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On Digitization Effects on Reconstruction of Geometric Properties of Regions

Reinhard Klette[1] and Jovisa Zunic[2]

Abstract

Representations of real regions by corresponding digital pictures cause an inherent loss of information. there are infinitely many different real regions with and identical corresponding digital picture. So, there are limitations in the reconstruction of the originals and their properties from digital pictures.
The problem which will be studied here is what is the impact of a digitization process on the efficiency in the reconstruction of the basic geometric properties

- position (usually described by the gravity centre or centrod),
- orientation (usually described by the axis of the least second moment),
- elongation (usually calculated as the ratio of the minimal and maximal second moments values w.r.t. the axis of least second moment),

of a planar convex region from the corresponding digital picture. Note that the size (area) estimation of the region (mostly estimated as the number of digital points belonging to the considered region) is a problem with an extensive history in number theory.
We start with smooth convex regions, i.e. whose boundaries have a continuous third order derivative and positive curvature (at every point), and show that if such a planar convex region is represented by a binary picture with resolution r, the mentioned features can be reconstructed with an absolute upper error bound of

$$\mathcal{O}\left(\frac{1}{r^{\frac{15}{11}-\epsilon}}\right) \approx \mathcal{O}\left(\frac{1}{r^{1.3636\ldots}}\right),$$

in the worst case. Since r is the number of pixels per unit, 1/r is the pixel size.
This result can be extended to regions which may be obtained from the previously described convex regions by finite applications of unions, intersections or set differences. The upper error bound remains the same which converges to zero with increase in grid resolution. The given description of the speed of convergence is very sharp.
Only smooth, curved regions are studied because if the considered region contains a straight section, the worst-case errors in the above estimations have 1 r as their order of magnitude. That is an trivial result.
The derivation is based on the estimation of the difference between the real moments (of the first and second order) and the corresponding discrete moments. The derived estimation can be a necessary mathematical tool in the evaluation of other procedures in the area of digital image analysis based on moment calculations.

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On Digitization Effects on Reconstruction of Geometric Properties of Regions

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Abstract

Representations of real regions by corresponding digital pictures cause an inherent loss of information. There are infinitely many different real regions with an identical corresponding digital picture. So, there are limitations in the reconstruction of the originals and their properties from digital pictures.

The problem which will be studied here is what is the impact of a digitization process on the efficiency in the reconstruction of the basic geometric properties

– position (usually described by the gravity center or centroid),

– orientation (usually described by the axis of the least second moment), and

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of a planar convex region from the corresponding digital picture. Note that the size (area) estimation of the region (mostly estimated as the number of digital points belonging to the considered region) is a problem with an extensive history in number theory.

We start with smooth convex regions, i.e. whose boundaries have a continuos third order derivative and positive curvature (at every point), and show that if such a planar convex region is represented by a binary picture with resolution r, then the mentioned features can be reconstructed with an absolute upper error bound of

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in the worst case. Since r is the number of pixels per unit, $\frac{1}{r}$ is the pixel size.

This result can be extended to regions which may be obtained from the previously described convex regions by finite applications of unions, intersections or set differences. The upper error bound remains the same which converges to zero with increase in grid resolution. The given description of the speed of convergence is very sharp.

Only smooth, curved regions are studied because if the considered region contains a straight section, the worst-case errors in the above estimations have $\frac{1}{r}$ as their order of magnitude. That is an trivial result.

The derivation is based on the estimation of the difference between the real moments (of the first and second order) and the corresponding discrete moments. The derived estimation can be a necessary mathematical tool in the evaluation of other procedures in the area of digital image analysis based on moment calculations.

Keywords: Digital images, geometric properties, moments, resolution, convergence.

1 Introduction and definitions

Because a quantization process is unavoidable, representations of real regions by computer pictures cause an inherent loss of information, even in case of using high spatial and scale resolutions. There are always infinitely many different real regions with an identical corresponding computer picture.

In this paper, our attention is focused on two-dimensional binary (digital) pictures and the limitation in the reconstruction of basic geometric properties of the original regions from their digital pictures caused by the digitization process. Some related work can be found in [11] and [16]. Noise effects to the efficiency in the reconstruction are not studied here – for some approaches see [10] and [3].

Since there are no further assumptions about the observed regions, the obtained results are simply given as function of the applied picture resolution – i.e. the number of pixels per unit.

In the diversity of different digitization models, we specify that for a set S its digitization $D(S)$ is defined to be the set of all grid points with integer coordinates which belong to the region occupied by the given set S:

$$(i,j) \in D(S) \quad \Leftrightarrow \quad (i,j) \in S, \quad \text{where } i \text{ and } j \text{ are integers.}$$

Image analysis of a binary picture usually starts with an estimation of the size and location of the region. Since the digitization of a region is assumed to be the set of digital points which fall into the area occupied by the region, it is straightforward to estimate the size of the region as the number of digital points inside of it. The difference between the area of a region and the number of digital points which belong to it, is a classical mathematical problem. A recent related result [7] from number theory could be used as the basic mathematical tool in our studies.

There are different ways to specify the position of a region. We describe the position of a region by its center of gravity or the centroid. In the continuous case, the centroid of S is defined as the point

$$\left(\frac{m_{1,0}(S)}{m_{0,0}(S)}, \frac{m_{1,0}(S)}{m_{0,0}(S)} \right),$$

where $m_{0,0}(S)$ is the zero-th order moment, while $m_{1,0}(S)$ and $m_{0,1}(S)$ are the first order moments.

In general, the (k,l)-moment, denoted by $m_{k,l}(S)$, of a planar set S is defined by

$$m_{k,l}(S) = \iint\limits_S x^k y^l \, dx \, dy.$$

The moment $m_{p,q}(S)$ has the order $p+q$. Three (and higher) dimensional moments and their orders are defined analogously.

In situations of image processing applications, the exact values of the moments $m_{p,q}(S)$ remain unknown. They are estimated by, so called, discrete moments $\mu_{k,l}(S)$ where

$$\mu_{k,l}(S) \quad = \quad \sum_{(i,j)\in D(S)} i^k \cdot j^l \quad = \quad \sum_{\substack{i,j \text{ are integers} \\ (i,j)\in S}} i^k \cdot j^l \quad ,$$

which can be calculated from the corresponding digital pictures of the regions. For example, the center of gravity of a set S is directly approximated from its digital image $D(S)$ as

$$\left(\frac{m_{1,0}(S)}{m_{0,0}(S)}, \quad \frac{m_{0,1}(S)}{m_{0,0}(S)}\right) \quad \approx \quad \left(\frac{\mu_{1,0}(S)}{\mu_{0,0}(S)}, \quad \frac{\mu_{0,1}(S)}{\mu_{0,0}(S)}\right) .$$

The next useful geometric property considered here is the orientation. The orientation of a region can be efficiently described by its, so called, axis of the least second moment. That is the line for which the integral of the square of the distance to points in the region is a minimum. That integral is

$$I(S,\varphi,\rho) = \iint_S r^2(x,y,\varphi,\rho)dxdy ,$$

where $r(x,y,\varphi,\rho)$ is the perpendicular distance from the point (x,y) to the line given in the form

$$x \cdot \cos\varphi - y \cdot \sin\varphi = \rho.$$

We are looking for the value of φ for which $I(S,\varphi,\rho)$ is the minimal possible, and by this angle we define the orientation of the region S. This φ-value will be denoted by $A(S)$, i.e.

$$\min_{\varphi,\rho} I(S,\varphi,\rho) \;=\; I(S,A(S),\overline{\rho}), \qquad \text{for some value of} \quad \overline{\rho} .$$

Accordingly we define the elongation of S (see [9]) as the ratio of the maximum and minimum values of $I(S,\varphi,\rho)$, i.e.

$$E(S) \;=\; \frac{\max_{\varphi,\ \rho} I(S,\varphi,\rho)}{\min_{\varphi,\ \rho} I(S,\varphi,\rho)} .$$

Primarily, we are interested in convex regions whose boundaries have a continuos third derivative and positive curvature (at every point). We show that if such a planar convex region is represented in a binary picture with resolution r, then the mentioned features can be reconstructed with an absolute upper error bound of

$$\mathcal{O}\left(\frac{1}{r^{\frac{15}{11}-\varepsilon}}\right) \;\approx\; \mathcal{O}\left(\frac{1}{r^{1.363636\ldots}}\right)$$

where r is the number of pixels per unit, i.e. the pixel size is $1/r$.

This result can be extended to regions which may be obtained from smooth planar convex regions by finite applications of unions, intersections or set differences. Such regions will be called smooth regions. The upper error bound remains the same which converges to zero with an increase in grid resolution. That is a worst-case analysis. Mathematical tools for the computation of the average (or expected) error due to quantization are developed in [11].

The obtained description of the speed of convergence can be interpreted to be very sharp. Namely, from the derivations of the results it can be concluded that they are sharp up to the Huxley's result [7], which is a strong mathematical result, related to the number of integer points inside of the strictly convex region of the form $r \cdot S$. It improves the previously best known upper bound of the rest term even for the famous "circle problem", i.e. when S is the unit circle [8].

In the case when the considered region contains a straight boundary section, the errors in the above estimations have $\frac{1}{r}$ as their order of magnitude (the worst-case situation). That is an trivial result and consequently, such situations are not considered in the paper. An analysis of the precision bounds by which a measure of the location can be determined on the image plane for the straight-edged regions can be found in [2].

Our derivations are based on the estimation of the difference between the real moments (of the first and second order) and the corresponding discrete moments. The method which has been used here can be extended to the estimation of the moments which have order bigger than two. Those estimations can be a useful tool for the evaluation of procedures in the area of digital image analysis based on moment calculations. Let us mention that the moment-concept in image analysis has ben introduced by Hu [6], since then a variety of new moment-types and moment-based methods have been developed and used, we mention several of them: object recognition [1], region representation [13], [22], determination of invariants [17], [18], motion estimation [14], similarity measurement [4].

While this paper studies the problems related to an asymptotic analysis of the moments behaviour there is also interest in the description of algorithms for a fast moment calculation because the use of moments is still limited due to computational complexity. Of course, only discrete moments are computable. For more details we refer to [10], [20].

Through the paper it is assumed that all appearing coordinates are positive. In other words, the origin is always placed in the lower-left corner of a studied digital picture.

The paper is organized as follows. The mathematical formulation of the problem and a recent result from number theory are given in Section 2. The asymptotic expressions of the discrete moments of the first order of a smooth region are derived in Section 3 while the asymptotic behaviour of the discrete moments of the second order is derived in Section 4. Section 5 is related to the location of the centroids of smooth regions. The central moments are considered in Section 6. Sections 7 and 8 give the error estimation in reconstruction of the orientation and elongation of smooth regions. Section 9 contains an example, concluding remarks and comments.

2 Necessary mathematics, Huxley's theorem

If nothing is known a-priori about the given smooth planar convex region then the precision in estimation can only be specified as a function of the grid resolution, i.e. of the number of pixels per unit. Of course, it can be expected that higher resolution enables a higher precision. But, the question is:

Which resolution preserves the required precision ?

The paper gives an answer.

Assume that $D_1(S)$ is a binary picture of region S for resolution $r_1 = 1$, i.e. one pixel per unit, and let $D_2(S)$ be the binary picture of the same region for resolution r_2, i.e. with r_2 pixels per unit. Then it follows that $D_2(S) = D_1(r_2 \cdot S)$, where $r_2 \cdot S$ is the dilation of S by factor r_2 . More precisely, for a real number r and a set S , the set $r \cdot S$ is defined to be

$$r \cdot S \; = \; \{(r \cdot x, \; r \cdot y) \; | \; (x, y) \in S\} \; .$$

In other words, for our purpose it is sufficient to consider regions of the form $r \cdot S$, which are digitized on the orthogonal grid. The study of $r \to \infty$ corresponds to the increase in picture resolution (for a general concept see [12]). For such an increase it is necessary to estimate the asymptotic behavior of the following expressions:

$$\frac{m_{1,0}(S)}{m_{0,0}(S)} - \frac{1}{r} \cdot \frac{\mu_{1,0}(r \cdot S)}{\mu_{0,0}(r \cdot S)} \qquad \text{and} \qquad \frac{m_{0,1}(S)}{m_{0,0}(S)} - \frac{1}{r} \cdot \frac{\mu_{0,1}(r \cdot S)}{\mu_{0,0}(r \cdot S)} \; , \tag{1}$$

which are the errors in the centroid location;

$$A(S) \; - \; \mathcal{A}(r \cdot S) \; , \tag{2}$$

where $\mathcal{A}(r \cdot S)$ is the angle which approximates $A(S)$ and and which can be calculated from $D(r \cdot S)$;

$$E(S) \; - \; \mathcal{E}(r \cdot S) \; , \tag{3}$$

where the approximation for $E(S)$ (calculated from $D(r \cdot S)$) is denoted by $\mathcal{E}(r \cdot S)$.

It is well known how to describe the centroid, orientation and elongation of S from the moments $m_{0,0}(S)$, $m_{1,0}(S)$, $m_{0,1}(S)$, $m_{2,0}(S)$, $m_{0,2}(S)$ and $m_{1,1}(S)$. It seems to be straightforward to approximate $m_{p,q}(S)$ by

$$\frac{1}{r^{p+q+2}} \cdot \mu_{p,q}(r \cdot S),$$

but the error term in this approximation determines the efficiency of the mentioned descriptions. So, we have to known the order of this error, (in the studied case) as a function of the applied picture resolution.

We cite the following result from number theory [7], which expresses $\mu_{0,0}(r \cdot S)$ for a smooth planar convex region.

Theorem 1 *If S is a convex region in the Euclidean plane, with C^3 boundary and positive curvature at every point of the boundary, then the number of lattice (digital) points belonging to $r \cdot S$ is*

$$\mu_{0,0}(r \cdot S) = r^2 \cdot P(S) + \mathcal{O}\left(r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}}\right) \quad ,$$

where $P(S)$ denotes the area of S, while $r \cdot S$ is the dilatation of S by factor r.

Lateron, because of simplicity, we will use a weaker result:

$$\mu_{0,0}(r \cdot S) = r^2 \cdot P(S) + \mathcal{O}\left(r^{\frac{7}{11}+\epsilon}\right) \quad , \qquad \text{for every} \quad \varepsilon > 0. \tag{4}$$

The preconditions of Theorem 1 can be relaxed to allow S to have a finite number of vertices (corners). The theorem can also be applied, e.g., to an intersection of the interiors of two convex curves (for details, see [7]). Our goal is to derive a "reasonable" asymptotic expression for $\mu_{1,0}(r \cdot S)$, $\mu_{0,1}(r \cdot S)$, $\mu_{2,0}(r \cdot S)$, $\mu_{0,2}(r \cdot S)$ and $\mu_{1,1}(r \cdot S)$ and by using these expressions to estimate (1), (2) and (3).

We will use the following definitions.

Definition 1 *For a smooth planar convex region S, a given integer k and a real number r, the set $(r \cdot S)(k)$ is defined as:*

$$(r \cdot S)(k) = \{(x,y) \quad | \quad (x,y) \in (r \cdot S) \quad \text{and} \quad x \leq k\} \quad .$$

Consequently, $D((r \cdot S)(k))$ is the set of digital points in the digitization of $r \cdot S$ lying in the closed half plane determined by $x \geq k$.

Definition 2 *For a smooth planar convex region S, a given integer k and a real number r, the digital point set $L(r \cdot S, k)$ is defined as*

$$L(r \cdot S, k) = \{(k,j) \quad | \quad (k,j) \in D(r \cdot S)\} \quad .$$

In other words, $L(r \cdot S, k)$ is a set of digital points in the digitization of $r \cdot S$ which belong to the vertical line $x = k$.

It follows that

$$D(r \cdot S) \quad = \bigcup_{k=-\infty}^{k=+\infty} L(r \cdot S, k) \quad = \bigcup_{k=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor} L(r \cdot S, k) \quad , \tag{5}$$

where x_{min} is the minimum x-value among all the points in set S, and x_{max} is the maximum x-value, i.e.

$$x_{min} = min\{x \quad | \quad (x,y) \in S\} \ , \qquad x_{max} = max\{x \quad | \quad (x,y) \in S\} \quad .$$

The values y_{min} and y_{max} are defined analogously. We need the following auxiliary lemma..

Lemma 1 *For a smooth planar convex region S, a given integer k and a real number r, the discrete $(0,0)$-moment of the set $(r \cdot S)(k)$ can be expressed as*

$$\mu_{0,0}((r \cdot S)(k)) \quad = \quad P((r \cdot S)(k)) \quad + \quad \frac{1}{2} \cdot \mu_{0,0}(L(r \cdot S, k)) \quad + \quad \mathcal{O}(r^{\frac{7}{11}+\varepsilon}).$$

Proof. Let $r \cdot \overline{S}$ be the region symmetrical to $r \cdot S$, with respect to the line $x = k$. Furthermore, the convex set $r \cdot S \cap r \cdot \overline{S}$ satisfies the conditions of Theorem 1, so the number of digital points belonging to $r \cdot S \cap r \cdot \overline{S}$ can be determined as $\mu_{0,0}(r \cdot S \cap r \cdot \overline{S}) = P(r \cdot S \cap r \cdot \overline{S}) + \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right)$. The statement follows because the set $r \cdot S \cap r \cdot \overline{S}$ is symmetrical with respect to the line $x = k$. □

3 Discrete moments of the first order

In this section we estimate the first order moments of a smooth region S from its digital picture as a function of the picture resolution.

Let us define three-dimensional sets V_i and V_i', for $i = \lceil r \cdot x_{min} \rceil, \lceil r \cdot x_{min} \rceil + 1, \ldots, \lfloor r \cdot x_{max} \rfloor - 1$.

Definition 3 *For a smooth planar convex region $r \cdot S$ and an integer i from the set $\{\lceil r \cdot x_{min} \rceil, \lceil r \cdot x_{min} \rceil + 1, \ldots, \lfloor r \cdot x_{max} \rfloor - 1\}$, we define 3D-sets*

$$V_i = \{(x,y,z) \quad | \quad (x,y) \in r \cdot S \quad and \quad x \geq i \quad and \quad i < z \leq i+1\}.$$

Definition 4 *For a smooth planar convex region $r \cdot S$ and an integer i from the set $\{\lceil r \cdot x_{min} \rceil, \lceil r \cdot x_{min} \rceil + 1, \ldots, \lfloor r \cdot x_{max} \rfloor - 1\}$, we also define 3D-sets*

$$V_i' = \{(x,y,z) \quad | \quad (x,y) \in r \cdot S \quad and \quad x \geq i \quad and \quad x < z \leq i+1\}.$$

The discrete moments $\mu_{1,0}(r \cdot S)$ and $\mu_{0,1}(r \cdot S)$ may be expressed as follows:

Theorem 2 *Let S be a convex planar region. Then the following asymptotical expressions hold:*

$$\mu_{1,0}(r \cdot S) \quad = \quad \sum_{\substack{i,j \ are \ integers \\ (i,j) \in r \cdot S}} i \quad = \quad \iint_{r \cdot S} x dx dy \quad + \quad \mathcal{O}\left(r^{\frac{18}{11}+\varepsilon}\right) \quad ,$$

and

$$\mu_{0,1}(r\cdot S) \quad = \quad \sum_{\substack{i,j \text{ are integers} \\ (i,j)\in r\cdot S}} j \quad = \quad \iint_{r\cdot S} y dx dy \quad + \quad \mathcal{O}\left(r^{\frac{18}{11}+\varepsilon}\right) \quad .$$

Proof. It holds that $\mu_{1,0}(r\cdot S)$ is equal to the number of digital points belonging to the $3D$ set C given by

$$C = \{(x,y,z) \quad | \quad (x,y)\in r\cdot S, \quad 0 < z \le x\} \quad = \quad C' \quad \cup \quad C''$$

where C' and C'' are defined as follows:

$$C' = \{(x,y,z) \quad | \quad (x,y)\in r\cdot S, \quad 0 < z < r\cdot x_{min}\}$$

and

$$C'' = \{(x,y,z) \quad | \quad (x,y)\in r\cdot S, \quad r\cdot x_{min} \le z \le x\}.$$

First, consider the number of digital points belonging to the set C'. From (4) it follows that

$$\begin{aligned} \mu_{0,0,0}(C') &= (\lceil r\cdot x_{min}\rceil - 1)\cdot\left(P(r\cdot S) + \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right)\right) = \\ &= vol(C') - r\cdot x_{min}\cdot P(r\cdot S) + (\lceil r\cdot x_{min}\rceil - 1)\cdot\left(P(r\cdot S) \quad + \quad \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right)\right) = \\ &= vol(C') + P(r\cdot S)\cdot(\lceil r\cdot x_{min}\rceil - r\cdot x_{min} - 1) + \mathcal{O}\left(r^{\frac{18}{11}+\varepsilon}\right). \end{aligned}$$

Now, let us calculate the number of digital points belonging to the set C''. According to Definitions 3 and 4 of the $3D$-sets V_i and V_i' it follows that

$$vol(C'') = \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor - 1} (vol(V_i) - vol(V_i')) + (\lceil r\cdot x_{min}\rceil - r\cdot x_{min})\cdot P(r\cdot S) + \mathcal{O}(r) =$$

$$= \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor - 1} vol(V_i) - \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor - 1} vol(V_i') + (\lceil r\cdot x_{min}\rceil - r\cdot x_{min})\cdot P(r\cdot S) + \mathcal{O}(r) =$$

$$= \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor - 1} P((r\cdot S)(i)) - \frac{1}{2}\cdot P(r\cdot S) + (\lceil r\cdot x_{min}\rceil - r\cdot x_{min})\cdot P(r\cdot S) + \mathcal{O}(r) =$$

$$= \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor - 1} \left(\mu_{0,0}((r\cdot S)(i)) - \frac{\mu_{0,0}(L(r\cdot S, i))}{2} + \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right)\right) - \frac{P(r\cdot S)}{2} +$$

$$+ (\lceil r\cdot x_{min}\rceil - r\cdot x_{min})\cdot P(r\cdot S) + \mathcal{O}\left(r^{\frac{18}{11}+\varepsilon}\right) \quad =$$

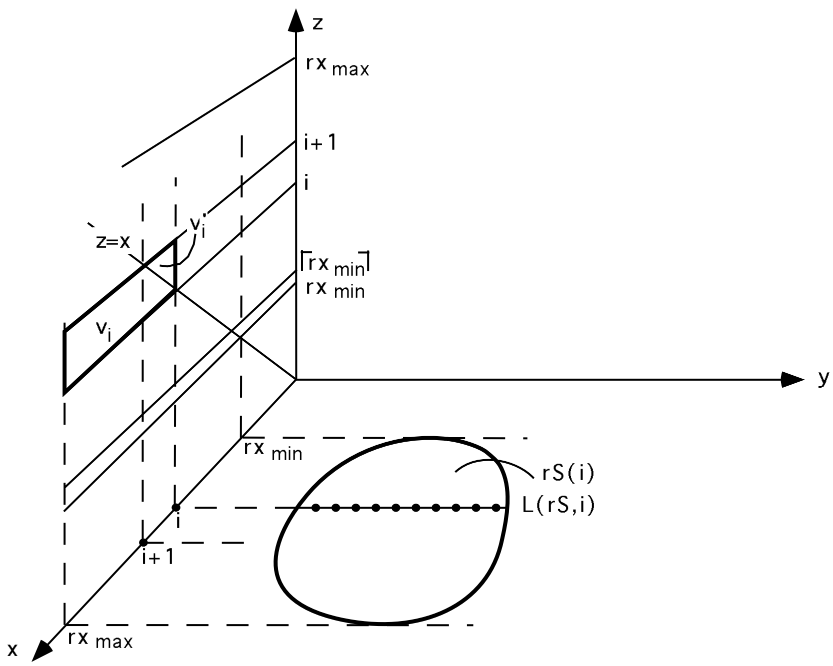

Figure 1: Decomposition for first order moment calculation.

$$= \quad (\mu_{0,0,0}(C'') - \frac{\mu_{0,0}(r\cdot S)}{2} - \frac{P(r\cdot S)}{2} + (\lceil r\cdot x_{min}\rceil - r\cdot x_{min})\cdot P(r\cdot S) + \mathcal{O}\left(r^{\frac{18}{11}+\varepsilon}\right) =$$

$$= \quad \mu_{0,0,0}(C'') - \mu_{0,0}(r\cdot S) \quad + \quad (\lceil r\cdot x_{min}\rceil - r\cdot x_{min})\cdot P(r\cdot S) + \mathcal{O}(r^{\frac{18}{11}+\varepsilon}).$$

Note that Lemma 1, (4), (5) and

$$\sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} vol(V_i') = \frac{1}{2}\cdot P(r\cdot S) + \mathcal{O}(r)$$

have been used. That gives

$$\mu_{0,0,0}(C'') = vol(C'') \quad + \quad \mu_{0,0}(r\cdot S) \quad - \quad (\lceil r\cdot x_{min}\rceil - r\cdot x_{min})\cdot P(r\cdot S) + \mathcal{O}(r^{\frac{18}{11}+\varepsilon}) \quad .$$

The sum of $\mu_{0,0,0}(C')$ and $\mu_{0,0,0}(C'')$ is the number of digital points in C. It follows that

$$\begin{aligned}
\mu_{1,0}(r\cdot S) &= \mu_{0,0,0}(C) = \mu_{0,0,0}(C') + \mu_{0,0,0}(C'') = \\
&= vol(C') + P(r\cdot S)\cdot(\lceil r\cdot x_{min}\rceil - r\cdot x_{min} - 1) + \mathcal{O}\left(x_{min}\cdot r^{\frac{18}{11}+\varepsilon}\right) + \\
&+ vol(C'') + \mu_{0,0}(r\cdot S) - (\lceil r\cdot x_{min}\rceil - r\cdot x_{min})\cdot P(r\cdot S) + \mathcal{O}(r^{\frac{18}{11}+\varepsilon}) = \\
&= vol(C) + \mathcal{O}\left(x_{min}\cdot r^{\frac{18}{11}+\varepsilon}\right) = vol(C) + \mathcal{O}\left(r^{\frac{18}{11}+\varepsilon}\right) = \\
&= m_{1,0}(r\cdot S) + \mathcal{O}\left(r^{\frac{18}{11}+\varepsilon}\right). \quad \Box
\end{aligned}$$

The above theorem has the following important consequence.

Theorem 3 *The moments of the first order* $m_{1,0}(S)$ *qnd* $m_{0,1}(S)$ *of a smooth convex region* S *can be estimated as follows:*

$$m_{1,0}(S) = \frac{1}{r^3}\cdot\mu_{1,0}(r\cdot S) + \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right) \quad \textit{and} \quad m_{0,1}(S) = \frac{1}{r^3}\cdot\mu_{0,1}(r\cdot S) + \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right).$$

Proof. By using the previous theorem we have:

$$\begin{aligned}
m_{1,0}(S) - \frac{1}{r^3}\cdot\mu_{1,0}(r\cdot S) &= m_{1,0}(S) - \frac{1}{r^3}\cdot\left(\iint\limits_{r\cdot S} x\,dxdy + \mathcal{O}\left(r^{\frac{18}{11}+\varepsilon}\right)\right) = \\
&= m_{1,0}(S) - \frac{1}{r^3}\cdot r^3\cdot m_{1,0}(S) + \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right) = \\
&= \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right). \quad \Box
\end{aligned}$$

4 Discrete moments of second order

In this section we give asymptotic expressions for the second order discrete moments of a smooth region $r\cdot S$. Obviously, the moments $\mu_{2,0}(r\cdot S)$ and $\mu_{0,2}(r\cdot S)$, because of symmetry, can be derived in an identical way, while the estimation of $\mu_{1,1}(r\cdot S)$ needs some modifications.

The following definitions of $3D$-sets W_i and W_i' are used.

Definition 5 *For a smooth planar convex region* $r\cdot S$ *and an integer* i *from the set* $\{\lceil r\cdot x_{min}\rceil, \lceil r\cdot x_{min}\rceil+1, \ldots, \lfloor r\cdot x_{max}\rfloor - 1\}$, *we define* $3D$*-sets*

$$W_i = \{(x,y,z) \mid (x,y)\in r\cdot S, \quad x\geq i, \quad i<z\leq i+1\}.$$

Definition 6 *For a smooth planar convex region $r\cdot S$ and an integer i from the set $\{\lceil r\cdot x_{min}\rceil, \lceil r\cdot x_{min}\rceil+1,\ldots,\lfloor r\cdot x_{max}\rfloor-1\}$, we also define 3D-sets*

$$W_i' = \{(x,y,z) \quad | \quad (x,y)\in r\cdot S, \quad x\geq i\,, \quad x^2 < z \leq i+1\}.$$

We start with the calculation of $\mu_{2,0}(r\cdot S)$. An auxiliary lemma is necessary.

Lemma 2

$$\sum_{\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor-1} vol(W_i') \;=\; \sum_{(i,j)\in r\cdot S} i \;+\; \mathcal{O}(r^2) \;=\; \mu_{1,0}(r\cdot S)+\mathcal{O}(r^2)$$

Proof. The boundary of $r\cdot S$ can be divided into two arcs of the form $y=y_1(x)$ and $y=y_2(x)$, such that $y_1(x)\leq y_2(x)$. Then it holds that

$$\begin{aligned}
\sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor-1} vol(W_i') &= \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor-1} \int_i^{i+1} dx \int_{y_1(x)}^{y_2(x)} dy \int_{x^2}^{(i+1)^2} dz = \\
&= \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor-1} \int_i^{i+1} dx \int_{\lceil y_1(i)\rceil}^{\lfloor y_2(i)\rfloor} dy \int_{x^2}^{(i+1)^2} dz \;+\; \mathcal{O}(r) = \\
&= \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor-1} \int_i^{i+1} (\lfloor y_2(i)\rfloor - \lceil y_1(i)\rceil)\cdot((i+1)^2-x^2)dx \;+\; \mathcal{O}(r) = \\
&= \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor-1} (\lfloor y_2(i)\rfloor - \lceil y_1(i)\rceil)\cdot\left(i+\frac{2}{3}\right) \;+\; \mathcal{O}(r) = \\
&= \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor-1} (\lfloor y_2(i)\rfloor - \lceil y_1(i)\rceil)\cdot i \;+\; \frac{2}{3}\cdot \sum_{\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor-1} (\lfloor y_2(i)\rfloor - \lceil y_1(i)\rceil) \;+\; \mathcal{O}(r) \;= \\
&= \sum_{(i,j)\in r\cdot S} i \;+\; \mathcal{O}(r^2) \;=\; \mu_{1,0}(r\cdot S) \;+\; \mathcal{O}(r^2)\,. \quad \Box
\end{aligned}$$

Let us note that

$$\sum_{\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor-1} (\lfloor y_2(i)\rfloor - \lceil y_1(i)\rceil)\cdot i$$

equals $\mu_{1,0}(r\cdot S)$ with error having order of magnitude upper bounded by

$$\max\{\; x \mid (x,y)\in r\cdot S\;\}\cdot(\text{perimeter of } \; r\cdot S) \;=\; \mathcal{O}(r^2) \quad ,$$

while

$$\sum_{\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} \left(\lfloor y_2(i)\rfloor - \lceil y_1(i)\rceil\right)$$

has $\mathcal{O}(r^2)$ as the order of magnitude .

The discrete moments $\mu_{2,0}(r\cdot S)$ and $\mu_{0,2}(r\cdot S)$ are evaluated by Theorem 4, while Theorem 6 evaluates $\mu_{1,1}(r\cdot S)$.

Theorem 4 *Let S be a convex planar region. Then the following asymptotical expressions hold:*

$$\mu_{2,0}(r\cdot S) \quad = \quad \sum_{\substack{i,j \text{ are integers} \\ (i,j)\in r\cdot S}} i^2 = \iint\limits_{r\cdot S} x^2 dx dy + \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right)$$

and

$$\mu_{0,2}(r\cdot S) \quad = \quad \sum_{\substack{i,j \text{ are integers} \\ (i,j)\in r\cdot S}} j^2 = \iint\limits_{r\cdot S} y^2 dx dy + \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right) \quad .$$

Proof. Let us notice that $\mu_{2,0}(r\cdot S)$ is equal to the number of digital points belonging to the $3D$ set B given by

$$B \quad = \quad \{(x,y,z) \quad | \quad (x,y)\in r\cdot S \quad \text{and} \quad 0 < z \le x^2\} \quad = \quad B' \quad \cup \quad B''$$

where B' and B'' are defined as follows:

$$B' = \{(x,y,z) \quad | \quad (x,y)\in r\cdot S \quad \text{and} \quad 0 < z \le \lceil r\cdot x_{min}\rceil^2\}$$

and

$$B'' = \{(x,y,z) \quad | \quad (x,y)\in r\cdot S \quad \text{and} \quad \lceil r\cdot x_{min}\rceil^2 < z \le x^2\}.$$

First, consider the number of digital points belonging to the set B'. From (4) it follows easily that

$$\mu_{0,0,0}(B') \quad = \quad \lceil r\cdot x_{min}\rceil^2 \cdot \left(P(r\cdot S) + \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right)\right) =$$

$$= \quad vol(B') \quad + \quad \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right).$$

Now, let us calculate the number of digital points belonging to set B''. According to Definitions 5 and 6 of the

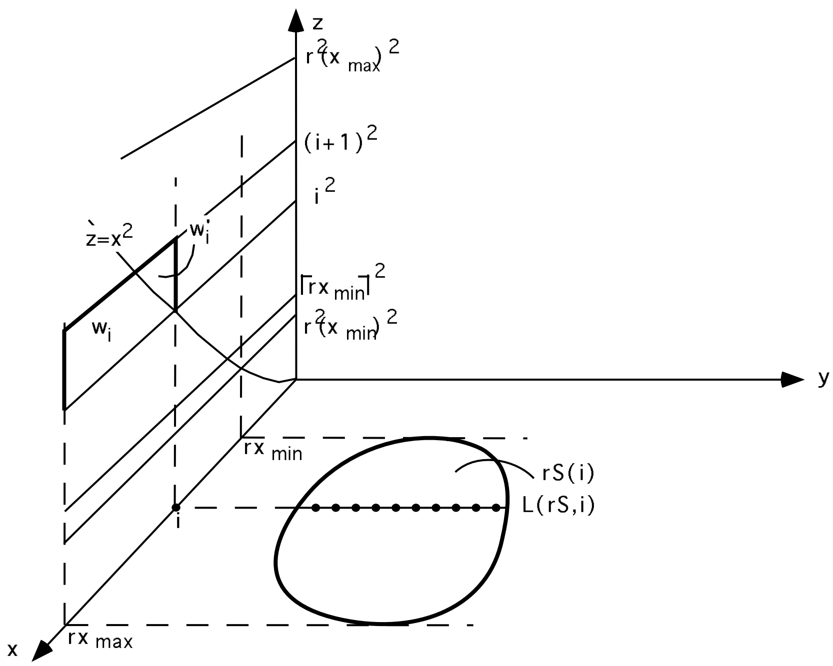

Figure 2: Decomposition for second order moment calculation.

3D-sets W_i and W_i', it follows that

$$vol(B'') = \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} (vol(W_i) - vol(W_i')) + \mathcal{O}(r^2) =$$

$$= \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} vol(W_i) - \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} vol(W_i') + \mathcal{O}(r^2) =$$

$$= \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} (2i+1)\cdot P((r\cdot S)(i)) - \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} vol(W_i') + \mathcal{O}(r^2) =$$

$$= \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} (2i+1)\cdot \left(\mu_{0,0}((r\cdot S)(i)) - \frac{1}{2}\cdot \mu_{0,0}(L(r\cdot S, i)) + \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right)\right) - \sum_{(i,j)\in r\cdot S} i + \mathcal{O}(r^2) =$$

$$= \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} (2i+1)\cdot \mu_{0,0}((r\cdot S)(i)) \quad - \quad \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} i\cdot \mu_{0,0}(L(r\cdot S,i)) \quad +$$

$$+\frac{1}{2}\cdot \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} \mu_{0,0}(L(r\cdot S,i) \quad +\mathcal{O}\left(r^{2+\frac{7}{11}+\varepsilon}\right) - \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} \mu_{0,0}(L(r\cdot S,i) + \mathcal{O}(r^2) =$$

$$= \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} (2i+1)\cdot \mu_{0,0}((r\cdot S)(i)) \quad - \quad \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} 2\cdot i\cdot \mu_{0,0}(L(r\cdot S,i)) \quad + \quad \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right) =$$

$$= \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} (2i+1)\cdot (\mu_{0,0}((r\cdot S)(i) - \mu_{0,0}(L(r\cdot S,i))) \quad + \quad \mathcal{O}\left(r^{2+\frac{7}{11}+\varepsilon}\right) =$$

$$= \mu_{0,0,0}(B'') + \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right)$$

Note that

$$\sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} (2i+1)\cdot (\mu_{0,0}((r\cdot S)(i) - \mu_{0,0}(L(r\cdot S,i)))$$

equals the number of digital points inside of B''. Lemma 1 and the equalities (4) and (5)also have been used. So, we have

$$\mu_{0,0,0}(B'') = vol(B'') \quad + \quad \mathcal{O}(r^{\frac{29}{11}+\varepsilon}) \quad .$$

The sum of $\mu_{0,0,0}(B')$ and $\mu_{0,0,0}(B'')$ is the number of digital points in B. Together with the already derived expression for $\mu_{0,0,0}(B')$ we have

$$\begin{aligned}
\mu_{2,0}(r\cdot S) &= \mu_{0,0,0}(B) = \mu_{0,0,0}(B') + \mu_{0,0,0}(B'') = \\
&= vol(B') + \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right) \quad + \quad vol(B'') \quad + \quad \mathcal{O}(r^{\frac{29}{11}+\varepsilon}) \quad = \\
&= vol(B) \quad + \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right) \quad = \\
&= m_{2,0}(r\cdot S) \quad + \quad \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right). \quad \Box
\end{aligned}$$

The following important theorem is a direct consequence of the previous statement.

Theorem 5 *Let a convex smooth region S be given. Then the moments $m_{2,0}(S)$ and $m_{0,2}(S)$ can be estimated by* $\frac{1}{r^4} \cdot \mu_{2,0}(r \cdot S)$ *and* $\frac{1}{r^4} \cdot \mu_{0,2}(r \cdot S)$, *respectively, within an* $\mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)$ *error.*

Proof. It holds

$$
\begin{aligned}
m_{2,0}(S) - \frac{1}{r^4} \cdot \mu_{2,0}(r \cdot S) &= m_{2,0}(S) - \frac{1}{r^4} \cdot \left(\iint_{r \cdot S} x^2 dxdy + \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right) \right) = \\
&= m_{2,0}(S) - \frac{1}{r^4} \cdot \iint_{r \cdot S} x^2 dxdy + \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right) = \\
&= m_{2,0}(S) - m_{2,0}(S) + \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right) = \\
&= \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right) . \quad \square
\end{aligned}
$$

It remains to estimate $\mu_{1,1}(r \cdot S)$.

Theorem 6 *Let S be a convex planar region. Then the following asymptotical expression holds,*

$$
\mu_{1,1}(r \cdot S) = \sum_{\substack{i,j \text{ are integers} \\ (i,j) \in r \cdot S}} i \cdot j = \iint_{r \cdot S} xydxdy + \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right) .
$$

Proof. Note that $\mu_{1,1}(r \cdot S)$ is equal to the number of digital points belonging to the $3D$ set E given by

$$
E = \{(x,y,z) \mid (x,y) \in r \cdot S \text{ and } 0 < z \leq x \cdot y\} = E' \cup E''
$$

where E' and E'' are defined as follows:

$$
E' = \{(x,y,z) \mid (x,y) \in r \cdot S \text{ and } 0 < z < r^2 \cdot z_{min}\}
$$

and

$$
E'' = \{(x,y,z) \mid (x,y) \in r \cdot S \text{ and } r^2 \cdot z_{min} \leq z \leq x \cdot y\}.
$$

Where z_{min} is defined to be

$$
z_{min} = \min\{ z \mid z = x \cdot y \text{ and } (x,y) \in S \}.
$$

Analogously, z_{max} is defined to be

$$
z_{max} = \max\{ z \mid z = x \cdot y \text{ and } (x,y) \in S \}.
$$

First, consider the number of digital points belonging to the set E'. From (6) it follows that

$$\begin{aligned}\mu_{0,0,0}(E') &= (\lceil r^2 \cdot z_{min} \rceil - 1) \cdot \left(P(r \cdot S) + \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right)\right) = \\ &= vol(E') - r^2 \cdot z_{min} \cdot P(r \cdot S) + (\lceil r^2 \cdot z_{min} \rceil - 1) \cdot \left(P(r \cdot S) \quad + \quad \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right)\right) = \\ &= vol(E') + P(r \cdot S) \cdot (\lceil r^2 \cdot z_{min} \rceil - r^2 \cdot z_{min} - 1) + \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right).\end{aligned}$$

Now let us calculate the number of digital points belonging to set E''. The following definition of $3D$-sets, ω_i and ω_i', is useful:

$$\omega_i \quad = \quad \{ \ (x, y, z) \mid (x, y) \in r \cdot S \quad \text{and} \quad x \cdot y \geq i \quad \text{and} \quad i < z < \min\{x \cdot y, i+1\} \ \}.$$

and

$$\omega_i' \quad = \quad \{ \ (x, y, z) \mid (x, y) \in r \cdot S \quad \text{and} \quad x \geq i \quad \text{and} \quad x \cdot y < z < i+1 \ \}.$$

Now, we can estimate the volume of E''.

$S_3(i)$ will denote the $2D$-region $\{ \ (x, y) \ \mid \ (x, y) \in r \cdot S \ \text{ and } \ x \cdot y \geq i \ \}$. It follows that

$$vol(E'') \quad = \quad \sum_{i=\lceil r^2 \cdot z_{min} \rceil}^{\lfloor r^2 \cdot z_{max} \rfloor - 1} vol(\omega_i) \quad + \quad (\lceil r^2 \cdot z_{min} \rceil - r^2 \cdot z_{min}) \cdot P(r \cdot S) + \mathcal{O}(r^2) =$$

$$= \sum_{i=\lceil r^2 \cdot z_{min} \rceil}^{\lfloor r^2 \cdot z_{max} \rfloor - 1} (P(S_3(i)) - vol(\omega_i')) \quad + \quad (\lceil r^2 \cdot z_{min} \rceil - r^2 \cdot z_{min}) \cdot P(r \cdot S) \quad + \quad \mathcal{O}(r^2) =$$

$$= \sum_{i=\lceil r^2 \cdot z_{min} \rceil}^{\lfloor r^2 \cdot z_{max} \rfloor - 1} P(S_3(i)) \quad - \quad \sum_{i=\lceil r^2 \cdot z_{min} \rceil}^{\lfloor r^2 \cdot z_{max} \rfloor - 1} vol(\omega_i') \quad + \quad (\lceil r^2 \cdot z_{min} \rceil - r^2 \cdot z_{min}) \cdot P(r \cdot S) \quad + \quad \mathcal{O}(r^2) =$$

$$= \sum_{i=\lceil r^2 \cdot z_{min} \rceil}^{\lfloor r^2 \cdot z_{max} \rfloor - 1} \left(\mu_{0,0}(S_3(i)) + \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right)\right) + (\lceil r^2 \cdot z_{min} \rceil - r^2 \cdot z_{min}) \cdot P(r \cdot S) + \mathcal{O}(r^2) =$$

$$= \mu_{0,0,0}(E'') \quad + \quad (\lceil r^2 \cdot z_{min} \rceil - r^2 \cdot z_{min}) \cdot P(r \cdot S) \quad + \quad \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right)$$

Thus,

$$\mu_{0,0,0}(E'') \quad = \quad vol(E'') \quad - \quad (\lceil r^2 \cdot z_{min} \rceil - r^2 \cdot z_{min}) \cdot P(r \cdot S) \quad + \quad \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon}\right).$$

The proof of the theorem is finished by summing $\mu_{0,0,0}(E')$ and $\mu_{0,0,0}(E'')$. So,

$$\begin{aligned}
\mu_{1,1}(r \cdot S) &= \mu_{0,0,0}(E') + \mu_{0,0,0}(E'') = \\
&= vol(E') + P(r \cdot S) \cdot \left(\lceil r^2 \cdot z_{min} \rceil - r^2 \cdot z_{min} - 1 \right) + \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon} \right) + \\
&\quad + vol(E'') - \left(\lceil r^2 \cdot z_{min} \rceil - r^2 \cdot z_{min} \right) \cdot P(r \cdot S) + \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon} \right) = \\
&= vol(E) + \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon} \right) = \\
&= m_{1,1}(r \cdot S) + \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon} \right). \quad \Box
\end{aligned}$$

The next theorem follows directly from the previous one.

Theorem 7 *Let a convex smooth region S be given. Then the moment $m_{1,1}(S)$ can be estimated by $\frac{1}{r^4} \cdot \mu_{1,1}(r \cdot S)$ within an $\mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon} \right)$ error.*

Proof. It holds

$$\begin{aligned}
m_{1,1}(S) - \frac{1}{r^4} \cdot \mu_{1,1}(r \cdot S) &= m_{1,1}(S) - \frac{1}{r^4} \cdot \left(\iint\limits_{r \cdot S} xydxdy + \mathcal{O}\left(r^{\frac{29}{11}+\varepsilon} \right) \right) = \\
&= m_{1,1}(S) - \frac{1}{r^4} \cdot \iint\limits_{r \cdot S} xydxdy + \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon} \right) = \\
&= m_{1,1}(S) - m_{1,1}(S) + \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon} \right) = \\
&= \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon} \right). \quad \Box
\end{aligned}$$

5 Efficiency in the reconstruction of gravity centers

Now, we are prepared to specify an upper error bound for the reconstruction of the gravity center of a smooth planar convex region. Note that the given estimates below are very sharp. They show that a reasonable high resolution ensures a location of the calculated gravity center within an error less than any arbitrary small fraction of the grid edge length $\frac{1}{r}$.

Theorem 8 *Let S be a smooth planar convex region. Then, for the gravity center position*

$$\left(\frac{m_{1,0}(S)}{m_{0,0}(S)}, \quad \frac{m_{0,1}(S)}{m_{0,0}(S)} \right)$$

the following error estimates hold:

$$\frac{m_{1,0}(S)}{m_{0,0}(S)} - \frac{1}{r} \cdot \frac{\mu_{1,0}(r \cdot S)}{\mu_{0,0}(r \cdot S)} = \frac{m_{1,0}(S)}{m_{0,0}(S)} - \frac{1}{r} \cdot \frac{\sum\limits_{\substack{i,j \text{ are integers} \\ (i,j) \in r \cdot S}} i}{\sum\limits_{\substack{i,j \text{ are integers} \\ (i,j) \in r \cdot S}} 1} = \mathcal{O}\left(\frac{1}{r^{\frac{15}{11}-\varepsilon}}\right)$$

and

$$\frac{m_{0,1}(S)}{m_{0,0}(S)} - \frac{1}{r} \cdot \frac{\mu_{0,1}(r \cdot S)}{\mu_{0,0}(r \cdot S)} = \frac{m_{0,1}(S)}{m_{0,0}(S)} - \frac{1}{r} \cdot \frac{\sum\limits_{\substack{i,j \text{ are integers} \\ (i,j) \in r \cdot S}} j}{\sum\limits_{\substack{i,j \text{ are integers} \\ (i,j) \in r \cdot S}} 1} = \mathcal{O}\left(\frac{1}{r^{\frac{15}{11}-\varepsilon}}\right) .$$

Proof. By using Theorem 2 it follows that

$$\frac{m_{1,0}(S)}{m_{0,0}(S)} - \frac{1}{r} \cdot \frac{\mu_{1,0}(r \cdot S)}{\mu_{0,0}(r \cdot S)} = \frac{r \cdot m_{1,0}(S) \cdot \left(r^2 \cdot m_{0,0}(S) + \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right)\right) - m_{0,0}(S) \cdot \left(r^3 \cdot m_{1,0}(S) + \mathcal{O}\left(r^{\frac{18}{11}+\varepsilon}\right)\right)}{r \cdot m_{0,0}(S) \cdot \left(r^2 \cdot m_{0,0}(S) + \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right)\right)}$$

$$= \frac{\mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)}{\left(m_{0,0}(S)\right)^2 + \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)} = \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right) . \quad \square$$

Several numerical examples are given in Table 1 in order to illustrate the previous theorem.

6 Central moments

The moments of the order one and two will vary for a given region depending on the spatial position of the region. Translation invariants (as should be orientation and elongation) are obtained by using the central moments for which the origin is at the centroid of the region. This normalization step is very common in moment based calculations. That is a reason for the following brief discussion of normalized moments.

Let the coordinates $\left(\frac{m_{1,0}(S)}{m_{0,0}(S)}, \frac{m_{1,0}(S)}{m_{0,0}(S)}\right)$ of the centroid of S be shortly denoted as $\overline{x}_c(S)$ and $\overline{y}_c(S)$. While the values $\frac{\mu_{1,0}(r \cdot S)}{\mu_{0,0}(r \cdot S)}$ and $\frac{\mu_{1,0}(r \cdot S)}{\mu_{0,0}(r \cdot S)}$ are denoted by $\overline{x}_d(r \cdot S)$ and $\overline{y}_d(r \cdot S)$, respectively. Then the expression

$$\overline{m}_{k,l}(S) = \iint\limits_S (x - \overline{x}_c(S))^k (y - \overline{y}_c(S))^l \, dx \, dy$$

is called the central (k,l)-moment of a given region S.

The discrete analogon for the central (k,l)-moment of a given region $r \cdot S$ is the so-called central discrete (k,l)-moment, which is equal to

$$\overline{\mu}_{k,l}(r \cdot S) \quad = \sum_{\substack{i,j \text{ are integers} \\ (i,j) \in r \cdot S}} (i - \overline{x}_d(r \cdot S))^k \cdot (j - \overline{y}_d(r \cdot S))^l \quad .$$

We show that central moments of second order of a given smooth region S can be recovered with the same $\mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)$ error bound from their corresponding central discrete moments. Obviously, the central moments of the first degree are equal to zero, i.e.

$$\overline{m}_{1,0}(S) \;=\; \overline{m}_{0,1}(S) \;=\; \overline{\mu}_{1,0}(r \cdot S) \;=\; \overline{\mu}_{1,0}(r \cdot S) \;=\; 0\;.$$

Second order moments are specified in the next theorem.

Theorem 9 *Let a smooth convex region S be given. Then the following differences*

a) $\overline{m}_{2,0}(S) - \frac{1}{r^4} \cdot \overline{\mu}_{2,0}(r \cdot S)$,

b) $\overline{m}_{0,2}(S) - \frac{1}{r^4} \cdot \overline{\mu}_{0,2}(r \cdot S)$,

c) $\overline{m}_{1,1}(S) - \frac{1}{r^4} \cdot \overline{\mu}_{1,1}(r \cdot S)$,

have an upper bound in $\mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)$.

Proof. We prove a). Theorems 3, 5 and 8 as well as the equations $\overline{m}_{2,0}(S) = m_{2,0}(S) - \overline{x}_c(S) \cdot m_{1,0}(S)$, i.e. $\overline{\mu}_{2,0}(r \cdot S) = \mu_{2,0}(S) - \overline{x}_d(r \cdot S) \cdot \mu_{1,0}(r \cdot S)$ will be used. It follows that

$$\overline{m}_{2,0}(S) - \frac{1}{r^4} \cdot \overline{\mu}_{2,0}(r \cdot S) = m_{2,0}(S) - \overline{x}_c(S) \cdot m_{1,0}(S) - \frac{1}{r^4} \cdot (\mu_{2,0}(r \cdot S) - \overline{x}_d(r \cdot S) \cdot \mu_{1,0}(r \cdot S)) =$$

$$= \left(m_{2,0}(S) - \frac{1}{r^4} \cdot \mu_{2,0}(r \cdot S)\right) - \left(\overline{x}_c(S) \cdot m_{1,0}(S) - \left(m_{1,0}(S) + \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)\right) \cdot \left(\overline{x}_c(S) + \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)\right)\right) =$$

$$= \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right) \;.$$

The items b) and c) can be proved analogously. □

7 Efficiency in the reconstruction of orientations

The usual practice in defining of the region's orientation is to choose the axes of the least second moment ([5] , [9]). More precisely, it is necessary to find the line, say in the polar-coordinates form $X \cdot \sin\varphi - Y \cdot \cos\varphi + \rho = 0$

as it is used usually in order to avoid numerical problems when the line is (nearly) vertical, for which the integral of the square of the distance to points in the region is a minimum. The squared distance $r(x,y,\rho,\varphi)$ from a point (x,y) to the specified line is $r^2(x,y,\rho,\varphi) = (x \cdot \cos\varphi \; - \; y \cdot \sin\varphi \; + \; \rho)^2$. That means the integral

$$I(S,\varphi,\rho) \;=\; \iint\limits_S (x \cdot \sin\varphi - y \cdot \cos\varphi + \rho)^2 dxdy$$

should be minimized with respect to ρ and φ. Differentiating with respect to ρ and setting the result to zero leads to

$$\overline{x}_c(S) \cdot \sin\varphi \; - \; \overline{y}_c(S) \cdot \cos\varphi \; + \; \rho \; = \; 0.$$

Thus the required line (the so-called axis of least second moments) passes through the centroid of S. This suggests a change of coordinates, i.e. a normalization by translation of S by the vector $(-\overline{x}_c(S), \; -\overline{y}_c(S))$. So, $I(S,\varphi,\rho)$ becomes

$$I(S,\varphi,\rho) = \overline{m}_{2,0}(S) \cdot \sin^2\varphi - 2 \cdot \overline{m}_{2,0}(S) \cdot \sin\varphi \cdot \cos\varphi + \overline{m}_{0,2}(S) \cdot \cos^2\varphi$$

or equivalently,

$$I(S,\varphi,\rho) = \frac{1}{2} \cdot (\overline{m}_{2,0}(S) + \overline{m}_{0,2}(S)) - \frac{1}{2} \cdot (\overline{m}_{2,0}(S) - \overline{m}_{0,2}(S)) \cdot \cos 2\varphi - \overline{m}_{1,1}(S) \cdot \sin 2\varphi \; . \tag{6}$$

Differentiating with respect to φ and setting the result to zero, we have ([5]) two solutions. The solution

$$\sin 2\varphi = \frac{2 \cdot \overline{m}_{1,1}(S)}{\sqrt{4 \cdot (\overline{m}_{1,1}(S))^2 + (\overline{m}_{2,0}(S) - \overline{m}_{0,2}(S))^2}} \; , \qquad \cos 2\varphi = \frac{\overline{m}_{2,0}(S) - \overline{m}_{0,2}(S)}{\sqrt{4 \cdot (\overline{m}_{1,1}(S))^2 + (\overline{m}_{2,0}(S) - \overline{m}_{0,2}(S))^2}} \tag{7}$$

leads to the desired minimum for $I(S,\varphi,\rho)$, while the solution

$$\sin 2\varphi = -\frac{2 \cdot \overline{m}_{1,1}(S)}{\sqrt{4 \cdot (\overline{m}_{1,1}(S))^2 + (\overline{m}_{2,0}(S) - \overline{m}_{0,2}(S))^2}} \; , \qquad \cos 2\varphi = -\frac{\overline{m}_{2,0}(S) - \overline{m}_{0,2}(S)}{\sqrt{4 \cdot (\overline{m}_{1,1}(S))^2 + (\overline{m}_{2,0}(S) - \overline{m}_{0,2}(S))^2}} \tag{8}$$

corresponds to the maximum value for $I(S,\varphi,\rho)$, unless $m_{2,0}(S) = m_{0,2}(S)$. The equality $m_{2,0}(S) = m_{0,2}(S)$ points out that S is to symmetrical in some sense. More details about the determination of orientations even in the case of "degenerate" images can be found in [15].

Consequently, the orientation $E(S)$ of the region S satisfies

$$\tan(2 \cdot E(S)) \;=\; \frac{2 \cdot \overline{m}_{1,1}(S)}{\overline{m}_{2,0}(S) - \overline{m}_{0,2}(S)} \; .$$

Now, we can estimate the error in approximating the orientation of a smooth region S by using the discrete analogon for the previous equality. Namely, if the orientation $E(S)$ of a smooth region S, given on the picture with the resolution r, is estimated by $\mathcal{E}(r \cdot S)$ where

$$\tan(2 \cdot \mathcal{E}(r \cdot S)) \;=\; \frac{2 \cdot \overline{\mu}_{1,1}(r \cdot S)}{\overline{\mu}_{2,0}(r \cdot S) - \overline{\mu}_{0,2}(r \cdot S)} \; ,$$

then the following theorem holds.

Theorem 10 *Let a smooth planar region S be given. If $m_{2,0}(S) \neq m_{0,2}(S)$ is assumed, then its orientation $A(S)$ can be recovered within error $\mathcal{O}(r^{-\frac{15}{11}+\varepsilon})$, by using*

$$\tan(2 \cdot A(S)) \approx \frac{2 \cdot \overline{\mu}_{1,1}(r \cdot S)}{\overline{\mu}_{2,0}(r \cdot S) - \overline{\mu}_{0,2}(r \cdot S)} .$$

Proof. By using Theorem 9 we have

$$\begin{aligned}
\tan(2 \cdot A(S)) - \tan(2 \cdot \mathcal{A}(r \cdot S)) &= \frac{2 \cdot \overline{m}_{1,1}(S)}{\overline{m}_{2,0}(S) - \overline{m}_{0,2}(S)} - \frac{2 \cdot \overline{\mu}_{1,1}(r \cdot S)}{\overline{\mu}_{2,0}(r \cdot S) - \overline{\mu}_{0,2}(r \cdot S)} = \\
&= \frac{2 \cdot \overline{m}_{1,1}(S)}{\overline{m}_{2,0}(S) - \overline{m}_{0,2}(S)} - \frac{2 \cdot \overline{m}_{1,1}(S) + \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)}{\overline{m}_{2,0}(S) - \overline{m}_{0,2}(S) + \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)} = \\
&= \frac{\mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)}{(m_{2,0}(S) - m_{0,2}(S))^2 + \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)} = \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right) . \quad \square
\end{aligned}$$

For a few numerical examples see the Table 1.

8 Reconstruction of elongation

As it is already mentioned the elongation of a region S is defined to be the ratio of the maximal (reached for the angle-value given by (8)) and the minimal (reached for the angle-value given by (7)) values of $I(S, \varphi, \rho)$. That is a definition such that, for example, the elongation of a circle has the value 1 (which is the minimal possible), while the elongation of a straight line segment is ∞.

If these calculated values are entered in (6) we have

$$\max_{\varphi,\rho} I(S, \varphi, \rho) = \frac{1}{2} \cdot (\overline{m}_{2,0}(S) + \overline{m}_{0,2}(S)) + \frac{1}{2} \cdot \sqrt{4 \cdot (\overline{m}_{1,1}(S))^2 + (\overline{m}_{2,0}(S) - \overline{m}_{0,2}(S))^2}$$

and

$$\min_{\varphi,\rho} I(S, \varphi, \rho) = \frac{1}{2} \cdot (\overline{m}_{2,0}(S) + \overline{m}_{0,2}(S)) - \frac{1}{2} \cdot \sqrt{4 \cdot (\overline{m}_{1,1}(S))^2 + (\overline{m}_{2,0}(S) - \overline{m}_{0,2}(S))^2} .$$

So, this suggests that the elongation $E(S)$ of a region S should be approximated by $\mathcal{E}(r \cdot S)$, where

$$\mathcal{E}(r \cdot S) = \frac{\overline{\mu}_{2,0}(r \cdot S) + \overline{\mu}_{0,2}(r \cdot S) + \sqrt{4 \cdot (\overline{\mu}_{1,1}(r \cdot S))^2 + (\overline{\mu}_{2,0}(r \cdot S) - \overline{\mu}_{0,2}(r \cdot S))^2}}{\overline{\mu}_{2,0}(r \cdot S) + \overline{\mu}_{0,2}(r \cdot S) - \sqrt{4 \cdot (\overline{\mu}_{1,1}(r \cdot S))^2 + (\overline{\mu}_{2,0}(r \cdot S) - \overline{\mu}_{0,2}(r \cdot S))^2}} .$$

We prove the following theorem.

Theorem 11 *If a smooth convex region S is given by a digital picture having the resolution r, then its elongation can be estimated by $\mathcal{E}(r\cdot S)$ where*

$$\mathcal{E}(r\cdot S) = \frac{\overline{\mu}_{2,0}(r\cdot S)+\overline{\mu}_{0,2}(r\cdot S)+\sqrt{4\cdot(\overline{\mu}_{1,1}(r\cdot S))^2+(\overline{\mu}_{2,0}(r\cdot S)-\overline{\mu}_{0,2}(r\cdot S))^2}}{\overline{\mu}_{2,0}(r\cdot S)+\overline{\mu}_{0,2}(r\cdot S)-\sqrt{4\cdot(\overline{\mu}_{1,1}(r\cdot S))^2+(\overline{\mu}_{2,0}(r\cdot S)-\overline{\mu}_{0,2}(r\cdot S))^2}}\ .$$

The error in the approximation $E(S)\approx\mathcal{E}(r\cdot S)$ has an upper bound in $\mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)$, i.e.

$$E(S) \ = \ \mathcal{E}(r\cdot S) \ + \ \mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)\ .$$

Proof. It is sufficient to apply the following equation. It holds

$$\frac{1}{r^4}\cdot\sqrt{4\cdot\left((\overline{\mu}_{1,1}(r\cdot S)\right)^2+\left(\overline{\mu}_{2,0}(r\cdot S)-\overline{\mu}_{0,2}(r\cdot S)\right)^2}=$$

$$=\sqrt{4\left(\frac{\overline{\mu}_{1,1}(r\cdot S)}{r^4}\right)^2+\left(\frac{\overline{\mu}_{2,0}(r\cdot S)-\overline{\mu}_{0,2}(r\cdot S)}{r^4}\right)^2}=$$

$$=\sqrt{4\left(\overline{m}_{1,1}(S)+\mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)\right)^2+\left(\overline{m}_{2,0}(S)-\overline{m}_{0,2}(S)+\mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)\right)^2}=$$

$$=\sqrt{4\cdot(\overline{m}_{1,1}(S))^2+\left(\overline{m}_{2,0}(S)-\overline{m}_{0,2}(S)\right)^2+\mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)}=$$

$$=\sqrt{4\cdot(\overline{m}_{1,1}(S))^2+\left(\overline{m}_{2,0}(S)-\overline{m}0,2(S)\right)^2}\cdot\sqrt{1+\mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)}=$$

$$=\sqrt{4\cdot(\overline{m}_{1,1}(S))^2+\left(\overline{m}_{2,0}(S)-\overline{m}_{0,2}(S)\right)^2}+\mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)\ ,$$

where Theorem 9 and $\sqrt{1+x}=1+\mathcal{O}(x)$ have been used. Then the rest of the proof follows from Theorem 9 and the use of elementary mathematics. □

9 An example and conclusions

The fact, that the digitization of real regions leads to some uncertainty, opens (at least) two kinds of question. We illustrate them with the following examples:

r	$x_c(S) - x_d(r \cdot S)$	$x_c(S) - y_d(r \cdot S)$	$A(S) - \mathcal{A}(r \cdot S)$	$E(S) - \mathcal{E}(r \cdot S)$	$r^{-\frac{15}{11}}$
1	-0.02	-0.0714285	——	——	1
2	-0.002	-0.7142857	——	——	0.3886015
3	-0.0533333	-0.0380952	——	-16.318875	0.2235528
π	-0.0292958	-0.0170624	——	-16.318875	0.2099270
$3\sqrt{2}$	0.0085954	-0.0133703	-88.062862	-6.4088558	0.1935815
5	-0.0109090	-0.0441558	-88..045099	-8.1739793	0.1113933
10	-0.0004878	-0.0153310	-89.501941	-1.1011864	0.0432876
23	-0.0030917	-0.0023867	-0.1137293	-0.0501661	0.0139026
100	-0.0002594	-0.0003305	-0.0181834	0.0605284	0.0018738
$100\sqrt{3}$	0.0003404	-0.0003389	-0.0059411	0.0181071	0.0008859
256	-0.0001566	-0.0001219	-0.0032078	0.0164456	0.0005200
100π	-0.0000937	-0.0001651	-0.0029504	0.0079204	0.0003933
500	-0.0000805	-0.0000495	-0.0009776	0.0086543	0.0002087
750	-0.0000389	-0.0000272	-0.0005014	0.0054513	0.0001200
1000	-0.0000312	-0.0000178	-0.0010762	0.0023119	0.0008111

Table 1: Errors between the values of the coordinates of the centroid, the orientation and elongation of the region S and their approximations calculated from the digitization of $r \cdot S$, for different values of r. The values $\mathcal{A}(S)$, $\mathcal{A}(2 \cdot S)$, $\mathcal{A}(3 \cdot S)$ and $\mathcal{A}(\pi \cdot S)$ as well as $\mathcal{E}(S)$ and $\mathcal{E}(2 \cdot S)$ cannot be calculated because dividing by zero appeared.

(i) Assume a given binary image representing a digital region. How to determine the set of all gravity centers of those regions where digitization leads to the same binary picture? This problem is already studied in relation to digital disks, i.e. binary pictures of circular regions, where the centers of gravity coincide with the midpoints of the circles ([19]).

(ii) What is an upper error bound for the approximation of the centroid of a region if the calculation is based on a binary image? In the case when the studied region is a circle (the centroid coincides with the midpoint of the circle), for an answer see [21].

In this paper the second kind of questions were studied. It is shown that the absolute error in the calculation of the centroids, orientations and elongations of smooth regions has an upper bound of $\mathcal{O}\left(\frac{1}{r^{15/11-\varepsilon}}\right)$, where ε is an arbitrary small positive number, while r is the picture resolution (i.e. the number of pixels per unit). An equivalent formulation of the studied problem is:

If smooth planar regions are considered, which picture resolution has to be used in order to satisfy a pre-specified precision in calculating the centroid, orientation and elongation ?

The paper answers that as follows: if α is the allowed error, then a picture resolution $\mathcal{O}\left(\alpha^{-\frac{11}{15}-\varepsilon}\right)$, i.e. the pixel size $\mathcal{O}\left(\alpha^{\frac{11}{15}+\varepsilon}\right)$, gives the garanty that the errors in the mentioned calculations are bounded above by α. The previous observation is summarized by the next theorem. The proof is omitted because it is trivial.

Theorem 12 *Let a positive number α be given. Then the area (size), centroid, orientation and elongation of a convex region S whit C^3 boundary having positive curvature at every point, or region which can be obtained as a finite number of unions, intersections and set differences of such regions, can be reconstructed within an absolute error less or equal to α if a suitable picture resolution r is applied. The order of magnitude of r is described by*

$$r = \mathcal{O}\left(\alpha^{-\frac{11}{15}-\varepsilon}\right) .$$

We give the following example which illustrates the obtained results. Let the region S be defined by the inequalities $y \geq x^3$ and $y \leq \sqrt{x}$. It is easy to obtain

$$m_{0,0}(S) = \frac{5}{12}, \quad m_{1,0}(S) = \frac{1}{5}, \quad m_{0,1}(S) = \frac{5}{28}, \quad m_{2,0}(S) = \frac{5}{42}, \quad m_{1,1}(S) = \frac{5}{48}, \quad m_{0,2}(S) = \frac{1}{10},$$

and

$$\overline{m}_{2,0}(S) = \frac{5}{42} - \frac{12}{125} = \frac{121}{5250}, \quad \overline{m}_{1,1}(S) = \frac{5}{48} - \frac{3}{35} = \frac{31}{1680}, \quad \overline{m}_{0,2}(S) = \frac{1}{10} - \frac{15}{196} = \frac{23}{980}.$$

This specifies that gravity center, orientation and elongation of S are

$$(x_c(S), y_c(S)) = \left(\frac{12}{25}, \frac{12}{28}\right) , \qquad A(S) = -\frac{1}{2}\arctan\frac{175}{2} \approx -44.67261^o, \qquad E(S) = \frac{6838 + 31\sqrt{30629}}{6838 - 31\sqrt{30629}}$$

Table 1 contains the values of $(x_d(r \cdot S),\ y_d(r \cdot S))$, $\mathcal{A}(r \cdot S)$ and $\mathcal{E}(r \cdot S)$ which approximate the position, orientation and elongation, when different picture resolutions are applied.

In this paper our attention was focused on the digitization effects on the possibility in the reconstruction of the basic geometric properties of smooth regions. The results are based on estimations of moments up to order two of the region from discrete data obtained by digitization. But there are many procedures in image processing which are based on moment calculations. So, the results obtained here can be used in the evaluation of the efficiency of such procedures.

r	$m_{0,0}(S) - \frac{\mu_{0,0}(r\cdot S)}{r^2}$	$m_{1,0}(S) - \frac{\mu_{1,0}(r\cdot S)}{r^2}$	$m_{0,1}(S) - \frac{\mu_{0,1}(r\cdot S)}{r^2}$	$r^{-\frac{15}{11}}$
1	-1.5833333	-0.8	-0.8214285	1
2	-0.3333333	-0.175	-0.1964285	0.3886015
3	-0.1388888	-0.0962962	-0.0806878	0.2235528
π	-0.0899392	-0.0580122	-0.0471893	0.2099270
$3\sqrt{2}$	-0.0277777	-0.0095131	-0.0178471	0.1393581
5	-0.0233333	-0.16	-0.0294285	0.1113933
10	0.0066666	0.003	-0.0034285	0.0432876
23	0.0083490	0.0027451	0.0026036	0.0139026
100	0.0004666	0.000116	0.0000624	0.0018738
$100\sqrt{3}$	0.0002666	-0.0000137	-0.0000268	0.0008859
256	0.0002848	0.0000714	0.0000713	0.0005200
100π	0.0001960	0.0000550	0.0000152	0.0003933
500	0.0000906	0.0000099	0.0000182	0.0002087
750	0.0000533	0.0000093	0.0000115	0.0001200
1000	0.0000586	0.0000151	0.0000176	0.0000811

Table 2: Errors between the values of the moments $m_{0,0}(S)$, $m_{1,0}(S)$ and $m_{0,1}(S)$ and their estimations from digitization of $r \cdot S$.

Accordingly the results as given in Theorems 2, 4 and 5 (and Theorems 6, 7, 8 and 9 as their direct consequences) are the main contributions of this paper. They are perhaps more important than the results of Theorems 8, 10 and 11 which are just their consequence. That is a reason for giving an illustration of these statements in the next two tables. Table 2 contains some numerical examples which show which is the difference between $m_{0,0}(S)$ and $\frac{\mu_{0,0}(r\cdot S)}{r^2}$ (showing the error in the size estimation and illustrating Huxley's result), and which is the difference between the first order moments and their estimations - a sharp theoretical upper bound is derived here. All the previously mentioned differences should be compared with $r^{-\frac{15}{11}}$.

Finally, Table 3 illustrates which is the efficiency in the second moment estimation if a given picture resolution r is applied. The errors in estimations have to be compared with $r^{-\frac{15}{11}}$.

r	$m_{2,0}(S) - \frac{\mu_{2,0}(r \cdot S)}{r^4}$	$m_{0,2}(S) - \frac{\mu_{0,2}(r \cdot S)}{r^4}$	$m_{1,1}(S) - \frac{\mu_{1,1}(r \cdot S)}{r^4}$	$r^{-\frac{15}{11}}$
1	-0.8809523	-0.9	-0.8958333	1
2	-0.1934523	-0.2125	-0.2083333	0.3886015
3	-0.1031746	-0.0851851	-0.0933641	0.2235528
π	-0.0657400	-0.0539897	-0.0600890	0.2099270
$3\sqrt{2}$	-0.0167548	-0.0203703	-0.0192901	0.1393581
5	-0.0233523	-0.0312	-0.0286333	0.1113933
10	0.0001476	-0.0048	-0.0029333	0.0432876
23	0.0015380	0.0013975	0.0012903	0.0139026
100	0.0001305	0.0000740	0.0001063	0.0018738
$100\sqrt{3}$	-0.0000279	-0.0000319	-0.0000302	0.0008859
256	0.0000551	0.0000579	0.0000546	0.0005200
100π	0.0000378	0.0000040	0.0000172	0.0003933
500	0.0000095	0.0000190	0.0000149	0.0002087
750	0.0000094	0.0000124	0.0000112	0.0001200
1000	0.0000112	0.0000134	0.0000116	0.0000811

Table 3: Errors between the values of the second order moments $m_{2,0}(S)$, $m_{0,2}(S)$ and $m_{1,1}(S)$ and their estimations from a digitization of $r \cdot S$.

The results of our paper are based on Huxley's theorem. This theorem is a strong mathematical result which is related to the number of integer points inside of a smooth planar convex curve γ and addresses an ancient mathematical problem. Gauss and Dirichlet knew that the area of a region bounded by curve γ estimates this number within an order $\mathcal{O}(s)$, where s is the length of the curve γ. The situation when γ is a circle is studied most carefully. Huxley's result [7] improves the previously best known upper bound ([8]) even in this case of a circle. From the proofs of Theorems 2, 4 and 6 it can be concluded that results derived here are sharp up to Huxley's result.

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Symbols and notations

- S – denotes a $2D$ region;
- r – denotes the picture resolution, i.e. $\frac{1}{r}$ is the pixel size ;
- $r \cdot S$ – is the dilation of the region S by factor r, i.e. $r \cdot S = \{(r \cdot x,\ r \cdot y) \mid (x, y) \in S\}$;
- $r \cdot \overline{S}$ – is the region symmetrical to $r \cdot S$ with respect to a specified (vertical) line $x = k$, where k is an integer;
- $(r \cdot S)(k)$ – is the set defined as: $(r \cdot S)(k) = \{(x, y) \mid (x, y) \in (r \cdot S) \text{ and } x \leq k\}$;
- $D(S)$ – is a digitization of a set S, defined to be the set of all grid points with integer coordinates which belong to the region occupied by the given set S: $(i, j) \in D(S) \Leftrightarrow (i, j) \in S$, where i and j are integers;
- $L(r \cdot S, k)$ – is the set of digital points defined as $L(r \cdot S, k) = \{(k, j) \mid (k, j) \in D(r \cdot S)\}$;
- $P(S)$ – is the area of the region S ;
- $vol(B)$ – is the volume of the $3D$-set B ;
- $m_{k,l}(S)$ – is the (k, l)-moment of a planar set S defined by $m_{k,l}(S) = \iint\limits_S x^k y^l \, dx \, dy$;
- $\mu_{k,l}(S)$ – denotes the discrete (k, l)-moment of the region S and it is defined by
$$\mu_{k,l}(S) = \sum_{(i,j) \in D(S)} i^k \cdot j^l = \sum_{\substack{i,j \text{ are integers} \\ (i,j) \in S}} i^k \cdot j^l \quad ;$$
- $\mu_{k,l}(r \cdot S)$ – denotes the discrete (k, l)-moment of the region $r \cdot S$, so,
$$\mu_{k,l}(r \cdot S) = \sum_{(i,j) \in D(r \cdot S)} i^k \cdot j^l = \sum_{\substack{i,j \text{ are integers} \\ (i,j) \in r \cdot S}} i^k \cdot j^l \quad ;$$
- $\mu_{0,0,0}(B)$ – denotes the three-dimensional zero-th order moments of a three dimensional set B and it is defined by $\mu_{0,0,0}(B) = \sum\limits_{(i,j,k) \in B} 1 = \sum\limits_{\substack{i,j,k \text{ are integers} \\ (i,j,k) \in B}} 1$;
- the gravity center (or centroid) of a region S is $\left(\frac{m_{1,0}(S)}{m_{0,0}(S)}, \ \frac{m_{0,1}(S)}{m_{0,0}(S)}\right)$; the gravity center (centroid) of the region S is also denoted by $(x_c(S), y_c(S))$;

- $\left(\frac{1}{r} \cdot \frac{\mu_{1,0}(S)}{\mu_{0,0}(S)}, \quad \frac{1}{r} \cdot \frac{\mu_{0,1}(S)}{\mu_{0,0}(S)}\right)$ – is the centroid approximation calculated from the digitization of $r \cdot S$, it is also denoted by $(x_d(r \cdot S), y_d(r \cdot S))$;

- $\overline{m}_{k,l}(S)$ – is the central (k,l)-moment of a planar set S defined by

$$\overline{m}_{k,l}(S) \quad = \quad \iint\limits_S (x - x_c(S))^k \cdot (y - y_c(S))^l \; dx\, dy;$$

- $\overline{\mu}_{k,l}(r \cdot S)$ is the central discrete moment for the region $r \cdot S$, and it equals

$$\overline{\mu}_{k,l}(r \cdot S) \quad = \quad \sum_{\substack{i,j \text{ are integers} \\ (i,j) \in r \cdot S}} (i - x_d(r \cdot S))^k \cdot (j - y_d(r \cdot S))^l \quad ;$$

- x_{min}, y_{min} and z_{min} are the minimal values of x, y and z coordinates, respectively, where (x,y,z) belong to some specified set. The values x_{max}, y_{max} and z_{max} are defined as the maximal ones;

- for a given set $r \cdot S$, C', C'', V_i, V_i', B', B'', W_i, W_i', E', E'', ω_i and ω_i' are three-dimensional sets defined as follows:

$$C' = \{(x,y,z) \quad | \quad (x,y) \in r \cdot S, \quad 0 < z < r \cdot x_{min}\} \;,$$

$$C'' = \{(x,y,z) \quad | \quad (x,y) \in r \cdot S, \quad r \cdot x_{min} \leq z \leq x\},$$

$$V_i = \{(x,y,z) \quad | \quad (x,y) \in r \cdot S, \quad x \geq i, \quad i < z \leq i+1\},$$

$$V_i' = \{(x,y,z) \quad | \quad (x,y) \in r \cdot S, \quad x \geq i \;, \quad x < z \leq i+1\},$$

$$B' = \{(x,y,z) \quad | \quad (x,y) \in r \cdot S, \quad 0 < z \leq \lceil r \cdot x_{min} \rceil^2\} \;,$$

$$B'' = \{(x,y,z) \quad | \quad (x,y) \in r \cdot S, \quad \lceil r \cdot x_{min} \rceil^2 < z \leq x^2\},$$

$$W_i = \{(x,y,z) \quad | \quad (x,y) \in r \cdot S, \quad x \geq i, \quad i < z \leq i+1\} \;,$$

$$W_i' = \{(x,y,z) \quad | \quad (x,y) \in r \cdot S, \quad x \geq i \;, \quad x^2 < z \leq i+1\},$$

$$E' = \{(x,y,z) \quad | \quad (x,y) \in r \cdot S, \quad 0 < z < r^2 \cdot z_{min}\} \;,$$

$$E'' = \{(x,y,z) \quad | \quad (x,y) \in r \cdot S, \quad r^2 \cdot z_{min} \leq z \leq x \cdot y\} \;,$$

$$\omega_i \quad = \quad \{\; (x,y,z) \mid (x,y) \in r \cdot S, \quad x \cdot y \geq i, \quad i < z < \min\{x \cdot y, i+1\} \;\},$$

$$\omega_i' \quad = \quad \{\; (x,y,z) \mid (x,y) \in r \cdot S, \quad x \geq i, \quad x \cdot y < z < i+1 \;\} \;.$$

- $r(x,y,\varphi,\rho)$ is the perpendicular distance from the point (x,y) to the line given in the form: $x \cdot \cos\varphi - y \cdot \sin\varphi = \rho$;

- $I(S,\varphi,\rho)$ – denotes the integral $\iint\limits_S r^2(x,y,\varphi,\rho) dx dy$;

– $A(S)$ – denotes the orientation of the region S and it is equal to this φ-value for which $\min_{\varphi,\rho} I(S,\varphi,\rho)$ reaches the minimum, i.e. $\min_{\varphi,\rho} I(S,\varphi,\rho) = I(S, A(S), \overline{\rho})$, for some $\overline{\rho}$;

– $\mathcal{A}(r \cdot S)$ – denotes the approximation of $A(S)$ calculated from the digitization of $r \cdot S$;

– $E(S)$ – denotes the elongation of the region S and it is $E(S) = \dfrac{\max_{\varphi,\ \rho} I(S,\varphi,\rho)}{\min_{\varphi,\ \rho} I(S,\varphi,\rho)}$;

– $\mathcal{E}(r \cdot S)$ – denotes the approximation of $E(S)$ calculated from the digitization of $r \cdot S$.