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# Criteria for Differential Equations in Computer Vision

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## Abstract

Differential equations (ODEs or PDEs) appear in many computer vision fields. Shape-from-shading, optical flow, optics, and 3D motion are examples of such fields. This report discusses theoretical criteria for the corresponding continuous problem, theoretical criteria for discrete numerical schemes, and experimental measurements for the implemented numerical schemes. These criteria are illustrated by discussing a shape-from-shading problem in which the reflectance map is linear.

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**Abstract:** Differential equations (ODEs or PDEs) appear in many computer vision fields. Shape-from-shading, optical flow, optics, and 3D motion are examples of such fields. This report discusses theoretical criteria for the corresponding continuous problem, theoretical criteria for discrete numerical schemes, and experimental measurements for the implemented numerical schemes. These criteria are illustrated by discussing a shape-from-shading problem in which the reflectance map is linear.

**Keywords:** Differential equations, evaluation criteria, linear shape-from-shading, well-posedness, convergence and stability

# 1 Introduction

Differential equations (ODEs or PDEs) appear in many computer vision fields. Shape-from-shading, optical flow, optics, and 3D motion are examples of such fields. Solving problems modeled by ODEs and PDEs can be accomplished by finding either an analytical solution, what is in general a difficult task, or by computing a numerical solution to the corresponding discrete scheme. Numerical solutions are usually more easily found with the aid of computer. In this report we consider the case of numerical schemes. The evaluation of numerical solution schemes may be based on

**(i)** theoretical criteria for the corresponding continuous problem,

**(ii)** theoretical criteria for the discrete numerical scheme, or

**(iii)** experimental measurements for the implemented numerical scheme.

Single-scheme analysis as well as a comparative analysis may be performed utilising any of these three types of criteria. *Well-posedness* questions for the corresponding continuous problems may be cited for type **(i)** evaluations. Upon derivation of a specific discrete numerical scheme (either sequential or parallel) *convergence and stability* issues need to be established. Most of the existing numerical schemes in computer vision have been so-far neither supplemented by pertinent convergence and stability analysis nor compared by using an appropriate evaluation procedure. The specification of *domains of influence* is a further example of type **(ii)** evaluations. Experimental analysis may be based on selected performance measures as different *error measures* within the domain of influence.

In this report we illustrate the study of numerical solution schemes for differential equations by discussing a special application. We report on theoretical and experimental results concerning the shape-from-shading problem in which the reflectance map is linear. The significance of this topic and the conclusions stemming out from this work is itemized in the closing section of this report.

We discuss and compare four different two-layer finite-difference based schemes derived for a linear shape-from-shading problem. The schemes are based on the combination of forward- and backward- difference derivative approximations and operate over a rectangular fixed grid. The evaluation analysis is based on both theoretical (convergence, stability, domain of influence) and experimental (performance of implemented schemes) criteria.

## 2 Linear Shape from Shading

Linear shape-from-shading problems arose in the study of the maria of the moon [Horn, 1986, Subsections 10.9 and 11.1.2] and in a local shape-from-shading analysis [Pentland, 1990]. If a small portion of an object surface, described by the graph of a function $u$, having reflectivity properties approximated by a linear reflectance map, is illuminated by a distant light source of unit intensity from direction $(a_1, a_2, -1)$, then the corresponding image function $\mathcal{E}(x_1, x_2)$ satisfies a *linear image irradiance equation* of the form

$$\left(a_1 \frac{\partial u}{\partial x_1}(x_1, x_2) + a_2 \frac{\partial u}{\partial x_2}(x_1, x_2) + 1\right)(a_1^2 + a_2^2 + 1)^{-1/2} = \mathcal{E}(x_1, x_2), \tag{1}$$

over an image domain $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : \mathcal{E}(x_1, x_2) \geq 0\}$. The problem consists in reconstructing the object surface $graph(u)$ based on given irradiance values $\mathcal{E}(x_1, x_2)$ and based on given or calculated light source parameters $a_1$ and $a_2$. These parameters and the grid resolution along the $x_1-$ and $x_2-$ axes characterise a specific *discrete linear shape-from-shading problem.* More general shape-from-shading problems are studied in [Horn, 1986, Klette *et al.*, 1998a, Klette *et al.*, 1998, Sethian, 1996].

Letting $E(x_1, x_2) = \mathcal{E}(x_1, x_2)(a_1^2 + a_2^2 + 1)^{1/2} - 1$, one can rewrite (1) into a transformed linear image irradiance equation

$$a_1 \frac{\partial u}{\partial x_1}(x_1, x_2) + a_2 \frac{\partial u}{\partial x_2}(x_1, x_2) = E(x_1, x_2). \tag{2}$$

In this report we evaluate four finite-difference based schemes derived from (2). Critical to our approach is the assumption that $u$ is given along some (not necessarily smooth) initial curve $\gamma$ in the image domain $\Omega$ or at the boundary of $\Omega$. A prior knowledge of pertinent boundary conditions is essential for other algorithms used in shape from shading and based on methods of *charactersitic strips* or *equal-height contour propagation* [Horn, 1986, Sethian, 1996]. These boundary conditions can be obtained, for example if we combine a single image shape recovery with the *photometric stereo technique* [Klette *et al.*, 1998, Kozera, 1991, Kozera, 1992, Onn and Bruckstein, 1990]. The latter is applicable only over the intersection of multiple images (*e.g.* over $\Omega = \Omega_1 \cap \Omega_2$) and does not require boundary conditions. As a side effect, apart from finding the function $u \in C^2(\Omega)$, missing Dirichlet boundary conditions are also recovered along the boundary $\partial\Omega$. These Dirichlet conditions constitute, in turn, a start-up curve $\gamma$ for each discussed finite-difference scheme to recover the unknown shape over the remaining non-overlapping parts of images (*i.e.* over $\Omega_1 \setminus \Omega$ and $\Omega_2 \setminus \Omega$). Alternatively, in certain cases (when the object is positioned on the plane parallel to $x_1x_2-$plane and has the so-called occluding boundary over $\partial\Omega$), one can also assume that $u_{|\partial\Omega} \equiv const$ along $\partial\Omega$. All presented here schemes provide the numerical solution of the following Cauchy problem:

Object surfaces $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ are considered over a rectangular image domain $\Omega = \{(x_1, x_2) \in \mathbf{R}^2 : -a \leq x_1 \leq a \text{ and } -b \leq x_2 \leq b\}$, with both $a$ and $b$ positive:

$$\begin{aligned} L(u(x_1, x_2)) &= E(x_1, x_2) \\ u(x_1, -b) &= f(x_1) \quad \text{for } -a \leq x_1 \leq a,\ sgn(a_1 a_2) \geq 0, \\ u(-a, x_2) &= g(x_2) \quad \text{for } -b \leq x_2 \leq b; \end{aligned} \tag{3}$$

where $Lu = a_1 u_{x_1} + a_2 u_{x_2}$, and the functions $f \in C([-a, a]) \cap C^2((-a, a))$ and $g \in C([-b, b]) \cap C^2((-b, b))$ satisfy $f(-a) = g(-b)$, $E \in C^2(\overline{\Omega})$, and $a_1$ and $a_2$ are light source parameters such that $(a_1, a_2) \neq (0, 0)$.

The case $sgn(a_1 a_2) \leq 0$ can be treated analogously. For details of this work, the interested reader is referred to [Kozera and Klette, 1997a, Kozera and Klette, 1997b].

## 3 Basic Notions and Theory for Finite-Difference Schemes

At first we recall some basic notions and results from the theory of finite-difference methods applied to PDEs [Van der Houwen, 1968, Chapter 1]. Assume that an interval $I = [-T, T]$ and a domain $G = G_1 \times G_2 \times ... \times G_m \subset \mathbf{R}^m$ (where each $G_i$ is a subinterval of $\mathbf{R}$) together with its boundary $\Gamma$ and $\bar{G} = G \cup \Gamma$

are given and that $(E_0(\bar{G}), \| \|_{E_0})$, $(E(\bar{G}), \| \|_{\bar{G}})$, $(E(\Gamma), \| \|_{\Gamma})$, and $(E(G), \| \|_G)$ are linear normed spaces of scalar or vector-valued functions, defined respectively, on the set of points $\bar{G}$, $\bar{G} \times I$, $\Gamma \times I$, and $G \times I$. Consider now the following problem:

$$U_t(x,t) + \sum_{i=1}^{m} D_i(x,t) U_{x_i}(x,t) = H(x,t),$$
$$U(\Gamma \times I) = \Psi(\Gamma), \ U(x,-T) = U_0(x), \tag{4}$$

where $(x,t) \in G \times I$, the scalar functions $U_0 \in E_0(\bar{G})$, $\Psi \in E(\Gamma)$, and a vector function $F(x,t) = (H(x,t), D(x,t)) \in E(G)$, for $D(x,t) = (D_1(x,t), \ D_2(x,t), \ldots, D_n(x,t))$. A problem of finding the inverse of a given mapping $\mathcal{L} : D_{\mathcal{L}} \to \Delta_{\mathcal{L}}$ of an unknown function $U \in D_{\mathcal{L}} = (E(\bar{G}), \| \|_{\bar{G}})$ onto a known element $(U_0, F, \Psi) \in \Delta_{\mathcal{L}} = (E_0(\bar{G}) \times E(G) \times E(\Gamma), \| \|_{\times})$, where $\|(U_0, F, \Psi)\|_{\times} = \|U_0\|_{E_0} + \|F\|_G + \|\Psi\|_{\Gamma}$, is called *an initial boundary value problem.*

**Definition 1** *An initial boundary value problem $\mathcal{L}U = (U_0, F, \Psi)$ is said to be* well-posed *with respect to norms in $E(\bar{G})$ and in $E_0(\bar{G}) \times E(G) \times E(\Gamma)$ if $\mathcal{L}$ has a unique inverse $\mathcal{L}^{-1}$ which is continuous at the point $(U_0, F, \Psi)$.*

Now we introduce *uniform grid sequence.* We replace the continuous interval $I = [-T, T]$ by a discrete set of points $[t_0 = -T, t_1, t_2, \ldots, \ t_M = T]$, where $t_{k+1} - t_k = \Delta t$, for each $k \in [0, \ldots, M-1]$, and $M\Delta t = 2T$. Furthermore assume a finite set of points $\Gamma_{\Delta t} \subset \Gamma$ and of points $G_{\Delta t} \subset G$ such that the fixed distance $\Delta x_i$, for $i \in [1, \ldots, m]$, between two consecutive points in the $x_i-$axis direction satisfies $\Delta x_i = \mathcal{A}_i \Delta t$, where $\mathcal{A}_i$ is a scaling factor such that, for some integer $N_i$, we have $N_i \mathcal{A}_i \Delta t = \mu(G_i)$, where $\mu(G_i)$ denotes the measure of $G_i$.

These three sets $\{t_k\}_{k=0}^{M}$, $G_{\Delta t}$, and $\Gamma_{\Delta t}$ of points constitute *a grid* $Q_{\Delta t}$ in $\bar{G} \times I$, *i.e.* $Q_{\Delta t} = \bar{G}_{\Delta t} \times \{t_k\}_{k=0}^{M}$, where $\bar{G}_{\Delta t} = G_{\Delta t} \cup \Gamma_{\Delta t}$. We assume that *a sequence of grids* $Q_{\Delta t}$ is defined in such a way that $\{Q_{\Delta t}\}$ is *dense* in $\bar{G} \times I$. The last requirement is satisfied when $\lim_{\Delta t \to 0^+} \mathcal{N}_i \mathcal{A}_i \Delta t = 0$ (for each $i \in [1, \ldots, m]$). Furthermore, we introduce the corresponding normed grid spaces

$$(E_0(\bar{G}_{\Delta t}), \| \|_{E_{0\Delta t}}), \quad (E(\bar{G}_{\Delta t}), \| \|_{\bar{G}_{\Delta t}}), \quad (E(\Gamma_{\Delta t}), \| \|_{\Gamma_{\Delta t}}), \quad (E(G_{\Delta t}), \| \|_{G_{\Delta t}}) \tag{5}$$

defined on the sets $\bar{G}_{\Delta t}$, $\bar{G}_{\Delta t} \times \{t_k\}_{k=0}^{M}$, $\Gamma_{\Delta t} \times \{t_k\}_{k=0}^{M}$, and $G_{\Delta t} \times \{t_k\}_{k=0}^{M}$, respectively.

The elements of these spaces are called *grid functions* and are denoted by lower case letters $u_0$, $u$, $\psi$, and $f$.

A mapping $\mathcal{R}_{\Delta t}$ of an unknown grid function $u$ of $(E(\bar{G}_{\Delta t}), \| \|_{\bar{G}_{\Delta t}})$ into the known element $(u_0, f, \psi)$ of $(E_0(\bar{G}_{\Delta t}) \times E(G_{\Delta t}) \times E(\Gamma_{\Delta t}), \| \|_{\Delta t_{\times}})$, where $\|(u_0, f, \psi)\|_{\Delta t_{\times}} = \|u_0\|_{E_{0\Delta t}} + \|f\|_{G_{\Delta t}} + \|\psi\|_{\Gamma_{\Delta t}}$ is defined for each grid $Q_{\Delta t}$, is called *a finite-difference scheme.*

Difference schemes can be described by the equation $\mathcal{R}_{\Delta t} u = (u_0, f, \psi)$, with the domain and range of $\mathcal{R}_{\Delta t}$ denoted by $D_{\mathcal{R}\Delta t}$ (called as *a discrete domain of influence*) and $\Delta_{\mathcal{R}\Delta t}$, respectively. It is assumed that both $D_{\mathcal{R}\Delta t}$ and $\Delta_{\mathcal{R}\Delta t}$ are linear spaces and $\mathcal{R}_{\Delta t}$ has a unique inverse $\mathcal{R}_{\Delta t}^{-1}$, which is continuous in $D_{\mathcal{R}\Delta t}$ for every $\Delta t \neq 0$.

**Definition 2** *For a given initial boundary value problem, a grid sequence and an associated finite-difference scheme we define that a set $D_I \subset \Omega$ is called* domain of influence, *where $D_I = cl\,(\bigcup D_{\mathcal{R}\Delta t})$.*

Let us now introduce the *discretisation operator* $[\ ]_{d(\Delta t)}$ which transforms a function $U \in E(\bar{G})$ to its discrete analogue $[U]_{d(\Delta t)}$ defined as $U$ reduced to the domain of the grid $Q_{\Delta t}$. In the same manner we define discretised elements $[U_0]_{d(\Delta t)} \in E_0(\bar{G}_{\Delta t})$, $[F]_{d(\Delta t)} \in E(G_{\Delta t})$, and $[\Psi]_{d(\Delta t)} \in E(\Gamma_{\Delta t})$. In this report we use the convention: $[U]_{d(\Delta t)} = u$, $[U_0]_{d(\Delta t)} = u_0$, $[F]_{d(\Delta t)} = f$, and $[\Psi]_{d(\Delta t)} = \psi$, where $f = (h, d)$. Moreover, it is also assumed that the norms on the grid sequence $\{Q_{\Delta t}\}$ *match* the corresponding norms from the related *continuous spaces i.e.*

$$\|u\|_{\bar{G}_{\Delta t}} \to \|U\|_{\bar{G}}, \quad \|u_0\|_{E_{0\Delta t}} \to \|U_0\|_{E_0}, \quad \|f\|_{G_{\Delta t}} \to \|F\|_G, \quad \|\psi\|_{\Gamma_{\Delta t}} \to \|\Psi\|_{\Gamma} \tag{6}$$

as $\Delta t \to 0$.

Now we introduce the evaluation criteria for numerical solution schemes. Assume that $\widetilde{U}$ is a solution to the initial boundary value problem $\mathcal{L}\widetilde{U} = (U_0, F, \Psi)$, and that $u$ is a solution to the corresponding discrete problem

$$\mathcal{R}_{\Delta t} u = (u_0, f, \psi). \tag{7}$$

If $\mathcal{R}_{\Delta t}$ is to be a "good approximation" of $\mathcal{L}$ we expect that the function $\widetilde{u} = [\widetilde{U}]_{d(\Delta t)}$, for some element $(\widetilde{u}_0, \widetilde{f}, \widetilde{\psi})$, satisfies a finite-difference equation $\mathcal{R}_{\Delta t}\widetilde{u} = (\widetilde{u}_0, \widetilde{f}, \widetilde{\psi})$ which closely relates to (7).

**Definition 3** *The value* $\|[\mathcal{L}\widetilde{U}]_{d(\Delta t)} - \mathcal{R}_{\Delta t}\widetilde{u}\|_{\Delta t_\times}$ *is called the* error of the approximation*, whereas the value* $\|u - \widetilde{u}\|_{\bar{G}_{\Delta t}}$ *is, in turn, called* the discretisation error*.*

**Definition 4** *We say that a difference scheme is* consistent *with an initial boundary value problem if the error of approximation converges to zero as* $\Delta t \to 0$*.*

**Definition 5** *We say that a difference scheme is* convergent *to the solution* $u$ *(if it exists) if the discretisation error converges to zero as* $\Delta t \to 0$*.*

**Definition 6** *We say that a linear finite-difference scheme is* R-F stable *(this definition follows Rjabenki and Filippov) if operators* $\{\mathcal{R}_{\Delta t}^{-1}\}$ *are uniformly bounded as* $\Delta t \to 0$*.*

Combining the Definitions 5.3 and 6.2 with the Theorem 5.1 in [Van der Houwen, 1968, Chapter 1] we have the following:

**Theorem 1** *A consistent and R-F stable finite-difference scheme is convergent to the solution of* $\mathcal{L}\widetilde{U} = (U_0, F, \Psi)$*, if such a solution exists.*

Of course, for a Cauchy problem (3) (with $a_2 \neq 0$), we have $m = 1$, $I = [-b, b]$, $x_2 = t$, $x_1 = x$, $G = (-a, a)$, $\Gamma = \{-a, a\}$ $U_0(x_1) = f(x_1)$, $\Psi(\Gamma \setminus \{a\}) = g(x_2)$, $H(x_1, x_2) = (1/a_2)E(x_1, x_2)$, and $D_1(x_1, x_2) = (a_1/a_2)$. If in turn, $a_2 = 0$ then the parameter $t$ is assigned to $x_1-$variable and further analysis is analogous to the preceding case. The continuous and discrete normed spaces defined above, are assumed to be equipped here with standard *maximum norms* $\| \|_\infty$ clearly satisfying the *compatibility conditions* (6) [Kozera and Klette, 1997a].

# 4 Evaluation of Different Finite-Difference Based Schemes

In this section we consider the problem (3) over a rectangular domain $\Omega$ with $a_2 \neq 0$. We assume a uniform grid $Q_{\Delta x_2}$ with $N_1 = M$, $\Delta x_2 = (2b/M)$ and $\Delta x_1 = (2a/M) = \mathcal{A}_1 \Delta x_2$, where $M \in [0, 1, \ldots, \infty]$ and $\mathcal{A}_1 = a/b$. It follows that $((a_1 \Delta x_2)\ /(a_2 \mathcal{A}_1)\Delta x_2)) = const$. In addition, we assume that a function $u$ is a $C^2$ solution to (2), and lastly that problem (3) is *well-posed* [Kozera and Klette, 1997a]. Note that $a_1$ and $a_2$ are the model parameters (light source parameters) of the linear problem.

## 4.1 Forward-Forward Finite-Difference Scheme

Applying forward-difference derivative approximations together with Taylor's formula yields

$$\left.\frac{\partial u}{\partial x_1}\right|_j^n = \frac{u_{j+1}^n - u_j^n}{\Delta x_1} + O(\Delta x_1) \quad \text{and} \quad \left.\frac{\partial u}{\partial x_2}\right|_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta x_2} + O(\Delta x_2), \tag{8}$$

for any $j, n \in \{1, \ldots, M-1\}$. Here $u_j^n$, $\left.\frac{\partial u}{\partial x_1}\right|_j^n$, and $\left.\frac{\partial u}{\partial x_2}\right|_j^n$ denote the values of $u$, $\frac{\partial u}{\partial x_1}$, and $\frac{\partial u}{\partial x_2}$, respectively, at the point $(x_{1j}, x_{2n})$ in the grid. $\Delta x_1$ and $\Delta x_2$ denote the distances between grid points in the respective directions. $M$ denotes the grid resolution. By substituting (8) into (2) at each grid point $(x_{1j}, x_{2n})$, we get

$$a_1 \frac{u_{j+1}^n - u_j^n}{\Delta x_1} + a_2 \frac{u_j^{n+1} - u_j^n}{\Delta x_2} + O(\Delta x_1, \Delta x_2) = E_j^n. \tag{9}$$

Denoting by $v$ an approximate of $u$, we obtain from (9) the following sequential two-level finite-difference *explicit* scheme

$$v_j^{n+1} = \left(1 + \frac{a_1 \Delta x_2}{a_2 \Delta x_1}\right) v_j^n - \frac{a_1 \Delta x_2}{a_2 \Delta x_1} v_{j+1}^n + \frac{\Delta x_2}{a_2} E_j^n, \tag{10}$$

with $j, n \in \{1, \ldots, M-1\}$. The following result holds [Kozera and Klette, 1997a]:

**Theorem 2** *Let $\alpha = (a_1 \Delta x_2)(a_2 \Delta x_1)^{-1}$ be a fixed constant. Then, numerical scheme* (10) *is R-F stable, if and only if $-1 \le \alpha \le 0$.*

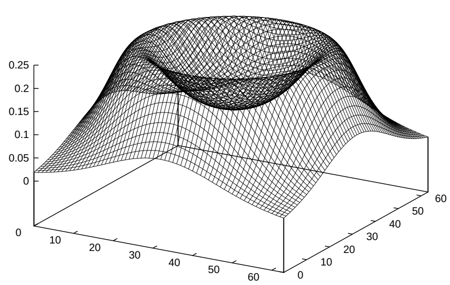
Consequently (by Theorem 1), for $-1 \le \alpha \le 0$, the sequence of functions $\{u_{\Delta x_2}\}$ (where each $u_{\Delta x_2}$ is a solution of (10) with $\Delta x_2$ temporarily fixed) is convergent to the solution of the Cauchy problem (3), while $\Delta x_2 \to 0$.

As mentioned before, given an initial boundary value problem (3), the scheme (10) recovers the unknown shape over a domain of influence $D_I$ which, for $a_1 \neq 0$ and $N_1 = M$, coincides with

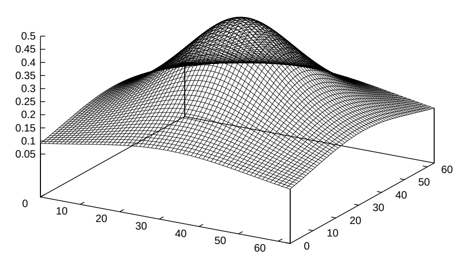
$$\Delta = \{(x_1, x_2) \in \mathbb{R}^2 : -a \le x_1 \le a, \quad \text{and} \quad -b \le x_2 \le (-b/a)x_1\}, \tag{11}$$

and for $a_1 = 0$ with the entire $\bar{\Omega}$. The scheme (10) has been tested for $a = b = \sqrt{2}$, with grid resolution $N_1 = M = 64$, $\Delta x_1 / \Delta x_2 = 1.0$, $a_1 = -0.5$, and $a_2 = 1.0$, and therefore with $\alpha = -0.5$.

A *volcano-like surface* represented by the graph of the function $u_v(x,y)$= $(1/(4(1 + (1 - x^2 - y^2)^2)))$ (see Fig. 1a) and a *mountain-like surface* represented by the graph of the function $u_m(x,y) = (1/(2(1 + x^2 + y^2)))$ (see Fig. 1b) were taken as test surfaces.




(a)




(b)

Figure 1: (a) The graph of the function $u_v(x,y) = (1/(4(1+(1-x^2-y^2)^2)))$ being a volcano-like surface. (b) The graph of the function $u_m(x,y) = (1/(2(1+x^2+y^2)))$ being a mountain-like surface.

The absolute errors between heights of the ideal and computed surfaces are presented in Fig. 2. For a value $\alpha \notin [-1, 0]$ an implementation of the numerical scheme (10) results in *instability* of (10), see [Kozera and Klette, 1997a].

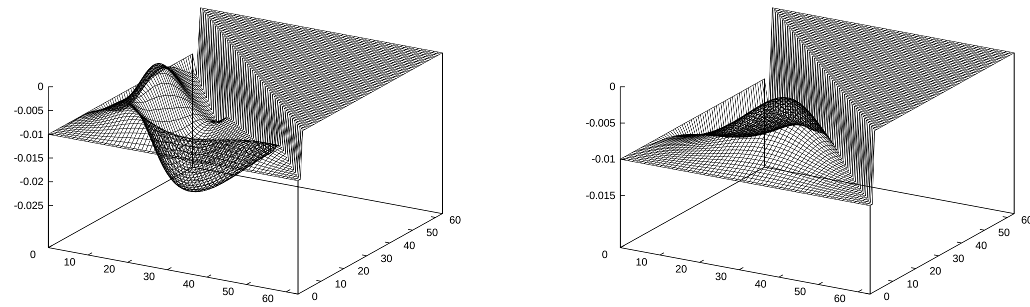
## 4.2 Backward-Forward Finite-Difference Scheme

Applying now a backward-difference derivative approximation to $u_{x_1}$

$$\left.\frac{\partial u}{\partial x_1}\right|_j^n = \frac{u_j^n - u_{j-1}^n}{\Delta x_1} + O(\Delta x_1),$$

and a forward-difference derivative approximation to $u_{x_2}$ leads to the corresponding two-level explicit finite-difference scheme

$$v_j^{n+1} = \left(1 - \frac{a_1 \Delta x_2}{a_2 \Delta x_1}\right) v_j^n + \frac{a_1 \Delta x_2}{a_2 \Delta x_1} v_{j-1}^n + \frac{\Delta x_2}{a_2} E_j^n, \tag{12}$$




(a) (b)

Figure 2: (a) The absolute error between volcano-like and computed surface for the forward-forward scheme. (b) The absolute error between mountain-like and computed surface for the forward-forward scheme.

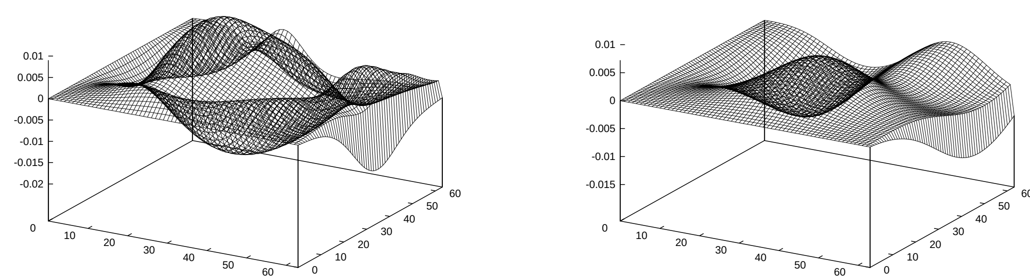
with $j, n \in \{1, \ldots, M-1\}$. The following stability and convergence result for the above finite-difference scheme holds [Kozera and Klette, 1997a]:

**Theorem 3** *Let $\alpha = (a_1 \Delta x_2)(a_2 \Delta x_1)^{-1}$ be a fixed constant. Then, numerical scheme* (12) *is R-F stable, if and only if $0 \leq \alpha \leq 1$.*

Consequently (by Theorem 1), for $0 \leq \alpha \leq 1$, the sequence of functions $\{u_{\Delta x_2}\}$ (where each $u_{\Delta x_2}$ is a solution of (12) with $\Delta x_2$ temporarily fixed) is convergent to the solution of the Cauchy problem (3), while $\Delta x_2 \to 0$.

As easily verified, the domain of influence $D_I$ of scheme (12) coincides with $\bar{\Omega}$, for arbitrary $\alpha$. Thus, assuming the goal of global shape reconstruction, it is clear that (12) provides a better reconstruction opposed to (10).

The scheme (12) has been tested for the same shapes as in the previous case. With $a = b = \sqrt{2}$, grid resolution $N_1 = M = 64$, $\Delta x_1 / \Delta x_2 = 1.0$, $a_1 = 0.5$, $a_2 = 1.0$, and thus with $\alpha = 0.5$, the absolute errors between heights of the ideal and computed surfaces are presented in Fig. 3.




(a) (b)

Figure 3: (a) The absolute error between volcano-like and computed surface for the backward-forward scheme. (b) The absolute error between mountain-like and computed surface for the backward-forward scheme.

## 4.3 Forward-Backward Finite-Difference Scheme

Applying now a forward-difference derivative approximation to $u_{x_1}$ and a backward-difference derivative approximation to $u_{x_2}$ leads to the following two-level explicit and *horizontal* finite-difference scheme:

$$v_{j+1}^n = \left(1 - \frac{a_2 \Delta x_1}{a_1 \Delta x_2}\right) v_j^n + \frac{a_2 \Delta x_1}{a_1 \Delta x_2} v_j^{n-1} + \frac{\Delta x_1}{a_1} E_j^n, \tag{13}$$

(for $a_1 \neq 0$), or otherwise to the following *vertical* two-level explicit scheme:

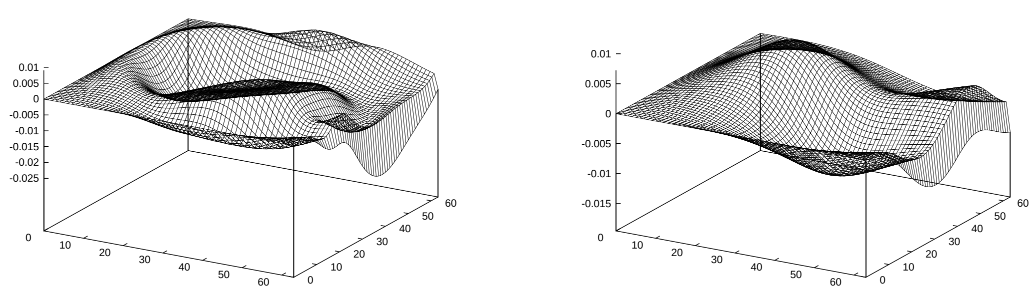
$$v_j^n = v_j^{n-1} + \frac{\Delta x_2}{a_2} E_j^n, \tag{14}$$

with $j, n \in \{1, \ldots, M-1\}$. Observe that for the scheme (13) the role of increment step $\Delta t$ is played by $\Delta x_1$, if an implicit scheme is not considered. Clearly, the shape reconstruction proceeds now sequentially along the $x_1-$axis direction (opposite to the previous cases). In a natural way, the boundary condition is represented by the function $f(x_1)$ and the corresponding initial condition by the function $g(x_2)$. We present now the next convergence result for the schemes (13) and (14) [Kozera and Klette, 1997a]:

**Theorem 4** *Let* $\tilde{\alpha} = (a_2 \Delta x_1)(a_1 \Delta x_2)^{-1}$ *be a fixed constant. Then, numerical scheme* (13) *is R-F stable, if and only if* $0 \leq \tilde{\alpha} \leq 1$. *Moreover, the numerical scheme* (14) *is unconditionally R-F stable.*

Consequently (by Theorem 1), for $0 \leq \tilde{\alpha} \leq 1$, the sequence of functions $\{u_{\Delta x_1}\}$, where each $u_{\Delta x_1}$ is a solution of (13) with $\Delta x_1$ temporarily fixed, is convergent to the solution of the Cauchy problem (3), while $\Delta x_1 \to 0$. Moreover, the sequence of computed solutions $\{u_{\Delta x_2}\}$ to (14) converges to the solution of the corresponding Cauchy problem (3), while $\Delta x_2 \to 0$.

For both schemes the respective domains of influence $D_I$ coincide with $\bar{\Omega}$. We discuss here only the performance of the scheme (13). It has been tested for the same sample surfaces as in the previous cases. With $a = b = \sqrt{2}$, grid resolution $N_1 = M = 64$, $\Delta x_1 / \Delta x_2 = 1.0$, $a_1 = 1.0$, $a_2 = 0.5$, and thus with $\tilde{\alpha} = 0.5$, the absolute errors between heights of the ideal and computed surfaces are presented in Fig. 4.



(a) (b)

Figure 4: (a) The absolute error between volcano-like and computed surface for the forward-backward scheme. (b) The absolute error between mountain-like and computed surface for the forward-backward scheme.

## 4.4 Backward-Backward Finite-Difference Scheme

Applying now backward-difference derivative approximations for both derivatives $u_{x_1}$ and $u_{x_2}$ we arrive at the following two-level *implicit* scheme:

$$v_j^n = \frac{1}{1+\alpha} v_j^{n-1} + \frac{\alpha}{1+\alpha} v_{j-1}^n + \frac{\Delta x_2}{a_2(1+\alpha)} E_j^n \tag{15}$$

(for $\alpha \neq -1$), or otherwise at the following two-level *explicit* scheme:

$$v_{j-1}^{n} = u_{j}^{n-1} + \frac{\Delta x_2}{a_2} E_j^n, \tag{16}$$

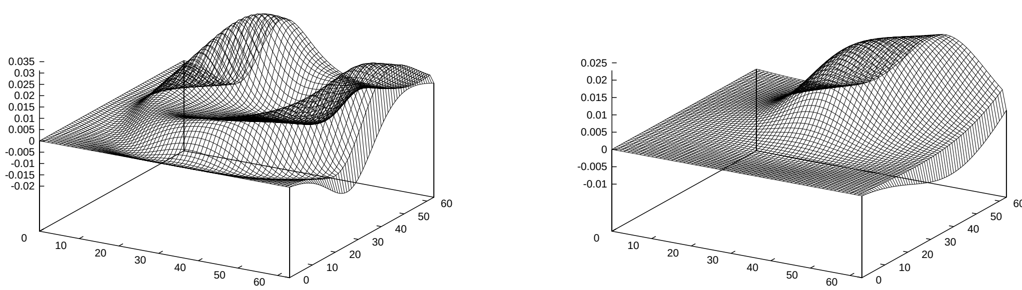
with $j, n \in \{1, \ldots, M-1\}$ and $\alpha = (a_1 \Delta x_2 / a_2 \Delta x_1)$.

It is clear that, as opposed to the last subsection, (15) cannot be reduced to the explicit iterative form by a mere change of the recovery direction. However, this can be achieved by using implicit approach [Kozera and Klette, 1997a, Subsections 3.4]. The following result for schemes (15) and (16) holds [Kozera and Klette, 1997a]:

**Theorem 5** *Let $\alpha = (a_1 \Delta x_2)(a_2 \Delta x_1)^{-1}$ be a fixed constant. Then, numerical scheme* (15) *is R-F stable, if and only if $\alpha \geq 0$. Moreover, the numerical scheme* (16) *is unconditionally R-F stable.*

Consequently (by Theorem 1), for $\alpha \geq 0$, the sequence of functions $\{u_{\Delta x_2}\}$, where each $u_{\Delta x_2}$ is a solution of (15) with $\Delta x_2$ temporarily fixed, is convergent to the solution of the Cauchy problem (3), while $\Delta x_2 \to 0$. Moreover, the sequence of computed solutions $\{u_{\Delta x_2}\}$ to (16) converges to the solution of the corresponding Cauchy problem (3), while $\Delta x_2 \to 0$.

The corresponding domain of influence $D_I$ for the scheme (15) covers the entire $\bar{\Omega}$, whereas for the scheme (16) coincides with (11). The scheme (15) has been tested for sample shapes as in the previous cases. With $a = b = \sqrt{2}$, grid resolution $N_1 = M = 64$, $\Delta x_1 / \Delta x_2 = 1.0$, $a_1 = 0.5$, $a_2 = 1.0$, and thus with $\alpha = 0.5$, the absolute errors between heights of the ideal and computed surfaces are presented in Fig. 5.




(a) (b)

Figure 5: (a) The absolute error between volcano-like and computed surface for the backward-backward scheme. (b) The absolute error between mountain-like and computed surface for the backward-backward scheme.

## 5 Conclusions

Four two-layer finite-difference based schemes are discussed in this report and the experimental results for two sample surfaces, a volcano-like and a mountain-like surface, are presented. *Convergence, stability, domain of influence,* and *maximum relative errors* are considered here as algorithmic features used as evaluating criteria. The table below collates both theoretical and experimental results. The corresponding *relative errors* listed in the last two columns (for the volcano-like and the mountain-like surfaces) represent the maximum of the ratio of the difference of heights between computed and ideal surfaces devided by the height of the ideal surface (volcano and mountain-like, respectively), for grid points within the domain of influence.

| Scheme | Stab./Conv. | Influence Domain | Error-V | Error-M |
|---|---|---|---|---|
| f-f | $-1 \leq \alpha \leq 0$ | $D_I = \Delta$ | $\leq 10\%$ | $\leq 3\%$ |
| b-f | $0 \leq \alpha \leq 1$ | $D_I = \Omega$ | $\leq 6\%$ | $\leq 2\%$ |
| f-b | $0 \leq \alpha^{-1} \leq 1$ | $D_I = \Omega$ | $\leq 8\%$ | $\leq 3\%$ |
| b-b | $0 \leq \alpha$ | $D_I = \Omega$ | $\leq 14\%$ | $\leq 5\%$ |

Finally, we itemize a few aspects of the presented results:

- In choosing a proper scheme theoretical criteria such as *stability, convergence* and *domain of influence*, or experimental criteria such as *relative error* can be used as evaluation criteria. For experimental errors for optical flow calculations (another case of applying numerical schemes for solving differential equations) see also [Klette *et al.*, 1998].
- *Stability and convergence* are intrinsic properties of a given scheme.
- The *domain of influence* depends on the choice of a given scheme, the geometry of $\Omega$ and available Dirichlet boundary conditions.
- A *complete convergence and stability analysis* of all considered schemes is reported in this report (as opposed to approach [Horn, 1986] or flawed results [Pentland, 1990]). Stability analysis provides also means to discuss noisy camera-captured input images [Kozera and Klette, 1997a].
- *Well-posedness* of the corresponding continuous Cauchy problem (3) is also established. For a complete proof see [Kozera and Klette, 1997a].
- As opposed to the classical base characteristic strips method [John, 1971] applied in computer vision by [Horn, 1986], all *two-layer schemes* introduced here operate on *fixed rectangular grid. Three-layer schemes* can also be investigated in future research.
- A *linear model of reflectance maps* can be applied to the satellite image interpretation or to local shading analysis [Horn, 1986, Pentland, 1990].
- The *linear case* helps to understand *a non-linear case.* Finite-difference schemes can also be applied to the *non-linear PDEs* [Rosinger, 1982] and therefore to *non-linear reflectance maps.*
- The single image finite-difference technique can be combined with the multiple image photometric stereo technique, if Dirichlet conditions are not *a priori* available (see Section 2).

A real image $\Omega$ may possess *invisible surface area i.e.* $\Omega_{black} \subset \Omega$, where $\mathcal{E} \equiv 0$. This work deals exclusively with the simulated images defined globally over $\Omega$. In particular, negative values of image function $\mathcal{E}$ over $\Omega_{black}$, were considered to be admissible according to the formula (1). If however, the simulated image function $\mathcal{E}$ is pre-defined as vanishing, whenever $a_1 u_{x_1} + a_2 u_{x_2} + 1 < 0$, the corresponding domain of influnece $D_I$ is clearly diminished. It depends no longer exclusively on a given finite-difference scheme, the geometry of $\Omega$ and the corresponding Dirichlet boundary conditions, but also on the choice of the specific illumination direction as well as the surface $graph(u)$. Stability and convergence results remain unchanged.

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