

Towards Experimental Studies of Digital Moment Convergence

Reinhard Klette¹ and Jovisa Zunic²

Abstract

Digital moments approximate real moments where the accuracy depends upon grid resolution. There are theoretical results about the speed of convergence. However, there is a lack of more detailed studies with respect to selected shapes of regions, or with respect to experimental data about convergence. This paper discusses moments for specific shapes of regions, and provides some initial experimental data about measured convergence of digital moments.

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Towards Experimental Studies of Digital Moment Convergence

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ABSTRACT

Digital moments approximate real moments where the accuracy depends upon grid resolution. There are theoretical results about the speed of convergence. However, there is a lack of more detailed studies with respect to selected shapes of regions, or with respect to experimental data about convergence. This paper discusses moments for specific shapes of regions, and provides some initial experimental data about measured convergence of digital moments.

Keywords: Digital geometry, digital regions, moments, multigrid convergence.

1. INTRODUCTION

Assume a planar set S and a Cartesian xy -coordinate system in the Euclidean plane. The (p, q) -moments of set S are defined by

$$m_{p,q}(S) = \iint_S x^p y^q dx dy ,$$

for integers $p, q \geq 0$. We say that the moment $m_{p,q}(S)$ has the *order* $p + q$.

As an example of such an integration, assume that set S is a unit square with vertices $(0, 0), (1, 0), (1, 1), (0, 1)$. Then we have

$$m_{p,q}(S) = \frac{1}{(p+1)(q+1)} .$$

Note that this square S is not in a centred position where the centroid would be assumed to coincide with the origin of our xy -coordinate system. The calculation of the integration is not difficult in this case. However, there are not many sets allowing such a simple and straightforward evaluation of moment integrals. We need such true moment values for experimental evaluations.

In image analysis and pattern recognition we have to deal with situations where real objects are given as digital sets $D(S)$ only, and the set S itself is unknown.¹³ For a set S , in this paper its digitization is defined to be the set of all grid points with integer xy -coordinates which belong to the region occupied by the given set S . This corresponds to the model used in number theory, see, e.g., the studies initiated by Gauss about the number of grid points within a circular region. For our studies we are interested to discuss such sets S in the Euclidean plane which allow to calculate the real moments, and these values will act in our experiments as true values to be compared with digital approximations of these real moments.

The exact value of a moment $m_{p,q}(S)$ remains unknown in image analysis where the set S is “visible” via a given digital image only. The moment value is approximated by the *discrete moment* $\mu_{p,q}(S)$ where

$$\mu_{p,q}(S) = \sum_{(i,j) \in D(S)} i^p \cdot j^q = \sum_{\substack{i,j \text{ are integers} \\ (i,j) \in S}} i^p \cdot j^q$$

assuming an ideal digitization of a set S . We model this process by considering set S in the Euclidean plane and ideal (i.e. error-free) digitization. The difference between the real moment and the discrete moment is the digitization error which is influenced by the chosen grid resolution used for digitizing the given set S . For modeling different grid resolutions we may chose one of the following approaches:

(i) We could assume that the set S remains fixed, and consider positive reals r , where $1/r$ is the distance between two neighboring grid points parallel to one of the coordinate axes, i.e. r would be the number of grid points per unit. Then a convergence of $r \rightarrow \infty$ is the way to specify increasing grid resolution by having more and more grid points per unit. That’s the way as used, e.g., in [8,16] for studying multigrid convergence problems.

(ii) Another way is to dilate set S by a factor r , $r > 0$, and to assume that the used grid in the Euclidean plane remains always the same, i.e. grid points are all points (x, y) having integer coordinates. For example, this way was been chosen by *C. Jordan*⁷ for studying the important problem of 3D volume estimation based on gridding techniques. Let $r \cdot S$ be the dilation of set S by factor r , i.e.

$$r \cdot S = \{(r \cdot x, r \cdot y) : (x, y) \in S\}.$$

Then a convergence of $r \rightarrow \infty$ is the way to specify increasing grid resolution by having larger and larger sets but a constant grid in the plane. Of course, for proper comparisons the values of the enlarged sets $r \cdot S$ have to be “scaled down” to the proper dimension.

We use approach (ii) in this paper. We reported in [9] on digitization errors for moments of order less than or equal to two. In [10] a more general result is reported for arbitrary orders $p, q \in \{0, 1, 2, \dots\}$ and a general family $\mathbf{F}_{f(r)}$ of sets specified by a an increasing function f which maps non-negative reals r into non-negative reals $f(r)$, and the following axioms: a bounded set S belongs to the family $\mathbf{F}_{f(r)}$ iff

- FR₁: set S is a planar convex set translated in the xy -coordinate system such that all points of S have nonnegative coordinates,
- FR₂: set S has a boundary having a continuous third derivative at every point (C^3 -boundary), and
- FR₃: the set S satisfies

$$|r^2 \cdot m_{0,0}(S) - \mu_{0,0}(r \cdot S)| = \mathcal{O}(f(r)),$$

for $r > 0$.

Then it holds for sets S of such a family $\mathbf{F}_{f(r)}$ defined by function $f(r)$ that

$$|m_{p,q}(r \cdot S) - \mu_{p,q}(r \cdot S)| = |r^{p+q+2} \cdot m_{p,q}(S) - \mu_{p,q}(r \cdot S)| = \mathcal{O}(f(r) \cdot r^{p+q}) \quad (1)$$

where $r \cdot S$ denotes the dilation of S by factor r , and r^{p+q+2} is the scaling factor to be used for comparing the discrete moments of a dilated set $r \cdot S$ with the real moments of the original set S .

As an illustration, let us mention here that the family of all convex bounded sets with C^3 boundary having at least one straight section on its boundary is contained in the family \mathbf{F}_r defined by function $f(r) = r$. This implies that the error term in the estimation of real moments from the corresponding discrete moments is $\mathcal{O}(r^{-1})$, see [10]. In view of our example in Subsection 2.1 below it holds that this estimate is the best possible for the family of all convex bounded sets with C^3 boundary having at least one straight section on its boundary

As another example, consider the family of all bounded sets with an C^3 boundary and positive curvature at any boundary point. This family of sets is contained in $\mathbf{F}_{f(r)}$ with

$$f(r) = r^{7/11} \cdot (\log r)^{47/22} \approx r^{7/11+\varepsilon},$$

see M.N.Huxley,⁴ and it is unknown whether this is the best possible upper bound in Equ. (1) for the family of all bounded sets with an C^3 boundary and positive curvature at any boundary point. As a consequence we know that the error term for moment estimations of bounded sets with an C^3 boundary and positive curvature at any boundary point, is upper bounded by $\mathcal{O}(r^{-15/11+\varepsilon})$, but a smaller upper bound might be valid as well.

Generally speaking, the number-theoretic function $f(r)$ describes the impact of the grid resolution on the precision in real moment estimation based on corresponding discrete moments. Namely, Equ. (1) specifies that

$$\frac{1}{r^{p+q+2}} \cdot \mu_{p,q}(r \cdot S)$$

approximates $m_{p,q}(S)$ within an error in $\mathcal{O}(f(r) \cdot r^{-2})$.

This paper addresses the issue of a more detailed study of multigrid convergence of moment estimates with respect to selected shapes of sets S , and with respect to experimental multigrid studies in general. Publications on discrete moments in image analysis normally focus on computation aspects, see [2,6,14,17,21,22]. We believe that the accuracy aspect, see [9,10,12,14], needs similar attention especially in cases of “complicated” shapes not yet studied in this paper, e.g. “curve-like” planar sets or sets with “high-curvature” boundaries.

2. TRUE AND ESTIMATED VALUES

We discuss four different sets for experimental evaluations of error bounds. The sets have to allow that real moments may be calculated, and the sets have also been chosen with respect of having a diversity in geometric shapes.

2.1. Square

We start with a centred set, i.e. having a centroid which coincides with the origin of our xy -coordinate system. In that case moments $m_{p,q}(S)$ are zero for all odd values of p if the given set is symmetric with respect to the x -axis, and zero for all odd values of q if the given set is symmetric with respect to the y -axis.

Assume real $a > 0$ and an $2a \times 2a$ square $S(a)$ which is centred and parallel to the coordinate axes, see Fig. 1. Then we have

$$\begin{aligned} m_{p,q}(S(a)) &= \int_{-a}^{+a} \int_{-a}^{+a} x^p y^q dx dy \\ &= \frac{(1 - (-1)^{p+1}) \cdot (1 - (-1)^{q+1})}{(p+1) \cdot (q+1)} \cdot a^{p+q+2} \end{aligned}$$

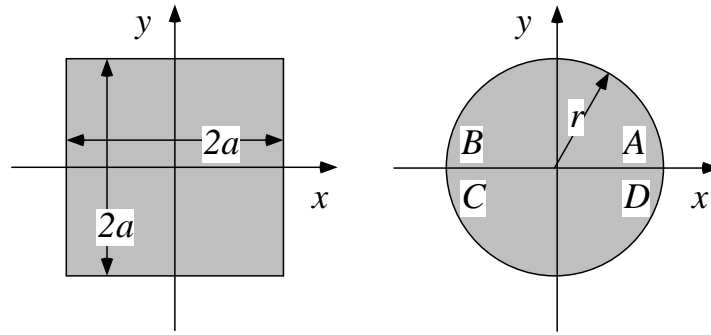


Figure 1. Square and circle in centred position.

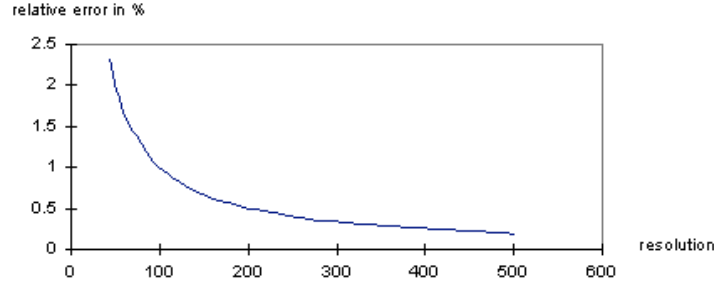


Figure 2. Relative errors for the square and zero-order moment estimation.

for $p, q \geq 0$. Note that $m_{p,q}(S(a)) = 0$ for odd p or q , as mentioned above for the general case. It follows that $m_{0,0}(S(a)) = 4a^2$, $m_{0,1}(S(a)) = m_{1,0}(S(a)) = m_{1,1}(S(a)) = 0$, $m_{0,2}(S(a)) = m_{2,0}(S(a)) = (4/3)a^4$, etc.

In the case of such a centred square it holds that the difference for the zero-order moment, i.e. between true and estimated contents of set S ,

$$m_{0,0}(S(a)) - \mu_{0,0}(S(a)) = a^2 \cdot m_{0,0}(S(1)) - \mu_{0,0}(S(a)) ,$$

varies between $-(4a+1)$ and $(4a+1)$, for $a = \lceil a \rceil - \varepsilon$ and $a = \lfloor a \rfloor + \varepsilon$ respectively, and ε small enough. That implies for arbitrary orders $p, q \geq 0$ that the real number

$$|m_{p,q}(S(a)) - \mu_{p,q}(S(a))| = |a^{p+q+2} \cdot m_{p,q}(S(1)) - \mu_{p,q}(S(a))|$$

belongs to the interval $[0, g(a)]$ where $g(a) = \mathcal{O}(a^{p+q+1})$, see Equ. (1). We illustrate the behavior of the error function in Fig. 2 for the zero-order case.

2.2. Circle

We consider at first the quarter circle A as illustrated in Fig. 1, and moment $m_{0,q}(A)$. Furthermore let q be odd, with $q = 2t - 1$. In this case the integration is simple, we have

$$\begin{aligned} m_{0,q}(A) &= \int_0^r \left(\int_0^{\sqrt{r^2-x^2}} y^q dy \right) dx \\ &= \frac{1}{q+1} \int_0^r (r^2 - x^2)^t dx \end{aligned}$$

for odd $q \geq 0$ and $q = 2t - 1$. For example, it follows

$$m_{0,1}(A) = \frac{1}{2} \int_0^r (r^2 - x^2) dx = \frac{1}{3} r^3 ,$$

$$m_{0,3}(A) = \frac{1}{2} \int_0^r (r^2 - x^2)^2 dx = \frac{4}{15} r^5 ,$$

etc. These values for set A can now be used to calculate values for sets B , C and D , see Fig. 1, as well. For q odd we have $m_{0,q}(B) = m_{0,q}(A)$ and $m_{0,q}(C) = m_{0,q}(D) = -m_{0,q}(A)$.

For example, for the half-circle $A \cup B$ we have $m_{0,1}(A \cup B) = (2/3)r^3$ and $m_{0,3}(A \cup B) = (8/15)r^5$. Of course, due to symmetries we have $m_{0,q}(A \cup D) = 0$ for the half-circle $A \cup D$, if q is odd, and $m_{0,q}(S) = 0$ for the full circle $S = A \cup B \cup C \cup D$, if q is odd.

The difference between the zero order discrete moment and the zero order moment (i.e. the contents) of a centred circle, or of a circle with a center at grid point position, with radius r is upper bounded by

$$r^{\frac{131}{208}},$$

as communicated in [5]. This (unpublished yet) result improves $r^{\frac{46}{73}+\varepsilon}$ which was previously the best known upper bound.

By using the same proof technique as was used in the proof of the Lemma 20 of [10], it can be shown that the differences

$$|m_{p,0}(C_1) - \mu_{p,0}(C_1)| \quad \text{and} \quad |m_{0,p}(C_1) - \mu_{0,p}(C_1)|$$

are upper bounded by

$$\mathcal{O}(r^{p+\frac{131}{208}}),$$

for a circle C_1 having radius r and a centre (a, b) , where a and b are integers greater or equal to r . That is the reason why the numerical values in Tab. 1 are chosen in such a way that the centre of the estimated circle has integer coordinates, i.e., r is chosen to be an integer. So far we do not have a similar theoretical result for the more general difference $|m_{p,q}(C_1) - \mu_{p,q}(C_1)|$. However, in our experiments we compared the error term with $r^{-\frac{285}{208}}$.

p	q	r	$m_{p,q}(S) - \frac{\mu_{p,q}(r \cdot S)}{r^{p+q+2}}$	$r^{-\frac{285}{208}}$
0	1	10	- 0.018407346	0.0426391
		50	+0.0039926536	0.00469985
		100	- 0.000007346	0.00181809
		500	+0.000200065359	0.000200397
	2	10	- 0.033609183	0.0426391
		50	+0.005779617	0.00469985
		100	- 0.000062583013	0.00181809
		500	+0.00029891708	0.000200397
1	1	10	- 0.028407346	0.0426391
		50	+0.0035926536	0.00469985
		100	- 0.00010734641	0.00181809
		500	+0.00019665359	0.0002000397
	2	10	- 0.043609183	0.0426391
		50	+0.005379617	0.00469985
		100	- 0.00016258301	0.00181809
		500	+0.00029491708	0.000200397
2	3	10	- 0.12981161	0.0426391
		50	+0.0099040164	0.00469985
		100	- 0.0012241628	0.00181809
		500	+0.0006085407	0.000200397
	4	10	- 0.22555667	0.0426391
		50	+0.016346123	0.00469985
		100	- 0.0021931319	0.00181809
		500	+0.0010141273	0.000200397

Table 1. Errors in approximating real moments $m_{p,q}(S)$ by $r^{-(p+q+2)} \cdot \mu_{p,q}(r \cdot S)$, for different resolutions r . The set S is the disc $(x-1)^2 + (y-1)^2 \leq 1$.

The exact values of the real moments of a disc $(x-1)^2 + (y-1)^2 \leq 1$ are (rounded to eight significant figures):

$$\begin{aligned} m_{0,1}(S) &= 3.1415927 & m_{0,2}(S) &= 3.9269908 \\ m_{1,1}(S) &= 3.1415927 & m_{1,2}(S) &= 3.9269908 \\ m_{2,3}(S) &= 6.6758844 & m_{2,4}(S) &= 9.8665644 \end{aligned}$$

These exact values are compared to discrete moments calculated for the digitized disc, see Tab. 1.

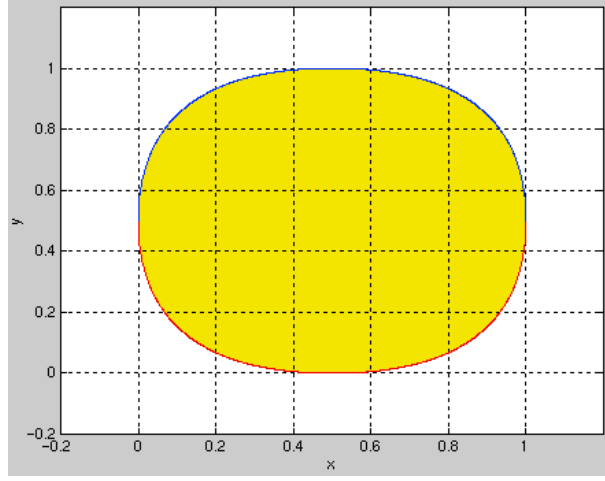


Figure 3. The isometric quartic boundary in the unit square.

2.3. Isometric Quartic Boundary Segments

We consider a more complex shape than the circle, defined by four isometric boundary segments, see Fig. 3. This set is defined as being the closed bounded region whose boundary consists of four segments of the following curve $r \cdot \gamma$, for different values of r . The algebraic curve γ of order 4 is defined as

$$\gamma : \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2} - \sqrt{1 - |2x - 1|} - \left|x - \frac{1}{2}\right|\right)^2.$$

The curve γ consists of four isometric quartic arcs, and it is also of number-theoretic interest, for example for the following two reasons:

(1) Let us consider such a convex lattice polygon P_m with vertices belonging to an $\mathbf{Z}_m = [0, m] \times [0, m]$ integer grid, defined by having a number of vertices which is the maximum of numbers of vertices of all convex lattice

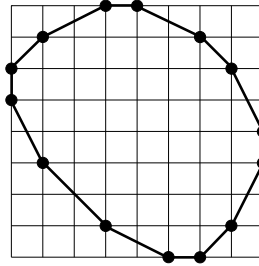


Figure 4. Example of a polygon P_8 .

p	q	r	$m_{p,q}(S) - \frac{\mu_{p,q}(r \cdot S)}{r^{p+q+2}}$	$r^{-\frac{15}{11}}$
0	0	1	+0.83333333	1
		2	- 0.41666667	0.38860157
		100	+0.0016333333	0.0018738174
		100 π	+0.00013884389	0.00039336492
	1	1	+0.41666667	1
		5	+0.09666667	0.1113933
		30 π	+0.00013212696	0.0020314779
		300	+0.0003944444	0.00041889713
1	1	2	- 0.104167	0.38860157
		e	+0.0434492583	0.25572916
		70	+0.0011394558	0.0030475948
		350	+0.000086394558	0.00033948165
	2	5	+0.0358452	0.1113933
		10	+0.00429524	0.0043287613
		121	+0.000266155	0.0014449011
		400	+0.0000715236	0.00028296738
2	3	7	+0.00899713	0.070403392
		14.5	- 0.00163832	0.026080527
		93.3	- 0.000170758	0.0020596705
		444	+0.000047717	0.00024543264
	4	12	+0.00463474	0.033758972
		27	+0.00178829	0.011172242
		121.22	- 0.000148198	0.0014113264
		500	+0.0000186481	0.00020873071

Table 2. Errors in approximating real moments $m_{p,q}(S)$ by $r^{-(p+q+2)} \cdot \mu_{p,q}(r \cdot S)$ for different resolutions r . The bounded set S is bounded by four segments of the curve γ .

polygons having vertices in the $[0, m] \times [0, m]$ integer grid. The number of vertices of P_m is a function in m , and the determination of this function has been studied for many years, see, for example [20,21]. An example of such a polygon P_m is shown in Fig. 4, for $m = 8$. The defined function is equal to

$$\frac{3}{\sqrt[3]{4\pi^2}} \cdot m^{\frac{2}{3}} + \mathcal{O}(m^{\frac{1}{3}} \cdot \log m) ,$$

see [1]. It turns out, see [23], that such polygons P_m , $m > 0$, converge to curve γ with respect to the Hausdorff metrics d for planar sets in the Euclidean plane, i.e.

$$\lim_{m \rightarrow \infty} d\left(\frac{1}{m} \cdot P_m, \gamma\right) = 0.$$

(2) A rather stronger result by A.M. Vershik (see [19], and also [3]) is as follows. The polygons P_m are special examples of **\mathbf{Z}_m -lattice polygons** defined by having a width and height of m . As $m \rightarrow \infty$ it holds that almost all convex \mathbf{Z}_m -lattice polygons, scaled by factor m^{-1} and (thus) lying in the square $[0, 1]^2$, are “very close” (in the Hausdorff metric sense) to this curve γ as defined above. The related central limit theorem is proved in [15].

Some numerical results for the estimation of moments of this bounded set having four segments of γ as its boundary, are given in Tab. 2. The exact values of the real moments of this set are (rounded to eight significant figures):

$$m_{0,0}(S) = 0.83333333 \qquad m_{0,1}(S) = 0.41666667$$

$$\begin{aligned}
m_{1,1}(S) &= 0.20833333 & m_{1,2}(S) &= 0.13184500 \\
m_{2,3}(S) &= 0.05772680 & m_{2,4}(S) &= 0.04330830
\end{aligned}$$

2.4. Parametrized Quartic Boundary Segments

We consider a family of parameterized sets. Any positive value of parameter m defines a bounded planar region defined by segments of the curve $y^2 \leq (mx^2 - m)^2$, see Fig. 5. This curve consists of four quartic arcs.

The curve is chosen again due to some number-theoretic background (besides the principal interest in having another geometric example for testing moment estimates). As mentioned above, the difference

$$f(r) = |m_{0,0}(r \cdot S) - \mu_{0,0}(r \cdot S)|$$

is upper bounded by $\mathcal{O}(r)$ also for cases where straight sections are allowed on the boundary of set S . It holds that the error term $f(r)$ can be smaller under additional assumptions about the boundary of S , see, for example, Huxley's result in [4]. Furthermore it is well-known¹¹ that, if we express $f(r)$ in the form $f(r) = \mathcal{O}(r^\alpha)$, then the statement $|m_{0,0}(rS) - \mu_{0,0}(rS)| = \mathcal{O}(r^\alpha)$ is false for $\alpha < 0.5$. The quartic arc $y = \sqrt{x}$ where $x \in [1, r]$ is an example of a curve whose length has the order of magnitude r and which passes trough exactly $\lfloor r^{0.5} \rfloor$ grid points.

We give numerical examples for sets defined by $m = 1$ and $m = 4$. The values given in Tables 3 and 4 correspond to the translated sets having points with positive coordinates only. The exact values of the real moments of the set in case $m = 1$ are (rounded to eight significant figures):

$$\begin{aligned}
m_{0,0}(S) &= 2.6666667 & m_{0,1}(S) &= 5.3333333 \\
m_{1,1}(S) &= 10.666667 & m_{1,2}(S) &= 22.552381 \\
m_{2,3}(S) &= 104.63492 & m_{2,4}(S) &= 240.54452
\end{aligned}$$

The exact values of the real moments of the set in case $m = 4$ are (rounded to seven significant figures):

$$\begin{aligned}
m_{0,0}(S) &= 10.666667 & m_{0,1}(S) &= 53.333333 \\
m_{1,1}(S) &= 106.66667 & m_{1,2}(S) &= 611.35238 \\
m_{2,3}(S) &= 8005.5873 & m_{2,4}(S) &= 53289.629
\end{aligned}$$

The values in both tables are produced for the same choices of r in order to illustrate the following:

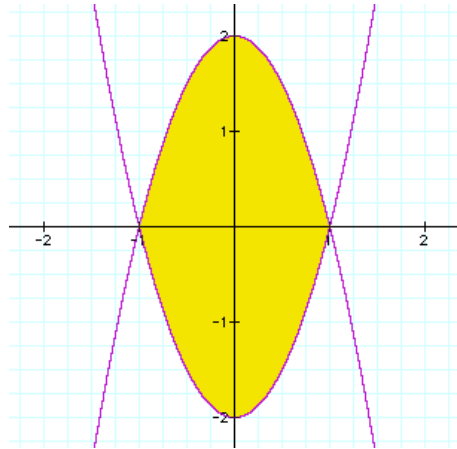


Figure 5. Four quartic boundary segments, example for $m = 2$.

p	q	r	$m_{p,q}(S) - \frac{\mu_{p,q}(r \cdot S)}{r^{p+q+2}}$	$r^{-\frac{15}{11}}$
0	0	10	- 0.02333333	0.043287613
		100	- 0.00023333	0.0018738174
		200	+0.00084166667	0.00072816839
		800	+0.00019010417	0.00010996157
	1	20	+0.002833333	0.016821634
		80	+0.010520833	0.0025402549
		160	+0.0045052083	0.00098714703
		320	+0.00017513021	0.00038360688
1	1	30	+0.03111111	0.0096770669
		90	+0.00014814815	0.0021633354
		270	+0.0017009602	0.00048361967
		540	+0.0014677641	0.00018793536
	2	25	- 0.15065137	0.012408468
		75	+0.011268443	0.0027739478
		225	- 0.0023672246	0.00062012378
		450	- 0.00059127473	0.00024098107
2	3	40	+0.53012504	0.0065369135
		160	+0.10820738	0.00098714703
		320	+0.04206529	0.00038360688
		640	+0.01614505	0.00014907024
	4	50	+0.51789184	0.0048219502
		200	+0.10489098	0.00072816839
		400	+0.038664549	0.00028296738
		1000	+0.014139002	0.000081113083

Table 3. Errors in approximating real moments $m_{p,q}(S)$ by $r^{-(p+q+2)} \cdot \mu_{p,q}(r \cdot S)$ for different resolutions r , where S is the bounded region whose boundary are four segments of the quartic curve $(y - 2)^2 = ((x - 2)^2 - 1)^2$.)

- 1) The impact of the size of real moments to be estimated: Namely, as it can be seen from the Tab. 4, if the real moments to be estimated have (relatively) big values then the required precision is not reached for small values of r as it happened in the previous examples. Of course, due to theoretical results any required precision can be reached, but higher resolutions r have to be used. It is perhaps more suitable that the *relative error* is used in such situations. Let us mention that if the usual relative error definition is used, and Equ. 1 is applied to such regions having no straight section on their boundary, then we have

$$\left| \frac{\mu_{p,q}(r \cdot S)}{m_{p,q}(r \cdot S)} - 1 \right| = \frac{|\mu_{p,q}(r \cdot S) - m_{p,q}(r \cdot S)|}{r^{p+q+2} \cdot m_{p,q}(S)} = \mathcal{O} \left(\frac{1}{r^{\frac{15}{11}-\varepsilon} \cdot m_{p,q}(S)} \right).$$

- 2) The impact of the elongation of the considered regions (for example, $m = 5$ corresponds to a higher elongation than $m = 2$): There are no related theoretical results yet. But it seems that an increase in elongations leads to an increase in errors in the “worst case” sense.

It could also be of interest to calculate errors in moment estimations for different values of m in such a way that both m and r tend to “infinity”, for example, combined cases such as $m = \log r$, $m = \sqrt{r}$, $m = r^2$, e.t.c.

p	q	r	$m_{p,q}(S) - \frac{\mu_{p,q}(r \cdot S)}{r^{p+q+2}}$	$r^{-\frac{15}{11}}$
0	0	10	- 0.023333333	0.043287613
		100	- 0.003833333	0.0018738174
		200	- 0.000958333	0.00072816839
		800	+0.00021510417	0.00010996157
	1	20	- 0.079166667	0.016821634
		80	+0.020052083	0.0025402549
		160	+0.01438821	0.00098714703
		320	+0.0067220052	0.00038360688
1	1	30	+0.01222222	0.0096770669
		90	+0.0037037037	0.0021633354
		270	+0.0058984911	0.00048361967
		540	+0.0043552812	0.00018793536
	2	25	- 0.94018097	0.012408468
		75	+0.10267252	0.0027739478
		225	- 0.01543446	0.00062012378
		450	- 0.020359072	0.00024098107
2	3	40	- 4.7117093	0.0065369135
		160	+3.0550335	0.00098714703
		320	+1.4338876	0.00038360688
		640	+0.60974317	0.00014907024
	4	50	- 65.552138	0.0048219502
		200	- 8.3047868	0.00072816839
		400	- 0.38911548	0.00028296738
		1000	+0.33176634	0.000081113083

Table 4. Errors in approximating real moments $m_{p,q}(S)$ by $r^{-(p+q+2)} \cdot \mu_{p,q}(r \cdot S)$ for different resolutions r , where S is the bounded region whose boundary are four segments of the quartic curve $(y - 5)^2 = (4(x - 2)^2 - 4)^2$.

3. CONCLUSION

This paper gave some examples in order to illustrate the error analysis in the estimation of real moments $m_{p,q}(S)$ by $r^{-(p+q+2)} \cdot \mu_{p,q}(r \cdot S)$. A few specific sets had been chosen. The error values are compared with a theoretical upper bound. This bound has been obtained for the error in such estimations¹⁰ where sets are assumed having a C^3 boundary and which are translated such that their points have positive coordinates only.

Our experimental results emphasize that probably a smaller error term than

$$r^{-\frac{15}{11}+\varepsilon}$$

can be obtained, but there are no theoretical results so far to support this hypothesis, except in case of the moments $m_{0,p}(C)$ and $m_{p,0}(C)$, where C is disk: in this case the error term is improved to

$$r^{-\frac{285}{208}}$$

A basic mathematical tool for the theoretical derivation of these error terms has been Huxley's results in [4] and his comments in [5].

The result from [18] suggests that the error term in Huxley's result can be $r^{\frac{3}{5}}$ or even smaller. Our experiments support this hypothesis.

It seems to be natural that there are some impacts of the position of the studied regions in the Euclidean plane on the precision of the estimation. We leave that as a problem for future research. Of course there is an influence of

the size of p and q to the precision in estimating real moments $m_{p,q}(S)$ estimation, see [12], but we have not yet considered this in our experiments so far.

References

- [1] D.M. Acketa, J.D. Žunić: On the maximal number of edges of digital convex polygons included into an (m, m) grid. *Journal of Combinatorial Theory Series A* **69** (1995) 358–368.
- [2] G.J. Agin: Vision systems. in: *Handbook of Industrial Robotics* (S.Y. Nof, ed.), John Wiley & Sons, New York (1985) 231–261.
- [3] I. Bárány: The limit shape of convex lattice polygons. *Discrete Comput. Geom.* **13** (1995) 279–295.
- [4] M.N. Huxley: Exponential Sums and Lattice Points. *Proc. London Math. Soc.* **60** (1990) 471–502.
- [5] M.N. Huxley: Exponential Sums and Lattice Points III. (personal communication)
- [6] X.Y. Jiang, H. Bunke: Ein konturbasierter Ansatz zur Berechnung von Momenten. in: *Mustererkennung 1991* (B. Radig, ed.), Proc. 13th DAGM Symposium, Springer, Berlin (1991) 143–150.
- [7] C. Jordan: Remarques sur les intégrales définies, *Journal de Mathématiques*, 4^e série, T. 8 (1892) 69–99.
- [8] R. Klette: Approximation and representation of 3D objects. in: *Advances in Digital and Computational Geometry* (eds.: R. Klette, A. Rosenfeld, F. Sloboda). Springer, Singapore (1998) 161–194.
- [9] R. Klette, J. Žunić: On errors in calculated moments of convex sets using digital images. Proc. “Vision Geometry VIII” **SPIE 3811** (1999) 105–116.
- [10] R. Klette, J. Žunić: Multigrid Convergence of Calculated Features in Image Analysis. accepted for *Journal of Mathematical Imaging and Vision*.
- [11] E. Krätzel: *Zahlentheorie*. VEB Deutscher Verlag der Wissenschaften, Berlin (1981).
- [12] S.X. Liao, M. Pawlak: On the accuracy of Zernike moments for image analysis. *IEEE Trans. PAMI* **20** (1998) 1358–1364.
- [13] R. Mukundan, K.R. Ramakrishnan: *Moment Functions in Image Analysis*. World Scientific, Singapore (1998).
- [14] R. Safaei-Rad, V.C. Smith, B. Benhabib, I. Tchoukanov: Application of moment and Fourier descriptors to the accurate estimation of elliptical-shape parameters. Proc. ASSP, Vol. 4, Toronto (1991) 2465–2468.
- [15] Ya.G. Sinai: Probabilistic approach to analyze the statistics of convex polygonal curves. *Funktsional Anal. Appl.* **28** (1994) 41–48 (paper in Russian, title translated into English).
- [16] F. Sloboda, B. Zlatko, R. Klette: On the topology of grid continua. Proc. “Vision Geometry VII” **SPIE 3454** (1998) 52–63.
- [17] N.J.C. Strachan, P. Nesvadba, A.R. Allen: A method for working out the moments of a polygon using an integration technique. *Pattern Recognition Letters* **11** (1990) 351–354.
- [18] H.P.F. Swinnerton-Dyer: The number of lattice points on a convex curve. *J. Number Theory* **6** (1974) 128–135.
- [19] A.M. Vershik: Limit shape of convex lattice polygons and related topics. *Funktsional Anal. Appl.* **28** (1994) 16–25 (paper in Russian, title translated into English).
- [20] K. Voss, R. Klette: On the maximal number of edges of convex digital polygons included into a square. *Computers and Artificial Intelligence* **1** (1982) 549–558.

- [21] K. Voss: *Discrete Images, Objects, and Functions in Z^n* . Springer, Berlin (1993).
- [22] M.F.Zakaria, L.J.Vroomen, P.J.A.Zsombor-Murray, J.M.H.VanKessel: Fast algorithm for the computation of moment invariants. *Pattern Recognition* **20** (1987) 639–643.
- [23] J.Žunić: Limit shape of convex lattice polygons having the minimal L_∞ diameter w.r.t. the number of their vertices. *Discrete Mathematics* **187** (1998) 245–254.