frac{x^2}{a^2}}}\sqrt{\frac{1-\frac{\left(1-\frac{c^2}{b^2}\right)}{1-\left(1-\frac{c^2}{a^2}\right)\frac{x^2}{a^2}}\frac{y^2}{b^2}}{1-\frac{1}{\left(1-\frac{x^2}{a^2}\right)}\frac{y^2}{b^2}}}\ \mathrm{dy}$$

$$= b\sqrt{1-\frac{x^2}{a^2}}\sqrt{\frac{1-\left(1-\frac{c^2}{a^2}\right)\frac{x^2}{a^2}}{1-\frac{x^2}{a^2}}}\int_0^1\sqrt{\frac{1-mt^2}{1-t^2}}\ \mathrm{dt}$$

$$= b\sqrt{1-\left(1-\frac{c^2}{a^2}\right)\frac{x^2}{a^2}}\ E(m),$$

where $s = x/a$, $t \;=\; y\big/\left[b\sqrt{1-s^2}\right]$,

$$m \;=\; \frac{\left(1-\frac{c^2}{b^2}\right)\left(1-s^2\right)}{1-\left(1-\frac{c^2}{a^2}\right)s^2} \tag{4}$$

and

$$E(m) \;=\; \int_0^1\sqrt{\frac{1-mt^2}{1-t^2}}\ \mathrm{dt}$$

is Legendre's complete elliptical integral $E(m)$ *of the second kind, see Milne-Thomson's notation in chapter 17 of.*[1] *Hence, the surface area of the ellipsoid is*

$$J_{area}(\mathbf{E}_{a,b,c}) \;=\; 8b\int_0^a\sqrt{1-\left(1-\frac{c^2}{a^2}\right)\frac{x^2}{a^2}}\ E(m)\ \mathrm{dx}$$

$$= 8ab\int_0^1\sqrt{1-\left(1-\frac{c^2}{a^2}\right)s^2}\ E(m)\ \mathrm{ds}\ .$$

That integral expression for $J_{area}(\mathbf{E}_{a,b,c})$ *can be evaluated numerically. However, Archimedes showed that, for* $a = b = c$, *the integrand is constant at* $\pi/2$. *But, as* $c/a \searrow 0$, *the integrand converges to* $\sqrt{1-s^2} = \sqrt{(1-s)(1+s)}$, *which has a singular derivative at* $s = 1$. *To get a smoother integrand, substitute* $u^2 \;=\; 1-s$, *which gives*

$$J_{area}(\mathbf{E}_{a,b,c}) \;=\; 16ab\int_0^1\sqrt{1-\left(1-\frac{c^2}{a^2}\right)s^2}\ E(m)\ u\ \mathrm{du}, \tag{5}$$

where $s = 1-u^2$, *and the parameter* m, $0 \le m \le 1$, *is given in Equ. (4). The integral* $E(m)$ *decreases from* $\pi/2$ *to 1 as m increases. The Arithmetic-Geometric Mean of Gauss is very efficient for evaluating both* $E(m)$ *and Legendre's complete elliptical integral* $K(m)$ *of the first kind,*

$$K(m) \;=\; \int_0^1\frac{\mathrm{dt}}{\sqrt{(1-t^2)(1-mt^2)}}$$

see Milne-Thomson, Section 17.6 in [1]. But if m is very close to 1, then the power series in $m_1 = 1-m$ *[5] should be used to evaluate* $K(m)$ *for* $m < 1$, *and to evaluate* $E(m)$ *for* $m \le 1$.

The integrand with respect to u in Equ. (5) is smooth enough for the integral to be evaluated by Romberg integration, for $c > 0$. As $c/a \searrow 0$ the ellipsoid converges to a 2–sided elliptical lamina, with surface area $2\pi ab$. Hence, for $c/a \ll 1$, the area should be evaluated as $2\pi ab$, plus some terms in c/a.

For our experimental studies of multigrid convergence, different ellipsoids are used to illustrate multigrid convergence. For surface area approaches which are not multigrid-convergent, it will be shown that they fail already for very elementary regular solids such as a cube or a sphere.

3. DIGITIZATION AND MULTIGRID CONVERGENCE

The problem of gridding based volume estimation was studied by *C. Jordan*[13] in 1892. Any grid point $(i, j, k) \in \mathcal{E}^3$ is assumed to be the center point of a cube with faces parallel to the coordinate planes and with edges of length 1. The boundary is part of this cube. Let Θ be a regular solid, i.e. it is contained in finitely many of such cubes. Dilate the set Θ with respect to an arbitrary point $p \in \mathcal{E}^3$ in a ratio $r : 1$. This transforms Θ into Θ_r^p. Let $l_r^p(\Theta)$ be the number of all cubes completely contained in the interior of Θ_r^p, and let $u_r^p(\Theta)$ be the number of all cubes having a non-empty intersection with Θ_r^p. Then it holds[13] that $r^{-3} \cdot l_r^p(\Theta)$ and $r^{-3} \cdot u_r^p(\Theta)$ always converge to limit values $L(\Theta)$ and $U(\Theta)$, respectively, for r to infinity, independent upon the chosen point p. Jordan called $L(\Theta)$ the *inner volume* and $U(\Theta)$ the *outer volume* of set Θ, or the *volume* $vol(\Theta)$ of Θ if $L(\Theta) = U(\Theta)$. Volume definition based on gridding techniques was studied, e.g., in [21,27].

The problem of area estimation of a set by the number of grid points contained in the considered set has an extensive history in number theory. It has already been studied by *C.F. Gauss* (1777–1855) for disks. *C.F. Gauss* and *P. Dirichlet* (1805–1859) knew already that the number of grid points inside of a planar convex curve γ estimates the area of the set bounded by this curve within an order of $\mathcal{O}(l)$, where l is the length of curve γ. The situation when γ is a circle is studied most carefully. *M.N. Huxley*'s result[11] from 1990 is recently the best known result for 3-smooth planar convex curves.

These two gridding approaches use different digitization models. Often we prefer Gauss' center-point digitization model: if a grid point $(i, j, k) \in \mathcal{E}^3$ belongs to Θ, being a center point of cube c then c is contained in the digital image of Θ. However, there might be cases where Jordan's scheme is more adequate, and the used scheme has to be named from case to case.

3.1. Multigrid Digitization

We assume an orthogonal grid with grid constant $0 < \vartheta \leq 1$ in n-space $\mathcal{E}^n$, $n \geq 1$, i.e. ϑ is the uniform spacing between grid points parallel to one of the coordinate axes. Furthermore, let $r \geq 1$ be the *grid resolution* defined as being the number of grid points per unit, i.e. any grid edge is of length $\vartheta = 1/r$.

In this paper we discuss the three-dimensional case only. We consider *r-grid points* $g_{i,j,k}^r = (\vartheta \cdot i, \vartheta \cdot j, \vartheta \cdot k)$ in $\mathcal{E}^3$ for integers i, j, k and $\vartheta = 1/r$. For $r = 1$ we simply speak about *grid points* (i, j, k) in $\mathcal{E}^3$.

DEFINITION 3.1. *For a set Θ in $\mathcal{E}^3$ its* digitization $D_r(\Theta)$ *is defined to be the set of all r-grid points contained in the given set Θ, i.e.*

$$D_r(\Theta) = \{g_{i,j,k}^r \ : \ g_{i,j,k}^r = (i/r, j/r, k/r) \in \Theta\} \ .$$

In the case $r = 1$ the digitization is denoted by $D(\Theta)$.

This is the model of *Gauss center-point digitization.* Let us define an r-cube $C_{i,j,k}^r$ as an *isothetic* cube, i.e. having faces only parallel to one of the coordinate planes, whose edges have length $1/r$ where the centroid of the cube is the r-grid point (i, j, k). Then *Jordan digitization* is defined as follows.

DEFINITION 3.2. *For a set Θ in $\mathcal{E}^3$ its inner and outer digitizations are defined by*

$$I_r(\Theta) = \{g_{i,j,k}^r \ : \ g_{i,j,k}^r = (i/r, j/r, k/r) \in \mathbf{I}_r(\Theta)\}$$

and

$$O_r(\Theta) = \{g_{i,j,k}^r \ : \ g_{i,j,k}^r = (i/r, j/r, k/r) \in \mathbf{O}_r(\Theta)\},$$

respectively, where $\mathbf{I}_r(\Theta)$ is the union of all cubes $C_{i,j,k}^r$ such that $C_{i,j,k}^r \subseteq \Theta^0$ and $\mathbf{O}_r(\Theta)$ is the union of all cubes $C_{i,j,k}^r$ such that $C_{i,j,k}^r \cap \Theta \neq \emptyset$.

The sets $D(\Theta)$ and $D_r(\Theta)$ (res. $I_r(\Theta)$ and $O_r(\Theta)$) are also called *digital sets*, and the set $\mathbf{D}_r(\Theta)$ defined by the union of all cubes $C^r_{i,j,k}$ such that $(i,j,k) \in D_r(\Theta)$ (res. $\mathbf{I}_r(\Theta)$ and $\mathbf{O}_r(\Theta)$) are isothetic polyhedra.

The *dilation* of a set $\Theta \subset \mathcal{E}^3$ by a factor $r \geq 1$ is defined to be

$$r \cdot \Theta \ = \ \{(r \cdot x, \ r \cdot y, \ r \cdot z) \ : \ (x,y,z) \in \Theta\} \ .$$

Following *Jordan*[13] this is a dilation with respect to the origin $(0,0,0)$, and other points in $\mathcal{E}^3$ could be chosen to be the fixpoint as well. Sometimes it may be more adequate to consider sets of the form $r \cdot \Theta$ (the preferred approach, e.g., by *Jordan* and *Minkowski*) digitized in the orthogonal grid with unit grid length, instead of sets Θ digitized in r-grids with $1/r$ grid length. The study of $r \to \infty$ corresponds to the increase in grid resolution, and this may be either a study of repeatedly dilated sets $r \cdot \Theta$ in the grid with unit grid length, or of a given set Θ in repeatedly refined grids. This is a general *duality principle for multigrid studies.*

3.2. Multigrid Convergence

Multigrid convergence § is one option for model-based evaluations of image analysis algorithms in general. This criterion is in common use in numerical mathematics. In general, algorithms may be judged according to criteria, such as methodological complexity of underlying theory, expected time for implementation, or run-time behavior and storage requirements of the implemented algorithm. Accuracy is an important criterion as well, and this can be modeled as convergence towards the true value for grid based calculations.

DEFINITION 3.3. *Let $\mathcal{F}$ be a family of sets $\mathbf{S}$, and $dig_r(\mathbf{S})$ a digital image of set $\mathbf{S}$, defined by a digitization mapping dig_r. Assume that a feature F, such as area, perimeter, or a moment, is defined for all sets in family $\mathcal{F}$. An estimator M is* multigrid convergent *for this family $\mathcal{F}$ and this digitization model dig_r iff there is a grid resolution $r_\mathbf{S} > 0$ for any set $\mathbf{S} \in \mathcal{F}$ such that the estimator value $M(dig_r(\mathbf{S}))$ is defined for any grid resolution $r \geq r_\mathbf{S}$, and $|M(dig_r(\mathbf{S}) - F(\mathbf{S})| < f(r)$ for a function f defined for real numbers, having positive real values only, and converging towards 0 if $r \to \infty$. The function f specifies the* convergence speed*.*

We do not discuss convergence speed in this paper. The function $f(r) = r^{-1}$ would specify *linear convergence*, and the function $f(r) = r^{-2}$ defines *quadratic convergence* - just to name two examples.

A surface area measurement technique is multigrid convergent for a class of regular solids iff it holds that for any set in this class the surface areas of approximating polyhedra of the digitized regular solid converge towards the true surface area of the regular solid if the grid resolution goes to infinity. In general it holds that the convergence of a gridding technique with respect to the Hausdorff-Chebyshev distance d_∞¶ between the given surface patch $\mathbf{A}$ and the approximated surface patch $\mathbf{B}_r$ does not imply multigrid convergence of surface areas of $\mathbf{B}_r$ towards the surface area of $\mathbf{A}$.

Classical results can be cited for multigrid convergent volume measurement, see, e.g. [27]. The convergence of these measurements towards the true value is illustrated in Fig. 2. We digitized a cube for different rotational positions, for $r \to \infty$. For consecutive grid constants $1/r$ we calculate the volume of the resulting isothetic polyhedron as number of r-cubes contained in this isothetic polyhedron times the volume $1/r^3$ of a single r-cube.

4. CLASSIFICATION OF POLYHEDRIZATION TECHNIQUES

Assume a regular solid Θ and a grid resolution r, we obtain $D_r(\Theta)$ or $I_r(\Theta)$ as the digitization of Θ. A polyhedrization technique is supposed to construct a polyhedron $\mathbf{P}_r$ approximating the surface of any regular solid Θ from a given digitization of this set, such as set $D_r(\Theta)$ or set $I_r(\Theta)$. W.l.o.G. let us select $D_r(\Theta)$ for the following discussion. The surface of a regular solid Θ is a Jordan surface. We assume that the resolution r is sufficiently high so that $\mathbf{P}_r$

§This is also called *multiresolution convergence* in other work. We prefer "multigrid" due to the pure geometric nature of the approach, compared to a traditional signal-theoretical interpretation of the word "resolution".

¶The Hausdorff-Chebyshev metric d_∞ is defined for sets of points A, B using the point metric d_∞ as introduced in [7] as generator,

$$d_\infty(A,B) = \max\left\{\sup_{p\in A}\inf_{q\in B} d_\infty(p,q), \sup_{p\in B}\inf_{q\in A} d_\infty(p,q)\right\},$$

where $d_\infty(p,q) = \max\{|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|\}$, for $p = (x_1,x_2,x_3)$ and $q = (y_1,y_2,y_3)$.

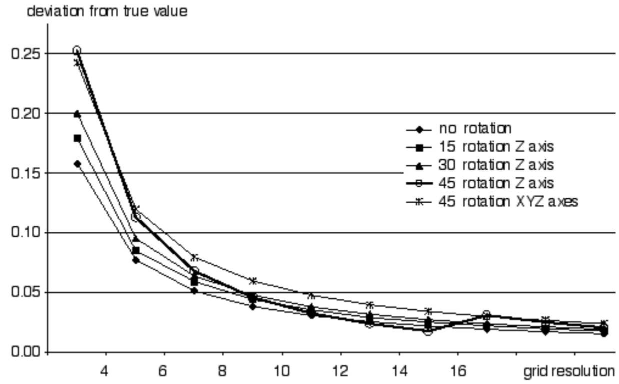

Figure 2. Convergence of measured volumes $(1+d)v$ of isothetic polyhedra towards the true volume v of the given cube, defining a *deviation* d or a *relative error* $|d|$.

becomes a simple polyhedron. The constructed polyhedron $\mathbf{P}_r$ is defined by polygons (faces), edges and vertices. For each polygon $\mathbf{G}$ of $\mathbf{P}_r$, there is a subset of $D_r(\Theta)$ such that only grid points in this subset have influence on the specification of polygon $\mathbf{G}$. This subset defines by inclusion a smallest ball $\mathbf{B}(\mathbf{G})$ with radius ρ, i.e. $\mathbf{B}(\mathbf{G})$ contains all the points in $D_r(\Theta)$ which have any impact on the position, orientation and size of $\mathbf{G}$. In other words, $\mathbf{B}(\mathbf{G})$ might be called the *ball of influence* for constructing the polygon labeled by $\mathbf{G}$. Assume a digitization $D_r(\Theta)$ being the input for the polyhedrization. Due to the finite number of calculated polygons, there is a maximum value $R(r,\Theta)$ of all radii ρ defined by balls of influence $\mathbf{B}(\mathbf{G})$, for any polygon $\mathbf{G}$ of the resulting polyhedrization $\mathbf{P}_r$.

DEFINITION 4.1. *A polyhedrization technique is called to be* local *if there exists a constant κ such that $R(r,\Theta) \leq \kappa/r$, for any regular solid Θ and any grid resolution r. If a polyhedrization method is not local then it is called* global.

We give several examples for local and global polyhedrization techniques as follows:

- **local polyhedrization technique** $\cdots$ the surface detection algorithm of voxel-based objects,[2] the marching cubes algorithm,[18] the marching tetrahedra algorithm,[6,24] the dividing cubes algorithm,[25] the discretized marching cubes algorithm,[20] and the algorithm for generation of discrete combinatorial polyhedra[14];
- **global polyhedrization technique** $\cdots$ minimum surface area polyhedra,[28] convex hull of $D_r(\Theta)$, discrete standard polyhedra.[8]

In the sequel we call a surface area calculation method local or global, depending upon the underlying polyhedrization technique assuming that the surface area calculation technique is defined by taking the surface area of the resulting polyhedra as its approximation value.

5. LOCAL TECHNIQUES

Let us consider the surface area of a resulting isothetic polyhedron $\mathbf{D}_r(\Theta)$ (see Fig. 3) using the algorithm published in [2] for tracing all faces on the surface (note: surface area measurement was not a subject in [2]). The figure signals convergence in all cases. However, the measured surface area values depend upon the given rotation of the cube, and the deviation d can be equal to 0 if the cube was in isothetic position, and about 0.90 (i.e. 90% error!) if it was rotated about 45° with respect to X, Y, and Z axes. These values are inappropriate for estimating a surface area of a cube if the rotation angle is unknown. A surface area measurement based on counts of faces on the surface of such isothetic polyhedra is *not related* to the true surface area, just to exclude this as a way of surface area estimation. A "classical" example is shown in Fig. 4) illustrating again that such a "staircase approach" fails if surface area measurement is concerned.

The use of a marching cubes algorithm[18] is one of the options of local approximations. Each local configuration of eight r-grid points, forming a cube, is treated according to a look-up table for defining triangular or planar surface

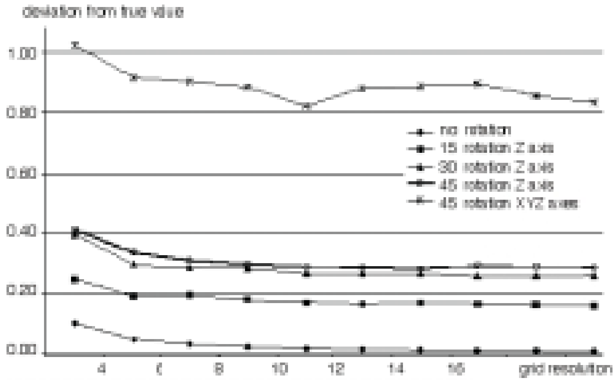

Figure 3. Each curve suggests "obvious convergence" of the measured surface areas of the isothetic polyhedra for a given rotational position of the cube towards a value $(1 + d)s$, where s denotes the true surface area of this cube.

patches within this cube. We may assume that a surface intersects such a local configuration in 2^8 different ways (i.e. no multiple intersections of grid edges), and these can be represented as fourteen major cases with respect to rotational symmetry. Alternatively a method developed by [31] calculates "contour chains" immediately without using a look-up table of all 2^8 different cases. The fourteen basic configurations originally suggested by [18] are topologically "incomplete". Occasionally they generate surfaces with holes. Such ambiguities of the marching cube look-up tables are discussed in [30]. See [28] for local situations of marching cubes configurations where at least two different topological interpretations are possible. Additional marching cubes configurations are also suggested in [15] from the viewpoint of combinatorial topology; all possible connectivities between any vertices of a marching cube are considered and additional configurations of marching cubes are proposed. More important, the calculated values of surface areas do not converge towards the true value as illustrated in Fig. 5. The total surface area is calculated based on a sum of all surface areas for the generated local (triangular) surface patches.

There are other techniques similar to the marching cubes algorithm. A dividing cubes algorithm[25,29] was proposed to improve the performance of the marching cubes algorithm by making use of hierarchical data structures such as octrees; the constructed polyhedra are equivalent to those by the marching cubes algorithm. A marching tetrahedra algorithm was suggested in [24]. By this algorithm, triangular surface patches are constructed referring to a look-up table for each local configuration of four r-grid points forming a tetrahedron instead of eight r-grid points forming a cube. Any tetrahedron is obtained by dividing a cube. Further studies on marching tetrahedra are reported in [6]. This algorithm generates more triangles than the marching cubes algorithm in general. On the other hand, a discretized marching cubes algorithm[20] was proposed for reducing the number of triangular surface patches. While the classical marching cubes algorithm determines every vertex of any triangular patch by interpolation on an edge

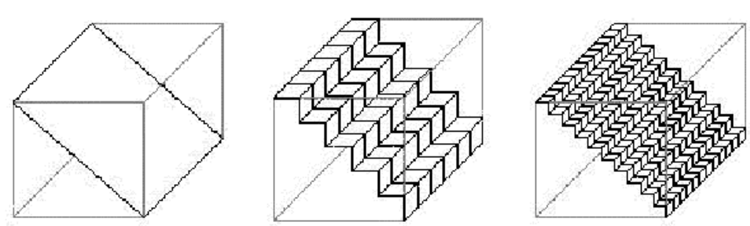

Figure 4. Assume a regular solid having a surface patch being a rectangle with slope $(45°, 45°, 0°)$ approximated by its digitization which forms a regular staircase for any grid resolution r. The total surface area of this staircase remains constant, i.e. it is always the same false value.

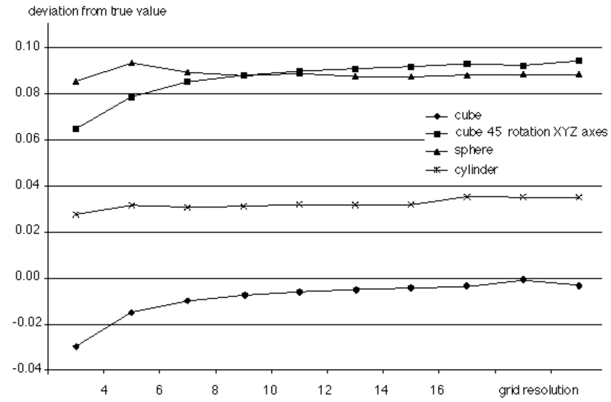

Figure 5. These curves indicate convergence of the measured surface areas towards values $(1+d)s$ where a marching cubes algorithm was applied to two cubes in different rotational positions, to a sphere and a cylinder

of a marching cube, this algorithm always sets such vertex to be the midpoint of an edge. Similarly, an algorithm for generation of discrete combinatorial polyhedra whose vertices are set to be only at r-grid points was proposed in.[14] This is based on the approach of combinatorial topology and there is a proof that any constructed polyhedra have no topological ambiguity.

For any technique, the constant κ is not more than $\frac{\sqrt{3}}{2}$. This is because we use a cube or a tetrahedron which is smaller than a cube to look up triangular surface patches and a cube is clearly included in a ball $\mathbf{B}(\mathbf{G})$ with radii $\frac{\sqrt{3}}{2r}$.

Error analysis of surface area calculation is studied when isothetic polyhedra are used for surface approximation in [17]; for a plane surface, the calculated surface area may be from 1 to $\sqrt{3}$ times the true value. This implies that deviation of calculated surface area from true values does not depend on the grid resolution r at all and is constant for a plane surface. In other words, the surface area errors between calculated values and true values are caused by the position and orientation of a plane surface and are not caused by the grid resolution r. For instance, if a plane has the normal vector $(1,1,1)$ and passes through an r-grid point, the calculated surface area becomes $\sqrt{3}S$ where S is the true value.

The similar analysis can be employed for the other local polyhedrization techniques such as the marching cubes algorithm. Because constructed surface patches are different between local polyhedrization techniques, the ratios between calculated surface area values and true values will be different for each polyhedrization technique. However, it is also true for the other local polyhedrization techniques that the surface area errors between calculated values and true values are caused by the position and orientation of a plane surface and are not caused by the grid resolution r. Actually, the number of normal vectors of triangular patches of a marching cubes algorithm is larger than that of an isothetic polyhedra generation. Thus, we expect that the surface area errors of a marching cube algorithm will be smaller than those of an isothetic polyhedron generation.

Because a cube is defined by six plane surfaces, we can extend our discussion of surface area errors for a plane surface to a cube. There are always errors around the corner of a cube and such errors depend on the grid resolution r.

6. GLOBAL TECHNIQUES

Local techniques exist in different implementations, and they are efficient with respect to computing time behavior. The crucial drawback as briefly discussed above is that they are not multigrid convergent to the true surface area value. There are also a few proposals on global techniques. But to our knowledge there are no theoretical results available about multigrid convergence, no efficient implementations (besides efficient convex hull algorithms), and even no experimental studies on multigrid behavior at all. For example, a global technique including experimental results (i.e. showing a few approximated surfaces, but not with respect to surface area) is contained in [8].

6.1. Minimum Surface Area Polyhedra

The surface area measurement approach introduced in [28] is a special approach towards global surface approximations. The Jordan digitization model is used in this case. Assume that both the inner interior $\mathbf{I}_r(\Theta)$ and the outer interior $\mathbf{O}_r(\Theta)$ of a given 3D object Θ are simply connected sets with respect to the given grid constant r. The set Θ remains constant in the sequel and we omit it from the formulas. Assume $\mathbf{I}_r \neq \emptyset$. Let $\partial \mathbf{I}_r$ be the surface of the inner interior $\mathbf{I}_r(\Theta)$, and $\partial \mathbf{O}_r$ be the surface of the isothetic polyhedron $\mathbf{O}_r$. Then it holds that

$$\emptyset \subset \mathbf{I}_r \subset \mathbf{O}_r^\circ \quad \text{and} \quad \partial \mathbf{I}_r \cap \partial \mathbf{O}_r = \emptyset,$$

and $\mathbf{O}_r \setminus \mathbf{I}_r^\circ$ is an isothetic polyhedron homomorphic with the torus, and for the Hausdorff-Chebyshev metric d_∞ it follows that

$$d_\infty(\partial \mathbf{I}_r, \partial \mathbf{O}_r) \geq 1/r.$$

Under the given assumptions it is possible to dilate $\mathbf{I}_r$ and to erode $\mathbf{O}_r$ (informally speaking, by adding or subtracting r-cubes) such that finally

$$d_\infty(\partial \mathbf{I}_r, \partial \mathbf{O}_r) = 1/r.$$

Note that these resulting sets $\mathbf{I}_r$ and $\mathbf{O}_r$ are not uniquely defined in general. However, the resulting *r-boundary* of Θ,

$$B_r(\Theta) = \mathbf{O}_r \setminus \mathbf{I}_r^\circ$$

is a *simple closed two-dimensional grid continuum* as defined in [28]. We denote it by $B_r(\Theta) = [S_1, S_2]$, where $S_1 = \partial \mathbf{I}_r$, and $S_2 = \partial \mathbf{O}_r$. It follows that $\partial B_r(\Theta) = S_1 \cup S_2$.

In analogy to the MLP technique in two dimensions it seems to be straightforward that the *surface area* of such a simple closed two-dimensional grid continuum $[S_1, S_2]$ in $\mathcal{R}^3$ should be defined to be the surface area of a minimum area polyhedral simple closed Jordan surface in $[S_1, S_2]$ containing S_1. Note that a proper inclusion $\emptyset \subset \mathbf{I}_r$ had been assumed. Thus an r-boundary is defined and there exist polyhedral simple closed Jordan surfaces in $[S_1, S_2]$ containing S_1. Let $\mathbf{P}$ be the set of all these polyhedral simple closed Jordan surfaces in $[S_1, S_2]$ containing S_1. The question arises whether the constraint of having a minimum surface area specifies one Jordan surface in $\mathbf{P}$ uniquely or not. Simple examples of configurations $[S_1, S_2]$ illustrate that the allowance of non-grid point vertices may reduce the surface area of polyhedra. Thus a limitation of the numbers of vertices, or a limitation on allowing r-grid point vertices only is required to stop a process of repeated "refinements" for reducing the surface area. However, this has impacts on the proof whether such a minimum-area polyhedron is uniquely defined or not. Furthermore, work on minimum-length polygons in 3D has shown that optimization seems to be unavoidable for vertex calculation[4] if non-r-grid points are allowed.

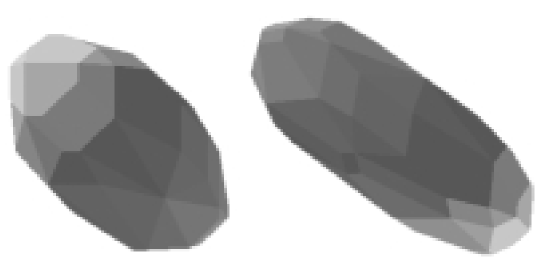

Figure 6. Visualization of a calculated convex hull of one of the rotated and digitized ellipsoid.

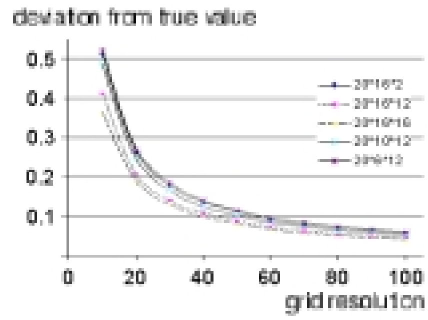

Figure 7. The relative errors of approximated surface area values of ellipsoids of different sizes, all rotated by 45 degrees about z- and y-axis. The grid value denotes the number of grid points within an interval defined by the maximum-value radius of the given ellipsoid.

6.2. Convex Hulls

We consider the calculation of the convex hull of a digitized ellipsoid, see Fig. 6. The calculated planar patches of the approximating convex polyhedron are not limited in size. The convex hull calculation is a global technique with respect to our definition given above.

The surface area of a convex polyhedra is well defined. We use these values for the estimation of the surface area of a given ellipsoid based on gridding techniques. Figure 7 shows the relative errors, i.e. the absolute difference between true surface area calculated based on the method described in Example 2, and the surface area of the convex hull of the digitized ellipsoid assuming a selected grid resolution and the Gauss center-point digitization scheme, normalized by dividing this difference by the true surface area value. These relative errors go to zero, see Fig. 7. Note that our experiments using marching-cube based surface area estimations produced relative errors above the 2% mark. Convex-hull based estimations passed the 1% threshold at grid resolutions close to 50 grid points.

Our experiments also allowed to study the behavior of convergence if the shape of the ellipsoid varies, e.g. if the minimum radius value (defining the *thickness* of the ellipsoid) is stepwise reduced increasing the curvature of the generated ellipsoids. See Fig. 8 for the resulting relative errors for four different grid values. These studies show that

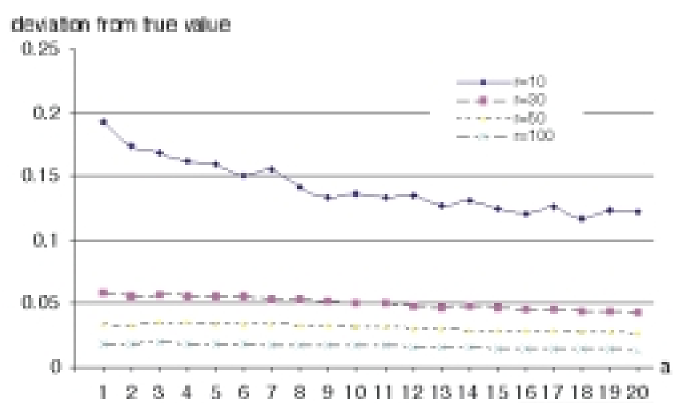

Figure 8. Ellipsoids of size $20 \times 20 \times a$ are digitized, with $a = 1, 2, ..., 20$ and grid resolutions $r = 10, 30, 50, 100$. The resulting relative errors show that the measured convergence is slightly faster if the ellipsoid is closer to being a sphere, i.e. if the curvature of the surface reduces.

the convergence to the true value is slightly slower for increased surface curvature. The convex hull calculation may be seen as a special case of DPS polyhedrization approaches defining global techniques.

7. CONCLUSIONS

Our hypothesis is that there is no local technique which allows multigrid-convergent surface area estimations just for the family of all convex sets in 3D Euclidean space. Local approximation techniques, such as marching cubes algorithms, also generate very large numbers of triangles what restricts their practical use for high resolution data, such as, e.g., in computer assisted radiology.

Global techniques should be studied with respect to the surface area estimation problem. We suggest the acronym *DPS* for *digital planar segment*, and a basic idea may be to design *DPS polyhedrization methods* which follow the ideas of DSS segmentations (*DSS* as an acronym of *digital straight segment*) of curves in the plane[10]: specify initial "surface voxels", merge "surface voxels" to obtain a maximum-size DPS (according to a chosen definition of digital planes, and following a chosen search strategy). A first example of a (global) DPS polyhedrization method is [8]. The convex hull calculation may also be seen as a special case of a DPS polyhedrization approach.

Convergence with respect to the Hausdorff-Chebyshev metric is not sufficient for surface area multigrid convergence. Most important, we need convergence theorems clarifying cases of multigrid convergence to the true value, for example for the family of all convex regular solids Θ being r-compact, i.e. having simply 4-connected sets $D_r(\Theta)$ for all $r \geq r_\Theta$. Experimental studies may help to evaluate the different polyhedrization techniques based on measures such as relative error, statistical deviation for $r = 0, 1, 2, 3, ...$, or a product measure of relative error times number of generated faces. The ellipsoids might be very useful for such experimental studies.

Acknowledgment

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References

[1] M. Abramowitz and I. Stegun (eds.): *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, 10th print, with corrections. Wiley, New York 1972.

[2] E. Artzy, G. Frieder, and G.T. Herman: The theory, design, implementation and evaluation of a three-dimensional surface detection algorithm. *CGIP* **15** (1981) 1–24.

[3] L.E.J. Brouwer: Beweis des Jordanschen Satzes für den n-dimensionalen Raum. *Math. Ann.* **71** (1912) 314–319.

[4] T. Buelow, R. Klette: Rubber band algorithm for estimating the length of digitized space-curves. Accepted paper for ICPR'2000, Barcelona, September 2000.

[5] A. Cayley: *An Elementary Treatise on Elliptic Functions.* Cambridge, 1876, p.54.

[6] A. Gueziec and R. Hummel: Exploiting triangulated surface extraction using tetrahedral decomposition. *IEEE Trans. on Visualization and Computer Graphics* **1** (1995) 328–342.

[7] F. Hausdorff: *Mengenlehre.* Walter de Gruyter & Co., Berlin (1927).

[8] J. Françon, and L. Papier: Polyhedrization of the boundary of a voxel object. *Discrete Geometry for Computer Imagery* (G. Bertrand, M. Couprie, L. Perroton, eds.) LNCS 1568, Berlin, Springer (1999) 425–434.

[9] O. Hölder: *Beiträge zur Potentialtheorie.* Diss. Tübingen, Stuttgart 1882, page 29.

[10] A. Hübler, R. Klette, K. Voss: Determination of the convex hull of a finite set of planar points within linear time. *EIK* **17** (1981) 121–139.

[11] M. N. Huxley: Exponential sums and lattice points. *Proc. London Math. Soc.* **60** (1990) 471–502.

[12] C. Jordan: *Cours d'analyse III* Paris, 1887.

[13] C. Jordan: Remarques sur les intégrales deéfinies, *Journal de Mathématiques*, 4^e série, T. 8 (1892) 69–99.

[14] Y. Kenmochi, A. Imiya, and A. Ichikawa: Boundary extraction of discrete objects. *Computer Vision and Image Understanding* **71** (1998) 281–293.

[15] Y. Kenmochi, K. Kotani, and A. Imiya, Marching Cubes Method with Connectivity. *Proceedings of 1999 International Conference on Image Processing* **4**, IEEE, (1999) 361–365.

[16] R. Klette: Approximation and representation of 3D objects. *Advances in Digital and Computational Geometry* (R. Klette, A. Rosenfeld, F. Sloboda, eds.), Springer, Singapore (1998) 161–194.

[17] A. Lenoir, R. Malgouyres, and R. Revenu: Fast computation of the normal vector field of the surface of a 3-d discrete object. *Discrete Geometry for Computer Imagery* (S. Miguet, A. Montanvert, S. Ubéda, eds.) LNCS 1176, Berlin, Springer (1999) 101–112.

[18] W. E. Lorensen, and H. E. Cline: Marching cubes: a high resolution 3D surface construction algorithm. *Computer Graphics* **21** (1987) 163–169.

[19] H. v. Mangoldt, and K. Knopp: *Einführung in die höhere Mathematik, III. Band, 12. Auflage*, Leipzig, S. Hirzel Verlag, 1965.

[20] C. Montani, R. Scanteni, and R. Scopigno: Discretized Marching Cubes. *Proceedings of Visualization '94* (D. Bergeron, A. Kaufman, eds.), IEEE, Computer Society Press (1994) 281–286.

[21] H. Minkowski: *Geometrie der Zahlen*, Teubner, Leipzig, 1910.

[22] P.A.P. Moran: The surface area of an ellipsoid. in: *Statistics and Probability: Essays in Honor of C.R. Rao* (eds.: G. Kallianpur, P.R. Krishnaiah, J.K. Gosh), North-Holland (1982), 511-518.

[23] G. Peano: Sulla definizione dell'area a'una superficie. *Atti della R. Accademia dei Lincei, Serie 4: Rendiconti, Rom* **6** (1890) 54–57. English translation: On the definition of the area of a surface. in: *Selected Works of Giuseppe Peano* (translated and edited, with a biographical sketch and bibliography, by H. C. Kennedy) George Allen & Unwin Ltd., London 1973, 137–140.

[24] J. C. Roberts: An overview of rendering from volume data including surface and volume rendering. TR, University of Kent at Canterbury (1993).

[25] H. Samet: *Applications of Spatial Data Structures* Addison Wesley, Reading (1993).

[26] H. A. Schwarz: Sur une définition erronée de l'aire d'une surface courbe, *Ges. math. Abhandl.* **2** (1890) 309–311.

[27] W. Scherrer: Ein Satz über Gitter und Volumen. *Mathematische Annalen* **86** (1922) 99–107.

[28] F. Sloboda, B. Zaťko and R. Klette: On the topology of grid continua. SPIE's 43rd Annual Meeting, 19-24 July 1998, San Diego, Conference 3454 (*Vision Geometry VII*), 52–63.

[29] J. Wilhelms and A. V. Gelder: Octrees for faster isosurface generation. *ACM Transactions on Graphics* **11** (1992) 201–227.

[30] J. Wilhelms and A. V. Gelder: Topological considerations in isosurface generation. Baskin Center for Computer Engineering and Information Sciences, University of California, Santa Cruz (1994) UCSC-CRL-94-31.

[31] G. Wyvill, C. McPheeters, and B. Wyvill: Data structures for soft objects. *The Visual Computer*, 2 (1986), 227–234.