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On the Topology of Grid Continua

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Abstract

One-dimensional and two-dimensional continua belong to the basic notions of set-theoretical topology and represent a subfield of the theory of dimensions developed by P. Urysohn and K. Menger. In this paper basic definitions and properties of grid continua in R^2 and R^3 are summarised. Particularly, simple one-dimensional grid continua in R^2 and in R^3, and simple closed two-dimensional grid continua in R^3 are emphasised. Concepts for measuring the length of one-dimensional grid continua, or the surface area of two-dimensional grid continua are introduced and discussed.

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Keywords: Digital geometry, one-dimensional grid continua in R^2 and R^3, two-dimensional grid continua in R^3, minimal polyhedral Jordan surface in R^3

1 Introduction

At the end of the 19th century C. Jordan and G. Peano introduced regular grids in order to define measurable sets in R^n. A topological unit of a regular grid in R^2 represents a square, and in R^3 a cube. The vertices of these squares or cubes have integer valued coordinates.

In digital image processing such units specify geometric locations of pixels or voxels, respectively. In the image processing context, regular grids in R^n were investigated in the past by several authors [3, 6, 7, 16]. Regular grids were also investigated in other disciplines as in computational physics [30].

The definition and study of curves or surfaces is one of the main subjects in mathematical analysis [9, 10, 11, 13, 25, 26, 27]. For example, fractal curves or surfaces illustrate the topological and geometrical complexity of curves or surfaces. It is one of the main problems in regular grid based computations to introduce and study curves or surfaces especially for the case of discrete (i.e. measurments are only given at grid point positions) and regular (i.e. regular grid in two-dimensional or higher-dimensional space) data.

Digital geometry aims "to study how geometrical properties can be determined from the pixel subsets themselves" [18]. Digital curves, digital straight lines and their properties were introduced in [15]. Digital surfaces were described in [12, 14, 17] and in [2]. These papers present different approaches to definitions and the study of *digital curves* or *digital surfaces*. They belong to the field of digital geometry.

Different *local approximation techniques* are also developed for calculating curves or surfaces in regular grid based computations. For example, in the digital image processing literature surfaces of three-dimensional volumes are also defined based on local decision criterions. A marching cube algorithm [8] takes eight voxels ("a cube of voxels") as a local input data configuration to specify triangles which are supposed to approximate the unknown surface. A marching cube algorithm determines the surface by analysing how the surface intersects a local configuration of eight voxels. Surfaces are assumed to intersect each grid edge (between two neighboring voxels) at most once. It follows that they can intersect such a local configuration in 2^8 different ways, and these can be represented as fourteen cases with respect to rotational symmetry. Alternatively contour chains can also be calculated in layers of the given volume data set immediately without using a look-up table [29].

The approach described in this paper is basically different from the concepts of a digital curve or of a digital surface in digital geometry, and from the concepts of local definitions of approximating surface patches as in marching cubes algorithms. One-dimensional and two-dimensional continua belong to the basic notions of set-theoretical topology and represent a subfield of the theory of dimensions developed by P. Urysohn and K. Menger [10, 11, 13, 26, 27]. They are also called *curves* and *surfaces*. In this paper the notion of simple one-dimensional grid continua in R^2 and R^3 and the notion of a simple two-dimensional grid continua in R^3 are introduced and their length and surface area, respectively, are defined. The length of simple one-dimensional grid continua in R^2 is based on the notion of a shortest polygonal Jordan curve in a polygonally bounded compact set and on the notion of a geodesic diameter of a polygon. The length of simple one-dimensional grid continua in R^3 is based on the notion of a shortest polygonal Jordan curve in a polyhedraly bounded compact set and on the notion of a geodesic diameter of a polyhedron. The surface area of a simple closed two-dimensional grid continuum in R^3 is based on the notion of a minimal polyhedral Jordan surface in a polyhedrally bounded compact set. The main advantages of the new approach are

(i) that the design of algorithms for approximating curves or surfaces can be based on results in topology which provide concepts for specifying fundamental features as length or surface area (and which also allow the study of convergence of feature calculations with respect to refined grid resolutions) [22, 24, 4], and

(ii) that experimental and algorithmic studies also have shown that the new approach allows time-efficient and space-efficient solutions (see [24] for the case of one-dimensional grid continua).

This paper is directed on introducing the notions of one-dimensional or two-dimensional grid continua as topological entities. See, e.g., [24] for a broader discussion of one-dimensional grid continua (also covering convergence aspects and proposals of algorithms).

Algorithms for measuring features as the length of a curve, the surface area, or the volume of a set in three-dimensional space should be consistent for different data sets of the same object taken at different spatial resolutions (e.g. different scan resolutions in confocal microscopy). This problem of feature convergence is of fundamental importance in image analysis [4, 5]. Since the length of a staircase function remains constant and does not converge towards the length of a diagonal straight line segment, similar statements can be formulated for using counts of all two-dimensional faces on the surface of three-dimensional cellular complexes (using an algorithm as, e.g., published in [1]) with respect to estimates of the surface area of an unknown simply-connected compact set in R^3 assuming that this set is represented (e.g. by means of *cube inclusion digitisation* [3]) by this cellular complex. However, the volumes of the cellular complexes (i.e. digitisations at different grid resolutions of an unknown simply-connected compact set in R^3) converge always towards the true value [19].

Topological ambiguities of local approximation techniques as of the marching cube look-up tables are discussed in the image analysis literature [28]. See Fig. 1 for local situations of marching cubes configurations where at least two different geometric interpretations are possible.

Furthermore, the fourteen basic configurations originally suggested in [8] are incomplete. Occasionally they generate "surfaces with holes". These local techniques also generate very large numbers of triangles what makes them practically unusable for high resolution data as given, e.g., in medical applications.

This paper is organised as follows: Section 2 introduces one-dimensional grid continua in the plane. Section 3 starts with a discussion of one-dimensional grid continua in the three-dimensional space, and also introduces the concept of two-dimensional grid-continua. The introduced notions are discussed with respect to topological properties. A few conclusions are given in Section 4.

2 Grid Continua in $\mathbf{R}^2$

Let us consider an orthogonal grid in R^2. For $p = 0, 1, \ldots,$ and for each tuple (w_1, w_2) of integers let

$$N^p_{(w_1,w_2)} := \{x \in R^2 \mid w_i\, 2^{-p} \le x_i \le (w_i + 1)\, 2^{-p},\ \ i = 1, 2\}. \quad (1)$$

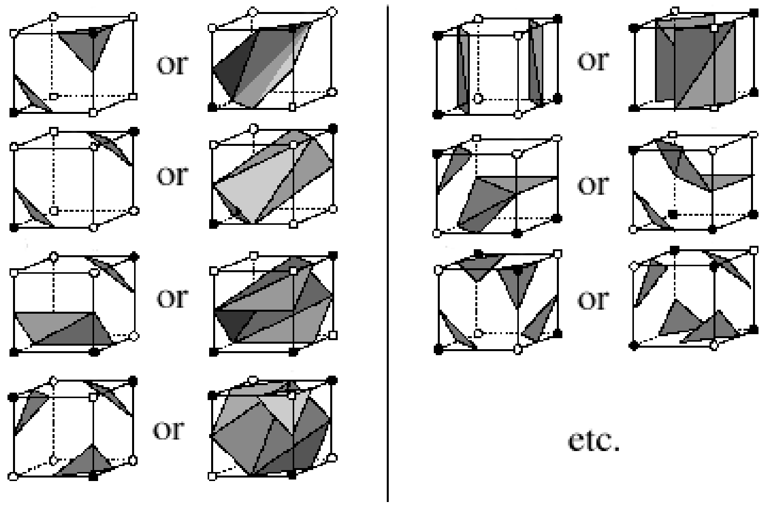

Figure 1: Local eight voxel configurations are insufficient for unique tolological specifications

$N^p_{(w_1,w_2)}$ represents the *topological unit* of an orthogonal grid in R^2. Let $A \subset \mathcal{Z}^2$ and

$$M_p := \bigcup_{(w_1,w_2)\in A} N^p_{(w_1,w_2)} \tag{2}$$

be a compact set, where $N^p_{(w_1,w_2)}$ is defined by (1). A set $M_p \subset R^2$ which consists of at least two $N^p_{(w_1,w_2)}$ elements is called *edge connected* if each element of $M_p \subset R^2$ possesses an edge connected neighbour. An edge connected set $M_p \subset R^2$ will be called a *planar grid continuum.* Important planar grid continua are simple one-dimensional planar grid continua [20, 24]:

Definition 1 *A planar grid continuum M_p is called a* simple closed planar one-dimensional grid continuum *if each element of M_p has exactly two edge connected neighbours (see Fig. 2).*

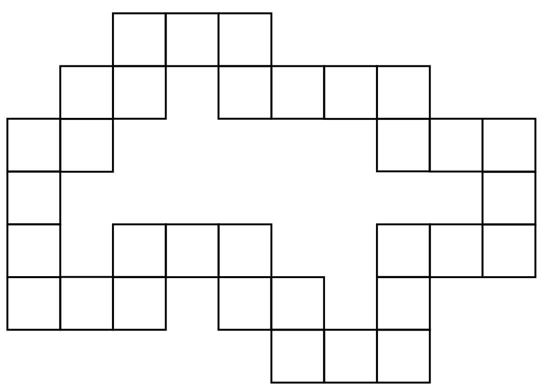

Figure 2: A simple closed planar one-dimensional grid continuum

Definition 2 *A planar grid continuum M_p is called a* simple open planar one-dimensional grid continuum *if there exist two elements of M_p which have exactly one edge connected neighbour and remaining elements of M_p, if any, have exactly two edge connected neighbours (see Fig. 3).*

A simple closed planar one-dimensional grid continuum M_p represents a polygonally bounded compact set with boundary $\partial M_p = L_1 \cup L_2$, where L_1, L_2 are simple closed polygonal Jordan curves, $L_1 \subset I(L_2)$, for which $\text{dist}(L_1, L_2) = 2^{-p}$, where 2^{-p} is the edge size of the $N^p_{(w_1,w_2)}$ element. $I(.)$ specifies the topological

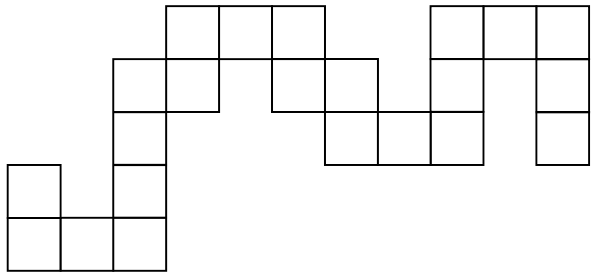

Figure 3: A simple open planar one-dimensional grid continuum

interior. A simple open planar one-dimensional grid continuum M_p represents a polygon with boundary $\partial M_p = L$, where L is a simple closed polygonal Jordan curve.

The *geodesic distance* between two points in a polygon is the length of the shortest path between these points. A polygon and the geodesic distance define a metric space. A *geodesic diameter* of a polygon is a shortest path internal to the polygon between two vertices of the polygon of maximal length. *A length* is associated to simple closed/open planar one-dimensional grid continua as follows [20, 24]:

Definition 3 *The* length of a simple closed planar one-dimensional grid continuum M_p *with boundary* $\partial M_p = L_1 \cup L_2$, *where* L_1, L_2 *are simple closed polygonal Jordan curves,* $L_1 \subset I(L_2)$, *is defined as the length of the corresponding shortest polygonal Jordan curve in* M_p *encircling* L_1 *(see Fig. 4).*

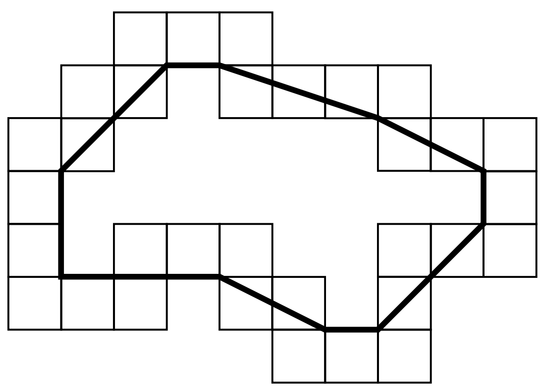

Figure 4: The length of a simple closed planar one-dimensional grid continuum

Definition 4 *The* length of a simple open planar one-dimensional grid continuum M_p *with boundary* $\partial M_p = L$, *where* L *is a simple closed polygonal Jordan curve, is defined as the length of the corresponding geodesic diameter of* M_p *(see Fig. 5).*

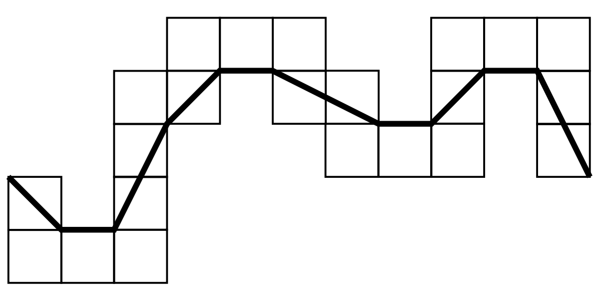

Figure 5: The length of a simple open planar one-dimensional grid continuum

Definition 5 *A simple open planar one-dimensional grid continuum M_p whose geodesic diameter is a line segment, is called a* straight line grid continuum *(see Fig. 6).*

The shortest polygonal Jordan curve in a simple closed planar one-dimensional grid continuum M_p, $\partial M_p = L_1 \cup L_2$, $L_1 \subset I(L_2)$, encircling L_1 is defined uniquely [24]. A geodesic diameter of a non-convex simple open planar one-dimensional grid continuum is also defined uniquely [20]. Algorithms for the shortest path problem solution in a polygon are summarised in [24]. The generalised Jordan curve theorem [25] holds for simple closed one-dimensional grid continua M_p:

Theorem 1 *Let $C_1, C_2 \subset R^2$ be two continua which do not cut R^2 such that $C_1 \cap C_2$ consists of two connected components. Then $C_1 \cup C_2$ cuts R^2 into two connected regions A, B.*

See Fig. 7 for an illustration of Theorem 1. In this example, $C_1 \cap C_2$ consists of two topological units (shaded squares) defining two connected components. A *continuum* is a connected compact set and a continuum $C \subset R^2$ *cuts* R^2 if there exist two points $x, y \in R^2 \setminus C$, which can not be connected by a path in $R^2 \setminus C$.

The following theorem [24] specifies a lower and an upper bound for the length estimation of convex Jordan curves in R^2:

Theorem 2 *Let M_p be a simple closed planar one-dimensional grid continuum with $\partial M_p = L_1 \cup L_2$, where $L_1 \subset I(L_2)$, and L_1, L_2 are both simple closed polygonal Jordan curves. Assume that M_p contains a convex Jordan curve $\gamma : [0, d(\gamma)] \to R^2$ encircling L_1 of length $d(\gamma)$ parametrised by arclength. Then*

$$d(CH(L_1)) \leq d(\gamma) < d(CH(L_1)) + 8 \cdot 2^{-p}, \, for \ p = 0, 1, \ldots,$$

where $d(CH(L_1))$ is the length of the boundary of the convex hull of L_1 and 2^{-p} is the edge size of the topological unit $N^p_{(w_1,w_2)}$.

In this case $d(CH(L_1))$ is identical to the length of the shortest polygonal simple closed Jordan curve in M_p encircling L_1. The theorem shows that the error of the length estimation of a planar convex Jordan curve for $p \to \infty$ tends to zero.

The following section introduces grid continua in R^3 starting with one-dimensional grid continua.

3 Grid Continua in $\mathbf{R}^3$

Let us consider an orthogonal grid in R^3. For $p = 0, 1, \ldots,$ and for each triple (w_1, w_2, w_3) of integers let

$$N^p_{(w_1,w_2,w_3)} := \{x \in R^3 \mid w_i \, 2^{-p} \leq x_i \leq (w_i + 1) \, 2^{-p}, \;\; i = 1, 2, 3\}. \tag{3}$$

$N^p_{(w_1,w_2,w_3)}$ represents the *topological unit* of an orthogonal grid in R^3. Let $A \subset \mathcal{Z}^3$ and

$$M_p := \bigcup_{(w_1,w_2,w_3) \in A} N^p_{(w_1,w_2,w_3)} \tag{4}$$

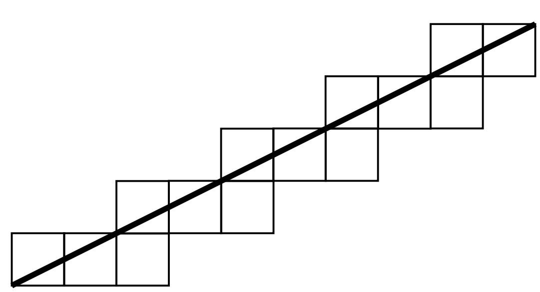

Figure 6: A straight line grid continuum in R^2

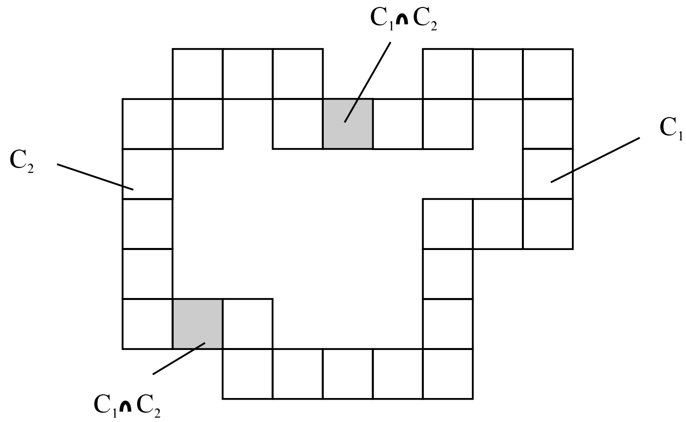

Figure 7: Illustration to the generalised Jordan curve theorem

be a compact set, where $N^p_{(w_1,w_2,w_3)}$ is defined by (3). A set $M_p \subset R^3$ which consists of at least two $N^p_{(w_1,w_2,w_3)}$ units is called *face connected* if each element of $M_p \subset R^3$ possesses a face connected neighbour. A face connected set $M_p \subset R^3$ will be called a *grid continuum* in R^3. Simple one-dimensional grid continua in R^3 are defined as follows [21]:

Definition 6 *A grid continuum $M_p \subset R^3$ is called a* simple closed one-dimensional grid continuum *if each element of M_p has exactly two face connected neighbours (see Fig. 8).*

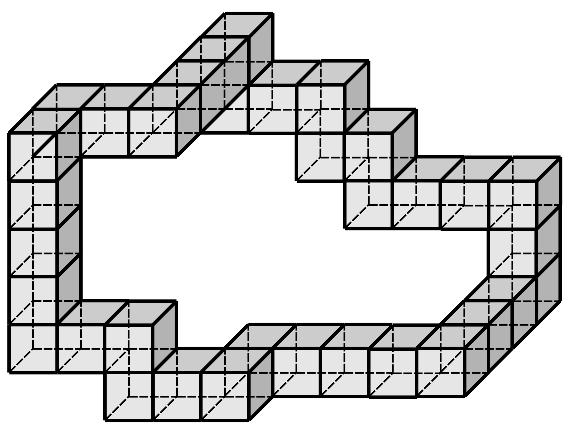

Figure 8: A simple closed one-dimensional grid continuum in R^3

Definition 7 *A grid continuum $M_p \subset R^3$ is called a* simple open one-dimensional grid continuum *if there exist two elements of M_p which have exactly one face connected neighbour and remaining elements of M_p, if any, have exactly two face connected neighbours (see Fig. 9).*

A simple closed one-dimensional grid continuum $M_p \subset R^3$ represents a polyhedrally bounded compact set which is homeomorphic with a torus. A simple open one-dimensional grid continuum $M_p \subset R^3$ represents a polyhedron with a polyhedral surface $\partial M_p = S$ homeomorphic with the unit sphere, i.e., S is a simple closed polyhedral Jordan surface.

The *geodesic distance* between two points in a polyhedron is the length of the shortest internal path between these points. A polyhedron whose boundary is homeomorphic with the unit sphere, and the geodesic distance define a metric space. A *geodesic diameter* of a polyhedron is a shortest path internal to the polyhedron between two vertices of the polyhedron of maximal length. A *length* is associated as follows [21] to simple closed/open one-dimensional grid continua in R^3:

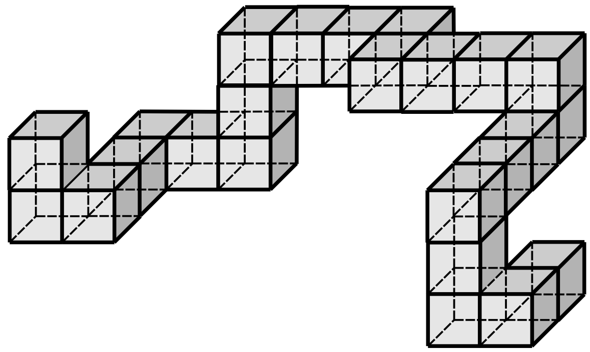

Figure 9: A simple open one-dimensional grid continuum in R^3

Definition 8 *The* length of a simple closed one-dimensional grid continuum $M_p \subset R^3$ *is defined as the length of the corresponding non-contractible shortest polygonal simple closed Jordan curve in* M_p *(see Fig. 10).*

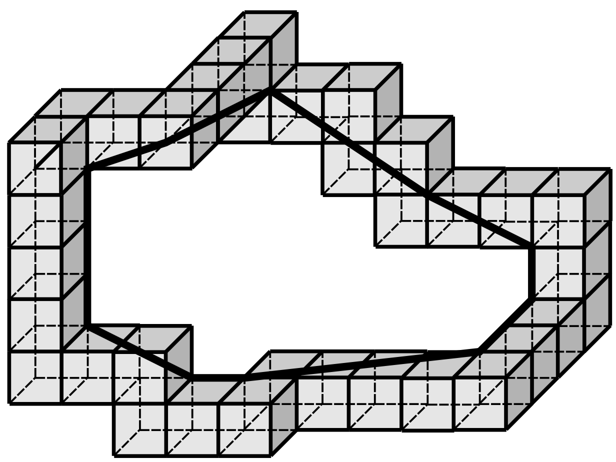

Figure 10: The length of a simple closed one-dimensional grid continuum in R^3

A *non-contractible shortest polygonal simple closed Jordan curve* in M_p means a shortest polygonal simple closed Jordan curve whose intersection with all elements of M_p is non-empty.

Definition 9 *The* length of a simple open one-dimensional grid continuum $M_p \subset R^3$ *is defined as the length of the corresponding geodesic diameter of* M_p *(see Fig. 11).*

Definition 10 *A simple open one-dimensional grid continuum* $M_p \subset R^3$*, whose geodesic diameter is a line segment, is called a* straight line grid continuum *(see Fig. 12).*

Definition 11 *A simple closed/open one-dimensional grid continuum* $M_p \subset R^3$ *is called a* simple closed/open flat one-dimensional grid continuum *in* R^3 *if all* $N^p_{(w_1,w_2,w_3)}$ *units of* M_p *lie on the same plane in* R^3*.*

A shortest non-contractible polygonal simple closed Jordan curve in a non-flat simple closed one-dimensional grid continuum in R^3 is defined uniquely [21]. A geodesic diameter of a non-flat simple open planar one-dimensional grid continuum $M_p \subset R^3$ is also defined uniquely [21].

A simple closed two-dimensional grid continuum in R^3 characterises a further class of two-dimensional grid contina $M_p \subset R^3$.

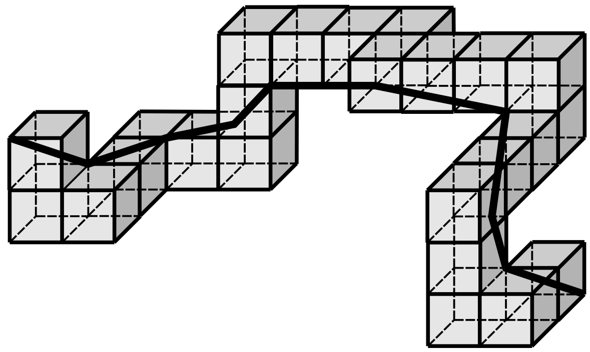

Figure 11: The length of a simple open one-dimensional grid continuum in R^3

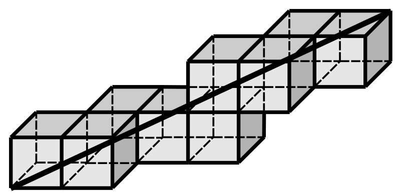

Figure 12: A straight line grid continuum in R^3

Definition 12 *A grid continuum $M_p \subset R^3$ with $\partial M_p = S_1 \cup S_2$, where S_1, S_2 are both non-empty, simple closed polyhedral Jordan surfaces with $S_1 \subset I(S_2)$, is called a* simple closed two-dimensional grid continuum *if*

$$dist(S_1, S_2) = 2^{-p},$$

where 2^{-p} is the edge size of the corresponding $N^p_{(w_1,w_2,w_3)}$ unit (see Fig. 13).

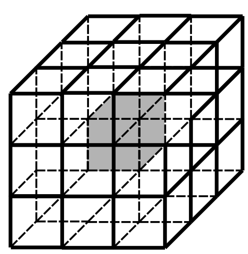

Figure 13: A simple closed two-dimensional grid continuum in R^3

The distance function *dist* is defined as the Hausdorff-Chebyshev distance between the sets S_1 and S_2.

A *surface area* is associated as follows to a simple closed two-dimensional grid continuum $M_p \subset R^3$:

Definition 13 *The* surface area *of a simple closed two-dimensional grid continuum $M_p \subset R^3$ with $\partial M_p = S_1 \cup S_2$, where S_1, S_2 are both simple closed polyhedral Jordan surfaces with $S_1 \subset I(S_2)$, is defined to be the surface area of the minimal polyhedral simple closed Jordan surface in M_p containing S_1 (see Fig. 14).*

The *minimal polyhedral simple closed Jordan surface S_{min}* means a simple closed polyhedral Jordan surface with the minimal surface area and has been introduced in [22] and [23]. See [4] for a brief discussion of this notion. The minimal polyhedral simple closed Jordan surface S_{min} in a two-dimensional simple closed grid continuum $M_p \subset R^3$ is defined uniquely [22] and it holds the following [23]:

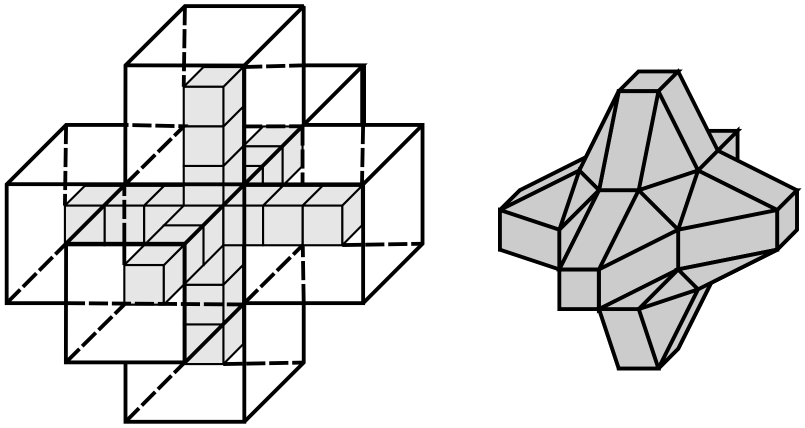

Figure 14: A simple closed two-dimensional grid continuum in R^3 and the corresponding minimal polyhedral simple closed Jordan surface

Theorem 3 *Let $M_p \subset R^3$ be a simple closed two-dimensional grid continuum with $\partial M_p = S_1 \cup S_2$, where S_1, S_2 are both simple closed polyhedral Jordan surfaces. Then all the vertices of $CH(S_1)$ belong to the vertices of the minimal polyhedral simple closed Jordan surface S_{min} and*

$$P_{S_{min}} \subseteq conv(S_1)$$

where $P_{S_{min}}$ is a polyhedron with $\partial P_{S_{min}} = S_{min}$ and $conv(S_1)$ is the convex hull of S_1 with $\partial conv(S_1) = CH(S_1)$.

See Fig. 15 for an illustration of an example. The minimal polyhedral simple closed Jordan surface S_{min} is either $CH(S_1)$ or its suitable topological deformation. The calculation of the minimal polyhedral simple closed Jordan surface specifies an interesting problem in computational geometry.

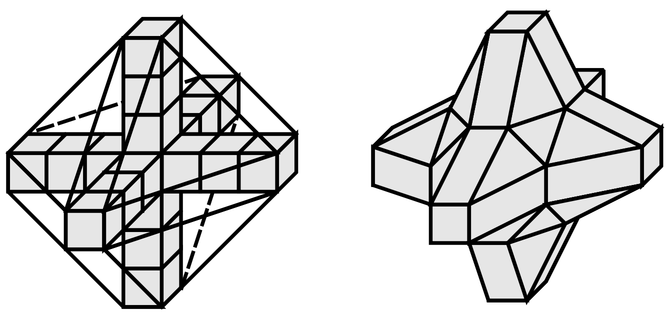

Figure 15: $CH(S_1)$ and the minimal polyhedral simple closed Jordan surface

4 Conclusions

Simple one-dimensional grid continua in R^2 or R^3, and simple closed two-dimensional grid continua in R^3 have been specified in this paper. The length of one-dimensional grid continua and the surface area of two-dimensional grid continua have been introduced. The length of an open or closed one-dimensional grid continuum in R^2 is based on the notion of the shortest polygonal simple closed Jordan curve in a polygonally bounded compact set, or on the notion of a geodesic diameter of a polygon, respectively. The length of an open or closed one-dimensional grid continuum in R^3 is based on the notion of the shortest polygonal simple closed Jordan curve in a polyhedrally bounded compact set, or on the notion of a geodesic diameter of a polyhedron, respectively. The surface area of a simple closed two-dimensional grid continuum in R^3 is based on the notion of the minimal polyhedral simple closed Jordan surface in a polyhedrally bounded compact set.

These notions allow to approximate, visualise and efficiently represent measurable simple closed/open one-dimensional continua in R^2 and R^3, and measurable simple closed two-dimensional continua in R^3 which are covered by elements of corresponding simple closed/open one-dimensional grid continua in R^2 and R^3 and by elements of a simple closed two-dimensional grid continuum in R^3, respectively. They represent the smoothest feasible approximation of corresponding measurable continua. The higher the grid point resolution the more accurate are length and surface area estimates.

The experiments with simple closed or open one-dimensional planar grid continua have shown that the shortest polygonal simple closed Jordan curves, or the geodesic diameters, respectively, were represented by $O(\sqrt{n})$ vertices, where n is the number of vertices of the corresponding grid continua [24]. A similar reduction factor is expected in the case of representation of surfaces related to simple closed two-dimensional grid continua in R^3 by the minimal polyhedral simple closed Jordan surfaces. This will enhance the applicability of algorithms for surface calculations compared to the "large numbers" of two-dimensional faces on surfaces of cellular complexes (as discussed in digital geometry), or compared to the "extremly large numbers" of triangles in local surface approximations resulting from a marching cube algorithm.

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