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Convergence of Calculated Features in Image Analysis

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This paper informs about number-theoretical and geometrical estimates of worst-case bounds for quantization errors in calculating features such as moments, moment based features, or perimeters in image analysis, and about probability-theoretical estimates of error bounds (eg. Standard derivations) for such digital approximations. New estimates (with proofs) and a review of previously known results are provided.

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Multigrid Convergence of Calculated Features in Image Analysis

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Abstract. This paper informs about estimates of worst-case bounds for quantization errors in calculating features such as moments, moment based features, or perimeters in image analysis, and about probability-theoretical estimates of error bounds (eg. standard deviations) for such digital approximations. New estimates (with proofs) and a review of previousely known results are provided.

1 Introduction

Representations of sets in euclidean spaces E^n by corresponding digital sets cause an inherent loss of information. There are infinitely many different sets in an euclidean space with an identical corresponding digital set. This paper studies resulting accuracy limitations of reconstructions of original sets, and of their features such as moments (eg. the area), perimeters and features derived from moments (eg. the centroid or the orientation).

1.1 History

The problem of area estimation of a set by the number of grid points contained in the considered set has an extensive history in number theory. It has already been studied by *C.F. Gauss* for disks. *C.F. Gauss* (1777–1855) and *P. Dirichlet* (1805–1859) knew already that the number of grid points inside of a planar convex curve γ estimates the area of the set bounded by this curve within an order of $\mathcal{O}(l)$, where l is the length of curve γ. The situation when γ is a circle is studied most carefully. *M.N. Huxley*'s result [7] from 1990 is a very good result for 3-smooth planar convex curves and improves even the best known worst-case error bound [9] previousely known for circles. Huxley's theorem is critical for all moment related results reported in this paper.

The important problem of volume estimation was studied by *C. Jordan* [11] based on gridding techniques. Any grid point $(i, j, k) \in E^3$ is assumed to be the centre point of a cube with faces parallel to the coordinate planes and with edges of length 1. The boundary is part of this cube. Let S be a set contained in finitely many of such cubes. Dilate the set S with respect to an arbitrary point $p \in E^3$ in a ratio $r : 1$. This transforms S into S_r^p. Let $l_r^p(S)$ be the number of all cubes completely contained in the interior of S_r^p, and let $u_r^p(S)$ be the number of all cubes having a non-empty intersection with S_r^p. Then it holds [11] that $r^{-3} \cdot l_r^p(S)$ and $r^{-3} \cdot u_r^p(S)$ always converge to limit values $L(S)$ and $U(S)$, respectively, for r to infinity, independent upon the chosen point p. Jordan called $L(S)$ the *inner volume* and $U(S)$ the *outer volume* of set S, or the *volume* $vol(S)$ of S if $L(S) = U(S)$. Volume definition based on gridding techniques was studied, eg., in [19, 22]. This paper will not focus on sets in 3D space, but the volume definition of 3D sets is used in proofs.

The problems of measuring the length of a curve or the perimeter of a 2D set, or the area of a closed or open surface of a 3D set based on gridding techniques have been studied in the context of digital image analysis, see, eg., [24] for the curve length problem and [14] for the surface area problem. However, *H. Minkowski* [19] proposed already solutions for both problems in his pioneering work on morphological operations. This paper discusses the convergence of one technique for grid-point based estimations of the length of a curve.

1.2 Multigrid digitization

We assume an orthogonal grid with grid constant $0 < \vartheta \leq 1$ in n-dimensional euclidean space E^n, $n \geq 1$, ie. ϑ is the uniform spacing between grid points parallel to one of the coordinate axes. Furthermore, let $r \geq 1$ be the *grid resolution* defined as being the number of grid points per unit, ie. any grid edge is of length $\vartheta = 1/r$.

In this paper we discuss the two-dimensional case only. We consider *r-grid points* $g_{i,j}^r = (\vartheta \cdot i, \vartheta \cdot j)$ in the euclidean plane, for integers i, j and $\vartheta = 1/r$. For $r = 1$ we simply speak about *grid points* (i, j) in the euclidean plane E^2.

Definition 1. For a set S in the euclidean plane its *digitization* $D_r(S)$ is defined to be the set of all r-grid points contained in the given set S, ie.

$$D_r(S) = \{g_{i,j}^r \ : \ g_{i,j}^r = (i/r, j/r) \in S\} \ .$$

In the case $r = 1$ the digitization is denoted by $D(S)$.

The sets $D(S)$ and $D_r(S)$ are also called *digital sets*. The *dilation* of a set $S \subset E^2$ by a factor $r \geq 1$ is defined to be

$$r \cdot S \ = \ \{(r \cdot x, \ r \cdot y) \ : \ (x, y) \in S\} \ .$$

Following *Jordan* [11] this is a dilation with respect to the origin $(0, 0)$, and other points in E^2 could be chosen to be the fixpoint as well. Sometimes it may be more adequate to consider sets of the form $r \cdot S$ (the preferred approach, eg., by *Jordan* and *Minkowski*) digitized in the orthogonal grid with unit grid length, instead of sets S digitized in r-grids with $1/r$ grid length. The study of $r \to \infty$ corresponds to the increase in grid resolution, and this may be either a study of repeatedly dilated sets $r \cdot S$ in the grid with unit grid length, or of a given set S in repeatedly refined grids. This is a general *duality principle for multigrid studies.*

1.3 Features

Assume a planar set S in the euclidean plane and a Cartesian xy-coordinate system in this plane. The (p,q)-*moments* of set S are defined by

$$m_{p,q}(S) = \iint_S x^p y^q \, dx \, dy \; ,$$

for integers $p, q \geq 0$. The moment $m_{p,q}(S)$ has the *order* $p+q$.

In image analysis, the exact values of moments $m_{p,q}(S)$ remain unknown. They are estimated by *discrete moments* $\mu_{p,q}(S)$ where

$$\mu_{p,q}(S) = \sum_{(i,j)\in D(S)} i^p \cdot j^q$$

which can be calculated from the corresponding digitized set $D(S)$ of set S. The grid constant ϑ has to be used as scaling factor if the approach involves repeatedly refined grids. The moment-concept has been introduced into image analysis by *M. Hu* [5].

We are interested in analyzing the accuracy of estimates of the following features of a set S. Note that ϑ has always to be used for scaling if the refined grid approach is taken, otherwise $r \cdot S$ replaces S if the dilation approach is prefered:

1. The *area* $A(S)$ of a planar set S, ie. the moment $m_{0,0}(S)$ of order zero, is estimated by the number of grid points in $D(S)$, ie. by the discrete moment $\mu_{0,0}(S)$.
2. The *perimeter* of S may be estimated by the perimeter value calculated by a *maximum-length digital straight segment approximation procedure*, see, eg., [15] and Fig.1 for an example. Another choice may be the *minimum-length polygon* approximation as discussed in [24, 25]. Both techniques have been compared in [15]. In this paper we discuss the digital straight segment (DSS) approximation technique. A complete discussion of multigrid behavior of the minimum-length polygon (MLP) approximation technique may be found in [24].

   Each r-grid point in $D_r(S)$ is a mid-point of an *r-grid square*. The *digital boundary* $B_r(S)$ is the boundary of the union of all those closed r-grid squares where the mid point is in S. A maximum-length digital straight segment approximation procedure [18, 20] calculates a sequence of DSS's connecting vertices of the digital boundary. Its result may vary with the allowed maximum Hausdorff distance [1] between the digital boundary $B_r(S)$

[1] The *Hausdorff distance* for sets of points A, B,

$$d_2(A,B) = \max\left\{ \max_{p\in A} \inf_{q\in B} d_2(p,q), \max_{p\in B} \inf_{q\in A} d_2(p,q) \right\} \; ,$$

generalizes the euclidean distance d_2 between points to a metric between sets of points.

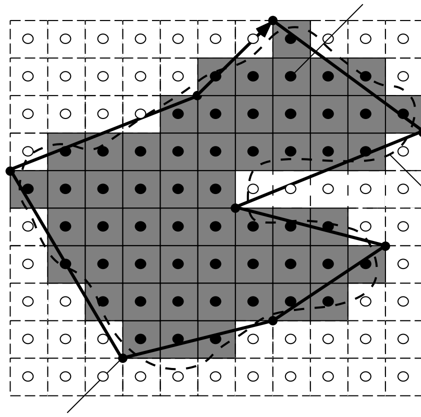

Fig. 1. [15] A simply-connected set S in the real plane, its digitization $D(S)$ and a (clockwise, starting at the uppermost-leftmost vertex) DSS approximation of its digital boundary which is used for perimeter estimation, and which consists of DSS's connecting vertices of grid squares.

and the calculated polygonal curve (in the "classical" paper [20] the threshold $1/r$, ie. the grid constant, has been used), the used orientation (clockwise or counter-clockwise), and the chosen inital vertex.

If the digital boundary $B_r(S)$ is not split into several poygonal curves (eg., in Fig. 3 there is a split into two polygonal curves) then the resulting DSS approximation is also a single (connected) polygonal curve. Its total length is used as an estimate of the perimeter of S.

3. For the *center of gravity* of a set S,

$$\left(\frac{m_{1,0}(S)}{m_{0,0}(S)}, \frac{m_{0,1}(S)}{m_{0,0}(S)}\right)$$

the estimate

$$\left(\frac{\mu_{1,0}(S)}{\mu_{0,0}(S)}, \frac{\mu_{0,1}(S)}{\mu_{0,0}(S)}\right)$$

is calculated from its digital set $D(S)$.

4. The orientation of a set S can be described by its axis of the least second moment. That is the line for which the integral of the squares of the distances to points in the digital set $D(S)$ is a minimum. That integral is

$$I(S, \varphi, \rho) = \iint_S r^2(x, y, \varphi, \rho) dxdy ,$$

where $r(x, y, \varphi, \rho)$ is the perpendicular distance from the point (x, y) to the line given in the form

$$x \cdot \cos\varphi - y \cdot \sin\varphi = \rho \ .$$

We are looking for the value of φ for which $I(S, \varphi, \rho)$ takes its minimum, and by this angle we define the *orientation* of the set S. This φ-value will be denoted by $or(S)$, i.e.

$$\min_{\varphi,\rho} I(S, \varphi, \rho) \ = \ I(S, or(S), \overline{\rho}), \text{ for some value of } \overline{\rho} \ .$$

Again, this feature is estimated by replacing integration and set S by a discrete addition and a digital set $D(S)$, respectively. With respect to applications note that this feature requires sets with "a main orientation", ie. $m_{2,0}(S) \neq m_{0,2}(S)$. For ("noisy") circular sets it would be distributed within the full 360 degree range.

5. The *elongation* of S (see [10, 31]) in direction φ is the ratio of maximum and minimum values of $I(S, \varphi, \rho)$, i.e.

$$E(S) \ = \ \frac{\max_{\varphi,\ \rho} I(S, \varphi, \rho)}{\min_{\varphi,\ \rho} I(S, \varphi, \rho)} \ .$$

It will be estimated by digital approximations of the I-function values as in case of the orientation of set S.

The feature *perimeter* is not defined by moments, and we list it here because of its general importance. It is known that the defined perimeter estimate converges towards the true value if a convex polygonal set is digitized with increasing grid resolution [18], see Theorem 12 below. This theorem comprises the theoretical fundamental for the given perimeter definition.

Central moments are also of common use in image analysis. Let

$$\left(\bar{x}_c\left(S\right), \bar{y}_c\left(S\right)\right) = \left(\frac{m_{1,0}(S)}{m_{0,0}(S)}, \ \frac{m_{0,1}(S)}{m_{0,0}(S)}\right)$$

and

$$\left(\bar{x}_d\left(S\right), \bar{y}_d\left(S\right)\right) = \left(\frac{\mu_{1,0}(S)}{\mu_{0,0}(S)}, \ \frac{\mu_{0,1}(S)}{\mu_{0,0}(S)}\right)$$

be the centroids of S and $D(S)$, respectively. Then,

$$\overline{m}_{p,q}(S) \ = \ \iint_S (x - \overline{x}_c(S))^p (y - \overline{y}_c(S))^q \ dx \ dy$$

are the *central* (p, q)*-moments* of order $p + q$, and

$$\overline{\mu}_{p,q}(S) \ = \ \sum_{(i,j) \in D(S)} (i - \overline{x}_d(S))^p \cdot (j - \overline{y}_d(S))^q$$

are the *discrete central* (p, q)*-moments* of order $p + q$. The central moments of order 1 are equal to zero.

1.4 Convergence of calculated features

Let **F** be a family of sets S in the euclidean plane, such as the family of all convex set, or the family of all straight line segments. Assume that feature M is defined for all sets in **F**.

Definition 2. We call an estimator $\hat{M}$ of M *convergent on* **F** iff

$$M(S) = \lim_{r \to \infty} \hat{M}(D_r(S)) ,$$

for all sets $S \in \mathbf{F}$ also meaning that $\hat{M}$ is defined on any digital set $D_r(S)$ if $S \in \mathbf{F}$, for all $r \geq r_S$ where r_S may exclude a finite range of r-values smaller than r_S.

This multigrid convergence approach has been used by different authors, see, eg., [12, 14, 18, 24, 27] for related work and references. Note that there is an equivalent definition for the dilation-based multigrid technique. Analyzing the accuracy of estimators means that the convergence (towards the true value, see Fig. 2 for a trivial convergence of the length of a curve towards a false value) needs to be satisfied, and that the speed of convergence may be analyzed by specifying the worst-case or probability-theoretical error bounds. Considerations with respect to statistical values require a probability model of the digitization process for the given family of sets.

The determination of worst-case error bounds is a problem typically studied in number theory. Assume that the complexity size of a problem is characterized by a non-negative real number. In the studies in this paper this number is the grid resolution r. Depending upon variable r a function $f(r) \geq 0$ may specify one important aspect of the problem such as a worst-case error defined for this grid resolution.

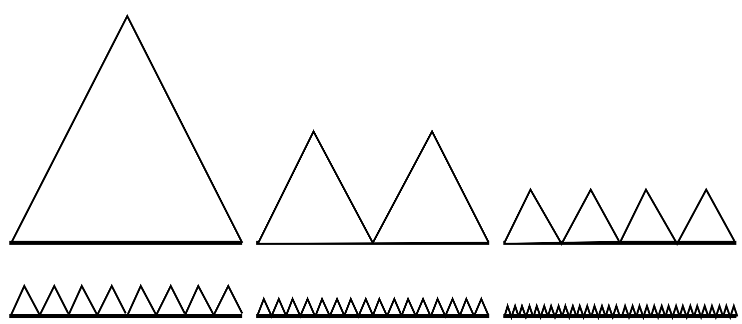

Fig. 2. The zig-zag curve approximates the straight line segment with respect to the Hausdorff distance. However, the length of the zig-zag curve remains constant and equal to twice the length of the approximated straight line segment assuming that we started with an equilateral triangle.

Definition 3. The function $f(r) \geq 0$ is in the asymptotic complexity class $\mathcal{O}(g(r))$ (we write $f(r) = \mathcal{O}(g(r))$) iff there exist a constant $c \geq 0$ and a constant $r_0 \geq 1$ such that $f(r) \leq c \cdot g(r)$, for all $r \geq r_0$. In this case we say that function $f(r)$ has the *worst-case bound* or *upper bound* $g(r)$.

The constant c is called the *asymptotic constant* of the given worst-case bound, and r_0 is its *validity parameter*. The values of the asymptotic constant and the validity parameter are of practical importance and its specification is often desirable, but unimportant for asymptotic characterizations. If f is an error measure then g specifies a *worst-case error bound.*

1.5 Organisation of the paper

The paper informs about new and previously published results in the area of multigrid feature calculations including number-theoretical, geometrical and probability-theoretical studies.

Besides the two-dimensional case we are also interested in one-dimensional digitizations (eg. straight line segments on a one-dimensional grid) and in digitizations of higher-dimensional sets such as volumes in 3D euclidean space. In the three-dimensional case the surface area specifies another interesting feature to be analyzed, see [19, 14]. The volume value may be estimated by $\mu_{0,0,0}(S)$, ie. the number of grid points contained in the given 3D set [11, 22]. However, this paper is limited to 1D and 2D problems.

The paper is organized as follows: Section 2 discusses number-theoretical worst-case, or upper error bounds for estimates of moments, and estimates of moment based features. The approach taken allows to cover different classes of sets, and the given (new) results specify sharp error bounds for moments of any order. Section 3 discusses the perimeter estimation problem and specifies an upper error bound for convex sets. This section is very much based on [18], but informs about an improved error bound compared to [18]. Section 4 finally informs on probability-theoretical error bounds based on work reported in [3]. Here, the contribution is that this previousely published work is formulated in the formal context of this paper. Finally, Section 5 concludes with a list of open problems.

2 Worst-case Error Bounds for Moment Estimates

In this section we consider worst-case error bounds in estimating real moments of sets $S \subset E^2$ from corresponding discrete moments.

Definition 4. A *planar n-smooth convex set* is a convex set in the euclidean plane whose boundary consists of a finite number of C^n arcs (ie., having continuous nth order derivatives) with positive curvature at every point on these arcs except arc endpoints, for $n \geq 0$.

Note that the claimed positive curvature excludes straight boundary segments. Let $\mathbf{F}_{nSC}$ be the family of all planar n-smooth convex sets, for $n \geq 0$. Throughout this Section 2 we assume that S is a planar 3-smooth convex set, or one obtained from planar 3-smooth convex sets by a finite number of intersections, set differences, or unions.

2.1 Huxley's theorem

Planar 3-smooth convex sets have been analyzed in [16, 17] for moments up to second order. A recent result [7] in number theory could be used as the basic mathematical tool in these studies. *M.N. Huxley*'s theorem [7] is a strong mathematical result, related to the number of grid points inside of a 3-smooth convex set of the form $r \cdot S$, where r is a positive real number.

Theorem 5. (Huxley 1990) *If S is a convex set in the euclidean plane with an C^3 boundary and a positive curvature at every point of the boundary, then the number of grid points belonging to $r \cdot S$ is*

$$\mu_{0,0}(r \cdot S) = r^2 \cdot A(S) + \mathcal{O}\left(r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}}\right) \ .$$

This theorem improves the previously best known upper bound of the error term even for the famous "circle problem", i.e. when S is the unit disk [9].

The precondition of Huxley's theorem can be relaxed. Actually it is sufficent to assume that $S \in \mathbf{F}_{3SC}$ holds, ie. S may have a finite number of "corners". This follows from the proof method used in [7]. This generalisation allows in consequence that the theorem can be applied, eg., to intersections of two planar 3-smooth convex sets or to difference sets of two planar 3-smooth convex sets. Note that the theorem also holds for (not necessarily connected) unions of a finite number of planar 3-smooth convex sets. Altogether this allows a generic definition of families $\mathbf{F}_{f(n)}$ of sets given in the next subsection.

In [16, 17] actually a weaker result (a conclusion of Huxley's theorem)

$$\mu_{0,0}(r \cdot S) = r^2 \cdot A(S) + \mathcal{O}\left(r^{\frac{7}{11}+\epsilon}\right) \ , \quad \text{for every } \ \varepsilon > 0$$

has been used for deriving upper error bounds for moments up to second order.

Theorem 6. (Klette/Žunić 1999) *If a set $S \in \mathbf{F}_{3SC}$ is digitized in a grid with grid resolution $r = 1/\vartheta$, then the absolute value of the difference*

$$m_{p,q}(S) \ - \ \tfrac{1}{r^{p+q+2}} \cdot \mu_{p,q}(r \cdot S) \ , \quad \textit{for } \ p + q \leq 2 \ ,$$

has a worst-case bound of

$$\mathcal{O}\left(\frac{1}{r^{\frac{15}{11}-\varepsilon}}\right) \ \approx \ \mathcal{O}\left(\frac{1}{r^{1.363636...}}\right) .$$

The same upper error bound holds for central and discrete central moments up to order 2, see [17], and this result can also be extended to sets in the closure $cl(\mathbf{F}_{3SC})$ which may be obtained from planar 3-smooth convex sets by finite applications of unions, intersections or set differences. The worst-case error bound remains the same which converges to zero with an increase in grid resolution r towards infinity.

So far it has been known that discrete moments or discrete central moments up to order 2 are convergent estimators of moments or central moments up to order 2 on the class $cl(\mathbf{F}_{3SC})$ of sets, respectively. This paper provides a more elegant and general proof for families $\mathbf{F}_{f(r)}$ of sets and moments of arbitrary order.

2.2 Families of sets defined by zero-order moments

The main result of this Section 2 is an upper error bound for estimating real moments (of an arbitrary order) based on corresponding discrete moments, for families $\mathbf{F}_{f(r)}$ of sets. These families are defined by differences between real and discrete moments of order zero.

Definition 7. Let a function $f(r) \geq 0$ be given, for $r \geq 0$. Then a class $\mathcal{C}$ of sets is a *closed subclass* of family $\mathbf{F}_{f(r)}$ iff it satisfies the following conditions:

(i) $\mathcal{C}$ is nonempty;
(ii) if $S \in \mathcal{C}$ then it satisfies

$$\mu_{0,0}(r \cdot S) - r^2 \cdot A(S) = \mathcal{O}\left(f(r)\right) \; ; \tag{1}$$

(iii) if a set S belongs to $\mathcal{C}$ then any isometric transformation of S belongs to $\mathcal{C}$ as well;
(iv) any set which can be represented by a finite number of unions, intersections and set-differences of sets from $\mathcal{C}$ also belongs to $\mathcal{C}$.

We give some comments on conditions (ii), (iii) and (iv). In (ii) we consider the difference between the area of $r \cdot S$ and the number of grid points inside of $r \cdot S$ when r tends to infinity, for an arbitrary set S from a subclass $\mathcal{C}$ of $\mathbf{F}_{f(r)}$. We assume that $\mathcal{O}(f(r))$ is a common upper bound for such differences of all sets in $\mathbf{F}_{f(r)}$. It will enable the calculation of a common upper error bound for moment estimations of sets in $\mathbf{F}_{f(r)}$.

Condition (iii) reflects a standard assumption in image analysis that a real set and the used digitization grid can be in arbitrary positions, i.e., translation, rotation or a different orientation of the grid axes are equivalent to the application of suitable chosen translations, rotations or reflection operations to the considered set. Formally speaking, $\mathcal{C}$ is closed with respect to isometric transformations.

Condition (iv) reflects situations in common image analysis where objects of interest are overlapping, grouping etc., which implies that their intersections, unions and consequently set differences should be allowed. Formally speaking, $\mathcal{C}$ is closed with respect to the applications of a finite number of unions, intersections and set-differences, ie. $cl(\mathcal{C}) = \mathcal{C}$.

The role of function $f(r)$ is actually to describe the impact of the applied grid resolution onto the worst-case error in real moment estimations. It will turn out that the precision in the reconstruction of moments is bounded above by $\mathcal{O}(r^{-1})$ for a set having a straight section on its boundary, and by

$$\mathcal{O}\left(\frac{(\log r)^{\frac{47}{22}}}{r^{\frac{15}{11}}}\right) \approx \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)$$

for all sets in $cl(\mathbf{F}_{3SC})$.

Throughout this Section 2 it can be assumed that the order of magnitude of $f(r)$ is between $r^{1/2}$ and r. Namely, our derivations are based on asymptotic estimations of the differences between the number of grid points inside of a convex set S (bounded by a curve γ) belonging to some specified class, and the area $A(S)$ of set S, see Equ. (1). *Gauss* and *Dirichlet* knew already that the area of a convex set bounded by curve γ estimates this number within an order $\mathcal{O}(l)$, where l is the length of the curve γ. That gives $f(r) = \mathcal{O}(r)$ as upper bound.

Under additional assumptions about the boundary of S the error term in Equ. (1) can be smaller. The curvature of the boundary plays an important role. If $f(r)$ is going to be expressed as $f(r) = \mathcal{O}(r^{\alpha})$ then α decrease if some additional smoothness conditions are assumed. It has been proved that Equ. (1) would be false for $\alpha < 0.5$, see [6, 13]. *Huxley*'s theorem bounds the error term in Equ. (1) by

$$\mathcal{O}\left(r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}}\right)$$

for a set S in the class $\mathbf{F}_{3SC}$.

We conclude this subsection with some examples. The closure $cl(\mathbf{F}_{3SC})$ of the class $\mathbf{F}_{3SC}$ with respect to finite numbers of unions, intersections or set-differences is a closed subclass of the family

$$\mathbf{F}_{r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}}}$$

because the conditions (i) ... (iv) are obviously satisfied.

A class of sets containing convex sets whose boundary consists of a finite number of C^3-arcs, also allowing straight line segments (ie. arcs with zero curvature), and which is closed with respect to finite numbers of unions, intersections or set-differences, is a closed subclass of the family $\mathbf{F}_r$. For details see [7]. Note that just one straight boundary segment makes already a difference compared to 3-smooth convex sets. Particularly, the class $cl(\mathcal{P})$ containing all convex polygons, and being closed with respect to finite numbers of unions, intersections or set-differences, is a closed subclass of the family $\mathbf{F}_r$.

2.3 Convergence theorems

The following theorems and the lemata in the Appendix are formulated for sets in families $\mathbf{F}_{f(r)}$. This was done for several reasons. It enables us to deal with two absolutely different situations (either there is at least one straight section on the boundary of the considered set or not) in a uniform way. Furthermore, it would support an immediate application of any possible theoretical improvement in the error bound in Equ. 1, ie. a better estimation of $f(r)$ will contribute to improved error bounds for estimations of moments of arbitrary order. As an illustration, *Swinnerton-Dayer*'s result [26] suggests that the exponent α in the error term $f(r) = r^{\alpha}$ of Equ. 1 should be $\frac{3}{5}$ or less, under the assumption that the boundary of set S consists of a finite number of C^3 arcs. If this can be shown then it would imply an improvement of the upper error bound for moment estimations. Finally this approach of using families $\mathbf{F}_{f(r)}$ has been chosen to enable direct applications of further results (from number theory) related to specific (closed) subclasses $\mathcal{C}$ of a family $\mathbf{F}_{f(r)}$.

The next two theorems are the main (new) results in this section which characterise the efficiency of estimations of real moments. As mentioned before, the curvature of the boundary of the considered set plays an important role. It makes an essential difference whether at least one straight section on the boundary is allowed (Theorem 8) or not (Theorem 9).

Theorem 8. *Let S be a convex set whose boundary consists of a finite number of C^3 arcs. Then $m_{p,q}(S)$ can be estimated by*

$$\frac{1}{r^{p+q+2}} \cdot \mu_{p,q}(r \cdot S) \quad \textit{within an error of} \quad \mathcal{O}\left(\frac{1}{r}\right) .$$

This error term is the best possible.

Proof. The upper bound is a direct consequence of Lemma 23 (in the Appendix) and the fact that a straight section on the boundary is allowed. In this case we have $f(r) = r$ [7] and

$$\max\left\{f(r),\ r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}}\right\} = \mathcal{O}(r) .$$

The following example shows that the established upper bound is the best possible. Let S be the unit square with vertices $(0,0), (1,0), (1,1), (0,1)$. Then we have

$$m_{p,q}(S) = \frac{1}{(p+1)(q+1)} .$$

For a given resolution r, the square S_r with vertices

$$(0,0),\ (1+\frac{1}{2r},0),\ (1+\frac{1}{2r},1+\frac{1}{2r}),\ (0,1+\frac{1}{2r}) ,$$

has the same digitization as S, i.e., $D_r(S) = D_r(S_r)$. But, the difference $m_{p,q}(S) - m_{p,q}(S_r)$ is equal to

$$\begin{aligned} m_{p,q}(S) - m_{p,q}(S_r) &= \int_0^1 x^p\,dx \int_1^{1+\frac{1}{2r}} y^q\,dy + \int_1^{1+\frac{1}{2r}} x^p\,dx \int_0^{1+\frac{1}{2r}} y^q\,dy \\ &= \frac{p+q+2}{2 \cdot (q+1) \cdot (p+1)} \cdot \frac{1}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) . \end{aligned}$$

In other words, for any choice of resolution r there exists a real square S_r such that the digitization of $r \cdot S_r$ coincides with the digitization of $r \cdot S$, while the real moments $m_{p,q}(S_r)$ differ from $m_{p,q}(S)$ by

$$\frac{p+q+2}{2 \cdot (q+1) \cdot (p+1)} \cdot \frac{1}{r} + \mathcal{O}\left(\frac{1}{r^2}\right) \ .$$

So, the error term $\mathcal{O}(\frac{1}{r})$ is the best possible. □

However, if S is 3-smooth and convex, ie. the boundary does not possess any straight segment, then the application of *Huxley*'s theorem finally leads to a reduced upper error bound.

Theorem 9. *Let a planar 3-smooth convex set S be given. Then $m_{p,q}(S)$ can be estimated by*

$$\frac{1}{r^{p+q+2}} \cdot \mu_{p,q}(r \cdot S) \quad \textit{within an error of} \quad \mathcal{O}\left(\frac{(\log r)^{\frac{47}{22}})}{r^{\frac{15}{11}}}\right) \approx \mathcal{O}\left(\frac{1}{r^{1.3636\ldots}}\right) \ .$$

Proof. The statement follows because of

$$S \in \mathbf{F}_{r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}}}$$

and Lemma 23 (in the Appendix). □

Remark. While in the case when at least one straight section is allowed on the boundary of S the error bound derived here is the best possible, in the case when the considered set is 3-smooth there is no guaranty whether the error bounds in *Huxley*'s Theorem, and consequently in Theorem 9 are the best possible or not. An improvement is expected to be characterised by an estimation of function $f(r)$ for a specific class of sets S.

Even in the case of the most frequently studied sets, the circles, an exact order of magnitude of $f(r)$ is unknown up to now. The best known upper bound is given by

$$\mathcal{O}\left(r^{\frac{46}{73}+\varepsilon}\right)$$

see [8] in the case when S is (before dilation) a unit circle with a midpoint at grid point location. On the other hand, in the case of such circles it is known that $f(r)$ has an order of magnitude described by the lower bound

$$\Omega\left(r^{\frac{1}{2}}\right) \ .$$

Just as a reminder, $g(r) = \Omega(h(r))$ implies that

$$\lim_{r \to \infty} \frac{g(r)}{h(r)} \neq 0 \ .$$

Let us consider the discrete moment $\mu_{1,0}(r \cdot S)$ where S is a circle with a center having integer coordinates. Then $\mu_{1,0}(r \cdot S)$ is equal to the number of grid points belonging to the 3D set

$$\{ (x, y, z) \ : \ (x, y) \in r \cdot S \ \wedge \ x \le z \} \ .$$

This number is half of the number of grid points belonging to the 3D set

$$\{ (x, y, z) \ : \ (x, y) \in r \cdot S \ \wedge \ z \le r \cdot (x_{min} + x_{max})\} \ .$$

Consequently,

$$\mu_{1,0}(r \cdot S) = \frac{r \cdot (x_{min} + x_{max})}{2} \cdot \left(r^2 \cdot A(S) + \Omega \left(r^{\frac{1}{2}} \right) \right) = r^3 \cdot m_{1,0}(S) + \Omega \left(r^{\frac{3}{2}} \right) .$$

In other words $\mathcal{O}(r^{-\frac{3}{2}})$ is a lower bound for the error in reconstructing $m_{1,0}(S)$ when S is a circle having a center with integer coordinates. For any circle in general position due to *Huxley*'s Theorem an upper error bound in moment estimation is given by

$$\mathcal{O}\left(\frac{(\log r)^{\frac{47}{22}})}{r^{\frac{15}{11}}} \right) \approx \mathcal{O}\left(\frac{1}{r^{1.3636\ldots}} \right)$$

as shown in [32].

2.4 Moment-based features

Let S be a set in $\mathbf{F}_{3SC}$. With respect to the listed features in Section 1 and Theorems 5 and 6 we may state the following upper error bounds for feature estimations (see also [17]):

1. An upper error bound for area estimates $\frac{1}{r^2} \, \mu_{0,0}\left(r \cdot S\right)$ is directly given by *Huxley*'s theorem, ie.

$$\left| A\left(S\right) - \tfrac{1}{r^2} \mu_{0,0}\left(r \cdot S\right) \right| \ = \ \mathcal{O}\left(\frac{1}{r^{\frac{15}{11} - \varepsilon}} \right) \ .$$

2. The same upper error bound holds for the estimates

$$\frac{1}{r} \cdot \frac{\mu_{1,0}\left(r \cdot S\right)}{\mu_{0,0}\left(r \cdot S\right)} \quad \text{and} \quad \frac{1}{r} \cdot \frac{\mu_{0,1}\left(r \cdot S\right)}{\mu_{0,0}\left(r \cdot S\right)}$$

of the coordinates

$$\frac{m_{1,0}\left(S\right)}{m_{0,0}\left(S\right)} \quad \text{and} \quad \frac{m_{0,1}\left(S\right)}{m_{0,0}\left(S\right)}$$

of the center of gravity.

3. For the estimate of the orientation only sets S with $m_{2,0}(S) \neq m_{0,2}(S)$ are relevant. Then S's orientation $or(S)$ can be recovered within an worst-case error of $\mathcal{O}(r^{-\frac{15}{11}+\varepsilon})$, by using the estimate [17]

$$\tan(2 \cdot or(S)) \ \approx \ \frac{2 \cdot \overline{\mu}_{1,1}(r \cdot S)}{\overline{\mu}_{2,0}(r \cdot S) - \overline{\mu}_{0,2}(r \cdot S)} \ .$$

4. The elongation of a 3-smooth convex set S can be estimated by $\mathcal{E}(r \cdot S)$ where
$$\mathcal{E}(r \cdot S) = \frac{t_1(r \cdot S) + \sqrt{t_2(r \cdot S)}}{t_1(r \cdot S) - \sqrt{t_2(r \cdot S)}} \ ,$$
where
$$t_1(A) = \overline{\mu}_{2,0}(A) + \overline{\mu}_{0,2}(A)$$
and
$$t_2(A) = 4 \cdot (\overline{\mu}_{1,1}(A))^2 + (\overline{\mu}_{2,0}(A) - \overline{\mu}_{0,2}(r \cdot S))^2 \ ,$$
for a planar set A. The error in the approximation $E(S) \approx \mathcal{E}(r \cdot S)$ has an upper error bound in $\mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)$, see [17].

Theorems 8 and 9 may be used to derive error bounds for sets with straight boundary segments, or for features defined by moments of higher order than just up to order two.

3 Worst-case Error Bounds for Perimeter Estimates

In this section we study the problem of estimating the perimeter of a convex polygonal set. In general a DSS approximation may be a set of polygonal curves calculated with respect to the digital boundary $B_r(S)$, see Fig. 3. For all $r \geq r_0$, a convex set S generates exactly one polygonal curve γ_r as DSS approximation being the boundary of a convex polygon. For these connected DSS approximations γ_r there exists a positive real ε_{DSS} such that the Hausdorff distance between $B_r(S)$ and γ_r does not exceed ε_{DSS}/r,
$$d_2(B_r(S), \gamma_r) \leq \frac{\varepsilon_{DSS}}{r} \ . \tag{2}$$
The value of ε_{DSS} is a general approximation constant, and the "classical" value is 1, see [20].

For the proof of the following theorem we cite two lemata as given and proved in [18]. The first lemma shows that the perimeter decreases if a convex, not necessarily polygonal set is eroded into a smaller convex polygonal set. Note that the assumption of convexity of set S is essential for proving this lemma.

Lemma 10. *Assume that a convex polygonal set S is contained in a convex set C of the euclidean plane, $S \subseteq C$. Then it follows that the perimeter of S is less or equal to the perimeter of C.*

An ε-*sausage* of a curve γ (as originally discussed in [19]) is the set of all points p whose Hausdorff distance $d_2(\{p\}, \gamma)$ is less or equal to ε. The border of any ε-sausage which is homeomorphic to the annulus splits into two curves, the shorter *inner boundary* and the longer *outer boundary* assuming that the boundary is rectifiable (ie. "shorter" and "longer" is defined in this case). The following second lemma from [18] shows how the perimeter of a convex polygonal set increases if the set is dilated towards the outer boundary of the ε-sausage of its boundary.

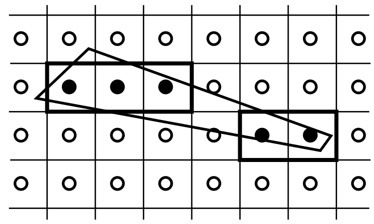

Fig. 3. A given polygonal convex set S and its digital boundary $B_r(S)$ which splits into two simple polygonal curves.

Lemma 11. *The length of the outer boundary of the ε-sausage of the boundary of a convex planar polygonal set S is equal to $Perimeter(S) + 2\pi\varepsilon$.*

Now we are prepared to state the following theorem about multigrid convergence of the estimated perimeter. Note that this theorem provides not only an asymptotic worst-case bound but even an explicit specification of the asymptotic constant. This theorem and most of its proof is basically a citation of work done by *V. Kovalevsky* and *S. Fuchs* reported in [18]. However, the theorem as stated below may not be found in [18]. Therefore we provide a proof in this paper. With respect to the given proof of this theorem the mentioned DSS approximation in this theorem needs to be such that it circumscribes a convex (!) polygonal set. This aspect was not explicitely mentioned in [18].

Theorem 12. (Kovalevsky/Fuchs 1992) *Let S be a convex polygonal set in the euclidean plane. Then there exists a grid constant r_0 such that for all $r \geq r_0$ it holds that the DSS approximation of $B_r(S)$ is connected and its length and the perimeter of S do not differ by more than*

$$\tfrac{2\pi}{r}\left(\varepsilon_{DSS} + 1/\sqrt{2}\right) .$$

Proof. At first note that there exists a grid constant r_0 such that for all $r \geq r_0$ it holds that the DSS approximation of $B_r(S)$ is connected. This connected DSS approximation will be denoted by γ_r.

Let $Bd(S)$ be the boundary of set S. For $B_r(S)$ it holds that

$$d_2(Bd(S), B_r(S)) \leq \frac{1}{r \cdot \sqrt{2}} , \qquad (3)$$

for all $r \geq r_0$. For proving this inequality assume that this Hausdorff distance would be greater than $(r \cdot \sqrt{2})^{-1}$. Then there exists at least one point p on $B_r(S)$ whose minimum euclidean distance to $Bd(S)$ is greater than $(r \cdot \sqrt{2})^{-1}$. Therefore the circle with centre p and radius $(r \cdot \sqrt{2})^{-1}$ would not contain any point of $Bd(S)$, ie. this circle is disjoint to set S.

Now assume that p is on the boundary of an r-grid square with mid-point g_{ij}^r. Then the r-grid point g_{ij}^r would be inside of this circle around point p with radius $(r \cdot \sqrt{2})^{-1}$ ie., it cannot be in the set S, ie. the r-grid point g_{ij}^r is not in $D_r(S)$. It follows that p cannot be a point on $B_r(S)$ which contradicts our assumption. Therefore such a point p does not exist and the distance must be always smaller than $(r \cdot \sqrt{2})^{-1}$. This concludes the proof of inequality (3).

This inequality (3), the inequality (2), and the general triangular inequality of a metric, satisfied by the Hausdorff distance, leads to the conclusion that

$$d_2(Bd(S), \gamma_r) \leq \frac{\varepsilon_{DSS}}{r} + \frac{1}{r \cdot \sqrt{2}} \, .$$

Let $\varepsilon = \varepsilon_{DSS}/r \; + \; 1/(r \cdot \sqrt{2})$. Then it follows that the perimeter of S and the length of γ_r differ by $2\pi\varepsilon$ at most. For showing this we assume that γ_r is the boundary of a convex (polygonal) set C, ie. it holds

$$d_2(Bd(S), Bd(C)) \leq \varepsilon \, . \tag{4}$$

We show that

$$|Perimeter(S) - Perimeter(C)| \leq 2\pi\varepsilon \, , \tag{5}$$

and this concludes the proof of the theorem.

According to the specified Hausdorff distance it follows that the boundary $Bd(C)$ lies in the ε-sausage of the boundary $Bd(S)$ of set S. Let $Bd_\varepsilon(S)$ be the outer boundary of the ε-sausage of $Bd(S)$. By Lemma 10 it follows that

$$Perimeter(C) \leq |Bd_\varepsilon(S)| \, .$$

By Lemma 11 it follows that

$$|Bd_\varepsilon(S)| = Perimeter(S) + 2\pi\varepsilon \, .$$

Therefore

$$Perimeter(C) \leq Perimeter(S) + 2\pi\varepsilon \, . \tag{6}$$

The Hausdorff distance is a metric and thus symmetric, ie. the boundary $Bd(S)$ lies also in the ε-sausage of the boundary $Bd(C)$. Let $Bd_\varepsilon(C)$ be the outer boundary of the ε-sausage of $Bd(C)$. By Lemma 10 it follows that

$$Perimeter(S) \leq |Bd_\varepsilon(C)| \, .$$

By Lemma 11 it follows that

$$|Bd_\varepsilon(C)| = Perimeter(C) + 2\pi\varepsilon \, .$$

Therefore

$$Perimeter(S) - 2\pi\varepsilon \leq Perimeter(C) \, . \tag{7}$$

From inequalities (6) and (7) it follows that

$$Perimeter(S) - 2\pi\varepsilon \leq Perimeter(C) \leq Perimeter(S) + 2\pi\varepsilon \, ,$$

what proves Equ. (5) and thus the theorem. □

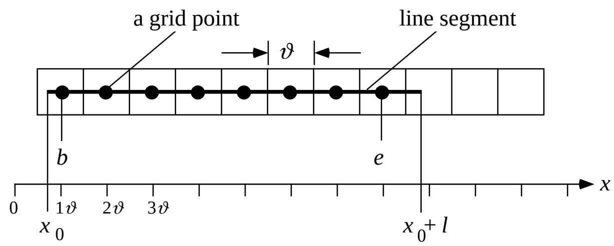

Fig. 4. Digitization of a line segment of length l in a one-dimensional grid of grid resolution $r = 1/\vartheta$.

If the set S is assumed to be more general than just polygonal convex then we may also conclude that

$$d_2(Bd(S), \gamma_r) \leq \frac{\varepsilon_{DSS}}{r} + \frac{1}{r \cdot \sqrt{2}} .$$

However, Equ. (5) is only valid for polygonal convex sets S and C, ie. the final conclusion about differences of perimeters cannot be drawn if S is not known to be polygonal convex (and thus C as well).

4 Probability-theoretical Error Bounds

The determination of probability-theoretical error bounds such as standard deviations or expected errors, is a problem in probability theory [2] or integral geometry [21]. A probability model is required for specifying the distribution of input data. In our case, the probability model has to specify how the digitization process performs on the chosen family of sets. For example, the standard deviation, ie. the square root of the variance, requires an analysis of the mean as well, and variance and mean are defined by the underlying probability model.

4.1 Digital straight line segments

Digital sets of specific shape (finite unions of digitized line segments) may be analysed based on results obtained for line digitizations. Digitization effects for measuring the area of a set have been studied in [28, 29, 30] utilizing this approach. The stochastic analysis of positional variations has also been discussed in Section 20.2 in [3]. Their analysis follows [4] who also studied digitizations of a line segment in a one-dimensional grid. Following the Definition 1, a grid point is in the digital set of a line segment if it is on the line segment (see Fig.4). Consider a line segment of length l going from x_0 to $x_0 + l$, see Fig.4. Let e be the integer denoting the rightmost grid point in the digitization of the given line segment, and let b be the integer denoting the leftmost grid point in the digitization of the given line segment.

The following *digitization model* has been used in [3]: For a uniform random variable q, $0 \leq q \leq 1$, let $x_0 + l = \vartheta \cdot (e^* - 0.5 + q)$, where $e^* = \lceil \frac{x_0+l}{\vartheta} - 0.5 \rceil$ is an integer. The assumed quantization model

$$e = \begin{cases} e^* & \text{with probability } q \\ e^* - 1 & \text{with probability } 1-q \end{cases} \tag{8}$$

specifies the relationship between e and e^*. For the line's left endpoint x_0 assume the representation $x_0 = \vartheta \cdot (b^* + 0.5 - s)$, where s is a uniform random variable with $0 \leq s \leq 1$, and $b^* = \lfloor \frac{x_0}{\vartheta} + 0.5 \rfloor$ is an integer. The assumed quantization model is

$$b = \begin{cases} b^* & \text{with probability } s \\ b^* + 1 & \text{with probability } 1-s \ . \end{cases} \tag{9}$$

This digitization model leads to the following theorem. This theorem may not be found exactly in this formulation in [3] but all the relevant material is specified there.

Theorem 13. (Haralick/Shapiro 1993) *Assume that line segments are digitized according to Equ. (8) and Equ. (9). Then* $\hat{e} = \vartheta \cdot (e + 0.5)$ *is an unbiased estimator for the position* $x_0 + l$*, and* $\hat{b} = \vartheta \cdot (b - 0.5)$ *is an unbiased estimator for the position* x_0*. The standard deviation of the estimated length of the line segment is given by*

$$S(\hat{e} - \hat{b}) = \frac{\vartheta}{\sqrt{3}} = \frac{1}{r\sqrt{3}}$$

and the standard deviation of the estimated centroid of the line segment is

$$S(\tfrac{\hat{b}+\hat{e}}{2}) = \frac{\vartheta}{\sqrt{12}} = \frac{1}{r \cdot \sqrt{12}} \ .$$

Proof. (Sketch only, see [3] for details) From Equ. (8) it follows that e has the mean $E(e) = \frac{x_0+l}{\vartheta} - 0.5$ and the variance $V(e) = \frac{1}{6}$ what means that $V(\hat{e}) = \frac{\vartheta^2}{6}$. From Equ. (9) it follows that b has the mean $E(b) = \frac{x_0}{\vartheta} + 0.5$ and the variance $V(b) = \frac{1}{6}$.

It follows that $\hat{e} - \hat{b}$ ($\hat{e}$ and $\hat{b}$ as defined in the theorem) is an unbiased estimator for the length of the line segment, $E(\hat{e} - \hat{b}) = E(\hat{e}) - E(\hat{b}) = (x_0 + l) - x_0 = l$. The variance of this estimator is $V(\hat{e} - \hat{b}) = V(\hat{e}) + V(\hat{b}) = \frac{\vartheta^2}{3}$.

For the centroid of the line it follows that the mean is equal to $E(\frac{\hat{b}+\hat{e}}{2}) = \frac{b+e}{2}$ and the variance is $V(\frac{\hat{b}+\hat{e}}{2}) = \frac{\vartheta^2}{12}$. □

4.2 Digital sets composed of parallel lines

Now assume a set S in the real plane which may be digitized by digitizing $N \geq 1$ line segments in different one-dimensional subgrids of a two-dimensonal orthogonal grid, all parallel to the x-coordinate axis. Line n begins at x_n and ends at $y_n = x_n + l_n$ with respect to the x-coordinate axis. The digitization of all of these line segments forms a digital set $D(S)$. Note that this set does not need to be connected.

We assume that there is no systematic alignment of the digitized line segments with the row and colum axes of the given orthogonal grid. Theorem 13 allows to derive error bounds for features of digital sets $D(S)$ composed of a set of N lines as specified as shown in [3]. This section briefly informs about what has been done there.

The (discrete) *column centroid* of such a set S is defined to be equal to

$$\bar{c}(S) = \frac{\sum\limits_{n=1}^{N} (y_n - x_n)\left(\frac{y_n + x_n}{2}\right)}{\sum\limits_{n=1}^{N} (y_n - x_n)} = \frac{\sum\limits_{n=1}^{N} l_n \cdot \left(\frac{y_n + x_n}{2}\right)}{\sum\limits_{n=1}^{N} l_n} .$$

This is considered to be an approximation of the moment $m_{1,0}(S)$ or of the discrete moment $\mu_{1,0}(S)$. A (discrete) *row centroid* $\bar{r}(S)$ could be defined in a similar way for connected runs of grid points.

The chosen definition of a column centroid suggests the estimator

$$\hat{c}(S) = \frac{\sum\limits_{n=1}^{N} (\hat{y}_n - \hat{x}_n)\left(\frac{\hat{y}_n + \hat{x}_n}{2}\right)}{\sum\limits_{n=1}^{N} (\hat{y}_n - \hat{x}_n)} .$$

It follows $E\left(\hat{c}(S)\right) \cong \bar{c}(S)$ for the mean and

$$\begin{aligned} V\left(\hat{c}(S)\right) &= \frac{\frac{\vartheta^2}{3} \sum\limits_{n=1}^{N} \left[\left(\frac{y_n + x_n}{2} - \bar{c}(S)\right)^2 + \frac{(y_n - x_n)^2}{4}\right]}{\left(\sum\limits_{n=1}^{N} (y_n - x_n)\right)^2} \\ &= \frac{\sum\limits_{n=1}^{N} \left[\left(\frac{y_n + x_n}{2} - \bar{c}(S)\right)^2 + \frac{l_n^2}{4}\right]}{3r^2 \left(\sum\limits_{n=1}^{N} l_n\right)^2} \end{aligned} \qquad (10)$$

for the variance. The standard deviation of the column centroid of set S follows as being the square root of this variance. This result of [3] simplifies for specific sets.

Example 1. (Haralick/Shapiro 1993) Assume a digital set $D(S)$ that is symmetric about a vertical axis running through its column centroid. Then it follows that $\frac{y_n + x_n}{2} = \bar{c}(S)$, for $n = 1, ..., N$, and

$$V\left(\hat{c}(S)\right) = \frac{\frac{\vartheta^2}{12} \sum\limits_{n=1}^{N} (y_n - x_n)^2}{\left(\sum\limits_{n=1}^{N} (y_n - x_n)\right)^2} = \frac{\sum\limits_{n=1}^{N} l_n^2}{12r^2 \left(\sum\limits_{n=1}^{N} l_n\right)^2} .$$

As a further example, consider a disk S with diameter $\vartheta \cdot N$, ie. N is the number of rows it takes to cover the disk. This leads to

$$V(\hat{c}(S)) = \frac{0.09 \cdot \vartheta^2}{N} = \frac{0.09}{r^2 \cdot N}$$

as a first-order approximation of the variance of the column centroid with standard deviation

$$S(\hat{c}(S)) = \frac{0.3 \cdot \vartheta}{\sqrt{N}} = \frac{0.3}{r \cdot \sqrt{N}} ,$$

and a similar result

$$S(\hat{r}(S)) = \frac{0.3 \cdot \vartheta}{\sqrt{Q}} = \frac{0.3}{r \cdot \sqrt{Q}}$$

holds for the estimated row centroid $\hat{r}(S)$ where Q is the number of colums it takes to cover the disk. If the disk is in symmetric position with respect to the grid then $N = Q$. In [3] also a rectangle oriented parallel to the rows of an image has been discussed as a further specific example.

5 Conclusions

For most of the given upper bounds it is not yet known whether they are optimum or not. The situation is sketched in Table 1. The upper error bounds for straight line segments are trivial. The analysis of standard deviations seems to be an open problem for the discussed 2D features in general. Of course, trivial bounds may be stated, eg. the given upper error bounds.

The curvature of the boundary is important for moment estimations. Let us just consider Lemma 16 in the Appendix. It is proved that the difference between the area of $(r \cdot S)(k)$ and the number of grid points inside of it is equal to half of the number of integer points on the vertical line $x = k$ plus an error term $\mathcal{O}(f(r))$. So, the considered difference is at least half of the number of integer points on the line $x = k$, ie. it has an order of magnitude equal to the grid resolution r. On the other hand, $(r \cdot S)(k)$ has only one straight section on its boundary. Another example which reaches the upper error bound $\mathcal{O}(r)$ was given in the proof of Theorem 8 for an example of a polygonal region (only straight boundary sections).

Note that the given error bounds in Table 1 are always related to one specific convergent estimator. It is also of interest whether there are better estimators supporting faster convergence. For example, if the perimeter R of a circle C : $(x-a)^2 + (y-b)^2 \leq R^2$ is estimated by

$$\frac{2}{r \cdot \mu_{0,0}(rC)} \cdot \sqrt{\mu_{2,0}(rC) \cdot \mu_{0,0}(rC) - (\mu_{1,0}(rC))^2}$$

then the error in the perimeter estimation is $\mathcal{O}(r^{-\frac{11}{15}+\varepsilon})$, see [33].

Table 1. Multigrid error bounds: the feature estimates are defined as specified in this paper.

SET TYPE	*FEATURE*	*UPPER BOUND*	*STAND.DEV.*
straight line segment	length	$\frac{1}{r}$	$\frac{1}{r\cdot\sqrt{3}}$
straight line segment	centroid	$\frac{1}{2\cdot r}$	$\frac{1}{r\cdot\sqrt{12}}$
simple polygon	moment (any order)	$\mathcal{O}\left(\frac{1}{r}\right)$	unknown
convex polygon	perimeter	$\frac{2\pi}{r}\left(\varepsilon_{DSS}+1/\sqrt{2}\right)$	unknown
3-smooth convex	area	$\mathcal{O}\left(r^{\frac{7}{11}}\cdot(\log r)^{\frac{47}{22}}\right)$	unknown
3-smooth convex	moment of any order	$\mathcal{O}\left(\frac{1}{r^{\frac{15}{11}-\varepsilon}}\right)$	unknown
3-smooth convex	center of gravity	$\mathcal{O}\left(\frac{1}{r^{\frac{15}{11}-\varepsilon}}\right)$	unknown
3-smooth convex	orientation	$\mathcal{O}\left(\frac{1}{r^{\frac{15}{11}-\varepsilon}}\right)$	unknown
3-smooth convex	elongation	$\mathcal{O}\left(\frac{1}{r^{\frac{15}{11}-\varepsilon}}\right)$	unknown

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Appendix: Lemata for Subsection 2.3

This appendix contains a few lemata which are used for proving the convergence theorems in Subsection 2.3. We use the following definitions.

Definition 14. For a planar set S, a given integer k and a real number r, the set $(r \cdot S)(k)$ is defined to be

$$(r \cdot S)(k) = \{(x,y) \ : \ (x,y) \in (r \cdot S) \ \wedge \ x \geq k\} \ .$$

Consequently, $D((r \cdot S)(k))$ is the set of grid points in the digitization of $r \cdot S$ lying in the closed half plane determined by $x \geq k$.

Definition 15. For a planar set S, a given integer k and a real number r, the grid point set $L(r \cdot S, k)$ is defined to be

$$L(r \cdot S, k) = \{(k,j) \ : \ (k,j) \in D(r \cdot S)\} \ .$$

In other words, $L(r \cdot S, k)$ is the set of grid points in the digitization of $r \cdot S$ which belongs to the vertical line $x = k$.

Lemma 16. *For a planar convex set S from a closed subclass $\mathcal{C}$ of a family $\mathbf{F}_{f(r)}$, and for a given integer k, the discrete $(0,0)$-moment of set $(r \cdot S)(k)$ can be expressed as*

$$\mu_{0,0}((r \cdot S)(k)) \ = \ A((r \cdot S)(k)) \ + \ \frac{1}{2} \cdot \mu_{0,0}(L(r \cdot S, k)) \ + \ \mathcal{O}(f(r)) \ .$$

Proof. Let $r \cdot \overline{S}$ be the set symmetrical to $r \cdot S$, with respect to the line $x = k$. It follows that the convex set $r \cdot S \cap r \cdot \overline{S}$ belongs to the closed subclass $\mathcal{C}$. So, the number of grid points belonging to $r \cdot S \cap r \cdot \overline{S}$ can be determined as

$$\mu_{0,0}(r \cdot S \cap r \cdot \overline{S}) = A(r \cdot S \cap r \cdot \overline{S}) + \mathcal{O}\left(f(r)\right)$$

The statement of the lemma follows because the set $r \cdot S \cap r \cdot \overline{S}$ is symmetrical with respect to the line $x = k$. □

Obviously, the moments $\mu_{p,0}(r \cdot S)$ and $\mu_{0,q}(r \cdot S)$, because of symmetry, can be derived in an identical way, while the estimation of $\mu_{p,q}(r \cdot S)$ where $p > 0$ and $q > 0$ needs some modifications. The following definitions of 3D-sets W_i and W_i' are used in the sequel:

Definition 17. For a planar convex set $r \cdot S$ and an integer i from the set $\{\lceil r \cdot x_{min} \rceil, \lceil r \cdot x_{min} \rceil + 1, \ldots, \lfloor r \cdot x_{max} \rfloor - 1\}$, we define 3D sets (see Fig. 5)

$$W_i = \{(x,y,z) \ : \ (x,y) \in r \cdot S \ \wedge \ x \geq i \ \wedge \ i^p < z \leq (i+1)^p\}$$

and

$$W_i' = \{(x,y,z) \ : \ (x,y) \in r \cdot S \ \wedge \ x \geq i \ \wedge \ x^p < z \leq (i+1)^p\} \ .$$

What follows is the calculation of discrete moments $\mu_{p,0}(r \cdot S)$. As a reminder, the function *vol* denotes the volume of a 3D set.

Lemma 18. *Let S be a convex set having a boundary consisting of a finite number of C^3 arcs. Then it holds that*

$$\sum_{\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} vol(W_i') = \sum_{\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} L(r \cdot S, i) \left((i+1)^p - i^p - \frac{p}{2} \cdot i^{p-1} \right) + \mathcal{O}(r^p) \ .$$

Proof. The boundary of $r \cdot S$ can be divided into two arcs of the form $y = y_1(x)$ and $y = y_2(x)$, such that $y_1(x) \leq y_2(x)$. Then it holds that

$$\sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} vol(W_i') = \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} \int_i^{i+1} dx \int_{x^p}^{(i+1)^p} dz \int_{y_1(x)}^{y_2(x)} dy$$

$$= \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} \int_i^{i+1} dx \int_{x^p}^{(i+1)^p} dz \left(\int_{y_1(x)}^{y_1(i)} dy + \int_{y_1(i)}^{\lceil y_1(i) \rceil} dy + \int_{\lceil y_1(i) \rceil}^{\lfloor y_2(i) \rfloor} dy \right.$$

$$\left. + \int_{\lfloor y_2(i) \rfloor}^{y_2(i)} dy + \int_{y_2(i)}^{y_2(x)} dy \right)$$

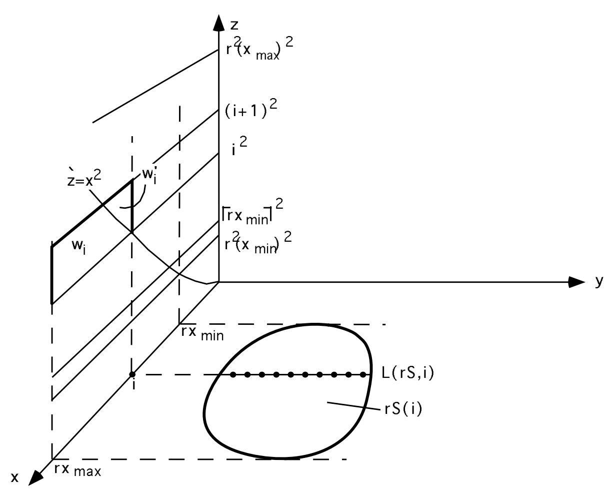

Fig. 5. Illustration of the decomposition for moment calculations with $p = 2$. For the general case exponent 2 needs to be replaced by exponent p.

$$= \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} \int_i^{i+1} dx \int_{x^p}^{(i+1)^p} dz \left(\int_{\lceil y_1(i)\rceil}^{\lfloor y_2(i)\rfloor} dy \quad + \quad \mathcal{O}(1) \right)$$

$$\left(\text{note that} \int\limits_{y_1(x)}^{y_1(i)} dy = \mathcal{O}(1) \quad \text{and} \quad \int\limits_{y_2(i)}^{y_2(x)} dy = \mathcal{O}(1) \right.$$

because $y_1(x)$ and $y_2(x)$ have a continuous first derivative)

$$= \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} \int_i^{i+1} (\lfloor y_2(i)\rfloor - \lceil y_1(i)\rceil) \cdot ((i+1)^p - x^p) dx \quad + \quad \mathcal{O}(r^p)$$

$$= \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} (\lfloor y_2(i)\rfloor - \lceil y_1(i)\rceil) \cdot \left((i+1)^p - i^p - \frac{p}{2} \cdot i^{p-1} + \mathcal{O}\left(i^{p-2}\right) \right) \; + \; \mathcal{O}(r^p)$$

$$= \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} L(r\cdot S, i) \cdot \left((i+1)^p - i^p - \frac{p}{2} \cdot i^{p-1} \right) \quad + \quad \mathcal{O}(r^p) \qquad \Box$$

The discrete moments $\mu_{p,0}(r\cdot S)$ and $\mu_{0,q}(r\cdot S)$ are evaluated by Lemma 19, while Lemma 21 evaluates $\mu_{p,q}(r\cdot S)$, where $p, q > 0$.

Lemma 19. *Let S be a convex planar set in $\mathbf{F}_{f(r)}$. Then the following (equivalent) asymptotic expressions are satisfied:*

$$\mu_{p,0}(r\cdot S) = \sum_{(i,j)\in D(r\cdot S)} i^p \; = \; \iint\limits_{r\cdot S} x^p dx dy \; + \; \mathcal{O}\left(f(r)\cdot r^p\right)$$

and

$$\mu_{0,q}(r\cdot S) = \sum_{(i,j)\in D(r\cdot S)} j^q \; = \; \iint\limits_{r\cdot S} y^q dx dy \; + \; \mathcal{O}\left(f(r)\cdot r^q\right) \quad .$$

Proof. Let us notice that $\mu_{p,0}(r\cdot S)$ is equal to the number of grid points belonging to the 3D set B given by

$$B \; = \; \{(x,y,z) \; : \; (x,y)\in r\cdot S \;\wedge\; 0 < z \le x^p\} \; = \; B' \,\cup\, B''$$

where B' and B'' are defined as follows:

$$B' = \{(x,y,z) \; : \; (x,y)\in r\cdot S \;\wedge\; 0 < z \le \lceil r\cdot x_{min}\rceil^p\}$$

and

$$B'' = \{(x,y,z) \; : \; (x,y)\in r\cdot S \;\wedge\; \lceil r\cdot x_{min}\rceil^p < z \le x^p\} \; ,$$

where x_{min} is defined to be

$$x_{min} = \min\{ \quad x \ : \ (x,y) \in S \quad \} .$$

Analogously, x_{max} is defined to be

$$x_{max} = \max\{ \quad x \ : \ (x,y) \in S \quad \} .$$

First, consider the number of grid points belonging to B'. It follows that

$$\mu_{0,0,0}(B') = \lceil r \cdot x_{min} \rceil^p \cdot (A(r \cdot S) + \mathcal{O}(f(r)))$$

$$= vol(B') \quad + \quad \mathcal{O}(f(r) \cdot r^p) \ .$$

Now we calculate the number of grid points belonging to B''. According to Definition 17 of the 3D sets W_i and W_i', and by using $\mathcal{O}(r^p)$ as a trivial estimation for the volume of the 3D set defined by

$$\{ \ (x,y,z) \ : \ (x,y) \in r \cdot S \ \wedge \ x \geq \lfloor r \cdot x_{max} \rfloor \ \wedge \ z \leq x^p \ \} ,$$

it follows that

$$vol(B'') = \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} (vol(W_i) - vol(W_i')) + \mathcal{O}(r^p)$$

$$= \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} vol(W_i) - \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} vol(W_i') + \mathcal{O}(r^p)$$

$$= \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} ((i+1)^p - i^p) \cdot A((r \cdot S)(i)) - \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} vol(W_i') + \mathcal{O}(r^p)$$

(by using Lemata 16 and 18, it follows)

$$= \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} ((i+1)^p - i^p) \cdot \Big(\mu_{0,0}((r \cdot S)(i))$$

$$- \frac{1}{2} \cdot \mu_{0,0}(L(r \cdot S, i)) + \mathcal{O}(f(r))\Big)$$

$$- \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} \mu_{0,0}(L(r \cdot S, i)) \cdot \left((i+1)^p - i^p - \frac{p}{2} \cdot i^{p-1}\right) \quad + \quad \mathcal{O}(r^p)$$

$$= \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} ((i+1)^p - i^p) \cdot (\mu_{0,0}((r \cdot S)(i)) - \mu_{0,0}(L(r \cdot S, i)) + \mathcal{O}(f(r)))$$

$$- \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} \frac{1}{2}\cdot \mu_{0,0}(L(r\cdot S,i))\cdot \left((i+1)^p - i^p - p\cdot i^{p-1}\right) + \mathcal{O}\left(r^p\right)$$

$$= \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} ((i+1)^p - i^p)\cdot (\mu_{0,0}((r\cdot S)(i)) - \mu_{0,0}(L(r\cdot S,i)))$$

$$+\mathcal{O}\left(f(r)\right)\cdot \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} \left(p\cdot i^{p-1} + \mathcal{O}(i^{p-2})\right)$$

$$- \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} \frac{1}{2}\cdot \mu_{0,0}(L(r\cdot S,i))\cdot \mathcal{O}\left(r^{p-2}\right) + \mathcal{O}\left(r^p\right)$$

$$= \mu_{0,0,0}(B'') + \mathcal{O}\left(r^p\cdot f(r)\right)$$

Note that for a fixed integer i, with $r\cdot x_{min} \le i \le r\cdot x_{max}$, it holds that

$$((i+1)^p - i^p)\cdot (\mu_{0,0}((r\cdot S)(i) - \mu_{0,0}(L(r\cdot S,i)))$$

equals the number of grid points belonging to W_i, and consequently

$$\sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} ((i+1)^p - i^p)\cdot (\mu_{0,0}((r\cdot S)(i) - \mu_{0,0}(\mu_{0,0}(L(r\cdot S,i))))$$

equals the number of grid points inside of B''.

The sum of $\mu_{0,0,0}(B')$ and $\mu_{0,0,0}(B'')$ is the number of grid points in B. Together with the already derived expression for $\mu_{0,0,0}(B')$, we have

$$\mu_{p,0}(r\cdot S) = \mu_{0,0,0}(B) = \mu_{0,0,0}(B') + \mu_{0,0,0}(B'')$$

$$= vol(B') + \mathcal{O}\left(r^p\cdot f(r)\right) \quad + \quad vol(B'') \quad + \quad \mathcal{O}(r^p\cdot f(r))$$

$$= vol(B) \ + \ \mathcal{O}\left(r^p\cdot f(r)\right) \quad = \quad m_{p,0}(r\cdot S) \ + \ \mathcal{O}\left(r^p\cdot f(r)\right). \quad \Box$$

It remains to estimate $\mu_{p,q}(r\cdot S)$, where $p,q>0$. The next definition and lemma are some analogons to Definition 14 and Lemma 16, respectively.

Definition 20. For a planar set S, given integers k, p, q, and a real number r, the set $(r\cdot S)(k,p,q)$ is defined to be

$$(r\cdot S)(k,p,q) = \{(x,y) \ : \ (x,y)\in (r\cdot S) \ \wedge \ x^p\cdot y^q \ge k\}\ .$$

Consequently, $D((r\cdot S)(k,p,q))$ is the set of grid points in the digitization of $r\cdot S$ lying in the closed part of the plane determined by $x^p\cdot y^q\ge k$.

Lemma 21. *For a planar convex set S in a closed subclass $\mathcal{C}$ of family $\mathbf{F}_{f(r)}$, and for given integers k, p, q, the discrete $(0,0)$-moment of the set $(r \cdot S)(k,p,q)$ can be expressed as*

$$\mu_{0,0}((r \cdot S)(p,q,k)) \;=\; A((r \cdot S)(p,q,k)) \;+\; \mathcal{O}\left(\max\left\{f(r),\; r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}}\right\}\right) .$$

Proof. Since the boundary of $(r \cdot S)(p,q,k)$ consists of a finite number of C^3-arcs the statement follows. Namely, if there is a straight section on the boundary of S then $f(r) = \mathcal{O}(r) = \max\left\{f(r),\; r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}}\right\}$, else, the strongest result which can be used (up to now) is

$$\mathcal{O}\left(\max\left\{f(r),\; r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}}\right\}\right) = \mathcal{O}\left(r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}}\right) . \quad \Box$$

Lemma 22. *Let S be a convex set in $\mathbf{F}_{f(r)}$. Then the following asymptotical expression holds,*

$$\mu_{p,q}(r \cdot S) = \iint\limits_{r \cdot S} x^p \cdot y^q \, dx dy + \mathcal{O}\left(r^{p+q} \cdot \left(\max\left\{f(r),\; r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}}\right\}\right)\right) .$$

Proof. Note that $\mu_{p,q}(r \cdot S)$ is equal to the number of grid points belonging to the 3D set E given by

$$E = \{(x,y,z) \;:\; (x,y) \in r \cdot S \;\wedge\; 0 < z \leq x^p \cdot y^q\} \;=\; E' \;\cup\; E''$$

where E' and E'' are defined as follows:

$$E' = \{(x,y,z) \;:\; (x,y) \in r \cdot S \;\wedge\; 0 < z < r^{p+q} \cdot z_{min}\}$$

and

$$E'' = \{(x,y,z) \;:\; (x,y) \in r \cdot S \;\wedge\; r^{p+q} \cdot z_{min} \leq z \leq x^p \cdot y^q\} ,$$

where z_{min} is defined to be

$$z_{min} \;=\; \min\{\, z \;:\; z = x^p \cdot y^q \;\wedge\; (x,y) \in S \,\} .$$

Analogously, z_{max} is defined to be

$$z_{max} \;=\; \max\{\, z \;:\; z = x^p \cdot y^q \;\wedge\; (x,y) \in S \,\} .$$

First, consider the number of grid points belonging to the set E'. From Equ. (1) it follows that

$$\begin{aligned}
\mu_{0,0,0}(E') &= (\lceil r^{p+q} \cdot z_{min} \rceil - 1) \cdot (A(r \cdot S) + \mathcal{O}(f(r))) \\
&= vol(E') - r^{p+q} \cdot z_{min} \cdot A(r \cdot S) \\
&\qquad + (\lceil r^{p+q} \cdot z_{min} \rceil - 1) \cdot (A(r \cdot S) \quad + \quad \mathcal{O}(f(r))) \\
&= vol(E') + A(r \cdot S) \cdot (\lceil r^{p+q} \cdot z_{min} \rceil - r^{p+q} \cdot z_{min}) + \mathcal{O}\left(r^{p+q} \cdot f(r)\right) .
\end{aligned}$$

$A(r \cdot S) = \mathcal{O}(r^2)$ and $p+q \geq 2$ have been used in this derivation. Now, let us calculate the number of grid points belonging to set E''. What follows is a definition of 3D-sets ω_i and ω_i', for $i = \lceil r^{p+q} \cdot x_{min} \rceil, \lceil r^{p+q} \cdot x_{min} \rceil + 1, \ldots, \lfloor r^{p+q} \cdot x_{max} \rfloor$:

$$\omega_i = \{ (x,y,z) : (x,y) \in r \cdot S \wedge x^p \cdot y^q \geq i \wedge i < z < \min\{x^p \cdot y^q, i+1\} \}$$

and

$$\omega_i' = \{ (x,y,z) : (x,y) \in r \cdot S \wedge i < x^p \cdot y^q \leq i+1 \wedge x^p \cdot y^q < z < i+1 \} .$$

Now, we can estimate the volume of E''. By using $\mathcal{O}(r^2)$ as a trivial upper bound for the volume of

$$\{(x,y,z) : (x,y) \in r \cdot S \wedge x^p \cdot y^q \leq \lceil r^{p+q} \cdot z_{min} \rceil \wedge x^p \cdot y^q \leq z \leq \lceil r^{p+q} \cdot z_{min} \rceil\}$$

it holds that

$$vol(E'')$$

$$= \sum_{i=\lceil r^{p+q} \cdot z_{min} \rceil}^{\lfloor r^{p+q} \cdot z_{max} \rfloor} vol(\omega_i) + \left(\lceil r^{p+q} \cdot z_{min} \rceil - r^{p+q} \cdot z_{min}\right) \cdot A(r \cdot S) + \mathcal{O}(r^2)$$

$$= \sum_{i=\lceil r^{p+q} \cdot z_{min} \rceil}^{\lfloor r^{p+q} \cdot z_{max} \rfloor} \left(A((r \cdot S)(i,p,q)) - vol(\omega_i')\right)$$

$$+ \left(\lceil r^{p+q} \cdot z_{min} \rceil - r^{p+q} \cdot z_{min}\right) \cdot A(r \cdot S) + \mathcal{O}(r^2)$$

$$= \sum_{i=\lceil r^{p+q} \cdot z_{min} \rceil}^{\lfloor r^{p+q} \cdot z_{max} \rfloor} A((r \cdot S)(i,p,q)) - \sum_{i=\lceil r^{p+q} \cdot z_{min} \rceil}^{\lfloor r^{p+q} \cdot z_{max} \rfloor} vol(\omega_i')$$

$$+ \left(\lceil r^{p+q} \cdot z_{min} \rceil - r^{p+q} \cdot z_{min}\right) \cdot A(r \cdot S) + \mathcal{O}(r^2)$$

$\Bigg($note that $\displaystyle\sum_{i=\lceil r^{p+q} \cdot z_{min} \rceil}^{\lfloor r^{p+q} \cdot z_{max} \rfloor} vol(\omega_i') \leq r^2 \cdot A(S)$ because the projections of ω_i' onto the xy-plane belong to $r \cdot S$)

$$= \sum_{i=\lceil r^{p+q} \cdot z_{min} \rceil}^{\lfloor r^{p+q} \cdot z_{max} \rfloor} \left(\mu_{0,0}((r \cdot S)(i,p,q)) + \mathcal{O}\left(\left(\max\left\{f(r),\ r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}}\right\}\right)\right)\right)$$

$$+ \left(\lceil r^{p+q} \cdot z_{min} \rceil - r^{p+q} \cdot z_{min}\right) \cdot A(r \cdot S) + \mathcal{O}(r^2)$$

$$= \mu_{0,0,0}(E'') \quad + \quad \left(\lceil r^{p+q} \cdot z_{min} \rceil - r^{p+q} \cdot z_{min} \right) \cdot A(r \cdot S)$$

$$+ \quad \mathcal{O} \left(r^{p+q} \cdot \left(\max \left\{ f(r), \; r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}} \right\} \right) \right)$$

Thus,

$$\mu_{0,0,0}(E'') \quad = \quad vol(E'') \quad - \quad \left(\lceil r^{p+q} \cdot z_{min} \rceil - r^{p+q} \cdot z_{min} \right) \cdot A(r \cdot S)$$

$$+ \quad \mathcal{O} \left(r^{p+q} \cdot \max \left\{ f(r), \; r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}} \right\} \right) .$$

The proof of the lemma is finished by adding $\mu_{0,0,0}(E')$ and $\mu_{0,0,0}(E'')$:

$$\mu_{p,q}(r \cdot S) \quad = \quad \mu_{0,0,0}(E') \quad + \quad \mu_{0,0,0}(E'')$$

$$= \quad vol(E') + A(r \cdot S) \cdot \left(\lceil r^{p+q} \cdot z_{min} \rceil - r^{p+q} \cdot z_{min} \right)$$

$$+ \mathcal{O} \left(r^{p+q} \cdot \max \left\{ f(r), \; r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}} \right\} \right)$$

$$+ \quad vol(E'') \; - \; \left(\lceil r^{p+q} \cdot z_{min} \rceil - r^{p+q} \cdot z_{min} \right) \cdot A(r \cdot S)$$

$$+ \quad \mathcal{O} \left(r^{p+q} \cdot \max \left\{ f(r), \; r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}} \right\} \right)$$

$$= \quad vol(E) \quad + \quad \mathcal{O} \left(r^{p+q} \cdot \max \left\{ f(r), \; r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}} \right\} \right)$$

$$= \quad m_{p,q}(r \cdot S) \quad + \quad \mathcal{O} \left(r^{p+q} \cdot \max \left\{ f(r), \; r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}} \right\} \right) . \quad \Box$$

The errors in estimating real moments $m_{p,q}(r \cdot S)$ by their discrete analogs $\mu_{p,q}(r \cdot S)$ are derived separately depending upon whether both p and q are strictly positive or not. It turns out that in both cases the error terms can be upper bounded by

$$\mathcal{O} \left(r^{p+q} \cdot \max \left\{ f(r), \; r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}} \right\} \right) \; .$$

The next lemma describes the efficiency in calculating real moments of an arbitrary order from the corresponding digital sets.

Lemma 23. *Assume a convex set S in $\mathbf{F}_{f(r)}$. Then $m_{p,q}(S)$ can be estimated by*

$$\frac{1}{r^{p+q+2}} \cdot \mu_{p,q}(r \cdot S)$$

within an error of

$$\mathcal{O} \left(\frac{\max \left\{ f(r), \; r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}} \right\}}{r^2} \right) \; .$$

Proof. It holds that

$$
\begin{aligned}
& m_{p,q}(S) \;-\; \frac{1}{r^{p+q+2}} \cdot \mu_{p,q}(r \cdot S) \\
&= m_{p,q}(S) - \frac{1}{r^{p+q+2}} \cdot \left(\iint\limits_{r \cdot S} x^p \cdot y^q \, dx dy \right. \\
&\qquad\quad \left. + \;\; \mathcal{O}\left(r^{p+q} \cdot \max\left\{ f(r),\; r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}} \right\} \right) \right) \\
&= m_{p,q}(S) \;-\; \frac{1}{r^{p+q+2}} \cdot \iint\limits_{r \cdot S} x^p \cdot y^q \, dx dy \;+\; \mathcal{O}\left(\frac{\max\left\{ f(r),\; r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}} \right\}}{r^2} \right) \\
&= m_{p,q}(S) \;-\; m_{p,q}(S) \;+\; \mathcal{O}\left(\frac{\max\left\{ f(r),\; r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}} \right\}}{r^2} \right) \\
&= \mathcal{O}\left(\frac{\max\left\{ f(r),\; r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}} \right\}}{r^2} \right) . \qquad \square
\end{aligned}
$$

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