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# Gravity Centers of Smooth Planar Convex Regions from Digital Pictures

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## Abstract

Representations of real objects by corresponding digital pictures cause an inherent loss of information. There are infinitely many different real shapes with an identical corresponding digital picture. The problem we are interested in is how effciently the gravity center (or centroid) of a planar convex region whose boundary has a continuos third derivative and positive curvature (at every point) can be reconstructed from its digital picture. We derive an absolute upper error bound if such a smooth planar convex region is represented in a binary picture with resolution *r*, where *r* is the number of pixels per unit. This result can be extended to regions which may be obtained from smooth planar convex regions by finite applications of unions, intersections or set differences. The upper error bound remains the same which converges to zero with increase in grid resolution.

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# Gravity Centers of Smooth Planar Convex Regions from Digital Pictures

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**Abstract:** Representations of real objects by corresponding digital pictures cause an inherent loss of information. There are infinitely many different real shapes with an identical corresponding digital picture. The problem we are interested in is how efficiently the gravity center (or centroid) of a planar convex region whose boundary has a continuos third derivative and positive curvature (at every point) can be reconstructed from its digital picture. We show that if such a smooth planar convex region is represented in a binary picture with resolution $r$, then the coordinates of its center of gravity can be reconstructed with an absolute upper error bound of

$$\mathcal{O}\left(\frac{1}{r^{\frac{15}{11}-\varepsilon}}\right)$$

where $r$ is the number of pixels per unit. This result can be extended to regions which may be obtained from smooth planar convex regions by finite applications of unions, intersections or set differences. The upper error bound remains the same which converges to zero with increase in grid resolution.

**Keywords:** Digital geometry, digital shapes, gravity center, centroid, convergence, grid resolution

## 1 Introduction

A *smooth planar convex region* is a convex set in the Euclidean plane whose boundary has a continuos third derivative (i.e. a $C^3$-boundary) and positive curvature at every point. The $(k,l)$-*moment*, denoted by $m_{k,l}(S)$, of a planar measurable set $S$ is defined by

$$m_{k,l}(S) = \iint\limits_S x^k y^l \; dx \; dy.$$

The *gravity center* or *centroid* of a set $S$ is defined as a point

$$\left(\frac{m_{1,0}(S)}{m_{0,0}(S)}, \quad \frac{m_{0,1}(S)}{m_{0,0}(S)}\right) = \left(\frac{\iint\limits_S x \; dx \; dy}{\iint\limits_S \; dx \; dy}, \quad \frac{\iint\limits_S y \; dx \; dy}{\iint\limits_S \; dx \; dy}\right) \quad .$$

Note that $m_{0,0}(S)$ is the area of set $S$, and it will be denoted by $P(S)$ in the rest of the paper.

In applications of image analysis and pattern recognition the real shapes are actually unknown. Real objects are acquired as binary images $D(S)$, i.e. as digital sets. A general convergence concept for refined grid resolutions is discussed in [4]. In the manifold of different digitization models, we specify that for a set $S$ its *digitization* is defined to be the set of all grid points with integer coordinates which belong to the region occupied by the given set $S$:

$$(i,j) \in D(S) \quad \Leftrightarrow \quad (i,j) \in S, \quad \text{where } i \text{ and } j \text{ are integers.}$$

For digital sets the moments $m_{k,l}(S)$ are replaced by *discrete moments* $\mu_{k,l}(S)$ where

$$\mu_{k,l}(S) \quad = \quad \sum_{(i,j)\in D(S)} i^k \cdot j^l \quad = \quad \sum_{\substack{i,j \text{ are integers} \\ (i,j)\in S}} i^k \cdot j^l \quad .$$

Consequently, the center of gravity of a set $S$ is approximated from its digital image as

$$\left(\frac{m_{1,0}(S)}{m_{0,0}(S)},\quad \frac{m_{0,1}(S)}{m_{0,0}(S)}\right) \quad\approx\quad \left(\frac{\mu_{1,0}(S)}{\mu_{0,0}(S)},\quad \frac{\mu_{0,1}(S)}{\mu_{0,0}(S)}\right) = \left( \frac{\sum\limits_{\substack{i,j \text{ are integers} \\ (i,j)\in S}} i}{\sum\limits_{\substack{i,j \text{ are integers} \\ (i,j)\in S}} 1}, \quad \frac{\sum\limits_{\substack{i,j \text{ are integers} \\ (i,j)\in S}} j}{\sum\limits_{\substack{i,j \text{ are integers} \\ (i,j)\in S}} 1} \right) \quad . \tag{1}$$

The fact is that the digitization of real objects causes an inherent loss of information. There are infinitely many different regions in the Euclidean plane which have the same digitization. So, the approximation (1) leads to the following questions:.

(i) Assume a given binary image of a region in the Euclidean plane. How to determine the set of all the gravity centers of all regions with the same binary picture? This problem is already studied in relation to digital disks, i.e. binary pictures of circular regions, where the centers of gravity coincide with the midpoints of the circles [5].

(ii) What is an upper error bound for the approximation of the gravity center of a region if the calculation is based on a binary image? An answer for circles is already given in [6].

In this paper we study the second question. First, we consider the situation where $S$ is a smooth planar convex region. The extension to regions which may be obtained from smooth planar convex regions by finite numbers of unions, intersections or set differences is straightforward. Such sets are called commonly *smooth regions*. Obviously, their boundaries consist of a finite number of $C^3$ arcs with positive curvature at every point within such an arc.

It will be shown that the absolute error in the approximation (1) of the gravity center of commonly smooth regions has an upper bound of $\mathcal{O}\left(\frac{1}{r^{15/11-\varepsilon}}\right)$, where $\varepsilon$ is an arbitrary small positive number, while $r$ is the *picture resolution* (i.e. the number of pixels per unit). Especially it holds that if the considered region has a straight line segment as a part of its boundary, then the precision is limited by $\mathcal{O}(\frac{1}{r})$.

An equivalent formulation of the studied problem is: *If a smooth planar convex region is considered, which picture resolution has to be used in order to satisfy the required precision in calculating the location of its gravity center?*

Through the paper it is assumed that all appearing coordinates are positive. In other words, the origin is always placed in the lower-left corner of a studied digital picture.

## 2 Definitions and Huxley's theorem

If nothing is known a-priori about the shape of the given smooth planar convex region then the precision in estimation can only be specified as a function of the grid resolution, i.e. of the number of pixels per unit.

Assume that $D_1(S)$ is a binary picture of the region $S$ for resolution $r_1 = 1$, i.e. one pixel per unit, and let $D_2(S)$ be the binary picture of the same region for resolution $r_2$, i.e. with $r_2$ pixels per unit. Then it follows that $D_2(S) = D_1(r_2 \cdot S)$, where $r_2 \cdot S$ is the dilation of $S$ by factor $r_2$ .

In other words, for our purpose it is sufficient to consider regions of a form $r \cdot S$ digitized on the orthogonal grid. The study of $r \to \infty$ corresponds to increase in picture resolution [4]. For such an increase we estimate the asymptotic behavior of the following two expressions:

$$\frac{1}{r} \cdot \left(\frac{m_{1,0}(r\cdot S)}{m_{0,0}(r\cdot S)} \quad - \quad \frac{\mu_{1,0}(r\cdot S)}{\mu_{0,0}(r\cdot S)}\right) \quad = \quad \frac{1}{r} \cdot \left( \frac{\iint\limits_{r\cdot S} x\ dx\ dy}{\iint\limits_{r\cdot S} dx\ dy} \quad - \quad \frac{\sum\limits_{\substack{i,j \text{ are integers} \\ (i,j)\in r\cdot S}} i}{\sum\limits_{\substack{i,j \text{ are integers} \\ (i,j)\in r\cdot S}} 1} \right) \quad , and \tag{2}$$

$$\frac{1}{r} \cdot \left( \frac{m_{0,1}(r \cdot S)}{m_{0,0}(r \cdot S)} - \frac{\mu_{0,1}(S)}{\mu_{0,0}(r \cdot S)} \right) = \frac{1}{r} \cdot \left( \frac{\iint\limits_{r \cdot S} y \, dx \, dy}{\iint\limits_{r \cdot S} dx \, dy} - \frac{\sum\limits_{\substack{i,j \text{ are integers} \\ (i,j) \in r \cdot S}} j}{\sum\limits_{\substack{i,j \text{ are integers} \\ (i,j) \in r \cdot S}} 1} \right) . \tag{3}$$

We cite the following result from number theory [1], which expresses $\mu_{0,0}(S)$ for a smooth planar convex region.

**Theorem 1** *If $S$ is a convex region in the Euclidean plane, with $C^3$ boundary and positive curvature at every point of the boundary, then the number of lattice points belonging to $r \cdot S$ is*

$$\mu_{0,0}(r \cdot S) = r^2 \cdot P(S) + \mathcal{O}\left(r^{\frac{7}{11}} \cdot (\log r)^{\frac{47}{22}}\right) \quad ,$$

*where $P(S)$ denotes the area of $S$, while $r \cdot S$ is the dilatation of $S$ by factor $r$.*

Later on we will use a weaker result:

$$\mu_{0,0}(r \cdot S) = r^2 \cdot P(S) + \mathcal{O}\left(r^{\frac{7}{11}+\epsilon}\right) \quad , \qquad \text{for every} \quad \varepsilon > 0. \tag{4}$$

The preconditions of Theorem 1 can be relaxed to allow $S$ to have a finite number of vertices (corners). The theorem can also be applied, e.g., to an intersection of the interiors of two convex curves (for details, see [1]).

Our goal is to derive a "reasonable" asymptotic expression for $\mu_{1,0}(r \cdot S)$, or $\mu_{0,1}(r \cdot S)$, respectively. We will use the following definitions.

**Definition 1** *For a smooth planar convex region $S$, a given integer $k$ and a real number $r$, the digital point set $(r \cdot S)(k)$ is defined as:*

$$(r \cdot S)(k) = \{(i,j) \quad | \quad (x,y) \in (r \cdot S) \quad \textit{and} \quad x \leq k\} \quad .$$

In other words, $D((r \cdot S)(k))$ is the set of digital points in the digitization of $r \cdot S$ lying in the closed half plane determined by $x \leq k$.

**Definition 2** *For a smooth planar convex region $S$, a given integer $k$ and a real number $r$, the digital point set $L(r \cdot S, k)$ is defined as*

$$L(r \cdot S, k) = \{(k,j) \quad | \quad (k,j) \in D(r \cdot S)\} \quad .$$

In other words, $L(r \cdot S, k)$ is a set of digital points in the digitization of $r \cdot S$ which belong to the vertical line $x = k$.

Now we define three-dimensional $(3D)$ sets $V_i$ and $V_i'$, for $i = \lceil r \cdot x_{min} \rceil, \lceil r \cdot x_{min} \rceil + 1, \ldots, \lfloor r \cdot x_{max} \rfloor - 1$.

**Definition 3** *For a smooth planar convex region $r \cdot S$ and an integer $i$ from the set $\{\lceil r \cdot x_{min} \rceil, \lceil r \cdot x_{min} \rceil + 1, \ldots, \lfloor r \cdot x_{max} \rfloor - 1\}$, we define $3D$-sets*

$$V_i = \{(x,y,z) \quad | \quad (x,y) \in r \cdot S, \quad x \geq i, \quad i < z \leq i+1\}.$$

**Definition 4** *For a smooth planar convex region $r \cdot S$ and an integer $i$ from the set $\{\lceil r \cdot x_{min} \rceil, \lceil r \cdot x_{min} \rceil + 1, \ldots, \lfloor r \cdot x_{max} \rfloor - 1\}$, we also define $3D$-sets*

$$V_i' = \{(x,y,z) \quad | \quad (x,y) \in r \cdot S, \quad x \geq i \quad x < z \leq i+1\}.$$

It follows that

$$D(r \cdot S) = \bigcup_{k=-\infty}^{k=+\infty} L(r \cdot S, k) = \bigcup_{k=\lceil r \cdot x_{min} \rceil}^{\lfloor r \cdot x_{max} \rfloor} L(r \cdot S, k) \quad , \tag{5}$$

where $x_{min}$ is the minimum $x$-value among all the points in set $S$, and $x_{max}$ is the maximum $x$-value, i.e.

$$x_{min} = min\{x \quad | \quad (x,y) \in S\} \quad , \quad x_{max} = max\{x \quad | \quad (x,y) \in S\} \quad .$$

The values $y_{min}$ and $y_{max}$ are defined analogously. We also use the following auxiliary lemma.

**Lemma 1** *For a smooth planar convex region $S$, a given integer $k$ and a real number $r$, the discrete $(0,0)$-moment of the set $(r\cdot S)(k)$ can be expressed as*

$$\mu_{0,0}((r\cdot S)(k)) \quad = \quad P((r\cdot S)(k)) \quad + \quad \frac{1}{2}\cdot \mu_{0,0}(L(r\cdot S,k)) \quad .$$

**Proof.** Let $r\cdot \overline{S}$ be the shape symmetrical to $r\cdot S$, with respect to the line $x=k$. Further, the convex set $r\cdot S\cap r\cdot \overline{S}$ satisfies the conditions of Theorem 1, so the number of digital points belonging to $r\cdot S\cap r\cdot\overline{S}$ can be determined as $\mu_{0,0}(r\cdot S\cap r\cdot \overline{S}) = P(r\cdot S\cap r\cdot\overline{S}) + \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right)$. The statement follows because $(r\cdot S\cap r\cdot\overline{S})$ is symmetrical with respect to the line $x=k$. QED

# 3 Discrete moments of first order

The discrete moments $\mu_{1,0}(r\cdot S)$ and $\mu_{0,1}(r\cdot S)$ may be expressed as follows:

**Theorem 2** *Let $S$ be a convex planar region. Then the following asymptotical expressions hold:*

$$\mu_{1,0}(r\cdot S) \quad = \sum_{\substack{i,j \text{ are integers}\\ (i,j)\in r\cdot S}} i = \iint\limits_{r\cdot S} x\,dxdy + \mathcal{O}\left(r^{\frac{18}{11}+\varepsilon}\right) \quad ,\textit{and}$$

$$\mu_{0,1}(r\cdot S) \quad = \sum_{\substack{i,j \text{ are integers}\\ (i,j)\in r\cdot S}} j = \iint\limits_{r\cdot S} y\,dxdy + \mathcal{O}\left(r^{\frac{18}{11}+\varepsilon}\right) \quad .$$

**Proof.** Let us notice that $\mu_{0,0}(r\cdot S)$ is equal to the number of digital points belonging to the $3D$ set $C$ given by

$$C=\{(x,y,z) \quad | \quad (x,y)\in r\cdot S, \quad 0<z\le x\} \quad = \quad C' \quad \cup \quad C''$$

where $C'$ and $C''$ are defined as follows:

$$C'=\{(x,y,z) \quad | \quad (x,y)\in r\cdot S, \quad 0<z<r\cdot x_{min}\}$$

and

$$C''=\{(x,y,z) \quad | \quad (x,y)\in r\cdot S, \quad r\cdot x_{min}\le z\le x\}.$$

First, consider the number of digital points belonging to the set $C'$. From (4) it follows that

$$\begin{aligned}
\mu_{0,0,0}(C') \quad &= \quad (\lceil r\cdot x_{min}\rceil - 1)\cdot\left(P(r\cdot S)+\mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right)\right) = \\
&= \quad vol(C') - r\cdot x_{min}\cdot P(r\cdot S) + (\lceil r\cdot x_{min}\rceil -1)\cdot\left(P(r\cdot S) \quad + \quad \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right)\right) = \\
&= \quad vol(C') + P(r\cdot S)\cdot(\lceil r\cdot x_{min}\rceil - r\cdot x_{min} - 1) + \mathcal{O}\left(x_{min}\cdot r^{\frac{18}{11}+\varepsilon}\right).
\end{aligned}$$

Now let us calculate the number of digital points belonging to set $C''$. According to Definitions 3 and 4 of the $3D$-sets $V_i$ and $V_i'$ it follows that

$$vol(C'') = \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} (vol(V_i)-vol(V_i')) + (\lceil r\cdot x_{min}\rceil - r\cdot x_{min})\cdot P(r\cdot S) + \mathcal{O}(r) =$$

$$= \quad \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} vol(V_i) - \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} vol(V_i') + (\lceil r\cdot x_{min}\rceil - r\cdot x_{min})\cdot P(r\cdot S) + \mathcal{O}(r) =$$

$$= \quad \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} P((r\cdot S)(i)) - \frac{1}{2}\cdot P(r\cdot S) + (\lceil r\cdot x_{min}\rceil - r\cdot x_{min})\cdot P(r\cdot S) + \mathcal{O}(r) =$$

$$= \sum_{i=\lceil r\cdot x_{min}\rceil}^{\lfloor r\cdot x_{max}\rfloor -1} (\mu_{0,0,0}((r\cdot S)(i)) - \frac{\mu_{0,0}(L(r\cdot S,i))}{2} + \frac{P(r\cdot S)}{2} + (\lceil r\cdot x_{min}\rceil - r\cdot x_{min})\cdot P(r\cdot S) +$$

$$+ \quad \mathcal{O}(r) \quad = \quad (\mu_{0,0,0}(C'') - \frac{\mu_{0,0}(r\cdot S)}{2} - \frac{P(r\cdot S)}{2} + (\lceil r\cdot x_{min}\rceil - r\cdot x_{min})\cdot P(r\cdot S) + \mathcal{O}(r) =$$

$$= \quad \mu_{0,0,0}(C'') - \mu_{0,0}(r\cdot S) \quad + \quad (\lceil r\cdot x_{min}\rceil - r\cdot x_{min})\cdot P(r\cdot S) + \mathcal{O}(r^{\frac{7}{11}+\varepsilon}).$$

Lemma 1, (4) and (5) have been used. That gives

$$\mu_{0,0,0}(C'') = vol(C'') \quad + \quad \mu_{0,0}(r\cdot S) \quad - \quad (\lceil r\cdot x_{min}\rceil - r\cdot x_{min})\cdot P(r\cdot S) + \mathcal{O}(r^{\frac{7}{11}+\varepsilon}) \quad .$$

The sum of $\mu_{0,0,0}(C')$ and $\mu_{0,0,0}(C'')$ is the number of digital points in $C$. It follows

$$\begin{aligned}
\mu_{1,0}(S) \quad &= \quad \mu_{0,0,0}(C) = \mu_{0,0,0}(C') + \mu_{0,0,0}(C'') = vol(C') + vol(C'') + P(r\cdot \mathcal{S}) + \\
&+ \quad P(r\cdot S)\cdot(\lceil r\cdot x_{min}\rceil - r\cdot x_{min} - 1) - P(r\cdot S)\cdot(\lceil r\cdot x_{min}\rceil - r\cdot x_{min}) + \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right) = \\
&= \quad vol(C) + \mathcal{O}\left(x_{min}\cdot r^{\frac{18}{11}+\varepsilon}\right) \quad = \quad vol(C) + \mathcal{O}\left(r^{\frac{18}{11}+\varepsilon}\right) \quad . \quad QED
\end{aligned}$$

# 4 Efficiency in the reconstruction of gravity centers

Now, we are prepared to specify an upper error bound for the reconstruction of the gravity center of a smooth planar convex region. Note that the given estimates below are very sharp and show that a reasonable high resolution ensures a location of the calculated gravity center within an error less than any arbitrary small fraction of the grid edge length $\frac{1}{r}$.

**Theorem 3** *Let $S$ be a smooth planar convex region. Then, for the gravity center position*

$$\left(\frac{1}{r}\cdot\frac{m_{1,0}(r\cdot S)}{m_{0,0}(r\cdot S)}, \quad \frac{1}{r}\cdot\frac{m_{0,1}(r\cdot S)}{m_{0,0}(r\cdot S)}\right)$$

*the following error estimates hold:*

$$\frac{1}{r}\cdot\frac{m_{1,0}(r\cdot S)}{m_{0,0}(r\cdot S)} - \frac{1}{r}\cdot\frac{\sum\limits_{\substack{i,j \text{ are integers}\\(i,j)\in r\cdot S}} i}{\sum\limits_{\substack{i,j \text{ are integers}\\(i,j)\in r\cdot S}} 1} = \mathcal{O}\left(\frac{1}{r^{\frac{15}{11}-\varepsilon}}\right) \quad , and$$

$$\frac{1}{r}\cdot\frac{m_{1,0}(r\cdot S)}{m_{0,0}(r\cdot S)} - \frac{1}{r}\cdot\frac{\sum\limits_{\substack{i,j \text{ are integers}\\(i,j)\in r\cdot S}} i}{\sum\limits_{\substack{i,j \text{ are integers}\\(i,j)\in r\cdot S}} 1} = \mathcal{O}\left(\frac{1}{r^{\frac{15}{11}-\varepsilon}}\right)$$

**Proof.** By using Theorem 2 it follows that

$$\frac{m_{1,0}(r\cdot S)}{m_{0,0}(r\cdot S)} - \frac{\sum\limits_{\substack{i,j \text{ are integers}\\(i,j)\in r\cdot S}} i}{\sum\limits_{\substack{i,j \text{ are integers}\\(i,j)\in r\cdot S}} 1} = \frac{\iint\limits_{r\cdot S} x\,dx\,dy}{\iint\limits_{r\cdot S} dx\,dy} - \frac{\iint\limits_{r\cdot S} x\,dx\,dy \quad + \quad \mathcal{O}\left(r^{\frac{18}{11}+\varepsilon}\right)}{\iint\limits_{r\cdot S} dx\,dy \quad + \quad \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right)} =$$

$$= \frac{\iint\limits_{r\cdot S} x\,dx\,dy \quad\cdot\quad \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right) \quad + \quad \iint\limits_{r\cdot S} dx\,dy \quad\cdot\quad \mathcal{O}\left(r^{\frac{18}{11}+\varepsilon}\right)}{\left(\iint\limits_{r\cdot S} dx\,dy\right)^2 \quad + \quad \iint\limits_{r\cdot S} dx\,dy \quad\cdot\quad \mathcal{O}\left(r^{\frac{7}{11}+\varepsilon}\right)} =$$

$$= \frac{x_{max}\cdot P(S)\cdot\mathcal{O}\left(r^{\frac{40}{11}+\varepsilon}\right) \quad + \quad r^2\cdot P(S)\cdot\mathcal{O}\left(r^{\frac{18}{11}+\varepsilon}\right)}{r^4\cdot P(S) \quad + \quad r^2\cdot P(S)\cdot\left(r^{\frac{18}{11}+\varepsilon}\right)} = \mathcal{O}\left(\frac{1}{r^{\frac{4}{11}-\varepsilon}}\right) \quad . \quad QED$$

## 5 An example and conclusions

We illustrate the obtained results by giving a solution to a problem considered in [6]. The problem is:

*With what accuracy an original circle can be reconstructed from given digital data ?*

In accordance with the paper [6], the answer is that the center position and the diameter of the original circle can be reconstructed with an upper error bound of $\mathcal{O}\left(r^{-\frac{15}{11}+\varepsilon}\right)$. Numerical data strongly confirm the obtained result. A few data are given in Table 1 ([6]), where $S$ is a circle with diameter $2 \cdot R$, while the grid edge length is assumed to be 1.

| $R$ | $a$ | $b$ | $a_{est} - a$ | $b_{est} - b$ |
|---|---|---|---|---|
| 2.5 | 9.4 | 6.2 | -0.03157 | 0.01052 |
| | 282.8 | 33.3 | -0.05001 | -0.04999 |
| | 2444422.4 | 33222.2 | -0.03157 | 0.01052 |
| 23.4 | 44.7 | 31.3 | 0.00041 | -0.00041 |
| | 29992.3 | 313.2 | 0.01144 | -0.05415 |
| | 3111331.3 | 229992.9 | -0.00325 | 0.00708 |
| 234.3 | 2282.4 | 339.9 | 0.00751 | -0.00053 |
| | 229992.9 | 323.2 | -0.00413 | 0.00511 |
| | 2882288.8 | 333322.2 | -0.00391 | 0.00391 |
| 2345.6 | 4424.2 | 3888.8 | -0.00217 | 0.00217 |
| | 299222.9 | 6464.4 | 0.00392 | -0.00594 |
| | 7444774.7 | 333311.3 | 0.00436 | -0.00436 |

Table 1: *Absolute errors between the values of the center coordinates* $(a, b)$ *and their estimations* $(a_{est}, b_{est})$ *from the digitization (on the integer grid) of the circle* $C: (x-a)^2 + (y-b)^2 \leq R^2$.

Moments have been widely used in shape recognition and identification. That is a reason for the ongoing strong interest in the computer vision community in (discrete) moment calculations (we cite [3], just for giving one example).

The results of our paper are based on Huxley's theorem. This is a strong mathematical result which is related to the number of integer points inside of a smooth planar convex curve $\gamma$ and addresses an ancient mathematical problem. Gauss and Dirichlet knew that the area of a region bounded by curve $\gamma$ estimates this number within an order $\mathcal{O}(s)$, where $s$ is the length of curve $\gamma$. The situation when $\gamma$ is a circle is studied most carefully. Huxley's result [1] improves the previously best known upper bound ([2]) even in this case of a circle. From the proof of Theorem 2 it can be concluded that our derived result is sharp up to Huxley's result.

## References


[1] M. N. Huxley: Exponential Sums and Lattice Points. *Proc. London Math. Soc.* 60 (1990) 471–502.

[2] H. Iwaniec, C. J. Mozzochi: On the Divisor and Circles Problems. *J. Number Theory* 29 (1988) 60–93.

[3] X. Jiang, H. Bunke: Simple and Fast Computation of Moments. *Pattern Recognition* 24 (1991) 801–806.

[4] R. Klette: Approximation and representation of 3D objects. *Advances in Digital and Computational Geometry* (R. Klette, A. Rosenfeld, F. Sloboda, eds.), Springer, Singapore (1998) 161–194.

[5] M. Worring, A. W. M. Smeulders: Digitized Circular Arcs: Characterization and Parameter Estimation. *IEEE Trans. PAMI* 17 (1995) 587-598.

[6] J. Žunić, N. Sladoje: A Characterization of Digital Disks by Discrete Moments. *Lecture Notes in Computer Science: Computer Analysis of Images and Patterns*, 1296 (1997) 582-589.