

Multigrid Convergence of Surface Approximations

Reinhard Klette ¹, Feng Wu ¹ and Shao-zheng Zhou ¹

Abstract

This report deals with multigrid approximations of surfaces. Surface area and volume approximations are discussed for regular grids (3D objects), and surface reconstruction for irregular grids (terrain surfaces). Convergence analysis and approximation error calculations are emphasized.

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Abstract: This report deals with multigrid approximations of surfaces. Surface area and volume approximations are discussed for regular grids (3D objects), and surface reconstruction for irregular grids (terrain surfaces). Convergence analysis and approximation error calculations are emphasized.

Keywords: Multigrid convergence, surface area, cellular complexes, marching cubes, minimum Jordan surface, greedy refinement, evaluation of algorithms

1 Introduction

This paper reports on multigrid convergence as a general approach for model-based evaluations of computer vision algorithms. This criterion is in common use in numerical mathematics. In general, algorithms may be judged according to criteria, such as methodological complexity of underlying theory, expected time for implementation, or run-time behaviour and storage requirements of the implemented algorithm. Accuracy is an important criterion as well, and this can be modeled as convergence towards the true value for grid based calculations.

Multigrid convergence is discussed in this paper for two different tasks. Both tasks are related to 3D surface approximation or 3D surface recovery. The existence of convergent algorithms is discussed for

- (i) the problems of volume and surface area measurement for Jordan sets or 3D objects based on *regular, orthogonal grids*: convergent volume measurement is known since the end of the 19th century, see, e.g. [Scherer, 1922], a provable convergent surface area measurement is a recent result in the theory of geometric approximations [Sloboda, 1997], and
- (ii) the problem of (incremental) 3D reconstructions of Jordan faces based on *irregular grids*: many techniques in computer vision are directed on reconstructing 3D surfaces; assuming that these techniques have successfully reconstructed a surface, then the next step is to represent this surface under special (e.g. incremental transmission) conditions [Zhou and Klette, 1997].

Jordan faces, surfaces and sets are proper models for discussing surface approximations. In this paper, a *3D object* in three-dimensional Euclidean space is a simply-connected compact set bounded by a (measurable) Jordan surface [Mangoldt and Knopp, 1965, Klette, 1998]. A Jordan surface is a finite union of Jordan faces.

3D objects are studied based on given digitizations in computer vision, image analysis, or object visualisation. Regular grids are of common use in computer vision or image analysis. Let $\mathbf{Z}_r = \{m \cdot 2^{-r} : m \in \mathbf{Z}\}$, where \mathbf{Z} is the set of integers and $r = 0, 1, 2, \dots$ specifies the grid constant 2^{-r} . The set \mathbf{Z}_r^n is the set of all *r-grid points* in *n-dimensional* Euclidean space. Each *r-grid point* (x_1, \dots, x_n) defines a topological unit (a *grid cube*)

$$\mathbf{C}_r(x_1, \dots, x_n) = \{(y_1, \dots, y_n) : (x_i - .5)2^{-r} \leq y_i \leq (x_i + .5)2^{-r}, i = 1, \dots, n\}.$$

Irregular grids, as Voronoi or Delaunay diagrams, are favoured for object visualisations. However, the vertices of these irregular grids are often restricted to be points in \mathbf{Z}_r^n .

In evaluations we have to compare truth against obtained results. A *true entity* is normally well-defined within mathematical studies, however in general not in image analysis applications. In this paper we propose and study a *mathematical problem*: A multigrid digitization model for 3D objects, either regular [Klette, 1985] or irregular [Zhou and Klette, 1997], assumes an ideal mapping of a given set (the “true 3D object” having the “true surface area”, the “true samples of a terrain map”, etc.) into a finite digital data set. The problem consists in analyzing the behavior of a given technique or algorithm assuming finer and finer grid resolution. For regular grids we reduce the grid constant 2^{-r} ($=$ side length of grid cubes). It converges towards zero for $r = 0, 1, 2, \dots$. For irregular grids we increase the number of vertices until we reach a given maximum of sample points. Assume that a Jordan face is given by values at these n vertices, and a specification of their neighbourhoods. The task consists in finding a series of approximation surfaces with m vertices ($m \in (i, i + 1, \dots, n)$; $i > 0$) with approximation error ε_m and $\varepsilon_1 \geq \varepsilon_2 \geq \dots \geq \varepsilon_m \geq 0$, for $m \leq n$. Convergence studies are directed on cases $n \rightarrow \infty$.

The convergence issue addressed in this paper corresponds not only to the continued progress in imaging technology. It is of special value also for understanding the *soundness of a chosen approach*. The convergence problem may be studied based on experimental evaluations of approaches, algorithms or implementations, using, e.g., synthetic 3D objects and a selected digitization model. We provide a few examples for this evaluation strategy in the following two sections. These examples may highlight the importance of convergence studies. Produced data sheets as the provided ones may be of value for practical situations. However a theoretical analysis leads (hopefully) to the complete answers as illustrated in the next section. Irregular approximations may be based on the suggested minimum Jordan surface constructions as well if the correct surface area of the reconstructed surface is desirable.

2 Regular Grids: Volume and Surface Area

Algorithms for measuring surface area and volume should be consistent for different data sets of the same object taken at different spatial resolutions. In our image analysis applications we do expect that measured surface areas and volume values converge towards proper values assuming an increase in spatial resolution. For example, volume or surface area measurement should not be influenced by the rotation angle of the given 3D object. Feature convergence is of fundamental importance in 3D object analysis.

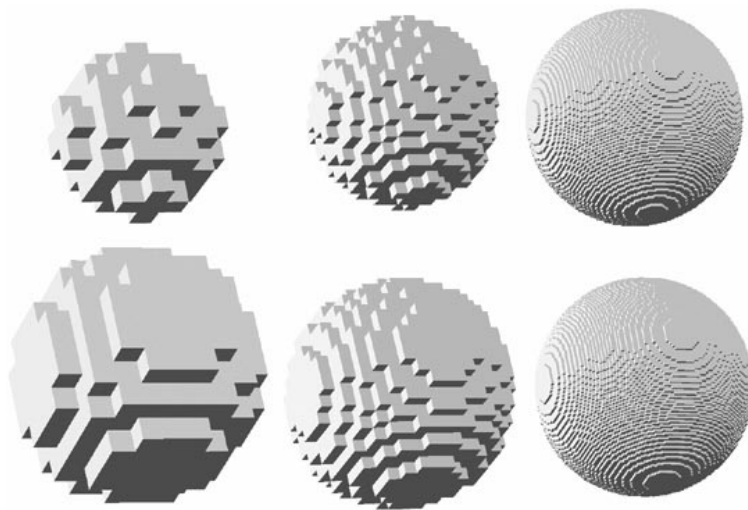


Figure 1: Grid cube inclusion (top row) and grid cube intersection (bottom row) digitization of a sphere assuming three different grid constants

2.1 Digitization

Grid cube inclusion (with respect to the topological interior of the given 3D object) and *grid cube intersection* digitizations are assumed for 3D objects having interior points (see Fig. 1). Grid cube inclusion digitization defines the *inner interior* $I_r^-(\Theta)$ of a given 3D object Θ , and grid cube intersection digitization defines the *outer interior* $I_r^+(\Theta)$. The resulting digital objects can be described as being grid continua, see, e.g., [Sloboda et al., 1998].

We consider synthetic objects. Assume, e.g., a cube Θ as a given three-dimensional set and a digitization of this cube with respect to a chosen grid constant. The resulting isothetic polyhedron $I_r^-(\Theta)$ contains all grid cubes \mathbf{C}_r which are completely inside of the given cube. An *isothetic polyhedral Jordan surface* is a polyhedral Jordan surface whose faces are coplanar either with the XY-, XZ-, or YZ-plane. An *isothetic polyhedron* is a polyhedron whose boundary is an isothetic polyhedral Jordan surface.

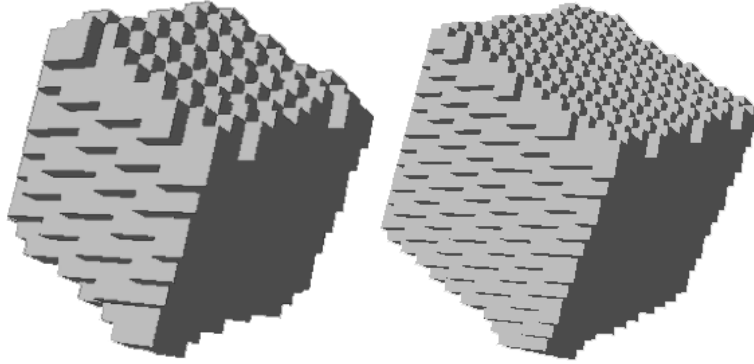


Figure 2: Two digitizations of a cube which is rotated about 45° with respect to X , Y , and Z axes

Figure 2 illustrates two different digitizations of the same cube which was rotated about 45° with respect to the x , y , and z axes. The same cube is now studied with respect to different rotations. Volume and surface area of 3D objects are invariant with respect to rotations.

2.2 Volume and Surface Area of Cellular Complexes

Classical results can be cited for grid based volume area measurement, see, e.g. [Scherer, 1922]. The convergence of these measurements towards the true value is illustrated in Fig. 3. For different rotational positions we digitize the cube for $r \rightarrow \infty$. For each grid constant 2^{-r} we calculate the volume of the resulting cellular complex as number of 3D cells contained in the cellular complex times the volume 2^{-3r} of a single 3D cell.

Now we consider the total surface area of all the two-dimensional surface cells of the resulting cellular complexes $I_r^-(\Theta)$ (Fig. 4) using the algorithm published in [Artzy et al., 1981] for visiting all 2D surface cells exactly once. The figure shows that there is “obvious convergence” in all cases. However, the measured surface area values depend upon the given rotation of the cube, and the deviation d can be equal to 0 if the cube was in isothetic position, and about 0.90 (i.e. 90% error!) if it was rotated about 45° with respect to X , Y , and Z axes. These values are inappropriate for estimating a surface area of a cube if the rotation angle is unknown. A surface area measurement based on counts of 2D surface cells of 3D cellular complexes is *not related* to the true surface area value. Since the length of a staircase function remains constant and does not converge towards the length of a diagonal straight line segment, similar statements can be said for using counts of two-dimensional faces on the surface of three-dimensional cellular complexes with respect of estimates of the surface area of the three-dimensional set represented by this cellular complex.

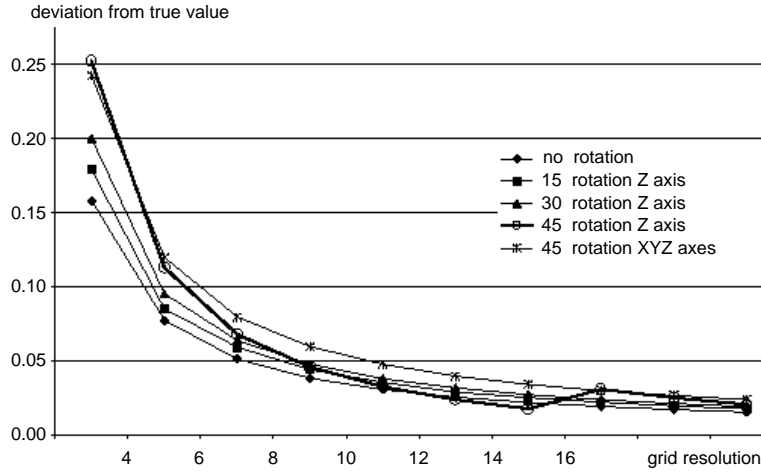


Figure 3: Measured convergence of volume values $(1 + d)s$ of cellular complexes towards the true value s , the volume of the given cube, where d is the deviation

2.3 Use of Local Approximations

The inclusion of diagonal elements into a simple local approximation approach as "8-neighborhood contours" in 2D, or local triangulations in 3D does not resolve this inconsistency. The use of a marching cubes algorithm [Lorensen and Cline, 1987] is one of the options of local approximations. Each elementary grid cube, defined by eight grid points, is treated according to a look-up table for defining triangular or planar surface patches within this elementary grid cube. A marching cubes algorithm determines the surface by deciding how the surface intersects a local configuration of eight voxels. A surface is assumed to intersect such a local configuration in 2^8 different ways (i.e. no multiple intersections of grid edges), and these can be represented as fourteen major cases with respect to rotational symmetry. Alternatively a method developed by [Wyvill et al., 1986] calculates the contour chains immediately without using a look-up table of all 2^8 different cases. The fourteen basic configurations originally suggested by [Lorensen and Cline, 1987] are incomplete. Occasionally they generate surfaces with holes.

Ambiguities of the marching cube look-up tables are discussed in [Wilhelms and Gelder, 1994]. See

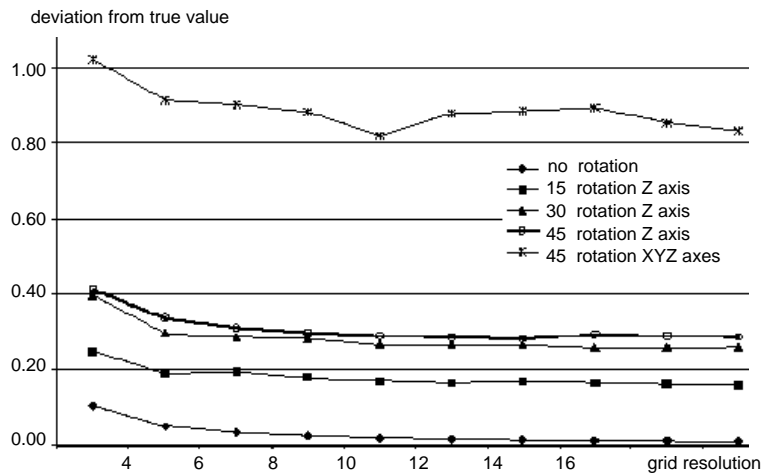


Figure 4: Each curve points out that there is "obvious convergence" of the measured surface area of all the two-dimensional surface faces of the cellular complex for a given rotational position of the cube towards a value $(1 + d)s$

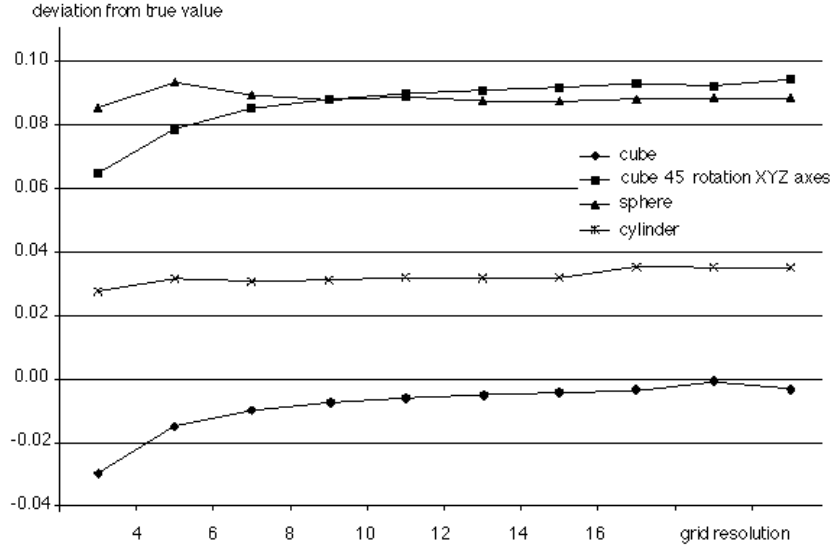


Figure 5: These curves show “obvious convergence” of the measured surface area towards values $(1+d)s$ where a marching cubes algorithm was applied to two cubes in different rotational positions, a sphere and a cylinder

[Sloboda et al., 1998] for local situations of marching cubes configurations where at least two different topological interpretations are possible. More important, the calculated values do not converge towards the true value as illustrated in Fig. 5. The surface area and the volume is calculated based on values for the different look-up table situations.

A marching tetrahedra algorithm was suggested in [Roberts, 1993]. It generates more triangles than the marching cubes algorithm in general. Trilinear interpolation functions were used in [Cheng, 1997] for the different basic cases of the marching cubes algorithm. In comparison to the linear marching cube algorithm [Heiden et al., 1991] the accuracy of the calculated surface area was slightly improved by using this trilinear marching cubes algorithm, which was confirmed for a few synthetic Jordan faces.

These local approximation techniques, such as marching cubes algorithms, also generate very large numbers of triangles what restricts their practical use for high resolution data, such as, e.g., in computer assisted radiology.

2.4 Minimum Jordan Surfaces

The surface area measurement approach introduced in [Sloboda, 1997], is basically different from the concepts in digital geometry, or from local surface approximation approaches. It is a special approach towards global surface approximations. Assume that both the inner interior $I_r^-(\Theta)$ and the outer interior $I_r^+(\Theta)$ of a given 3D object Θ are simply connected sets with respect to the given grid constant 2^{-r} . Assume $I_r^-(\Theta) \neq \emptyset$. Let $S_r^-(\Theta)$ be the surface of the isothetic polyhedron $I_r^-(\Theta)$, and let $S_r^+(\Theta)$ be the surface of the isothetic polyhedron $I_r^+(\Theta)$. Then it holds that

$$\emptyset \subset I_r^-(\Theta) \subset I(I_r^+(\Theta)) \quad \text{and} \quad S_r^-(\Theta) \cap S_r^+(\Theta) = \emptyset,$$

and $I_r^+(\Theta) \setminus I(I_r^-(\Theta))$ is an isothetic polyhedron homomorphic with the torus.

Furthermore, let d_∞ be the Hausdorff metric [Hausdorff, 1927, page 100]. It follows that

$$d_\infty(S_r^-(\Theta), S_r^+(\Theta)) \geq 2^{-r}.$$

Under the given assumptions the constraint $d_\infty(S_r^-(\Theta), S_r(\Theta)) = 2^{-r}$ leads to a uniquely defined isothetic polyhedron $I_r(\Theta)$, with $I_r^-(\Theta) \subset I_r(\Theta) \subseteq I_r^+(\Theta)$. Let $S_r(\Theta)$ be the surface of $I_r(\Theta)$. The difference set

(r -boundary of Θ)

$$B_r(\Theta) = I_r(\Theta) \setminus I(I_r^-(\Theta))$$

is a *simple closed two-dimensional grid continuum* as defined in [Sloboda et al., 1998]. We denote it by $B_r(\Theta) = [S_1, S_2]$, where $\partial B_r(\Theta) = S_1 \cup S_2$ with $S_1 = S_r^-(\Theta)$ as inner simple closed polyhedral surface, and $S_2 = S_r(\Theta)$ as outer closed polyhedral surface. Note the proper inclusion $\emptyset \subset I_r^-(\Theta)$.

The *surface area of a simple closed two-dimensional grid continuum* $[S_1, S_2]$ in \mathbb{R}^3 is defined to be the surface area of a minimum area polyhedral simple closed Jordan surface in $[S_1, S_2]$ containing S_1 . The following two theorems were proved in [Sloboda, 1997]:

Theorem 1 *Assume a simple closed two-dimensional grid continuum $[S_1, S_2]$. Then there exists a uniquely defined polyhedral simple closed Jordan surface in $[S_1, S_2]$ containing S_1 with minimum surface area.*

We call this the *minimum Jordan surface* of $B_r(\Theta) = [S_1, S_2]$. Thus a Jordan set Θ and a grid resolution $r \geq r_0$ (such that both the inner interior $I_r^-(\Theta)$ and the outer interior $I_r^+(\Theta)$ are simply connected sets) uniquely define a minimum Jordan surface $MJS_r(\Theta)$ having a surface area of $J_{area}(MJS_r(\Theta))$.

Theorem 2 *For any smooth Jordan set $\Theta \subset \mathbb{R}^3$ it holds that*

$$J_{area}(\partial\Theta) = \lim_{r \rightarrow \infty} J_{area}(MJS_r(\Theta))$$

where $MJS_r(\Theta)$ is the minimum Jordan surface for resolution $r \geq r_0$.

The theorem is also valid for Jordan surfaces which possess a finite number of edges. A polyhedron has its surface area well defined. Altogether this specifies a sound (i.e. convergence and convergence towards the proper value) procedure for calculating the surface area of a digitized Jordan set.

3 Reconstruction of Multiresolution Terrain Surfaces

Multiresolution terrain surfaces are especially useful for fast rendering, real-time display, and progressive transmission. The general problem of reconstructing surfaces of 3D objects is restricted to a situation where only Jordan faces (terrain surfaces or height maps) have to be reconstructed. However, accuracy may be modeled by convergence considerations as well. We herein propose and discuss a greedy refinement approach for the reconstruction of multiresolution terrain surfaces or the progressive reconstruction of terrain surfaces.

3.1 Brief Review of Techniques

The problem of triangulating a set of points to produce a surface is a well researched topic in computer graphics and computational geometry. We mainly explore ways of triangulating a set of points to represent, visualize and transmit terrain surfaces in multiresolutions. A terrain surface can be modelled in much simpler ways comparing with a generic 3D surface. It can be represented by a single-valued bivariate function over the domain of the model. The reconstruction of terrain surface is referred to as a $2\frac{1}{2}D$ modelling problem. A terrain is mathematically described by a height function: $\Phi : \mathbf{D} \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$. In practical applications, the function Φ is sampled at a finite set of points $P = \{p_1, \dots, p_n\} \subset \mathbf{D}$. In this case the function Φ can be defined piecewise over a subdivision Σ of \mathbf{D} with vertices in P . The main goal is to reconstruct terrain surfaces at high speed, from an initial coarse resolution to full resolution. In the context of this paper we are interested in evaluating the quality increase during this process of approaching full resolution, and in convergence properties assuming that the number n of sample points goes to infinity.

There are various algorithms for terrain simplification or polygonal simplification. They can be categorized as:

(i) *simple uniform grid methods* as cellular complexes or marching cubes (see section above): They are simple for representation but impossible for real-time display or fast rendering. Downsampling can be used to represent simplified models, but the quality is not desirable.

(ii) *hierarchical subdivision methods*: They include quad-tree, k-d tree, and hierarchical triangulation data structures. However, it seems difficult for them to maintain the continuity of the surface where patches of surfaces with different resolutions meet.

(iii) *feature methods*: Using local features for simplification like curvatures does not produce results with globally desirable quality. See also the integration problem as stated in [Klette, 1998].

(iv) *decimation methods* [Ciampalini et al., 1997]: Those algorithms simply remove the point whose absence adds the smallest error to the approximation. The main advantage for decimation methods is that they can remove several points in one step. However, their retriangulation seems complicated and not efficient enough.

(v) *refinement techniques* [Garland and Heckbert, 1995, Cignoni et al., 1997]: They start with a minimal approximation, then progressively refine it by adding the point which will introduce the minimal sum of approximation errors to the approximation, and executing Delaunay retriangulation. Delaunay retriangulations are necessary because they are essential for the future numerical interpolation or retrieval of elevation values and for minimizing the aliasing problems in terrain surfaces' visualization or display.

Refinement techniques are especially suitable for the reconstruction of surfaces with continuous resolutions.

3.2 Greedy Refinement

We designed a refinement algorithm following the general greedy refinement approach with Delaunay retriangulation. Let P be a finite set of points in \mathbf{R}^2 . The main idea of the greedy refinement algorithm can be described as follows. An initial triangulation T is constructed first, whose vertex set is composed of all extreme points of the convex hull of P . The triangulation or mesh is then refined through the iterative insertion of new vertices, one at a time (at each iteration, select the point p which will introduce the minimal sum of all approximation errors). The triangulation or mesh is updated accordingly by Delaunay retriangulation. This refinement process continues until the specified goal (e.g. error threshold) is met. A pseudo code for such a greedy refinement algorithm is as follows:

```
GreedyRefinement(var SetOfVertices P, var Triangulation T,
                 var ErrorMeasureHeap H)
begin
  construct initial triangulation T;
  calculate approximation errors;
  construct heap H of error measure values;
  while (GoalNotMet)
    SELECTION: pop out the minimum value from H;
               from P remove the corresponding vertex p,
               which introduced the minimum error measure;
    INSERTION: insert vertex p into triangulation T;
    RETRIANGULATION: add newborn cells on triangulation T
                     and delete dead cells from T;
                     retriangulate T;
    RECALCULATION: calculate error values of newborn cells;
                  update error values of other living cells;
                  push error values of newborn cells in H;
end
```

For measuring the approximation error we take the absolute value of the difference between the interpolated elevation value $S(x, y)$ and the actual elevation value $\Phi(x, y)$ at vertex $\mathbf{p} = (x, y)$ as the

approximation error $\varepsilon(x, y) = |\Phi(x, y) - S(x, y)|$. Based on our data structure, for every cell Δ containing a certain set of points, say $\{p_0, \dots, p_k\}$, we try to select a point p_i , with $0 \leq i \leq k$, whose selection as a new triangulation point will introduce the minimal sum σ of approximation errors of the remaining vertices inside of cell Δ . Then the new error measure value of cell Δ is equal to

$$\sigma(\Delta) = \varepsilon_0 + \dots + \varepsilon_{i-1} + \varepsilon_{i+1} + \dots + \varepsilon_k.$$

Greedy refinement methods use data structures such as *quedge* [Guibas and Stolfi, 1985] and *facet-edge* [Cignoni et al., 1997]. Our greedy refinement algorithm is based on a very straightforward data structure:

```

Class Mesh
{
    Heap ErrorMeasures;
    Array Cells;
    ...
}

Class Cell
{
    Vertex v0,v1,v2;
    Neighbors n0,n1,n2;
    Array points;
    ErrorMeasure e;
    ErrorMeasureOfOtherCell other;
    ...
}

```

Every cell has three vertices, three possible neighbors, points which are included inside, its error measure value, and the sum of error measure values of all the other cells. The mesh is composed of cells. The mesh's array stores all pointers of cells (dead and living). The heap includes pairs of (ErrorMeasure e, long p), where p points to a cell's position in the Array of Cells. The heap data structure makes the greedy refinement more efficient.

3.3 Approximation Example

Figure 6 shows an original terrain model which is created by the bivariate function $\Phi(x, y) = \frac{1}{2}(\sin(3x)^4 + \cos(2y)^4 + \sin(x + 4y)^3 - \cos(xy)^5) + 1.0$, where $x \in [0, 1]$, and $y \in [0, 1]$.

Figure 7 indicates the refinement result refined with 15% percent of the vertices of the original model. It is evident that Figure 7 keeps the important features like peaks and valleys in Figure 6.

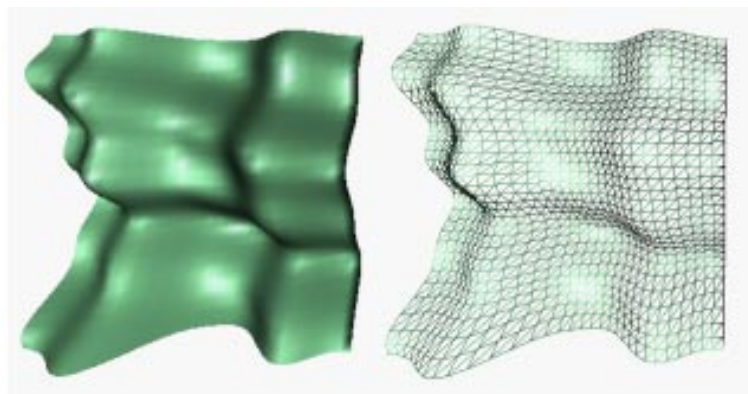


Figure 6: Original terrain surface with $32 \times 32 = 1024$ vertices

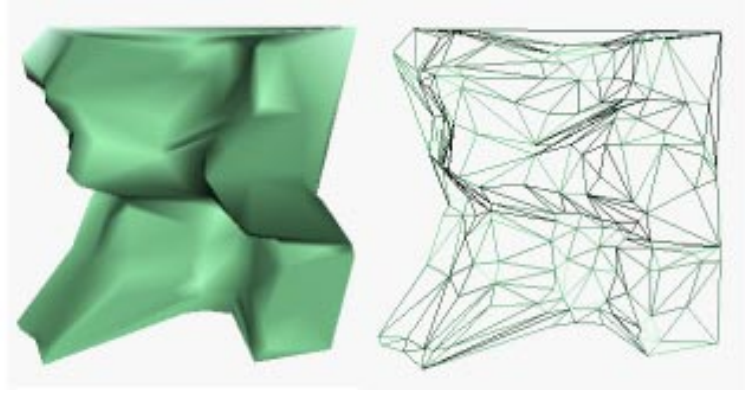


Figure 7: Approximation using 15% percent of the original model's vertices

Figure 8 indicates the refinement result refined with 30% percent of vertices of the original model. Figure 8 has all features in Figure 6. They are very similar in shape. It is enough to represent the original one for fast rendering and real-time display with this simplified model.

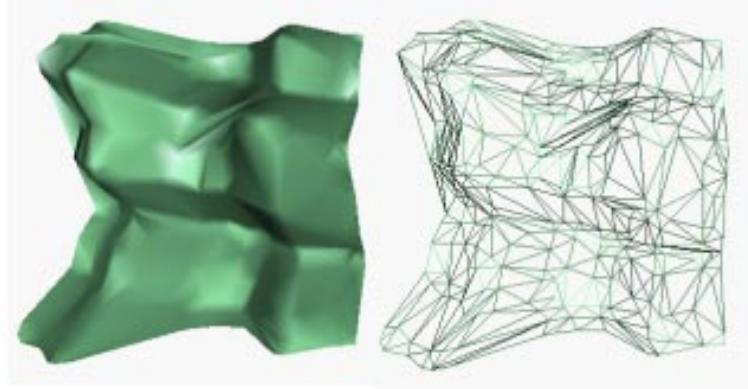


Figure 8: Approximation using 30% percent of the original model's vertices

From the Figure 9, it is clear that the approximation errors are initially reduced drastically when the vertex number increases, but slowly after 30% percent of vertices has been added. However, this behavior depends on the chosen global approximation error measure.

A more general study would require to analyse such algorithms for classes of surface functions Φ , and for increases in the number n of sampling points with $n \rightarrow \infty$.

4 Conclusion

Surface measurement for 3D objects Θ can be based on calculating minimum Jordan surfaces as shown in [Sloboda, 1997]. Marching cubes algorithms do not lead to convergent approximations of minimum Jordan surfaces, and surfaces of isothetic polyhedrons $I_r^-(\Theta)$ or $I_r^+(\Theta)$ are unrelated to the true surface area value.

Surface reconstructions of Jordan faces (e.g. terrain models), given by just a finite number n of surface points, can be obtained by incremental reconstructions based on irregular grids where the selection procedure of points, and the surface approximation strategy decides about the error vs percentage of vertices ratio. We proposed a revised greedy refinement approach to progressively reconstruct terrain surfaces. We employed a very straightforward data structure. Based on greedy refinement and our data

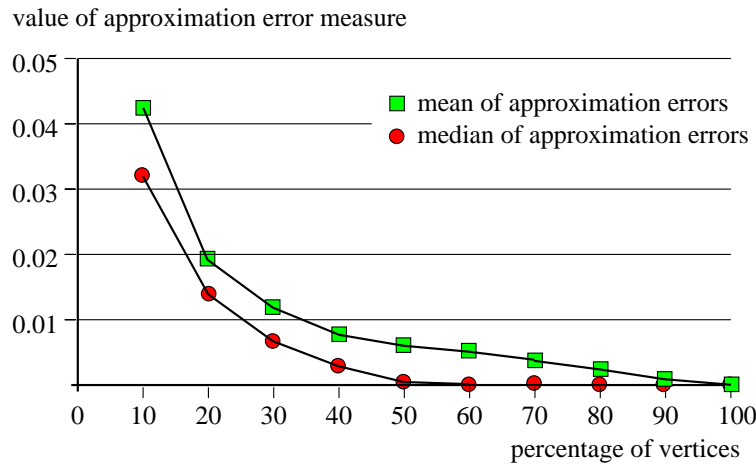


Figure 9: Approximation errors vs percentage of original model's vertices

structure, we can reconstruct a family of terrain surfaces with continuous resolutions, which are necessary for fast-rendering and real-time display. This will support more detailed studies of convergence behaviour (i.e. convergence with respect to an increase in the number of given sample points) in the future.

Assuming that the original n sample points form a regular $N \times N$ grid and that the given surface (terrain) is non-convex, a local triangulation is not convergent towards the true surface area value if $N \rightarrow \infty$. This follows by similar experiments as illustrated for the marching cubes algorithm. The calculation of minimum Jordan surfaces may be suggested instead. Accurate surface area calculations are, e.g., of crucial importance in GIS (geographic information systems).

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