Computer Science Department of The University of Auckland
CITR at Tamaki Campus (http://www.citr.auckland.ac.nz)

CITR-TR-44 May 1999

On Errors in Calculated Moments of Convex Sets Using Digital Images

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Abstract

Moments have been widely used in shape recognition and identification. In general, the (k,l)-moment, denoted by $m_{k,l}(S)$, of a planar measurable set S is defined by

$$m_{k,l}(S) = \iint_S x^k y^l \, dx \, dy .$$

We assume situations in image analysis and pattern recognition where real objects are acquired (by thresholding, segmentation etc.) as binary images D(S), i.e. as digital sets or digital regions. For a set S, in this paper its digitization is defined to be the set of all grid points with integer coordinates which belong to the region occupied by the given set S. Since in image processing applications, the exact values of the moments $m_{p,q}(S)$ remain unknown, they are usually approximated by discrete moments $u_{k,l}(S)$ where

$$\mu_{k,l}(S) = \sum_{(i,j)\in D(S)} i^k \cdot j^l = \sum_{\substack{i,j \text{ are integers} \\ (i,j)\in S}} i^k \cdot j^l .$$

Moments of order up to two (i.e. k - l <= 2) are frequently used and our attention is focused on them, i.e. on the limitation in their estimation from the corresponding digital picture. In this paper it is proved that

$$m_{k,l}(S) - \mu_{k,l}(S) = \mathcal{O}\left(\frac{1}{r^{\frac{15}{11}-\varepsilon}}\right) \approx \mathcal{O}\left(\frac{1}{r^{1.363636\ldots}}\right) ,$$

for k + l <= 2, where S is a convex set in the plane with a boundary having continuous third derivative and positive curvature at every point, while r is the number of pixels per unit (i.e. 1/r is the size of the pixel).

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On Errors in Calculated Moments of Convex Sets Using Digital Images

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ABSTRACT

Moments have been widely used in shape recognition and identification. In general, the (k,l)-*moment*, denoted by $m_{k,l}(S)$, of a planar measurable set S is defined by

$$m_{k,l}(S) \quad = \quad \iint\limits_S x^k y^l \, dx \, dy \quad .$$

We assume situations in image analysis and pattern recognition where real objects are acquired (by thresholding, segmentation etc.) as binary images $D(S)$, i.e. as digital sets or digital regions. For a set S, in this paper its digitization is defined to be the set of all grid points with integer coordinates which belong to the region occupied by the given set S. Since in image processing applications, the exact values of the moments $m_{k,l}(S)$ remain unknown, they are usually approximated by *discrete moments* $\mu_{k,l}(S)$ where

$$\mu_{k,l}(S) \quad = \quad \sum_{(i,j)\in D(S)} i^k \cdot j^l \quad = \quad \sum_{\substack{i,j \text{ are integers} \\ (i,j)\in S}} i^k \cdot j^l \quad .$$

Moments of order up to two (i.e. $k+l \leq 2$) are frequently used and our attention is focused on them, i.e. on the limitation in their estimation from the corresponding digital picture. In this paper it is proved that

$$m_{k,l}(S) - \frac{1}{r^{k+l+2}} \cdot \mu_{k,l}(r \cdot S) = \mathcal{O}\left(\frac{1}{r^{\frac{15}{11}+\varepsilon}}\right) \approx \mathcal{O}\left(\frac{1}{r^{1.363636\ldots}}\right)$$

for $k+l \leq 2$, where S is a convex set in the plane with a boundary having continuous third derivative and positive curvature at every point, r is the number of pixels per unit (i.e. $1/r$ is the size of the pixel), while $r \cdot S$ denotes the dilation of S by factor r.

Keywords: Digital geometry, digital regions, moments, multigrid convergence.

1. PRELIMINARIES

Moments have been widely used in shape recognition and identification. This is a reason for the ongoing strong interest in the computer vision community in moment calculations. In general, the (k,l)-*moment*, denoted by $m_{k,l}(S)$, of a planar measurable set S is defined by

$$m_{k,l}(S) \quad = \quad \iint\limits_S x^k y^l \, dx \, dy \quad .$$

The moment $m_{k,l}(S)$ has the order $k+l$.

In applications of image analysis and pattern recognition real objects (to be modelled as being sets in the Euclidean plane) are acquired (say, by image segmentation) as binary images $D(S)$, i.e. as digital sets or digital regions. In the manifold of different digitization models (see, e.g. 9 for a *midpoint scheme* and a *four-vertex scheme*), we specify that for a set S its digitization (see Fig. 1) is defined to be the set of all grid points with integer coordinates which belong to the region occupied by the given set S (simple *point-inclusion scheme*):

$$(i,j) \in D(S) \quad \Leftrightarrow \quad (i,j) \in S, \quad \text{where } i \text{ and } j \text{ are integers.}$$

Since in image processing applications, the exact values of the moments $m_{k,l}(S)$ remain unknown, they are usually replaced by so-called *discrete moments* $\mu_{k,l}(S)$ where

$$\mu_{k,l}(S) \quad = \quad \sum_{(i,j)\in D(S)} i^k \cdot j^l \quad = \quad \sum_{\substack{i,j \text{ are integers} \\ (i,j)\in S}} i^k \cdot j^l \quad ,$$

which can be calculated from corresponding digital regions $D(S)$ of a given set S.

The moments of order up to two are often used and our attention is focused on them, i.e. on the limitation in their estimation from the corresponding digital picture. If there are no further a-priori assumptions or constraints about the considered regions, the precision in estimation depends only on the applied picture resolution (grid constant). Assume a convex set S whose boundary has continuous third derivatives and positive curvature at every point (a *smooth planar convex region*). For such a set it is shown that the errors in estimating real moments (with order up to two) by corresponding discrete moments, are upper bounded by

$$\mathcal{O}\left(\frac{1}{r^{\frac{15}{11}-\varepsilon}}\right) \approx \mathcal{O}\left(\frac{1}{r^{1.363636\ldots}}\right)$$

where r is the number of pixels per unit, i.e. $1/r$ is the size of the pixel. This is a further result within the general area of multigrid convergence[6] where an increase in grid resolution is modelled by an increase of parameter r.

The result can be extended easily to sets T which can be represented by a finite number of unions, intersections or set differences of convex sets S satisfying the considered boundary constraint. Such regions T are commonly called

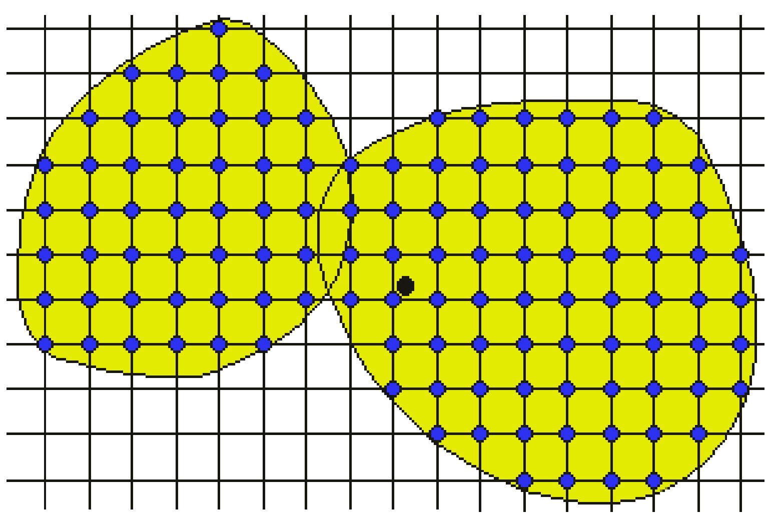

Figure 1. Union of two smooth convex planar sets and the centroid of this union. The centroid may be approximated based on the given digital region. In the shown example, the grid resolution is characterised by $r = 6$ grid points per unit. Increased grid resolutions allow better accuracy in approximating the centroid.

smooth regions. If the boundary of such a region T has straight sections then the considered errors have $1/r$ as the order of magnitude. That is a trivial result.

The established upper bound can be understood as being optimum and can be applied as a useful tool in evaluation of procedures which are based on moment calculations. Let us mention that the moment-concept in image analysis was introduced by Hu,[3] and since then a variety of new moment-types and moment-based methods have been developed and used. We cite a few of them: object recognition,[1] reconstruction of geometric properties of regions,[5,7] determination of invariants,[11] motion estimation,[10] and similarity measurement.[2]

The method used here can be extended to the estimation of moments of arbitrary order.

In this paper it is assumed that all appearing coordinates of picture points are positive. The origin is always assumed to be in the lower-left corner of an image coordinate system.

The paper is organized as follows. The mathematical formulation of the problem and a recent result from number theory are given in Section 2. The asymptotic expressions of discrete moments of second order are derived in Section 3. Section 4 contains an example and concluding remarks.

2. NECESSARY MATHEMATICS

An accurate estimation of the number of grid points inside or on a given circle is a classical problem in mathematics, and initial results are due to C.F. Gauss (1777-1855). A recent result by M.N. Huxley[4] states a very accurate estimation for smooth planar convex sets. If nothing is known a-priori about the given smooth planar convex region then the precision in estimation can only be specified as a function of the grid resolution which is modelled in this paper by the number of pixels per unit. Of course, it can be expected that higher resolution enables a higher precision. But, the question is: *Which resolution ensures the required precision?* This paper gives an answer for the case of moment approximation.

Assume that $D_1(S)$ is a binary picture of region S for resolution $r_1 = 1$, i.e. one pixel per unit, and let $D_2(S)$ be the binary picture of the same region for resolution r_2, i.e. with r_2 pixels per unit. Then it follows that $D_2(S) = D_1(r_2 \cdot S)$, where $r_2 \cdot S$ is the dilation of S by factor r_2. More precisely, for a real number r and a set S, the set $r \cdot S$ is defined to be

$$r \cdot S \;=\; \{(r \cdot x,\; r \cdot y) \;\mid\; (x, y) \in S\}\ .$$

In other words, for our purpose it is sufficient to consider regions of the form $r \cdot S$, which are digitized on the orthogonal grid. The study of $r \to \infty$ corresponds to the increase in picture resolution (for the multigrid convergence concept see also 6). It seems to be straightforward to approximate $m_{p,q}(S)$ by

$$\frac{1}{r^{p+q+2}} \cdot \mu_{p,q}(r \cdot S) \quad ,$$

but the error term in this approximation determines the efficiency of the mentioned descriptions. So, we have to known the order of the following errors

$$m_{0,0}(S) - \frac{1}{r^2} \cdot \mu_{0,0}(r \cdot S), \qquad m_{1,0}(S) - \frac{1}{r^3} \cdot \mu_{1,0}(r \cdot S), \qquad m_{0,1}(S) - \frac{1}{r^3} \cdot \mu_{0,1}(r \cdot S) \tag{1}$$

$$m_{2,0}(S) - \frac{1}{r^4} \cdot \mu_{2,0}(r \cdot S), \qquad m_{1,1}(S) - \frac{1}{r^4} \cdot \mu_{1,1}(r \cdot S) \quad \text{and} \quad m_{0,2}(S) - \frac{1}{r^4} \cdot \mu_{0,2}(r \cdot S) \tag{2}$$

as a function of the applied picture resolution. We cite the following result from number theory[4], which expresses $\mu_{0,0}(r \cdot S)$ for a smooth planar convex region.

THEOREM 2.1. (Huxley 1990) *If S is a convex region in the Euclidean plane, with C^3 boundary and positive curvature at every point of the boundary, then the number of lattice (digital) points belonging to $r \cdot S$ is*

$$\mu_{0,0}(r \cdot S) = r^2 \cdot P(S) + \mathcal{O}\left(r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}}\right) \quad ,$$

where $P(S)$ denotes the area of S, while $r \cdot S$ is the dilatation of S by factor r.

Later on, because of simplicity, we use a weaker result

$$\mu_{0,0}(r \cdot S) = r^2 \cdot P(S) + \mathcal{O}\left(r^{\frac{7}{11}+\epsilon}\right) \quad , \qquad \text{for every} \quad \varepsilon > 0. \tag{3}$$

The preconditions of Theorem 2.1 can be relaxed to allow S to have a finite number of vertices (corners). In the paper related to the reconstruction of gravity centers[8] the authors prove the following.

THEOREM 2.2. *The moments $m_{1,0}(S)$ and $m_{0,1}(S)$ of a smooth convex region S can be estimated as follows:*

$$m_{1,0}(S) = \frac{1}{r^3} \cdot \mu_{1,0}(r \cdot S) \; + \; \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right) \quad \textit{and} \quad m_{0,1}(S) = \frac{1}{r^3} \cdot \mu_{0,1}(r \cdot S) \; + \; \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right) \; .$$

Our goal here is to derive a "reasonable" asymptotic expression for $\mu_{2,0}(r \cdot S)$, $\mu_{0,2}(r \cdot S)$ and $\mu_{1,1}(r \cdot S)$ and by using these expressions to estimate (2). We use the following definitions.

DEFINITION 2.3. *For a smooth planar convex region S, a given integer k and a real number r, the set $(r \cdot S)(k)$ is defined as* $\quad (r \cdot S)(k) = \{(x, y) \mid \quad (x, y) \in (r \cdot S) \quad \textit{and} \quad x \geq k\} \quad .$

Consequently, $D((r \cdot S)(k))$ is the set of digital points in the digitization of $r \cdot S$ lying in the closed half plane determined by $x \geq k$.

DEFINITION 2.4. *For a smooth planar convex region S, a given integer k and a real number r, the digital point set $L(r \cdot S, k)$ is defined as* $\quad L(r \cdot S, k) = \{(k, j) \mid \quad (k, j) \in D(r \cdot S)\}$.

In other words, $L(r \cdot S, k)$ is a set of digital points in the digitization of $r \cdot S$ which belong to the vertical line $x = k$. It follows that

$$D(r \cdot S) \quad = \bigcup_{k=-\infty}^{k=+\infty} L(r \cdot S, k) \quad = \bigcup_{k=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor} L(r \cdot S, k) \quad , \tag{4}$$

where x_{min} is the minimum x-value among all the points in set S, and x_{max} is the maximum x-value, i.e.

$$x_{min} = min\{x \mid (x, y) \in S\} \; , \qquad x_{max} = max\{x \mid (x, y) \in S\} \; .$$

The values y_{min} and y_{max} are defined analogously. We need the following auxiliary lemma.

LEMMA 2.5. *For a smooth planar convex region S, a given integer k and a real number r, the discrete* $(0, 0)$*-moment of the set $(r \cdot S)(k)$ can be expressed as*

$$\mu_{0,0}((r \cdot S)(k)) \quad = \quad P((r \cdot S)(k)) \quad + \quad \frac{1}{2} \cdot \mu_{0,0}(L(r \cdot S, k)) \quad + \quad \mathcal{O}(r^{\frac{7}{11}+\varepsilon}).$$

Proof. Let $r \cdot \overline{S}$ be the region symmetrical to $r \cdot S$, with respect to the line $x = k$. Furthermore, the convex set $r \cdot S \cap r \cdot \overline{S}$ satisfies the conditions of Theorem 2.1, so the number of digital points belonging to $r \cdot S \cap r \cdot \overline{S}$ can be determined as

$$\mu_{0,0}(r \cdot S \cap r \cdot \overline{S}) = P(r \cdot S \cap r \cdot \overline{S}) + \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right) \quad .$$

The statement follows because the set $r \cdot S \cap r \cdot \overline{S}$ is symmetrical with respect to the line $x = k$. $\square$

3. DISCRETE MOMENTS UP TO SECOND ORDER

In this section we give asymptotic expressions for the second order discrete moments of a smooth region $r \cdot S$. Obviously, the moments $\mu_{2,0}(r \cdot S)$ and $\mu_{0,2}(r \cdot S)$, because of symmetry, can be derived in an identical way, while the estimation of $\mu_{1,1}(r \cdot S)$ needs some modifications. The following definitions specify $3D$-sets W_i and W_i'.

DEFINITION 3.1. *For a smooth planar convex region $r \cdot S$ and an integer i from the set*

$$\{\lceil r \cdot x_{min} \rceil, \lceil r \cdot x_{min} \rceil + 1, \ldots, \lfloor r \cdot x_{max} \rfloor - 1\},$$

we define $3D$-sets

$$W_i = \{(x, y, z) \quad | \quad (x, y) \in r \cdot S, \quad x \geq i, \quad i^2 < z < (i+1)^2\} \quad .$$

DEFINITION 3.2. *For a smooth planar convex region $r \cdot S$ and an integer i from the set*

$$\{\lceil r \cdot x_{min} \rceil, \lceil r \cdot x_{min} \rceil + 1, \ldots, \lfloor r \cdot x_{max} \rfloor - 1\},$$

we also define $3D$-sets

$$W_i' = \{(x, y, z) \quad | \quad (x, y) \in r \cdot S, \quad x \geq i \ , \quad x^2 < z \leq (i+1)^2\}.$$

We start with the calculation of $\mu_{2,0}(r \cdot S)$. An auxiliary lemma is necessary.

LEMMA 3.3. $$\sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} vol(W_i') \quad = \quad \sum_{(i,j) \in r \cdot S} i \quad + \quad \mathcal{O}(r^2) \quad = \quad \mu_{1,0}(r \cdot S) \quad + \quad \mathcal{O}(r^2) \ .$$

Proof. The boundary of $r \cdot S$ can be divided into two arcs of the form $y = y_1(x)$ and $y = y_2(x)$, such that $y_1(x) \leq y_2(x)$. Then it holds that

$$\sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} vol(W_i') = \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} \int_i^{i+1} dx \int_{y_1(x)}^{y_2(x)} dy \int_{x^2}^{(i+1)^2} dz$$

$$= \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} \int_i^{i+1} dx \int_{\lceil y_1(i) \rceil}^{\lfloor y_2(i) \rfloor} dy \int_{x^2}^{(i+1)^2} dz \quad + \quad \mathcal{O}(r^2)$$

$$= \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} \int_i^{i+1} (\lfloor y_2(i) \rfloor - \lceil y_1(i) \rceil) \cdot ((i+1)^2 - x^2) dx \quad + \quad \mathcal{O}(r^2)$$

$$= \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} (\lfloor y_2(i) \rfloor - \lceil y_1(i) \rceil) \cdot i \quad + \quad \frac{2}{3} \cdot \sum_{\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} (\lfloor y_2(i) \rfloor - \lceil y_1(i) \rceil) \quad + \quad \mathcal{O}(r^2)$$

$$= \sum_{(i,j) \in r \cdot S} i \quad + \quad \mathcal{O}(r^2) \quad = \quad \mu_{1,0}(r \cdot S) \quad + \quad \mathcal{O}(r^2) \ .$$

□

Let us note that

$$\sum_{\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} (\lfloor y_2(i) \rfloor - \lceil y_1(i) \rceil) \cdot i$$

equals $\mu_{1,0}(r \cdot S)$ with an error having an order of magnitude upper bounded by

$$\max\{\ x \mid (x,y) \in r \cdot S\ \} \cdot (\text{perimeter of } \ r \cdot S) \quad = \quad \mathcal{O}(r^2) \quad ,$$

while

$$\sum_{\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} (\lfloor y_2(i) \rfloor - \lceil y_1(i) \rceil) = \mathcal{O}(r^2) \ .$$

The discrete moments $\mu_{2,0}(r \cdot S)$ and $\mu_{0,2}(r \cdot S)$ are evaluated by Theorem 3.4, while Theorem 3.5 evaluates $\mu_{1,1}(r \cdot S)$.

Theorem 3.4. *Let S be a smooth planar convex region. Then the following asymptotical expressions hold:*

$$\mu_{2,0}(r \cdot S) \quad = \quad \sum_{\substack{i,j \text{ are integers} \\ (i,j) \in r \cdot S}} i^2 \quad = \quad \iint\limits_{r \cdot S} x^2 dx dy \quad + \quad \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right) \qquad \textit{and}$$

$$\mu_{0,2}(r \cdot S) \quad = \quad \sum_{\substack{i,j \text{ are integers} \\ (i,j) \in r \cdot S}} j^2 \quad = \quad \iint\limits_{r \cdot S} y^2 dx dy \quad + \quad \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right) \quad .$$

Proof. Let us notice that $\mu_{2,0}(r \cdot S)$ is equal to the number of digital points belonging to the $3D$ set B given by

$$B \quad = \quad \{(x,y,z) \quad | \quad (x,y) \in r \cdot S \quad \text{and} \quad 0 < z \le x^2\} \quad = \quad B' \quad \cup \quad B''$$

where B' and B'' are defined as follows:

$$B' = \{(x,y,z) \quad | \quad (x,y) \in r \cdot S \quad \text{and} \quad 0 < z \le \lceil r \cdot x_{min} \rceil^2\} \quad \text{and}$$

$$B'' = \{(x,y,z) \quad | \quad (x,y) \in r \cdot S \quad \text{and} \quad \lceil r \cdot x_{min} \rceil^2 < z \le x^2\}.$$

First, consider the number of digital points belonging to the set B'. From (3) it follows easily that

$$\mu_{0,0,0}(B') \quad = \quad \lceil r \cdot x_{min} \rceil^2 \cdot \left(P(r \cdot S) + \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right)\right) = vol(B') \quad + \quad \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right) .$$

Now, let us calculate the number of digital points belonging to set B''. See Fig. 2 for the used decomposition. According to Definitions 3.1 and 3.2 of the $3D$-sets W_i and W'_i, it follows that

$$vol(B'') = \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} vol(W_i) - \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} vol(W'_i) + \mathcal{O}(r^2)$$

$$= \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} (2i+1) \cdot P((r \cdot S)(i)) - \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} vol(W'_i) + \mathcal{O}(r^2)$$

$$= \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} (2i+1) \cdot \mu_{0,0}((r \cdot S)(i)) \quad - \quad \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} i \cdot \mu_{0,0}(L(r \cdot S, i))$$

$$+ \frac{1}{2} \cdot \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} \mu_{0,0}(L(r \cdot S, i)) \ + \ \mathcal{O}\left(r^{2+\frac{7}{11}+\varepsilon}\right) - \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} i \cdot \mu_{0,0}(L(r \cdot S, i)) + \mathcal{O}(r^2)$$

$$= \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} (2i+1) \cdot \mu_{0,0}((r \cdot S)(i)) \quad - \quad \sum_{i=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor - 1} 2 \cdot i \cdot \mu_{0,0}(L(r \cdot S, i)) \quad + \quad \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right)$$

$$= \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} (2i+1)\cdot(\mu_{0,0}((r\cdot S)(i)) - \mu_{0,0}(L(r\cdot S,i))) \quad + \quad \mathcal{O}\left(r^{2+\frac{7}{11}+\varepsilon}\right)$$

$$= \quad \mu_{0,0,0}(B'') + \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right).$$

Note that

$$\sum_{\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} (2i+1)\cdot(\mu_{0,0}((r\cdot S)(i)) - \mu_{0,0}(L(r\cdot S,i)))$$

equals the number of digital points inside of B''. Lemma 2.5 and the equalities (3) and (4) also have been used. So, we have

$$\mu_{0,0,0}(B'') = vol(B'') \quad + \quad \mathcal{O}(r^{\frac{29}{11}+\varepsilon}) \quad .$$

The sum of $\mu_{0,0,0}(B')$ and $\mu_{0,0,0}(B'')$ is the number of digital points in B. Together with the already derived expression for $\mu_{0,0,0}(B')$ we have

$$\mu_{2,0}(r\cdot S) = \mu_{0,0,0}(B) = \mu_{0,0,0}(B') + \mu_{0,0,0}(B'') = vol(B') + vol(B'') + \mathcal{O}(r^{\frac{29}{11}+\varepsilon})$$

$$= \quad vol(B) \quad + \quad \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right) \quad = \quad m_{2,0}(r\cdot S) \quad + \quad \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right).$$

□

The following important theorem is a direct consequence of the previous statement.

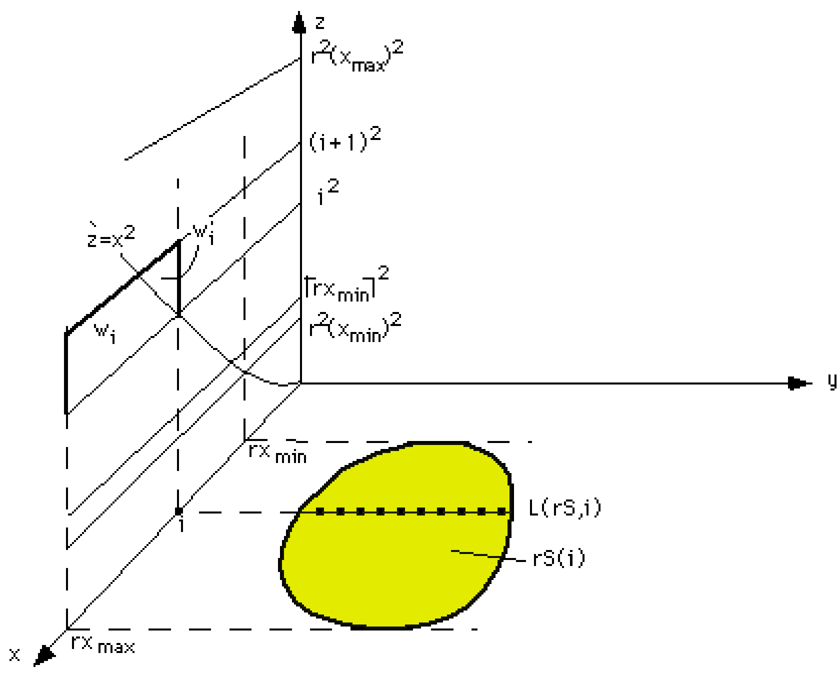

Figure 2. Decomposition for second order moment calculation.

THEOREM 3.5. *Let a convex smooth region S be given. Then the moments $m_{2,0}(S)$ and $m_{0,2}(S)$ can be estimated by*

$$\frac{1}{r^4} \cdot \mu_{2,0}(r \cdot S) \quad \text{and} \quad \frac{1}{r^4} \cdot \mu_{0,2}(r \cdot S),$$

respectively, within an $\mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)$ error.

Proof. It holds

$$m_{2,0}(S) \;-\; \frac{1}{r^4} \cdot \mu_{2,0}(r \cdot S) \;=\; m_{2,0}(S) - \frac{1}{r^4} \cdot \left(\iint\limits_{r \cdot S} x^2 dxdy \;+\; \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right)\right)$$

$$= \;\; m_{2,0}(S) \;-\; \frac{1}{r^4} \cdot \iint\limits_{r \cdot S} x^2 dxdy \;+\; \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right) \;=\; m_{2,0}(S) \;-\; m_{2,0}(S) \;+\; \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)$$

$$= \;\; \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right).$$

□

It remains to estimate $\mu_{1,1}(r \cdot S)$.

THEOREM 3.6. *Let S be a smooth planar convex region region. Then the following asymptotical expression holds,*

$$\mu_{1,1}(r \cdot S) \;=\; \sum_{\substack{i,j \text{ are integers} \\ (i,j) \in r \cdot S}} i \cdot j \;=\; \iint\limits_{r \cdot S} xydxdy \;+\; \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right) \quad .$$

Proof. Note that $\mu_{1,1}(r \cdot S)$ is equal to the number of digital points belonging to the $3D$ set E given by

$$E = \{(x,y,z) \;\mid\; (x,y) \in r \cdot S \;\text{ and }\; 0 < z \le x \cdot y\} \;=\; E' \;\cup\; E''$$

where E' and E'' are defined as follows:

$$E' = \{(x,y,z) \;\mid\; (x,y) \in r \cdot S \;\text{ and }\; 0 < z < r^2 \cdot z_{min}\}$$

and

$$E'' = \{(x,y,z) \;\mid\; (x,y) \in r \cdot S \;\text{ and }\; r^2 \cdot z_{min} \le z \le x \cdot y\}.$$

Where z_{min} is defined to be

$$z_{min} \;=\; \min\{ \; z \;\mid\; z = x \cdot y \;\text{ and }\; (x,y) \in S \; \}.$$

Analogously, z_{max} is defined to be

$$z_{max} \;=\; \max\{ \; z \;\mid\; z = x \cdot y \;\text{ and }\; (x,y) \in S \; \}.$$

First, consider the number of digital points belonging to the set E'. From (3) it follows that

$$\mu_{0,0,0}(E') \;=\; (\lceil r^2 \cdot z_{min} \rceil - 1) \cdot \left(P(r \cdot S) + \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right)\right)$$

$$=\; vol(E') + P(r \cdot S) \cdot (\lceil r^2 \cdot z_{min} \rceil - r^2 \cdot z_{min} - 1) + \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right).$$

Now let us calculate the number of digital points belonging to set E''. The following definition of $3D$-sets, ω_i and ω_i', is useful:

$$\omega_i \quad = \quad \{\ (x,y,z) \mid (x,y) \in r \cdot S \quad \text{and} \quad x \cdot y \geq i \quad \text{and} \quad i < z \leq \min\{x \cdot y, i+1\}\ \}$$

and

$$\omega_i' \quad = \quad \{\ (x,y,z) \mid (x,y) \in r \cdot S \quad \text{and} \quad i < xy \leq i+1 \quad \text{and} \quad i < z \leq x \cdot y\ \}.$$

Now, we can estimate the volume of E''. $S_3(i)$ denotes the $2D$-region $\{\ (x,y) \mid \ (x,y) \in r \cdot S \quad \text{and} \quad x \cdot y \geq i \quad \}$. It follows that

$$\begin{aligned}
vol(E'') \quad &= \sum_{i=\lceil r^2 \cdot z_{min} \rceil}^{\lfloor r^2 \cdot z_{max} \rfloor - 1} vol(\omega_i) \quad + \quad \left(\lceil r^2 \cdot z_{min} \rceil - r^2 \cdot z_{min}\right) \cdot P(r \cdot S) + \mathcal{O}(r^2) \\
&= \sum_{i=\lceil r^2 \cdot z_{min} \rceil}^{\lfloor r^2 \cdot z_{max} \rfloor - 1} P(S_3(i)) \ - \sum_{i=\lceil r^2 \cdot z_{min} \rceil}^{\lfloor r^2 \cdot z_{max} \rfloor - 1} vol(\omega_i') \ + \ \left(\lceil r^2 \cdot z_{min} \rceil - r^2 \cdot z_{min}\right) \cdot P(r \cdot S) \ + \ \mathcal{O}(r^2) \\
&= \sum_{i=\lceil r^2 \cdot z_{min} \rceil}^{\lfloor r^2 \cdot z_{max} \rfloor - 1} \left(\mu_{0,0}(S_3(i)) + \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right)\right) + \left(\lceil r^2 \cdot z_{min} \rceil - r^2 \cdot z_{min}\right) \cdot P(r \cdot S) + \mathcal{O}(r^2) \\
&= \mu_{0,0,0}(E'') \quad + \quad \left(\lceil r^2 \cdot z_{min} \rceil - r^2 \cdot z_{min}\right) \cdot P(r \cdot S) \quad + \quad \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right) .
\end{aligned}$$

Thus,

$$\mu_{0,0,0}(E'') \quad = \quad vol(E'') \quad - \quad \left(\lceil r^2 \cdot z_{min} \rceil - r^2 \cdot z_{min}\right) \cdot P(r \cdot S) \quad + \quad \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right).$$

The proof of the theorem is finished by calculating the sum of $\mu_{0,0,0}(E')$ and $\mu_{0,0,0}(E'')$. So,

$$\begin{aligned}
\mu_{1,1}(r \cdot S) \quad &= \quad \mu_{0,0,0}(E') \quad + \quad \mu_{0,0,0}(E'') \\
&= \quad vol(E') + P(r \cdot S) \cdot \left(\lceil r^2 \cdot z_{min} \rceil - r^2 \cdot z_{min} - 1\right) + \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right) \\
&\quad + \ vol(E'') \quad - \quad \left(\lceil r^2 \cdot z_{min} \rceil - r^2 \cdot z_{min}\right) \cdot P(r \cdot S) \quad + \quad \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right) \\
&= \quad vol(E) \quad + \quad \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right) = \quad m_{1,1}(r \cdot S) \quad + \quad \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right).
\end{aligned}$$

□

The next theorem follows directly from the previous one.

Theorem 3.7. *Let a convex smooth region S be given. Then the moment $m_{1,1}(S)$ can be estimated by $\frac{1}{r^4} \cdot \mu_{1,1}(r \cdot S)$ within an $\mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)$ error.*

Proof. It holds

$$m_{1,1}(S) \;-\; \frac{1}{r^4}\cdot\mu_{1,1}(r\cdot S) \;=\; m_{1,1}(S) - \frac{1}{r^4}\cdot\left(\iint\limits_{r\cdot S} xydxdy \;+\; \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right)\right)$$

$$= \quad m_{1,1}(S) \;-\; \frac{1}{r^4}\cdot\iint\limits_{r\cdot S} xydxdy \;+\; \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right) \;=\; m_{1,1}(S) \;-\; m_{1,1}(S) \;+\; \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)$$

$$= \quad \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right) \; .$$

□

4. CENTRAL MOMENTS

The moments of order one and two will vary for a given region depending on the spatial position of the region. Translation invariants (as orientation and elongation invariants) are obtained by using central moments for which the origin is at the centroid of the region. This normalization step is very common in moment based calculations. That is a reason for our following brief discussion of normalized moments. Let the coordinates

$$\left(\frac{m_{1,0}(S)}{m_{0,0}(S)}, \frac{m_{0,1}(S)}{m_{0,0}(S)}\right)$$

of the centroid of S be shortly denoted as $\overline{x}_c(S)$ and $\overline{y}_c(S)$. While the values

$$\frac{\mu_{1,0}(r\cdot S)}{\mu_{0,0}(r\cdot S)} \quad \text{and} \quad \frac{\mu_{0,1}(r\cdot S)}{\mu_{0,0}(r\cdot S)}$$

are denoted by $\overline{x}_d(r\cdot S)$ and $\overline{y}_d(r\cdot S)$, respectively. Then the expression

$$\overline{m}_{k,l}(S) \quad = \quad \iint\limits_S (x-\overline{x}_c(S))^k\,(y-\overline{y}_c(S))^l \; dx\; dy$$

is called the *central* (k,l)*-moment* of a given region S.

The discrete analogue for the central (k,l)-moment of a given region $r\cdot S$ is the so-called *central discrete* (k,l)*-moment*, which is equal to

$$\overline{\mu}_{k,l}(r\cdot S) \quad = \quad \sum_{\substack{i,j \text{ are integers}\\ (i,j)\in r\cdot S}} (i-\overline{x}_d(r\cdot S))^k\cdot(j-\overline{y}_d(r\cdot S))^l \quad .$$

We show that central moments of second order of a given smooth region S can be recovered with the same $\mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)$ error bound from their corresponding central discrete moments. Obviously, the central moments of the first degree are equal to zero. Second order central moments are specified in the next theorem.

THEOREM 4.1. *Let a smooth convex region S be given. Then the following differences*

$$\overline{m}_{2,0}(S) - \frac{1}{r^4}\cdot\overline{\mu}_{2,0}(r\cdot S), \quad \overline{m}_{0,2}(S) - \frac{1}{r^4}\cdot\overline{\mu}_{0,2}(r\cdot S), \quad \text{and} \quad \overline{m}_{1,1}(S) - \frac{1}{r^4}\cdot\overline{\mu}_{1,1}(r\cdot S),$$

have an upper bound in $\mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)$.

Proof. We estimate the first difference. Theorems 2.2 and 3.5 as well as the equations $\overline{m}_{2,0}(S) = m_{2,0}(S) - \overline{x}_c(S) \cdot m_{1,0}(S)$, i.e. $\overline{\mu}_{2,0}(r \cdot S) = \mu_{2,0}(r \cdot S) - \overline{x}_d(r \cdot S) \cdot \mu_{1,0}(r \cdot S)$ will be used. It follows

$$\overline{m}_{2,0}(S) - \frac{1}{r^4} \cdot \overline{\mu}_{2,0}(r \cdot S) = m_{2,0}(S) - \overline{x}_c(S) \cdot m_{1,0}(S) - \frac{1}{r^4} \cdot (\mu_{2,0}(r \cdot S) - \overline{x}_d(r \cdot S) \cdot \mu_{1,0}(r \cdot S))$$

$$= \left(m_{2,0}(S) - \frac{\mu_{2,0}(r \cdot S)}{r^4} \right) - \left(\overline{x}_c(S) \cdot m_{1,0}(S) - \left(m_{1,0}(S) + \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon} \right) \right) \cdot \left(\overline{x}_c(S) + \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon} \right) \right) \right)$$

$$= \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon} \right) .$$

The remaining two differences may be proved analogously. □

5. AN EXAMPLE AND CONCLUSIONS

In this paper it is shown that the absolute error in the calculation of the moments of second order of smooth regions has an upper bound of

$$\mathcal{O}\left(\frac{1}{r^{15/11-\varepsilon}} \right),$$

where ε is an arbitrary small positive number, while r is the grid resolution (i.e. the number of pixels per unit). An equivalent formulation of the studied problem is: *If smooth planar regions are considered, which picture resolution has to be used in order to satisfy a pre-specified precision in calculating the second moments ?* The paper gives the following answer.

THEOREM 5.1. *Let a positive number α be given. Then the second-order moments of a convex region S with C^3 boundary having positive curvature at every point, or of a region which can be obtained by a finite number of unions, intersections and set differences of such regions, can be reconstructed within an absolute error less or equal to α if a suitable grid resolution*

$$r = \mathcal{O}\left(\alpha^{-\frac{11}{15}-\varepsilon} \right)$$

is applied.

r	$m_{2,0}(S) - \frac{\mu_{2,0}(r \cdot S)}{r^4}$	$m_{0,2}(S) - \frac{\mu_{0,2}(r \cdot S)}{r^4}$	$m_{1,1}(S) - \frac{\mu_{1,1}(r \cdot S)}{r^4}$	$r^{-\frac{15}{11}}$
1	-0.8809523	-0.9	-0.8958333	1
2	-0.1934523	-0.2125	-0.2083333	0.3886015
π	-0.0657400	-0.0539897	-0.0600890	0.2099270
$3\sqrt{2}$	-0.0167548	-0.0203703	-0.0192901	0.1393581
10	0.0001476	-0.0048	-0.0029333	0.0432876
23	0.0015380	0.0013975	0.0012903	0.0139026
100	0.0001305	0.0000740	0.0001063	0.0018738
$100\sqrt{3}$	-0.0000279	-0.0000319	-0.0000302	0.0008859
256	0.0000551	0.0000579	0.0000546	0.0005200
100π	0.0000378	0.0000040	0.0000172	0.0003933
500	0.0000095	0.0000190	0.0000149	0.0002087
1000	0.0000112	0.0000134	0.0000116	0.0000811

Table 1. Errors between the values of the second order moments $m_{2,0}(S)$, $m_{0,2}(S)$ and $m_{1,1}(S)$ and their estimations from corresponding digital pictures with resolution r.

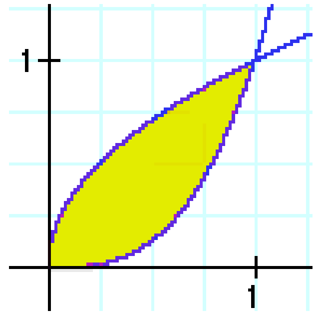

Figure 3. Region S as used in the example (see Table 1).

We give an example which illustrates the obtained results. Let the region S be defined by inequalities $y \geq x^3$ and $y \leq \sqrt{x}$. The exact values for the second order moments are

$$m_{2,0}(S) = \frac{5}{42}, \quad m_{1,1}(S) = \frac{5}{48} \quad \text{and} \quad m_{0,2}(S) = \frac{1}{10} \ .$$

Table 1 illustrates which is the efficiency in the second moment estimation if a given picture resolution r is applied. The errors in estimations have to be compared with $r^{-\frac{15}{11}}$.

The results of our paper are based on Huxley's theorem. This theorem is a strong mathematical result which is related to the number of integer points inside of a smooth planar convex curve γ and addresses an ancient mathematical problem. From the proofs of Theorems 2.2, 3.4 and 3.6 it can be concluded that results such as derived here are optimum with respect to Huxley's result.

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