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Measurements of Arc Length's by Shortest Polygonal Jordan Curves

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Abstract

Arc length's may be measured by numeric integration if the curves are given by analytic formulas. This report deals with an alternative way. A shortest polygonal Jordan curve may be used to estimate arc length's. This theoretically known result is illustrated in this report. Measured convergence of error values is discussed for a few examples of curves. A new error measure (approximation effectiveness) is proposed and discussed.

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Abstract: Arc length's may be measured by numeric integration if the curves are given by analytic formulas. This report deals with an alternative way. A shortest polygonal Jordan curve may be used to estimate arc length's. This theoretically known result is illustrated in this report. Measured convergence of error values is discussed for a few examples of curves. A new error measure (approximation effectiveness) is proposed and discussed.

Keywords: Multigrid convergence, arc length, shortest polygonal Jordan curve, yin-yang symbol

1 Introduction

A few moments in the long history of parametric and topological definitions of planar curves are discussed in [Sloboda et al., 1992, Sloboda et al., 1998a]. The definition of curves was actually one of the harder problems in mathematics.

We start with parametric curves. A *planar Jordan curve* γ is given by a parameterized path $\phi : [a, b] \rightarrow \mathbf{R}^2$ with $a \neq b$, $\phi(a) = \phi(b)$, and $\phi(s) \neq \phi(t)$, for all $a \leq s < t < b$. It holds that

$$\gamma = \{(x, y) : \phi(t) = (x, y) \wedge a \leq t \leq b\}. \tag{1}$$

A *planar Jordan arc* γ is defined by a subinterval $[c, d]$ with $a \leq c < d \leq b$. A Jordan curve is also called a *closed Jordan arc*. A *rectifiable Jordan arc* γ has a bounded *arc length*

$$d(\gamma) = \sup_{\{t_0,\ldots,t_n\}:c=t_1<\ldots<t_n=d} \sum_{i=1}^{n} \|\phi(t_i) - \phi(t_{i-1})\| \quad < \ \infty. \tag{2}$$

Initially C. Jordan defined 1883 a curve γ in parametric form as

$$\gamma = \{(x, y) : x = \alpha(t) \wedge y = \beta(t) \wedge 0 \leq t \leq b\}. \tag{3}$$

However, G. Peano constructed in 1890 a curve satisfying equation (3) and filling the whole unit square. See [Skarbek 1992] for applications of this *Peano curve*. Equation (1), due to C. Jordan in 1893, may be prefered to exlude the Peano curve. However, for analytical arc length calculations equation (3) is in common use, and the arc length is equal to

$$d(\gamma) = \int_c^d \sqrt{\frac{d\alpha(t)}{dt}^2 + \frac{d\beta(t)}{dt}^2}\, dt, \tag{4}$$

assuming C^1-functions α and β.

These definitions are relevant for curves which possess parametric forms. Not all curves possess parametric forms, and some may have a parametric form but it may remain a problem to find out about it. Especially in image analysis we have to deal with curves which are given in digitized pictorial form, and where a parametric description is typically not a final goal of the analysis process (see, e.g., the yin-yang symbol in Fig. 1). Here, topological approaches for curve definitions may be more relevant in general.

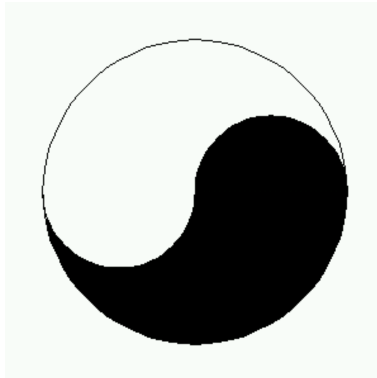

Figure 1: In Chinese philosophy and religion the two principles of yin and yang interact to influence all creatures' destinies. The dark portion, yin, represents the feminine aspect, and the light portion, yang, represents the masculine. The composition of this symbol is based on two circles touching at the center of a larger circle. We study the border of yin.

A *planar Cantor curve* γ is defined to be a connected compact set in $\mathbf{R}^2$ which does not possess internal points. Such a planar Cantor curve is a continouos image of a straight line segment if and only if each point of the set possesses an arbitrary small connected neighborhood within the set. More details about the history of topological curve definitions may be found in [Sloboda et al., 1992, Sloboda et al., 1998a].

It is one of the main problems in regular grid based computations to introduce and study curves especially for the case of discrete (i.e. measurements are only given at grid point positions) and regular (i.e. regular grid in two-dimensional space) data [Sloboda et al., 1998]. This cited paper deals with *simple closed one-dimensional grid continua* specifying a topological definition of curves in the orthogonal grid, and with *simple open one-dimensional grid continua* specifying a topological definition of arcs in the orthogonal grid. The *length* of simple one-dimensional grid continua in $\mathbf{R}^2$ is identified with the length of a shortest (polygonal) Jordan curve in a polygonally bounded compact set (case of a closed continua) or with the length of a geodesic diameter of a polygon (case of an open continua).

In this report we illustrate the use of a shortest Jordan curve algorithm as a tool for calculating the length of given curves or arcs. For example, we apply the algorithm to calculate the classical π value (Archimedes approximated π based on regular n-gons, $n \leq 96$). Another application is the approximation of the yin boundary in the yin-yang symbol. Furthermore we apply the algorithm to calculate the length for an oscillation curve (there exists approximation formulas for this curve).

2 Definitions

The arc length of a curve γ will be approximated based on digitizations at different grid resolution r. A curve is considered to be the boundary of a *2D region* (i.e. a simply-connected compact set in $\mathbf{R}^2$ having interior points). The digitization of such a 2D region Θ leads to digital representations of its boundary $\gamma = \partial\Theta$.

Let $\mathbf{Z}_r = \{m \cdot 2^{-r} : m \in \mathbf{Z}\}$, where $\mathbf{Z}$ is the set of integers and $r = 0, 1, 2, ...$ specifies the *grid constant* 2^{-r}. The set $\mathbf{Z}_r^2$ is the set of all *r-grid points* in two-dimensional Euclidean space $\mathbf{R}^2$. Each r-grid point (x_1, x_2) defines a topological unit (a *grid square*)

$$\mathbf{C}_r(x_1, x_2) = \{(y_1, y_2) : (x_i - .5)2^{-r} \leq y_i \leq (x_i + .5)2^{-r}, i = 1, 2\}.$$

Digitization of 2D regions is defined with respect to these topological units, see [Klette, 1985].

Grid square inclusion (with respect to the topological interior of a given 2D region) and *grid square intersection* digitizations are assumed for 2D regions having interior points. Grid square inclusion digitization defines the *inner interior* $I_r^-(\Theta)$ of a given 2D region Θ, and grid square intersection digitization defines

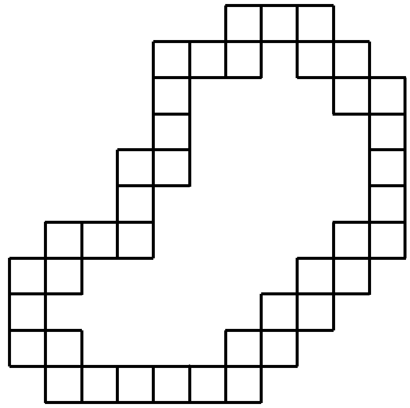

Figure 2: Simple closed planar one-dimensional grid continuum.

the *outer interior* $I_r^+(\Theta)$. The resulting isothetic polygons can be described as being grid continua, see, e.g., [Sloboda et al., 1998]. A *planar grid continuum* is an edge-connected finite union of topological units $\mathbf{C}_r$. An *isothetic polygonal Jordan curve* is a polygonal Jordan curve whose edges are parallel either with the x- or with the y-axis. An *isothetic polygon* is a polygon whose boundary is an isothetic polygonal Jordan curve.

We explain the arc length measurement approach introduced in [Sloboda et al., 1992]. It is a special approach towards global curve approximations. Assume that both the inner interior $I_r^-(\Theta)$ and the outer interior $I_r^+(\Theta)$ of a given 2D region Θ are edge-connected sets with respect to the given grid constant 2^{-r} (which defines the length of a grid edge). Assume $I_r^-(\Theta) \neq \emptyset$. Let $C_r^-(\Theta)$ be the boundary of the isothetic polygon $I_r^-(\Theta)$, and let $C_r^+(\Theta)$ be the boundary of the isothetic polygon $I_r^+(\Theta)$. Then it holds that

$$\emptyset \subset I_r^-(\Theta) \subset I(I_r^+(\Theta)) \quad \text{and} \quad C_r^-(\Theta) \cap C_r^+(\Theta) = \emptyset,$$

and $I_r^+(\Theta) \setminus I(I_r^-(\Theta))$ is an isothetic polygon.

Furthermore, let d_∞ be the Hausdorff-Chebyshev metric for sets A, B of points in $\mathbf{R}^2$, i.e.,

$$d_\infty(A,B) = max\{\max_{\mathbf{p}\in A} \min_{\mathbf{q}\in B} d_\infty(\mathbf{p},\mathbf{q}), \max_{\mathbf{p}\in B} \min_{\mathbf{q}\in A} d_\infty(\mathbf{p},\mathbf{q})\}, \tag{5}$$

where $d_\infty(\mathbf{p},\mathbf{q}) = max\{|x_1 - x_2|, |y_1 - y_2|\}$, for $\mathbf{p} = (x_1, x_2)$ and $\mathbf{q} = (y_1, y_2)$. It follows that

$$d_\infty(C_r^-(\Theta), C_r^+(\Theta)) \geq 2^{-r}.$$

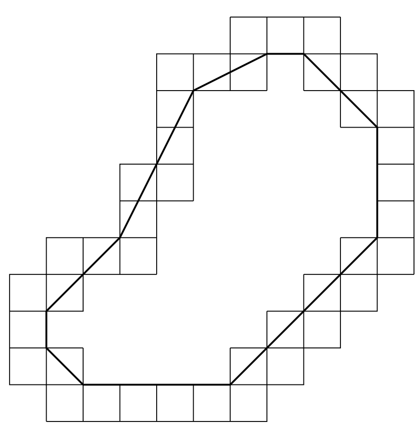

Figure 3: Shortest polygonal Jordan curve in a given simple closed planar one-dimensional grid continuum.

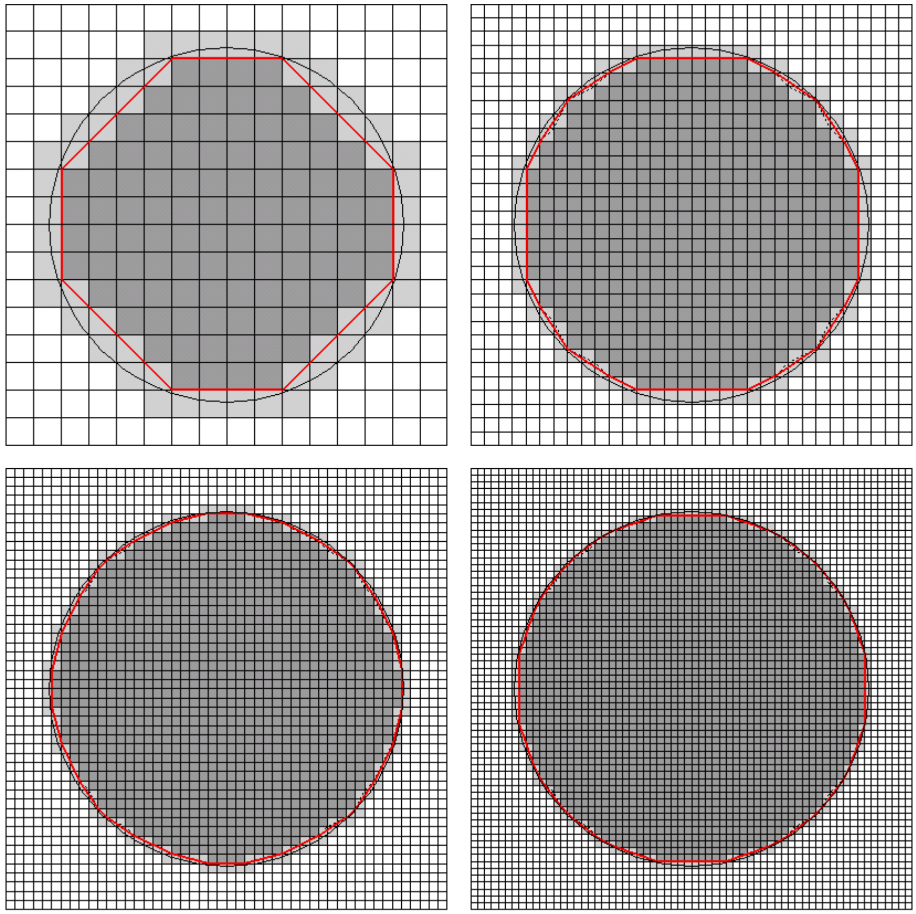

Figure 4: Approximation of a circle at various resolutions by inner and outer interiors: 16×16 top left, 32×32 top right, 48×48 bottom left, and 64×64 bottom right.

Under the given assumptions the constraint $d_\infty(C_r^-(\Theta), C_r(\Theta)) = 2^{-r}$ leads to a uniquely defined isothetic polygon $I_r(\Theta)$, with $I_r^-(\Theta) \subset I_r(\Theta) \subseteq I_r^+(\Theta)$. The difference set ($r$-*boundary* of Θ)

$$B_r(\Theta) = I_r(\Theta) \setminus I(I_r^-(\Theta))$$

is a *simple closed planar one-dimensional grid continuum* as defined in [Sloboda et al., 1998], i.e. each topological unit $\mathbf{C}_r$ in B_r has exactly two edge-connected neighbors (see Fig. 2). We denote it by $B_r(\Theta) = [C^-, C^+]$, where $\partial B_r(\Theta) = C^- \cup C^+$ with $C^- = C_r^-(\Theta)$ as inner simple closed polyhedral surface, and C^+ is the boundary of $I_r(\Theta)$ as outer closed polyhedral surface. Note the proper inclusion $\emptyset \subset I_r^-(\Theta)$.

The *length of a simple closed planar one-dimensional grid continuum* $B_r = [C^-, C^+]$ in $\mathbf{R}^2$ is defined to be the length of the shortest polygonal Jordan curve in B_r encircling C^- (see Fig. 3).

Such a shortest polygonal Jordan curve is uniquely defined [Sloboda et al., 1998a], for $B_r = [C^-, C^+]$. Let C_r denote this uniquely defined shortest polygonal Jordan curve. First linear time algorithms for calculating shortest polygonal Jordan curves for a given simple closed planar one-dimensional grid continuum were published in [Sloboda et al., 1992]. We have designed and implemented a new linear time algorithm which will be reported in a forthcoming report (which runs very time-efficient even for extremly high grid resolutions). This algorithm was used for the measurements which are reported in the following section.

3 Measurements of Arc Length's

Three curves are dealt with in this section. We illustrate the shortest polygonal Jordan curve calculation as a tool for arc length approximation.

3.1 Calculating the π Value

A circle is approximated by gridding techniques generating the inner and outer interiors for different grid resolutions. For the resulting one-dimensional grid continua $[C_r^-, C_r^+]$ the shortest polygonal Jordan curves are calculated. The resulting multigrid curve approximation allows to calculate the estimated arc length at different grid resolution.

Table 1 shows the calculated π values corresponding to the different grid resolutions. Clearly the calculated π value in this approximation converges towards the true value of π as grid resolution increases.

Resolution	32	64	128	256	512	1024
$\|M_r\|$	200	408	824	1640	3272	6552
$\|C_r\|$	20	32	48	72	144	280
E_r	0.0210	0.0065	0.0024	0.0013	0.0006	0.0002
θ_r	0.4207	0.2091	0.1171	0.0941	0.0843	0.0543
π_r	3.0755	3.1211	3.1339	3.1375	3.1398	3.1410

Table 1: Approximation errors for calculating π.

In Table 1, $|M_r|$ and $|C_r|$ denote the number of vertices of M_r, and the number of vertices of the shortest polygonal Jordan curve C_r, respectively. Value E_r denotes the absolute length error in relation to the length $2\pi R$ of the circle calculated using $\pi = 3.1415926...$.

Definition 1 *The value* $\theta_r = E_r \times |C_r|$ *represents the* measure of effectiveness of the appoximation.

The value π_r represents the approximated π value.

Two sequences of measured values provided in table 1 are visualized in Fig. 5, and Fig. 6. Figure 5 illustrates the convergence towards π. For grid constant 2^{-10} we have already an error less than 0.0002.

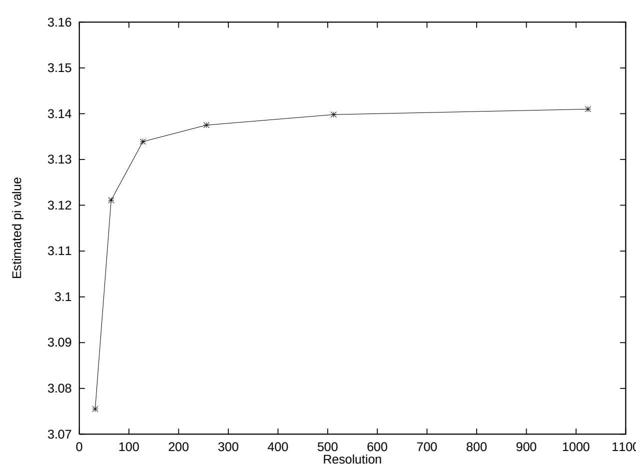

Figure 5: Convergence towards π.

Figure 6 illustrates that the number of vertices $|C_r|$ increases not fast enough such that the effectiveness value is effected in converging towards zero. The number of vertices $|C_r|$ is about in the order of the root of the $|M_r|$ value.

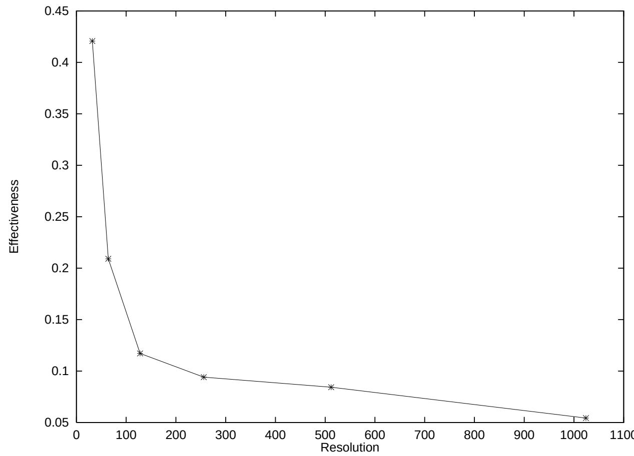

Figure 6: Circle approximation effectiveness

The shortest polygonal Jordan technique is a very efficient approximation approach for calculating the π value.

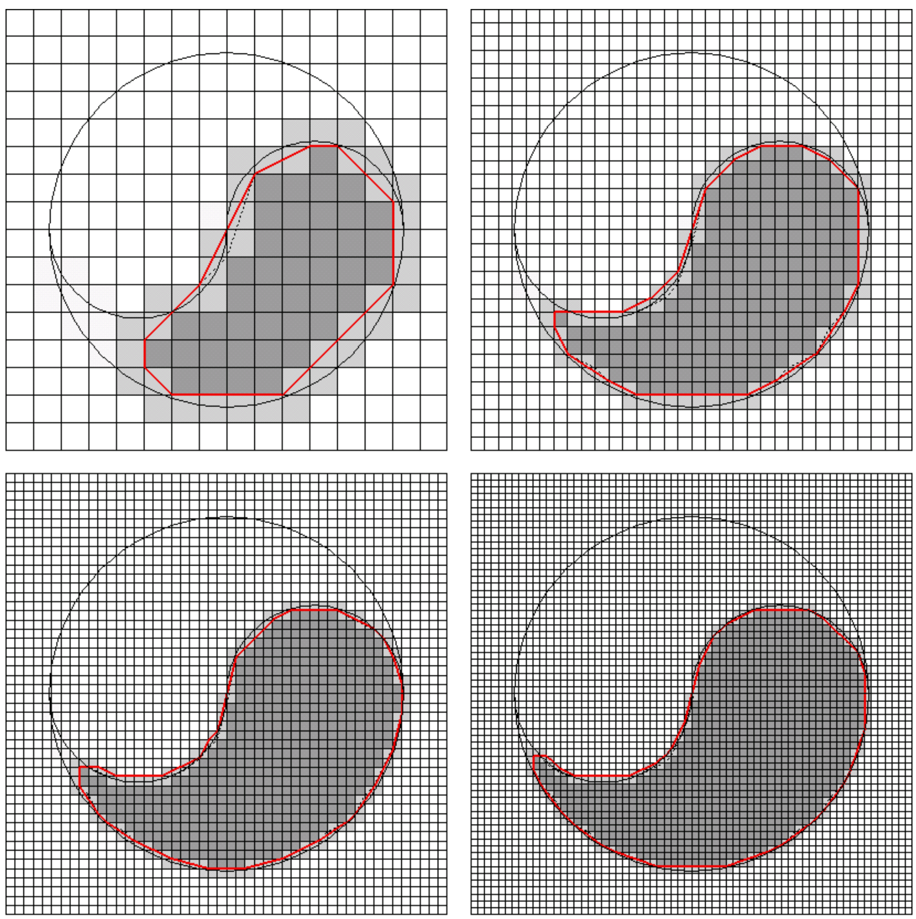

Figure 7: Approximation of yin curve at various resolutions by inner and outer interiors: 16×16 top left, 32×32 top right, 48×48 bottom left, and 64×64 bottom right.

3.2 Appoximation of the Yin Borderline in Yin-Yang

Now we approximate the yin curve in the yin-yang symbol by the shortest polygonal Jordan technique and take the length of the resulting polygonal Jordan curves as approximations of the length of the yin curve. Fig. 7 shows approximations of the yin curve with different grid resolutions. Note that the actual length of the yin curve is knwn and so the approximation error E_r.

We call the upper left of the yin curve its *tail part*. The tail part is omitted in the digitization no matter how high the resolution is. This results from shrinking of the outer interior until a simple one-dimensional border B_r is reached.

Table 2 shows the approximation accuracy and effectiveness with respect to different grid resolutions 2^{-r}. In Tab. 2, $|M_r|$, $|C_r|$, E_r, and θ_r are as defined for Tab. 1.

Resolution	32	64	128	256	512	1024
$\|M_r\|$	174	366	758	1560	3166	6402
$\|C_r\|$	21	36	57	96	161	286
E_r	0.1883	0.1240	0.0962	0.0597	0.0433	0.0303
θ_r	3.9548	4.4649	5.4820	5.7278	7.0384	8.6546

Table 2: Approximation errors for calculating the arc length of the yin curve.

Two sequences of measured values provided in table 2 are visualized in Fig. 8, and Fig. 9. Figure 8 illustrates the convergence of the approximation error E_r towards 0. For grid constant 2^{-10} we have an error less than 0.0303. This higher error in comparison to the circle situation is due to the digitization behavior at the tail part of the yin curve.

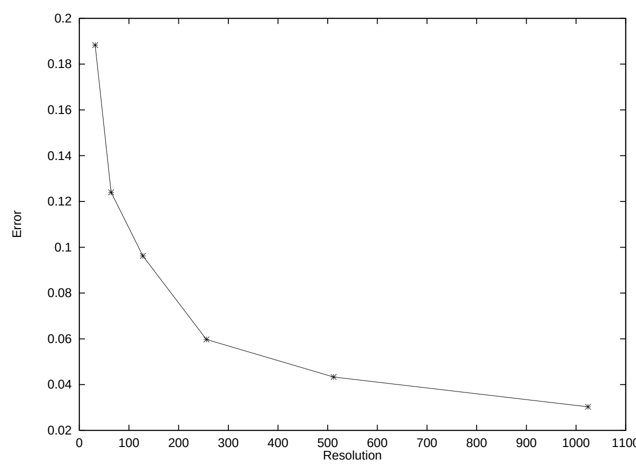

Figure 8: Convergence of the approximation error towards 0.

Figure 9 illustrates that besides again the number of vertices $|C_r|$ increases very slowly (again the number of vertices $|C_r|$ is about in the order of the root of the $|M_r|$ value) we have worse effectiveness due to slower approximation error convergence towards 0 in comparison to the circle case.

However, the shortest polygonal Jordan technique is still an efficient approximation approach for calculating the length of the yin curve especially with respect to the number of vertices of C_r.

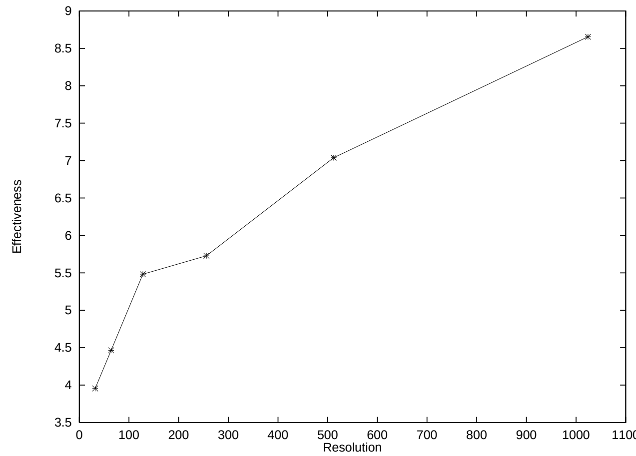

Figure 9: Yin curve approximation effectiveness

3.3 Measurement of an Oscillation Curve

An interesting combination of trigonometric and algebraic functions is given by the well-known *sinc* function:

$$y = sin(ax)/(bx) \ . \tag{6}$$

This function equals a/b at $x = 0$ and shows increasingly damped oscillations as x moves away from the y-axix (see Fig. 10).

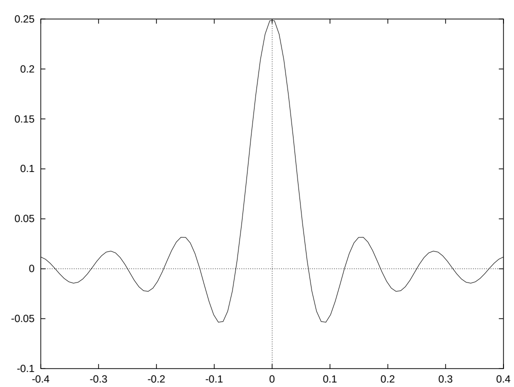

Figure 10: Oscillation curve $y = sin(16 * \pi * x)/(64 * \pi * x)$

In mathematics we know about different ways to approximate the arc length of the curve defined by function (6). However we do not have an exact value and thus we will not specify either the error of approximation nor the approximation effectiveness. However, the shortest Jordan curve technique provides a way to approximate the curve's length. We approximate the oscillation curve defined by Equation 6 where $a = 16$, $b = 64$, and $-0.4 \leq x \leq 0.4$, i.e. $y = sin(16 * \pi * x)/(64 * \pi * x)$.

Resolution	32	64	128	256	512	1024
$\|M_r\|$	164	376	808	1632	3280	6568
$\|C_r\|$	12	30	59	103	144	213
$length_r$	0.970091	1.258189	1.395002	1.437993	1.457203	1.464747

Table 3: Approximation of the arc length of the oscillation curve.

Figure 11 shows the inner and outer interior approximations for different grid resolutions 2^{-r}.

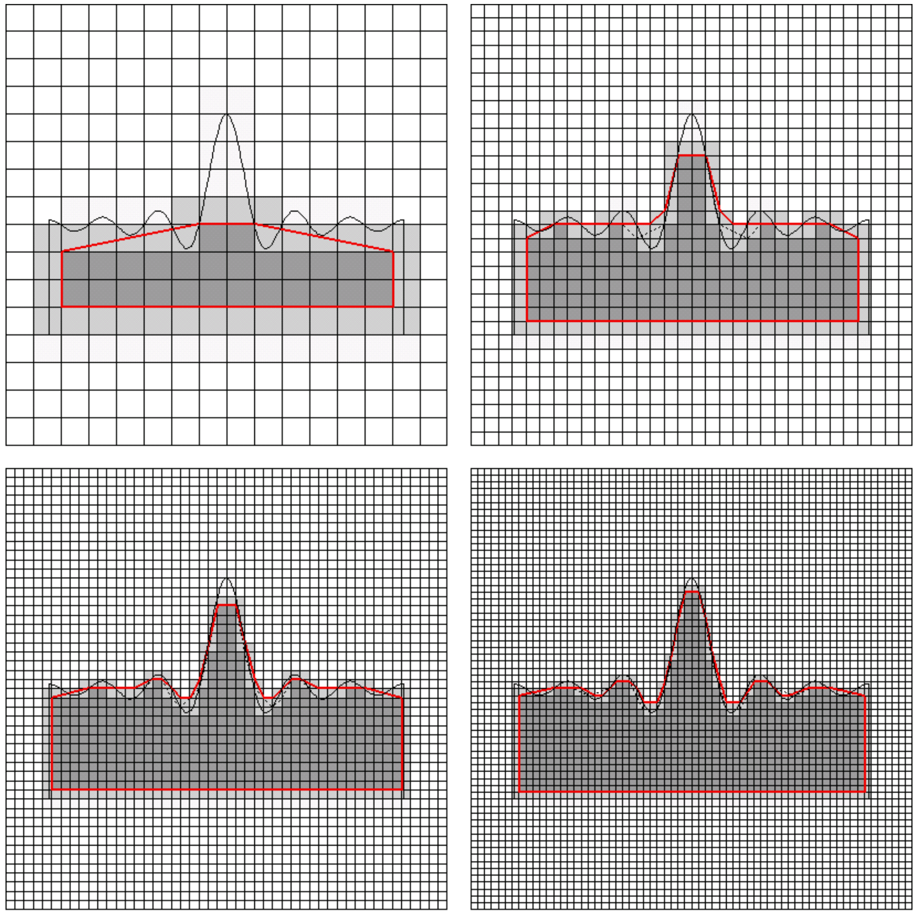

Figure 11: Approximation of the chosen oscillation curve at various resolutions by inner and outer interiors: 16×16 top left, 32×32 top right, 48×48 bottom left, and 64×64 bottom right.

Table 3 shows the approximation with respect to different grid resolutions. The values $|M_r|$ and $|C_r|$ are as defined above, and $length_r$ is the calculated length of the curve C_r at grid constant 2^{-r}.

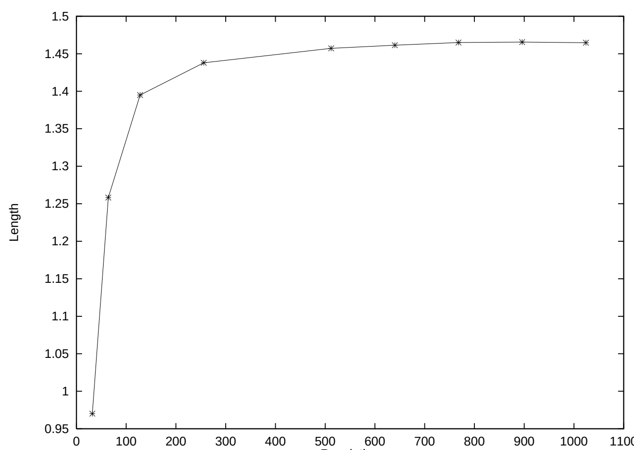

Figure 12: Convergence of the approximation error towards 0.

The sequence of measured length values provided in table 3 is visualized in Fig. 12.

The Fig. 12 illustrates the convergence towards a certain value. It follows from theorems stated in [Sloboda et al., 1998] that the length of the shortest polygonal Jordan curve converges towards the true length of this curve.

4 Conclusion

This report illustrates measurements achieved with the minimum polygonal Jordan curve technique. It shows that this technique can be used for given parametric curves as the circle, constructed curves as the yin curve (where we have not specified a parametric form in this report, however it was actually used to calculate the length of the curve), or "difficult" curves as the sinc curve where the exact arc length value is unknown. Further interesting curve examples might be the *arbelos* or the *salinon* as introduced by Archimedes more than two thousand years ago. The introduced measure of approximation effectiveness is considered to be a tool for analysing and comparing the approximation behaviour of different approximation techniques. For example, this measure allows to discuss the relation between the number n of approximating (inner or outer) n-gons (as used by Archimedes) and the approximation error. This measure can be used for further approximation techniques (e.g. straight line segment sequence approximations of borders of digital objects in the plane as suggested in digital geometry). The applied algorithm will be reported in a forthcoming report.

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