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Evaluation of Algorithms for Linear Shape from Shading

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Abstract

We analyse different sequential algorithms for the recovery of object shape from a single shading pattern generated under the assumption of a linear reflectance map. The algorithms are based on the finite difference approximation of the derivatives. They operate on a rectangular discrete image (or part of it) and use the height of the sought-after surface along a curve in the image (image boundary) as initial data. The evaluation of different numerical schemes is achieved by comparing stability, convergence, and domains of influence of each scheme in question. The relative difficulty of handling a linear case indicates that the case of non-linear reflectance maps is far from being trivial.

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1 Introduction

In this paper, we present some results concerning the shape-from-shading problem in which the reflectance map is linear. Such a special case arises *e.g.* in the study of the maria of the moon (see [1, Subsections 10.9 and 11.1.2]). If a small portion of a surface, described by the graph of a function u, having reflectivity properties approximated by a linear reflectance map, is illuminated by a distant point source of unit power in direction $(a_1, a_2, -1)$, then the corresponding image $\mathcal{E}(x_1, x_2)$ satisfies a *linear image irradiance equation* of the following form

$$\left(a_1 \frac{\partial u}{\partial x_1}(x_1, x_2) + a_2 \frac{\partial u}{\partial x_2}(x_1, x_2) + 1\right)(a_1^2 + a_2^2 + 1)^{-1/2} = \mathcal{E}(x_1, x_2), \tag{1}$$

over $\Omega = \{(x, y) \in \mathbb{R}^2 : \mathcal{E}(x_1, x_2) > 0\}$. Letting $E(x_1, x_2) = \mathcal{E}(x_1, x_2)(a_1^2 + a_2^2 + 1)^{1/2} - 1$, one can rewrite (1) as a *transformed* linear image irradiance equation

$$a_1 \frac{\partial u}{\partial x_1}(x_1, x_2) + a_2 \frac{\partial u}{\partial x_2}(x_1, x_2) = E(x_1, x_2). \tag{2}$$

In this paper we evaluate different finite difference algorithms for a direct shape recovery modelled by the equation (2). The original idea of this work is an extension of Kozera work in [4, 5], where the convergence analysis of the finite difference scheme based on central difference approximation of the derivatives has been discussed. We continue to investigate here the issue of the stability and the convergence of different algorithms based on the combination of the forward and backward derivative approximations. Convergence, stability, and domain of influence, will be considered here as algorithmic features and used in this paper for evaluating shape reconstruction algorithms based on finite difference schemes. Critical to our approach is the assumption that u is given along some (not necessarily smooth) initial curve γ in the image (image boundary). The algorithms provide the numerical solution of the following Cauchy problem (for $u \in C(\bar{\Omega}) \cap C^2(\Omega)$) considered over a rectangle Ω:

$$L(u(x_1, x_2)) = E(x_1, x_2) \tag{3a}$$

$$u(x_1, 0) = f(x_1) \quad 0 \le x_1 \le a, \text{ for } \operatorname{sgn}(a_1 a_2) \ge 0, \tag{3b}$$

$$u(x_1, b) = f(x_1) \quad 0 \le x_1 \le a, \text{ for } \operatorname{sgn}(a_1 a_2) < 0, \tag{3c}$$

$$u(0, x_2) = g(x_2) \quad 0 \le x_2 \le b; \tag{3d}$$

here $Lu = a_1 u_{x_1} + a_2 u_{x_2}$, and functions $f \in C([0, a]) \cap C^2((0, a))$ and $g \in C([0, b]) \cap C^2((0, b))$ satisfy $f(0) = g(0)$, $E \in C^2(\overline{\Omega})$, and a_1 and a_2 are constants such that $(a_1, a_2) \neq (0, 0)$. To simplify consideration we will assume that $\mathrm{sgn}(a_1 a_2) \geq 0$ and $a_2 \neq 0$ and therefore only the case $(3)(a, b, d)$ will be considered. The remaining cases can be treated analogously.

For a full version of this paper containing more experimental results and both convergence and well-posedness proofs an interested reader is referred to Kozera and Klette [6]. Also a more detailed information about general shape-from-shading problem can be found *e.g.* in [1], Horn and Brooks [2], or Klette *et al.* [3].

2 Basic Notions and Theory for Finite Difference Schemes

We recall now standard definitions and results from the theory of solving linear partial differential equations with the aid of the finite difference method (see Van der Houwen [8, Chapter 1]).

Assume that an interval $[0, T]$ and a domain $G \subset \mathbb{R}$ together with its boundary Γ and $\bar{G} = G \cup \Gamma$ are given and that $(E_0(\bar{G}), \| \, \|_{E_0})$, $(E(\bar{G}), \| \, \|_{\bar{G}})$, $(E(\Gamma), \| \, \|_G)$, and $(E(G), \| \, \|_G)$ are linear normed spaces of scalar (vector) functions, defined respectively, on the set of points $\bar{G}$, $\bar{G} \times [0, T]$, $\Gamma \times [0, T]$, and $G \times [0, T]$. Consider now the following problem

$$U_t(x,t) + D(x,t)U_x(x,t) = H(x,t), \quad U(\Gamma \times [0,T]) = \Psi(\Gamma), \quad \text{and} \quad U(x,0) = U_0(x), \tag{4}$$

where $(x, t) \in G \times [0, T]$, the scalar functions $U_0 \in E_0(\bar{G})$, $\Psi \in E(\Gamma)$, and the vector function $F(x,t) = (H(x,t), D(x,t)) \in E(G)$. An initial boundary value problem $\mathcal{L}U = (U_0, F, \Psi)$ may be interpreted as a mapping of the unknown function U onto the triple of functions (U_0, H, Ψ), or if we want to include dependence of U on the vector coefficient D, as a mapping onto a triple (U_0, F, Ψ).

More precisely, problem of finding the inverse of a given mapping $\mathcal{L} : D_{\mathcal{L}} \to \Delta_{\mathcal{L}}$ of an unknown function $U \in D_L = (E(\bar{G}), \| \, \|_{\bar{G}})$ onto a known element $(U_0, F, \Psi) \in \Delta_{\mathcal{L}} = (E_0(\bar{G}) \times E(G) \times E(\Gamma), \| \, \|_{\times})$, where $\|(U_0, F, \Psi)\|_{\times} = \|U_0\|_{E_0} + \|F\|_G + \|\Psi\|_{\Gamma}$, will be called *initial boundary value problem.*

The initial boundary value problem $\mathcal{L}U = (U_0, F, \Psi)$ is said to be *well-posed* with respect to the norms in $E(\bar{G})$ and in $E_0(\bar{G}) \times E(G) \times E(\Gamma)$ if $\mathcal{L}$ has a unique inverse $\mathcal{L}^{-1}$ which is continuous at the point (U_0, F, Ψ).

We shall now introduce the definition of the *uniform grid sequence.* We replace the continuous interval $[0, T]$ by a discrete set of points $[t_0 = 0, t_1, t_2, \ldots, t_M = T]$, where $t_{i+1} - t_i = \Delta t$ (for each $i \in [0, \ldots, M-1]$) and $M\Delta t = T$, together with a finite set of points $\Gamma_{\Delta t} \subset \Gamma$ such that the distance between two consecutive points in the X-axis direction satisfies $\Delta x = \mathcal{N}(\Delta t)$ and $N\mathcal{N}(\Delta t) = \mu(G)$, where $\mu(G)$ denotes the measure of G. These three sets of points constitute *a grid* or *net* $Q_{\Delta t}$ in

$\bar{G} \times [0,T]$ *i.e.* $Q_{\Delta t} = \bar{G}_{\Delta t} \times \{t_k\}_{k=0}^{N}$, where $\bar{G}_{\Delta t} = G_{\Delta t} \cup \Gamma_{\Delta t}$. We assume that *a sequence of nets* $Q_{\Delta t}$ is defined in such a way that $\lim_{\Delta t \to 0^+} Q_{\Delta t}$ is dense in $\bar{G} \times [0,T]$. The last requirement is satisfied when $\lim_{\Delta t \to 0^+} \mathcal{N}(\Delta t) = 0$. Furthermore, we introduce the corresponding normed grid spaces

$$(E_0(\bar{G}_{\Delta t}), \| \ \|_{E_0 \Delta t}), \quad (E(\bar{G}_{\Delta t}), \| \ \|_{\bar{G}_{\Delta t}}), \quad (E(\Gamma_{\Delta t}), \| \ \|_{\Gamma_{\Delta t}}), \quad \text{and} \quad (E(G_{\Delta t}), \| \ \|_{G_{\Delta t}}) \tag{5}$$

defined on the sets $\bar{G}_{\Delta t}$, $\bar{G}_{\Delta t} \times \{t_k\}_{k=0}^{N}$, $\Gamma_{\Delta t} \times \{t_k\}_{k=0}^{N}$, and $G_{\Delta t} \times \{t_k\}_{k=0}^{N}$, respectively. The elements of these spaces are called *net functions* and will be denoted by lower case letters u_0, u, ψ, and f.

A mapping $\mathcal{R}_{\Delta t}$ of an unknown net function u of $(E(\bar{G}_{\Delta t}), \| \ \|_{\bar{G}_{\Delta t}})$ into the known element (u_0, f, ψ) of $(E_0(\bar{G}_{\Delta t}) \times E(G_{\Delta t}) \times E(\Gamma_{\Delta t}), \| \ \|_{\Delta t_\times})$, where $\|(u_0, f, \psi)\|_{\Delta t_\times} = \|u_0\|_{E_0 \Delta t} + \|f\|_{G_{\Delta t}} + \|\psi\|_{\Gamma_{\Delta t}}$ is defined for each net $Q_{\Delta t}$, will be called *a finite difference scheme*. Difference schemes can be described by the equation $\mathcal{R}_{\Delta t} u = (u_0, f, \psi)$, with the domain and range of $\mathcal{R}_{\Delta t}$ denoted by $D_{\mathcal{R}\Delta t}$ (called as *a discrete domain of influence*) and $\Delta_{\mathcal{R}\Delta t}$, respectively. It will be assumed that both $D_{\mathcal{R}\Delta t}$ and $\Delta_{\mathcal{R}\Delta t}$ are linear spaces and $\mathcal{R}_{\Delta t}$ has *a unique inverse* $\mathcal{R}_{\Delta t}^{-1}$, which is *continuous* in $D_{\mathcal{R}\Delta t}$ for every $\Delta t \neq 0$. We can also define a set $D_I \subset \Omega$ called *a domain of influence* as $D_I = cl\,(\bigcup D_{\mathcal{R}\Delta t})$ (where symbol *cl* denotes the set closure operation) which clearly depends on a given initial boundary value problem, grid sequence and associated finite difference scheme.

Let us now introduce the *discretisation operator* $[\ \ ]_{d(\Delta t)}$ which transforms a function $U \in E(\bar{G})$ to its discrete analogue $[U]_{d(\Delta t)}$ defined as U reduced to the domain of the net $Q_{\Delta t}$. In the same manner we can define discretised elements $[U_0]_{d(\Delta t)} \in E_0(\bar{G}_{\Delta t})$, $[F]_{d(\Delta t)} \in E(G_{\Delta t})$, and $[\Psi]_{d(\Delta t)} \in E(\Gamma_{\Delta t})$. In this paper we shall use the convention:

$$[U]_{d(\Delta t)} = u, \quad [U_0]_{d(\Delta t)} = u_0, \quad [F]_{d(\Delta t)} = f, \quad \text{and} \quad [\Psi]_{d(\Delta t)} = \psi,$$

where $f = (h, d)$. Moreover, it is also assumed that the norms on the grid sequence $\{Q_{\Delta t}\}_{\Delta t}$ *match* the corresponding norms from the related *"continuous spaces" i.e.*

$$\|u\|_{\bar{G}_{\Delta t}} \to \|U\|_{\bar{G}}, \quad \|u_0\|_{E_0 \Delta t} \to \|U_0\|_{E_0}, \quad \|f\|_{G_{\Delta t}} \to \|F\|_G, \quad \text{and} \quad \|\psi\|_{\Gamma_{\Delta t}} \to \|\Psi\|_\Gamma, \tag{6}$$

as $\Delta t \to 0$.

We shall now introduce the basic definitions. Assume now that $\widetilde{U}$ is a solution to the initial boundary value problem $\mathcal{L}\widetilde{U} = (U_0, F, \Psi)$, and that u is a solution to the corresponding discrete problem

$$\mathcal{R}_{\Delta t} u = (u_0, f, \psi). \tag{7}$$

If $\mathcal{R}_{\Delta t}$ is to be a good approximation of $\mathcal{L}$ we should expect that the function $\widetilde{u} = [\widetilde{U}]_{d(\Delta t)}$, for some element $(\widetilde{u}_0, \widetilde{f}, \widetilde{\psi})$, satisfies a finite difference equation $\mathcal{R}_{\Delta t}\widetilde{u} = (\widetilde{u}_0, \widetilde{f}, \widetilde{\psi})$ which closely relates to (7). The value $\|[\mathcal{L}\widetilde{U}]_{d(\Delta t)} - \mathcal{R}_{\Delta t}\widetilde{u}\|_{\Delta t_\times}$ is called the *error of approximation.* The value $\|u - \widetilde{u}\|_{\bar{G}_{\Delta t}}$ is in turn called *the discretisation error.*

Definition 1. We say that a difference scheme is *consistent* with an initial boundary value problem if the error of approximation converges to zero as $\Delta t \to 0$. We say also that a difference scheme is *convergent* to the solution u (if it exists) if the discretisation error converges to zero as $\Delta t \to 0$.

Finally, we shall define a notion of stability for the linear difference schemes in the sense of Rjabenki and Filippov (see also [8]).

Definition 2. A linear difference scheme is *R-F stable* if operators $\{\mathcal{R}_{\Delta t}^{-1}\}$ are uniformly bounded as $\Delta t \to 0$.

The natural question arises here about the relationship between the stability and convergence of the consistent difference schemes. Combining the Definition 5.3 and 6.2 with the Theorem 5.1 (see Van der Houwen [8]) we have the following:

Theorem 1 *A consistent and R-F stable finite-difference scheme is convergent to the solution of* $\mathcal{L}\widetilde{U} = (U_0, F, \Psi)$ *(if such solution exists).*

Of course for a Cauchy problem (3)(a,b,d), we have $T = b$, $x_2 = t$, $G = (0, a)$, $\Gamma = \{0\}$, $U_0(x_1) = f(x_1)$, $\Psi(\Gamma) = g(x_2)$, $H(x_1, x_2) = (1/a_2)E(x_1, x_2)$, and $D(x_1, x_2) = (a_1/a_2)$. The corresponding normed spaces are assumed to be here as follows:

$$E_0(\bar{G}) = \{U_0 : [0, a] \to \mathbb{R} : U_0 \in C([0, a]) \cap C^2((0, a))\}$$

with $\|U_0\|_{E_0} = max_{x_1 \in [0,a]}|U_0(x_1)|$,

$$E(\bar{G}) = \{(E, (a_1/a_2)) : [0, a] \times [0, b] \to \mathbb{R}^2 : E, (a_1/a_2) \in C([0, a] \times [0, b]) \cap C^2((0, a) \times (0, b))\}$$

with $\|(E, (a_1/a_2))\|_{\bar{G}} = max_{(x_1,x_2) \in [0,a]\times[0,b]}|E(x_1, x_2)| + max_{(x_1,x_2) \in [0,a]\times[0,b]}|(a_1/a_2)(x_1, x_2)|$,

$$E(\Gamma) = \{g : \{0\} \times [0, b] \to \mathbb{R} : g \in C(\{0\} \times [0, b]) \cap C^2(\{0\} \times (0, b))\}$$

with $\|g\|_G = max_{x_2 \in [0,b]}|g(0, x_2)|$, and

$$E(G) = \{U : (0, a) \times [0, b] \to \mathbb{R} : U \in C([0, a] \times [0, b]) \cap C^2((0, a) \times (0, b))\}$$

with $\|U\|_G = max_{(x_1,x_2)\in[0,a]\times[0,b]}|U(x_1,x_2)|$.

In a similar manner we can introduce discrete analogues of the above case of *"continuous infinity norms"* in the corresponding grid spaces (5). It is clear that such discrete analogues satisfy *compatibility conditions* (6).

3 Evaluation of Different Finite-Difference Schemes

We shall pass now to the derivation of a number of finite difference schemes for the linear shape-from-shading problem defined by (3)(a,b,d). We assume here that $\Delta x_2 = (b/M)$, $\Delta x_1 = (a/M)$ (where $M \in [0,1,\ldots,\infty]$; so $M = N$), $((a_1\Delta x_2)/(a_2\mathcal{N}(\Delta x_2))) = const$, and that a function u is a C^2 solution to (2). For the sake of convenience we assume that problem (3) is well-posed (for sufficient conditions assuring well-posedness of (3) see [6]).

3.1 Forward-Forward Finite Difference Approximation

Applying *forward difference* approximations together with Taylor's formula yields

$$\left.\frac{\partial u}{\partial x_1}\right|_j^n = \frac{u_{j+1}^n - u_j^n}{\Delta x_1} + O(\Delta x_1) \quad \text{and} \quad \left.\frac{\partial u}{\partial x_2}\right|_j^n = \frac{u_j^{n+1} - u_j^n}{\Delta x_2} + O(\Delta x_2), \tag{8}$$

for any $j,n \in \{1,\ldots,M-1\}$; here u_j^n, $\left.\frac{\partial u}{\partial x_1}\right|_j^n$, and $\left.\frac{\partial u}{\partial x_2}\right|_j^n$ denote the values of u, $\frac{\partial u}{\partial x_1}$, and $\frac{\partial u}{\partial x_2}$, respectively, at the point (x_{1j}, x_{2n}) in the grid; Δx_1 and Δx_2 denote the distances between grid points in the respective directions; M denotes the density of the grid. By substituting (8) into (2) at each point (x_{1j}, x_{2n}), we get

$$a_1\frac{u_{j+1}^n - u_j^n}{\Delta x_1} + a_2\frac{u_j^{n+1} - u_j^n}{\Delta x_2} + O(\Delta x_1, \Delta x_2) = E_j^n. \tag{9}$$

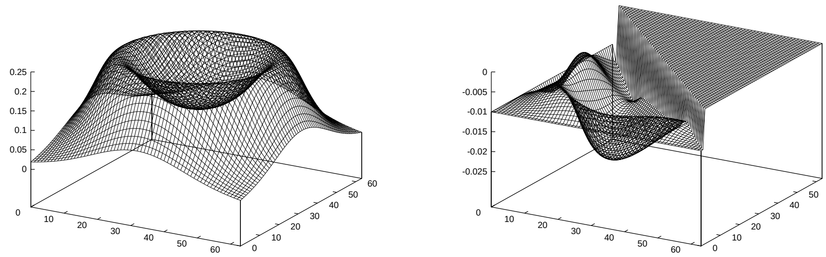
Denoting by v an approximate of u, we obtain from (9) the following finite difference equation

$$a_1\frac{v_{j+1}^n - v_j^n}{\Delta x_1} + a_2\frac{v_j^{n+1} - v_j^n}{\Delta x_2} = E_j^n.$$

This leads to the sequential two-level *explicit* scheme

$$v_j^{n+1} = \left(1 + \frac{a_1\Delta x_2}{a_2\Delta x_1}\right)v_j^n - \frac{a_1\Delta x_2}{a_2\Delta x_1}v_{j+1}^n + \frac{\Delta x_2}{a_2}E_j^n \tag{10}$$

with $j,n \in \{1,\ldots,M-1\}$. The above formula is an example of the so-called *explicit iterative canonical form*, (see [6]) for which a single value of v at level $n+1$, depends explicitly on the values of v from the preceding levels. We are ready now to establish the following result (for a full proof see [6]):



(a) (b)

Fig. 1. (a) The graph of the function $u_v(x,y) = (1/(4(1+(1-x^2-y^2)^2)))$ being a volcano-like surface. (b) The graph of the function $u_m(x,y) = (1/(2(1+x^2+y^2)^2))$ being a mountain-like surface.

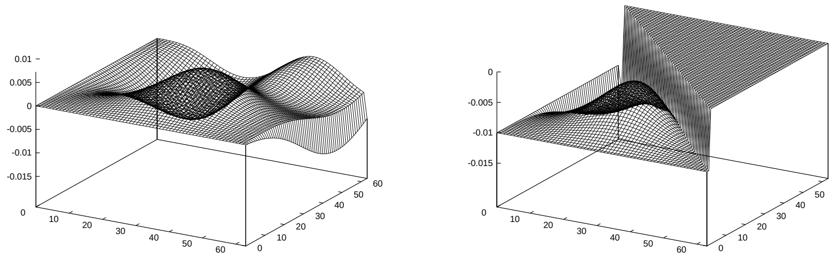
Theorem 2 *Consider the problem* (3) *over a rectangle* Ω*. Let* $\alpha = (a_1\Delta x_2)(a_2\Delta x_1)^{-1}$ *be a fixed constant. Then, numerical scheme* (10) *is R-F stable, if and only if* $-1 \le \alpha \le 0$*. Consequently (by Th.* 1*), for* $-1 \le \alpha \le 0$*, the sequence of functions* $\{u_{\Delta x_2}\}$ *(where each* $u_{\Delta x_2}$ *is a solution of* (10) *with* Δx_2 *temporarily fixed) is convergent to the solution of the Cauchy problem* (3)*, while* $\Delta x_2 \to 0$*.*

As mentioned before, given an initial boundary value problem (3), the scheme (10) recovers the unknown shape over a domain of influence which, for $a_1 \neq 0$ and $M = N$, coincides with

$$D_I = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \le x_1 \le a, \quad \text{and} \quad 0 \le x_2 \le (-b/a)x_1 + b\}, \tag{11}$$

and for $a_1 = 0$ with the entire $\bar{\Omega}$.

The algorithm (10) has been tested on a number of commonly encountered shapes. For example, with $\Delta x_1/\Delta x_2 = 1.0$, $a_1 = -0.5$, and $a_2 = 1.0$, and thus $\alpha = -0.5$, the volcano-like surface represented by the graph of the function $u_v(x,y) = (1/(4(1+(1-x^2-y^2)^2)))$ (see Figure 1a) and for the mountain-like surface represented by the graph of the function $u_m(x,y) = (1/(2(1+x^2+y^2)^2))$ (see Figure 1b) were taken as test surfaces. The absolute errors between heights of the ideal and computed surfaces are presented in Figure 2. It should also be noted that for $\alpha \notin [-1, 0]$ an implementation of numerical scheme (10), for both volcano-like and mountain-like surfaces, resulted in instability of (10) (see [6]).



(a) (b)

Fig. 2. (a) The absolute error between volcano-like and computed surface for the forward-forward scheme. (b) The absolute error between mountain-like and computed surface for the forward-forward scheme.

3.2 Backward-Forward Finite Difference Approximation

Applying now a *backward difference* approximation to u_{x_1}

$$\left.\frac{\partial u}{\partial x_1}\right|_j^n = \frac{u_j^n - u_{j-1}^n}{\Delta x_1} + O(\Delta x_1),$$

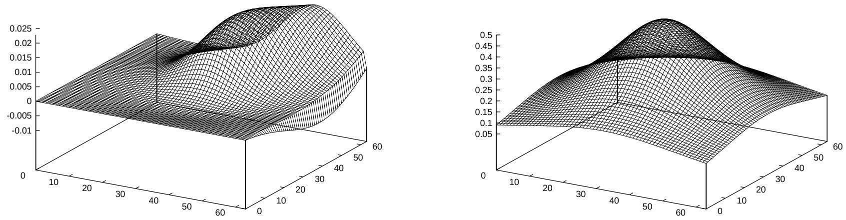
and a *forward difference* approximation to u_{x_2} leads to the corresponding two-level *explicit* finite difference scheme

$$v_j^{n+1} = \left(1 - \frac{a_1 \Delta x_2}{a_2 \Delta x_1}\right) v_j^n + \frac{a_1 \Delta x_2}{a_2 \Delta x_1} v_{j-1}^n + \frac{\Delta x_2}{a_2} E_j^n, \tag{12}$$

with $j, n \in \{1, \ldots, M-1\}$. We shall now establish the corresponding stability and convergence result for the latter finite difference scheme (for a full proof see [6]).

Theorem 3 *Consider the problem* (3) *over a rectangle* Ω*. Let* $\alpha = (a_1 \Delta x_2)(a_2 \Delta x_1)^{-1}$ *be a fixed constant. Then, numerical scheme* (12) *is R-F stable, if and only if* $0 \leq \alpha \leq 1$*. Consequently (by Th.* 1*), for* $0 \leq \alpha \leq 1$*, the sequence of functions* $\{u_{\Delta x_2}\}$ *(where each* $u_{\Delta x_2}$ *is a solution of* (12) *with* Δx_2 *temporarily fixed) is convergent to the solution of the Cauchy problem* (3)*, while* $\Delta x_2 \to 0$*.*

As easily verified the domain of influence of scheme (12) coincides with $\bar{\Omega}$, for arbitrary α. Thus given the criterion of deriving a global shape reconstruction algorithm, it is clear that (12) provides a better reconstruction means as opposed to (10). Of course the last observation is based on the assumption that the Cauchy problem (3)(a,b,d) is here considered.



(a) (b)

Fig. 3. (a) The absolute error between volcano-like and computed surface for the backward-forward scheme. (b) The absolute error between mountain-like and computed surface for the backward-forward scheme.

The algorithm (12) has been tested for the same shapes as in the previous case. With $\Delta x_1/\Delta x_2 = 1.0$, $a_1 = 0.5$, and $a_2 = 1.0$, and thus $\alpha = 0.5$, the absolute errors between heights of the ideal and computed surfaces are presented in Figure 3.

3.3 Forward-Backward Finite Difference Approximation

Applying now a *forward difference* approximation to u_{x_1} and a *backward difference* approximation to u_{x_2} leads to the following two-level *explicit horizontal* scheme

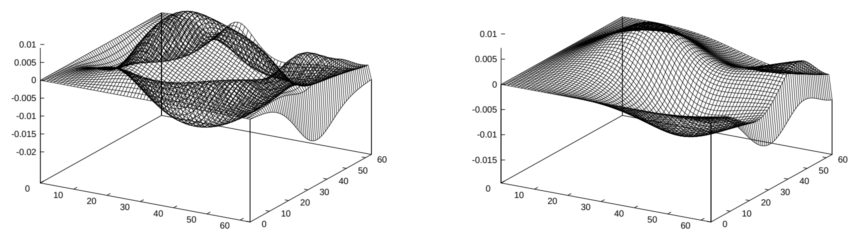
$$v_{j+1}^n = \left(1 - \frac{a_2 \Delta x_1}{a_1 \Delta x_2}\right) v_j^n + \frac{a_2 \Delta x_1}{a_1 \Delta x_2} v_j^{n-1} + \frac{\Delta x_1}{a_1} E_j^n, \tag{13}$$

(for $a_1 \neq 0$), or otherwise to the following *vertical* two-level *explicit* scheme

$$v_j^n = v_j^{n-1} + \frac{\Delta x_2}{a_2} E_j^n, \tag{14}$$

with $j, n \in \{1, \ldots, M-1\}$. Observe that for the scheme (13) the role of increment step Δt is played by Δx_1 (if we do not want to deal with *implicit schemes*). Clearly, the shape reconstruction proceeds here sequentially along X_1-axis direction (opposite to the so far presented cases). In a natural way, the boundary condition is represented by the function $f(x_1)$ and the corresponding initial condition by the function $g(x_2)$. We shall present now the next convergence result for the schemes (13) and (14) (for a full proof see [6]).

Theorem 4 *Consider the problem* (3) *over a rectangle* Ω. *Let* $\tilde{\alpha} = (a_2 \Delta x_1)(a_1 \Delta x_2)^{-1}$ *be a fixed constant. Then, numerical scheme* (13) *is R-F stable, if and only if* $0 \leq \tilde{\alpha} \leq 1$. *Consequently (by Th.* 1*), for* $0 \leq \tilde{\alpha} \leq 1$, *the sequence of functions* $\{u_{\Delta x_1}\}$ *(where each*



(a) (b)

Fig. 4. (a) The absolute error between volcano-like and computed surface for the forward-backward scheme. (b) The absolute error between mountain-like and computed surface for the forward-backward scheme.

$u_{\Delta x_1}$ *is a solution of* (13) *with* Δx_1 *temporarily fixed) is convergent to the solution of the Cauchy problem* (3), *while* $\Delta x_1 \to 0$. *Moreover, numerical scheme* (14) *is R-F stable and its sequence of computed solutions* $\{u_{\Delta x_2}\}$ *converges to the solution of the corresponding Cauchy problem* (3), *while* $\Delta x_2 \to 0$.

A simple inspection shows, that for both schemes the domains of influence coincide with $\bar{\Omega}$. For the sake of brevity we discuss here only the performance of the scheme (14). It has been tested for the same sample surfaces as in the previous cases. With $\Delta x_1/\Delta x_2 = 1.0$, $a_1 = 1.0$, and $a_2 = 0.5$, and thus $\widetilde{\alpha} = 0.5$, the absolute errors between heights of the ideal and computed surfaces are presented in Figure 4.

3.4 Backward-Backward Finite Difference Approximation

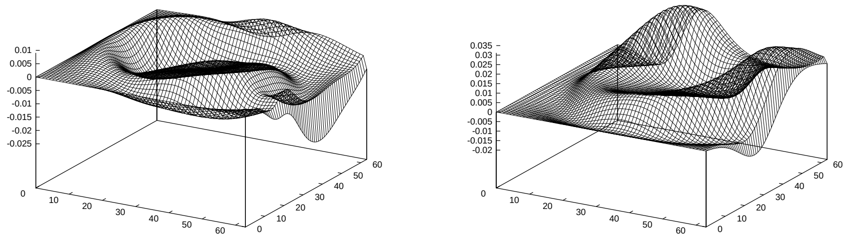
Applying now *backward difference* approximation for both derivatives u_{x_1} and u_{x_2} we arrive at the following two-level *implicit* scheme

$$v_j^n = \frac{1}{1+\alpha} v_j^{n-1} + \frac{\alpha}{1+\alpha} v_{j-1}^n + \frac{\Delta x_2}{a_2(1+\alpha)} E_j^n \tag{15}$$

(for $\alpha \neq -1$), or otherwise at the following two-level *explicit* scheme

$$v_{j-1}^n = u_j^{n-1} + \frac{\Delta x_2}{a_2} E_j^n, \tag{16}$$

with $j, n \in \{1, \ldots, M-1\}$ and $\alpha = (a_1 \Delta x_2 / a_2 \Delta x_1)$. It is clear that (15) (due to its symmetry) cannot be straightforward reduced to the canonical explicit iterative form by a mere change of the "evolution direction" (as presented in the last subsection). This implicit scheme can still be however transformed



(a) (b)

Fig. 5. (a) The absolute error between volcano-like and computed surface for the backward-backward scheme. (b) The absolute error between mountain-like and computed surface for the backward-backward scheme.

to such explicit form (see Subsection 3.4 in [6]). We present now a theorem establishing the stability and convergence of the schemes (15) and (16) (for a full proof see [6]).

Theorem 5 *Consider the problem* (3) *over a rectangle* Ω. *Let* $\alpha = (a_1 \Delta x_2)(a_2 \Delta x_1)^{-1}$ *be a fixed constant. Then, numerical scheme* (15) *is R-F stable, if and only if* $\alpha \geq 0$. *Consequently (by Th.* 1*), for* $\alpha \geq 0$, *the sequence of functions* $\{u_{\Delta x_2}\}$ *(where each* $u_{\Delta x_2}$ *is a solution of* (15) *with* Δx_2 *temporarily fixed) is convergent to the solution of the Cauchy problem* (3)*, while* $\Delta x_2 \to 0$. *Moreover, numerical scheme* (16) *is R-F stable and its sequence of computed solutions* $\{u_{\Delta x_2}\}$ *converges to the solution of the corresponding Cauchy problem* (3)*, while* $\Delta x_2 \to 0$.

An easy inspection shows that the domain of influence for the scheme (15) covers the entire $\bar{\Omega}$, whereas for (16) coincides with (11). We discuss briefly the corresponding experimental results only for the scheme (15). It has been tested for the same functions as previously. With $\Delta x_1/\Delta x_2 = 1.0$, $a_1 = 0.5$, and $a_2 = 1.0$, and thus $\alpha = 0.5$, the absolute errors between heights of the ideal and computed surfaces are presented in Figure 5.

Conclusions

A number of algorithms based on the finite difference method applied to linear shape from shading is here presented and analysed. The evaluation of different numerical schemes is achieved by comparing stability, convergence, and domains of influence of each scheme in question. The relative difficulty of handling a linear case indicates that the case of non-linear reflectance maps is far from being trivial. It should be pointed out, however, that a finite difference technique can also be applied for the non-linear PDEs (see *e.g.* Rosinger [7]). We conclude this paper by remarking that the base characteristic direction (a_1, a_2) coincides with the x_2-axis (x_1-axis) direction only if $a_1 = 0$ and $a_2 \neq 0$ ($a_2 = 0$ and $a_1 \neq 0$). Thus the algorithms discussed in this paper are essentially different from Horn's one (see [1, Subsection 11.1.2]). Moreover, in comparison with the method of characteristic strips used by Horn, all schemes presented in this paper are supplemented by a full convergence analysis.

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