Border and Surface Tracing - Theoretical Foundations

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Abstract

In this paper we define and study digital manifolds of arbitrary dimension, and provide (in particular) a general theoretical basis for curve or surface tracing in picture analysis. The studies involve properties such as one-dimensionality of digital curves and $(n-1)$-dimensionality of digital hypersurfaces that makes them discrete analogs of corresponding notions in continuous topology. The presented approach is fully based on the concept of adjacency relation and complements the concept of dimension as common in combinatorial topology. This work appears to be the first one on digital manifolds based on a graph-theoretical definition of dimension. In particular, in the n-dimensional digital space, a digital curve is a one-dimensional object and a digital hypersurface is an $(n-1)$-dimensional object, as it is in the case of curves and hypersurfaces in the Euclidean space. Relying on the obtained properties of digital hypersurfaces, we propose a uniform approach for studying good pairs defined by separations and obtain a classification of good pairs in arbitrary dimension. We also discuss possible applications of the presented definitions and results.

Index Terms

digital geometry, digital topology, discrete dimension, digital manifold, digital curve, digital hypersurface, good pair.

I. Introduction

A. Objectives and Motivation

The main objective of this work is to provide intuitively reasonable and mathematically sound definitions for digital manifolds and, on this basis, to further develop the theory of good pairs of adjacency relations.

In continuous topology, dimension is the basic property that determines a curve or a surface. For example, the following (simple) definition is used in mathematics since the early 19th century: A curve is a one-dimensional continuum (Urysohn 1923 [50], Menger 1932 [37]). [1] Obviously, this formal definition perfectly matches the intuitive notion of a curve. For example, without any doubt, the set of points in Figure 1 (left) is a curve, while those in

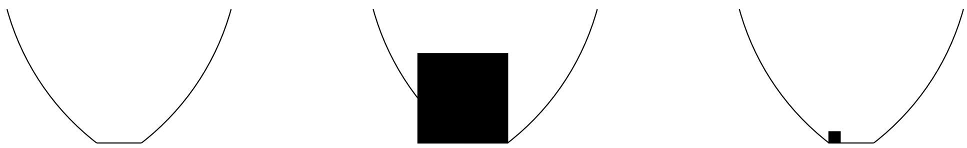

Fig. 1. *Left:* A set of points that is a curve in the real plane. *Middle, Right:* Sets of points that are not curves since each of them contains as a proper subset a square that has the topological dimension two.

Figures 1 (middle, right) are not. However, it is well-known that, due to the very nature of the discrete world, considerations in discrete spaces are more complex both from a formal and an intuitive point of view. Various attempts to establish notions and results, analogous to those known for continuous sets, have often faced difficulties

[1] *A continuum* in $\mathbb{R}^2$ is a nonempty subset of a topological space that is compact (closed and bounded) and topologically connected.

due to ambiguities (sometimes called “paradoxes”) that do not exist in continuous Euclidean space. In Section III-A, this known situation is briefly recalled by an example comparing the Jordan-Veblen curve theorem with a discrete counterpart. (For a more comprehensive treatment of this issue, see Section 1.1.4 of [26].)

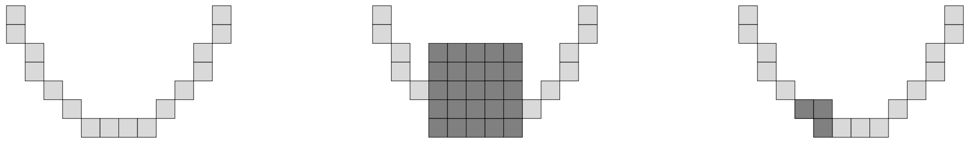

Fig. 2. We assume 0-adjacency. *Left:* This set of pixels satisfies all known definitions of a digital curve. *Middle:* This set of pixels does not satisfy any definition of a digital curve because of the 5×5 (gray) block. *Right:* This set of pixels contains a small (gray) block. Is it a digital 0-curve?

Difficulties in establishing a sound analogy between continuous and discrete spaces have caused (since the late 1960s) a large number of diverse formal definitions of digital curves or digital surfaces (see Section I-C). All these require a curve (surface) to be a connected set of pixels (voxels), and implicitly they rest on the idea that a curve is one-dimensional (a surface two-dimensional). For example, according to all the known definitions, the digital object in Figure 2 (left) is a digital curve, while the one in Figure 2 (middle) is not. Basically, a reasonable definition assures that a curve (surface) does not contain any “block” of pixels (voxels) that contributes a two-dimensional (three-dimensional) subset. Sometimes a set, being a digital curve (or surface) according to some definition, may contain a small block as shown in Figure 2 (right). Usually this is considered fine with respect to its visual appearance, but may expose “theoretical defects” that are not present in the continuous case.

From this point of view it remains unclear (with respect to recent literature) whether the set of pixels in Figure 2 (right) represents a digital curve or not (e.g., with respect to 0-adjacency).

Our answer to that question is based on the understanding that, from a theoretical point of view, even the “smallest defect” that results from certain imperfections in a definition is as “bad” as a “big one.” In mathematics, it has taken about 40 years since the first (erroneous) definition of a curve at the end of 19th century by Jordan (which he corrected later on), to the final completion of the basics of a theory of curves by Urysohn and Menger.

In this paper, we strictly follow the approach to use appropriate dimensionality in definitions of digital curves or surfaces. More than 30 years ago, there has been the publication [39] by Mylopoulos and Pavlidis defining the dimension of digital objects in the context of picture analysis (following combinatorial topology [2]). But this remained fairly unnoticed in the picture analysis community. The monograph [26] recalled this work. This paper is further detailing the importance of the dimensionality approach in digital geometry in particular, and in picture analysis in general.

As a benefit, we think that this way the notion of a digital curve or surface will better conform to human intuition (by analogy to the continuous case) and will also gain more mathematical rigor. The latter is beyond doubt important for the purposes of developing machine intelligence. The former is also desirable (although sometimes human intuition might be misleading[2]).

While one can still argue whether or not our formal approach can provide significant advantages regarding curve and surface visualization, we consider it well-justified by other, more important motivations. For example, our definitions allow the classification of digital curves or surfaces with respect to possible types of gaps they may possess. This is important for practical applications.

For example, it is important for ray tracing or understanding of the topology of digitized 3D sets. Assume that an unknown closed continuous surface Γ has been digitized, e.g., by a tomography scanner. Let M be the resulting digital set of voxels. If now the border $\partial(M)$ of M is determined in a way to constitute a digital surface satisfying the proposed definitions, one will have information about the type of possible gaps in this surface. The requirement for gap-freeness of $\partial(M)$ is important when a discrete model of a surface is traced through digital rays (e.g., for visualization or illumination purposes), since the penetration of a ray through the surface causes a false hole in it. Knowledge about the type of gaps of $\partial(M)$ may predetermine the usage of an appropriate type of digital rays for tracing the border in order to avoid wrong conclusions about the topology of the original continuous 3D set having the frontier Γ. Then, for the purposes of surface reconstruction, one will be able to faithfully model the geometry of the original 3D set. This is of importance for 3D imaging (e.g.) in medicine (e.g., organ and tumor measurements in CT images, beating heart, or lung simulations), bioinformatics (e.g., protein binding simulations), robotics (e.g., motion planning), or engineering (e.g., finite elements stress simulations). A small hole in a heart surface created by imperfections of the synthetic representation, while possibly insignificant (or simply unnoticeable) for visualization, renders the synthetic surface useless for blood flow simulation. Finite element simulations may yield incorrect results if surfaces have singularities. It is of importance to have sound mathematical definitions that can assure correctness of key topological properties of synthetic surfaces or volumes.

As a second example, we mention applications where 3D volumes are mapped into digital skeletons. We briefly discuss such an application in Section V.

Furthermore, the proposed definitions and obtained theoretical results about digital surfaces allowed us to determine and classify all possible good pairs of adjacency relations defined by separation. The usefulness of the theory of good pairs to picture analysis has been recognized quite early by experts in the field; see [45]. Good pairs are well-studied and understood for digital spaces of dimension two, and partially for dimension three. Our approach allows to do so for digital spaces of arbitrary dimension.

B. Some Precursory Remarks

Combinatorial topology (see, e.g., [2]) introduced the concept of dimension for topologies defined on discrete sets (e.g., poset topologies). We recall basic definitions of digital geometry before specifying the dimension of sets of pixels or voxels.

[2]Remember that, led by their intuition, people until not so long ago believed that the Earth is flat and is the center of the Universe.

A regular orthogonal grid subdivides $\mathbb{R}^n$ into n-dimensional hypercubes (e.g., unit squares for $n = 2$ or unit cubes for $n = 3$, also called n-*cells* for short) defining a class $\mathbb{C}_n^{(n)}$. Let $\mathbb{C}_n^{(k)}$ be the class of all k-dimensional facets of n-dimensional hypercubes, for $0 \leq k < n$. The grid-cell space $\mathbb{C}_n$ is the union of all these classes $\mathbb{C}_n^{(k)}$, for $0 \leq k \leq n$. The *grid cell topology* (see [26] for reviewing material) is defined on $\mathbb{C}_n$; open or closed sets of n-cells correspond to $(n-1)$- or 0-connected regions in the graph-theoretical model of adjacencies between n-cells; see [25]. (This is a general justification for the alternative use of 4- or 8-adjacency in binary pictures, as proposed in [17].)

The grid cell topology (in particular the alternative use of $(n-1)$- or 0-adjacency for binary pictures) supports "topologically sound" picture analysis, and it is an example of a poset topology. We could consider to apply the notion of dimension within n-dimensional grid cell topology, as known from combinatorial topology. A single n-cell is here an n-dimensional subset of the Euclidean space. However, this is *not* equivalent to dimensions as considered in this article. For example, a simple $(n-1)$-path of n-cells would be an n-dimensional set in the grid cell topology, but it will be one-dimensional with respect to the notion of dimension as used below!

Digital 2D or 3D pictures can be considered to be defined on $\mathbb{C}_n^{(n)}$, for $n = 2$ or $n = 3$, using either graph-theoretical concepts such as neighborhoods or adjacencies, or a particular digital topology (typically the grid cell topology). Picture analysis benefits from the grid cell topology (using a maximum-label rule for multi-level or multi-channel pictures), and more generally (as we will show), from a theory of *good pairs* [3] because this allows to specify *separation theorems* which form a theoretical justification for any border or surface tracing algorithm. Such separation theorems have been often studied with respect to particular digital topologies or graph-theoretical concepts (see, e.g., [26]).

C. Good Pairs

In this paper we study digital manifolds, in particular digital curves and hypersurfaces. On this basis, we define good pairs of adjacency relations in grid-cell spaces $\mathbb{C}_n^{(n)}$ ($n \geq 2$), equipped with adjacencies A_α (e.g., $\alpha = 0, 1$ for $n = 2$, and $\alpha = 0, 1, 2$ for $n = 3$)[4]. Our study of good pairs is directed on the understanding of separability properties: which kind of sets, defined by one type of adjacency, allow to separate sets defined by another type of adjacency. These separating sets can be defined in the form of digital curves in 2D, or as digital surfaces in 3D. In this way, studies of good pairs and of (separating) surfaces are directly related to one-another.

Informally speaking, a good pair combines two adjacency relations on $\mathbb{C}_n^{(n)}$ which appear to be "suitable for picture analysis." The reason for suggesting the first good pairs (α, β) in [17], with (α, β) equal to (1,0) or (0,1), were observations in [45]. (A_α is the adjacency relation for 1*s*, which are the "object" pixels with value 1, and A_β is the adjacency relation for 0*s* which are the "non-object" pixels, also defining the "background.") The benefit of two alternative adjacencies was then formally shown in [42]: (1,0) or (0,1) define region adjacency graphs for

[3]The name was created for the oral presentation of [28]. Note that the same term has been used already with different meaning in topology.

[4]In 2D, 0- and 1-adjacency correspond to 8- and 4-adjacency, respectively, while in 3D, 0-, 1-, and 2-adjacency correspond to 26-, 18- and 6-adjacency, respectively. The latter are traditionally used within the grid-point model on $\mathbb{Z}^3$.

binary pictures which form a rooted tree. This simplifies topological studies of binary pictures because it allows to specify a separation theorem (typically formulated in terms of 4- or 8-paths, and 4- or 8-holes). A good pair of adjacency relations can also be "combined" into s-adjacency (see [25], [26]) which can be used for "topologically sound" multi-level or multi-channel picture analysis.

Digital surfaces have been studied frequently over the years. For example, [22] defines digital surfaces in $\mathbb{Z}^3$ based on adjacencies of 3-cells. A mathematical framework (based on a notion of "moves") for defining and processing digital manifolds is proposed in [12].

For obtaining α-surfaces by digitization of surfaces in $\mathbb{R}^3$, see [13]. It is proved in [36] that there is no local characterization of a 26-connected subset S of $\mathbb{Z}^3$ such that its complement $\overline{S}$ consists of two 6-components and every voxel of S is adjacent to both of these components. [36] defines a class of 18-connected surfaces in $\mathbb{Z}^3$, proves a surface separation theorem for those surfaces, and studies their relationship to the surfaces defined in [38]. [4] introduces a class of "strong" surfaces and proves that both the 26-connected surfaces of [38] and the 18-connected surfaces of [36] are strong. For further studies on 6-surfaces, see [11]. Digital surfaces in the context of arithmetic geometry are studied in [5]. For various other topics related to digital manifolds we also refer to [9].

D. Digital Topologies

A *digital topology* on $\mathbb{C}_n$ is defined by a family of open subsets that satisfy a number of axioms (see, e.g., Section 6.2 in [26]). For all $n \geq 2$ we have at least two digital topologies, known as *grid cell topology* (first time formulated in 1935 within an exercise in [2] for 2D; in general, the nD case is an example of a poset topology) and as *grid point topology* (as specified in 1970 [52] for nD). Digital topologies on $\mathbb{C}_n$ can be mapped by isomorphisms into digital topologies on $\mathbb{C}_n^{(n)}$; for example, the grid cell topology is isomorphic to the topology of incidence grids, also isomorphic to the Khalimsky topology, and a special case of a topology of Euclidean, or of abstract complexes (see, e.g., [26] for reviewing material).

For example, the topology of incidence grids is one possible approach for considering 2D or 3D picture analysis: frontiers of closed sets of n-cells define hypersurfaces, consisting of $(n-1)$-cells, which separate interior from exterior. An axiomatic approach to digital topology in relation to some other approaches is presented in [16].

A separation theorem for the Khalimsky topology is proved in [31]. For discrete combinatorial surfaces, see [19]. The approximation of n-dimensional manifolds by graphs is studied in [47], [48], with a special focus on topological properties of such graphs defined by homotopy, and on homology or cohomology groups. Approximation of boundaries of finite sets of grid points (in n dimensions) based on "continuous analogs" is proposed and studied in [34]. [21] discusses local topological configurations (stars) for surfaces in incidence grids.

Frontiers in cell complexes (and related topological concepts such as components and fundamental groups) are studied in [1]. For characterizations of, and algorithms for curves and surfaces in frontier grids, see [20], [33], [44], [49]. [14] defines curves in incidence grids.

[18] shows that there are two digital topologies on $\mathbb{C}_2^{(2)}$, five on $\mathbb{C}_3^{(3)}$, and [29] shows that there are 24 on $\mathbb{C}_4^{(4)}$ (all up to homeomorphisms). The product of all nD digital topologies with the 1D alternating topology of [52] is an

$(n+1)$D digital topology, and we also have always the $(n+1)$D grid cell topology. This gives at least $n+20$ digital topologies for all $n \geq 4$. However, many of those digital topologies (due to different degrees of non-uniformity) have no relevance for being applied in computer analysis of regularly sampled nD data. In applied picture analysis there seems to be a general preference for adjacency-based algorithms compared to topology-based algorithms.

E. Digital Topologies versus Good Pairs

Good pairs may induce a digital topology on $\mathbb{C}_n$ (and not vice-versa in general). For example, from [26] we know that the good pair (1,0) is equivalent to regarding 1-components of 1*s* as open regions and 0-components of 0*s* as closed regions in $\mathbb{C}_2$ (or vice versa, for the good pair (0,1)). According to [25], this can be generalized to arbitrary $n \geq 2$: the good pairs $(0, n-1)$ or $(n-1, 0)$ are equivalent to the grid cell topology in $\mathbb{C}_n$, if both models are using identical total orders of values of n-cells.

The present paper provides a complete characterization of good pairs, showing that there are exactly $n+1$ good pairs on $\mathbb{C}_n^{(n)}$. Together with the lower bound for numbers of digital topologies, as given in the previous subsection for $n \leq 4$, we thus know that there are (for $n \geq 3$) more digital topologies than good pairs. As already mentioned, some of the known digital topologies (for $n = 3$ and $n = 4$) seem to be irrelevant for practical use. On the other hand, all the defined good pairs can be considered to be of practical relevance, also covering the grid cell topology as stated above.

In fact, digital topologies can be, for example, product topologies which introduce certain type of "alternating pattern" into the nD grid (see [26]). In applied picture analysis, any type of non-uniform treatment of grid points, just due to their geometric location, faces opposition. On the other hand, the known good pairs do not define dependencies from locations of grid points (pixels, voxels, and so forth), but from picture values at those grid points. This has been accepted in picture analysis since its beginning; see the work of 1967 by Duda, Hart, and Munson [17] where the alternating use of 0- and 1-adjacency has been proposed for white or black pixels. As mentioned in [40], a remarkable observation in a related context is that the 1-neighborhood cannot be used for consistent estimation of the Euler number, while $(1, 2)$ is considered to be a good pair of adjacency relations.

F. Results and Structure of the Paper

In this paper we present alternative definitions of digital hypersurfaces, partially following ideas already published in some of the references cited above. In short, a digital α-hypersurface is composed by (closed) α-curves; two such curves are either disjoint and non-adjacent, or disjoint but adjacent, or they have overlapping portions. The main contributions of the paper are as follows ($n \geq 2$):

- We define digital manifolds in arbitrary dimensions, as the definitions involve the notion of *dimension of a digital object* [39]. Thus a digital curve is a one-dimensional digital manifold, while a digital hypersurface in nD is an $(n-1)$-dimensional manifold, in conformity to continuous topology (see, e.g., the topological definitions of curves by Urysohn and Menger, as discussed in [26]). To our knowledge of the available literature, this is the first work involving dimensionality in defining these notions in digital geometry.

- We show that there are two and only two basic types of α-hypersurfaces, one for $\alpha = n-2$ and one for $\alpha = n-1$. For $\alpha = n-2$, a hypersurface S has $(n-2)$-gaps which appear on $(n-2)$-manifolds that build S and, possibly, between adjacent or overlapping pairs of such $(n-2)$-manifolds. Moreover, S is $(n-1)$-gapfree [5] and $(n-1)$-minimal. For $\alpha = n-1$, the hypersurface S is 0-gapfree and 0-minimal.
- We investigate combinatorial properties of digital hypersurfaces, showing that a digital hypersurface can define a matroid.
- Relying on the obtained properties of digital hypersurfaces, we study good pairs of adjacency relations in arbitrary dimension. We define nD good pairs through separation by digital hypersurfaces and show that there are exactly $n+1$ such good pairs. We also provide a short review and comments on some other approaches for defining good pairs which have been communicated elsewhere.

The paper is organized as follows. In the next Section II we recall some basic definitions and facts. We also prove a lemma that characterizes one-dimensionality of digital sets and that is used in the further sections. In Section III we presents our basic results about digital manifolds. In Section IV we provide a characterization of good pairs of adjacency relations defined by separation by digital hypersurfaces. We conclude with some remarks in Section VI.

II. Preliminaries

We start with recalling basic definitions; notations follow [26]. In particular, the grid point space $\mathbb{Z}^n$ allows a refined representation by an incidence grid defined on the cellular space $\mathbb{C}_n$ introduced above.

A. Some Basic Definitions

Elements in $\mathbb{C}_n^{(k)}$ are k-cells, for $0 \leq k \leq n$. An m-dimensional facet of a k-cell is an m-cell, for $0 \leq m \leq k-1$. Two k-cells are called *m-adjacent* if they share an m-cell. Two k-cells are *properly m-adjacent* if they are m-adjacent but not $(m+1)$-adjacent.

A digital object D is a finite set of n-cells. In dimension two these are usually called *pixels* and in dimension three *voxels*. An *m-path* in D is a sequence of n-cells from D such that every two consecutive n-cells are m-adjacent. The *length* of a path is the number of n-cells it contains. A *proper m-path* is an m-path in which at least two consecutive n-cells are not $(m+1)$-adjacent. Two n-cells of a digital object D are *m-connected* (in D) iff there is an m-path in D between them. A digital object D is *m-connected* iff there is an m-path connecting any two n-cells of D. D is *properly m-connected* iff it contains two n-cells such that all m-paths between them are proper. An *m-component* of D is a maximal (by inclusion, i.e., non-extendable) m-connected subset of D.

Let M be a subset of a digital object D. If $D \setminus M$ is not m-connected then the set M is said to be *m-separating* in D. (In particular, the empty set m-separates any set D which is not m-connected.)

[5]This was also called “tunnel-free” in earlier publications (e.g., in [3], [41]). The Betti number β_1 defines the number of tunnels in topology. Informally speaking, the location of a tunnel cannot be uniquely identified in general; there is only a unique way to count the number of tunnels. Locations of gaps are identified by defining sets. There are sets (e.g., knots) which have a tunnel (i.e., $\beta_1 > 0$) but no gap (in the sense of [3], [41]).

Let M be an m-separating digital object in D such that $D \setminus M$ has exactly two m-components. An *m-simple cell* (or *m-simple point*) of M (with respect to D) is an n-cell c such that $M \setminus \{c\}$ is still m-separating in D. An m-separating digital object in D is *m-minimal* (or *m-irreducible*) if it does not contain any m-simple cell (with respect to D).

For a set of n-cells D, by $\overline{D}$ we denote the complement of D to the whole digital space $\mathbb{C}_n^{(n)}$, and by $card(D)$ its cardinality. By $V(D)$ we denote the union (in $\mathbb{R}^n$) of all elements of D considered as hypercubes in $\mathbb{R}^n$.

$J^+(A)$ is the outer Jordan digitization (also called *supercover*) of a set $A \subseteq \mathbb{R}^n$, which consists of all n-cells intersected by A.

By $N_\alpha(c)$ we denote the *unit α-ball* (also called the *α-neighborhood* of c) with center c consisting of all α-neighbors of c. Furthermore, let $A_\alpha(c) = N_\alpha(c) \setminus \{c\}$ be the *α-adjacency set* of c.

For a given set $D = \{c_1, c_2, \ldots, c_m\} \subseteq \mathbb{C}_n^{(n)}$, we define its *$\alpha$-adjacency graph* $G_\alpha(D, E)$ where D is the set of vertices, and $E = \{(c_i, c_j) : c_i \text{ and } c_j \text{ are } \alpha\text{−adjacent}\}$ the set of edges.

Next, we recall the definition of Hausdorff distance of two sets. Given a metric space E with a metric d, let $\mathcal{E}$ be a family of closed nonempty subsets of E. For every $x \in E$ and every $A \in \mathcal{E}$, let $d(x, A) = \inf\{d(x, y) : y \in A\}$. Then, given two sets $A, B \in \mathcal{E}$, $H_d(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(A, b) : b \in B\}\}$ is called the *Hausdorff distance* between A and B.

A set $P(b, a_1, a_2, \ldots, a_n, \omega) = \{x \in \mathbb{Z}^n | -\frac{\omega}{2} \leq b + \sum_{i=1}^n a_i x_i < \frac{\omega}{2}\}$ is a *digital hyperplane*[6] with coefficients $a_1, a_2, \ldots, a_n$, b and *thickness* ω. A digital hyperplane with thickness $\omega = |a|_{\max} = \max\{|a_1|, |a_2|, \ldots, |a_n|\}$ is called *naive*, and one with thickness $\omega = \sum_{i=1}^n |a_i|$ is called *standard*. For $n = 2$ and 3 one obtains a definition of a digital line and digital plane, respectively (see Figures 6, right, and 8).

B. Gaps

A *gap* is an important notion in discrete geometry and topology. Usually, gaps are defined through separability as follows: Let a digital object M be m-separating but not $(m-1)$-separating in a digital object D. Then M is said to have *k-gaps* for any $k < m$. A digital object without m-gaps is called *m-gapfree*. See Figure 3.

Although the above definition has been used in a number of papers by different authors, one can reasonably argue that it requires further refinement. Consider, for instance, the following example. Let M_1 and M_2 be two digital objects that are subsets of a superset D, and assume that $M_1 \cap M_2 = \emptyset$ (we may think that M_1 and M_2 are "far away" from each other). In addition, assume that M_1 has a k-gap with respect to an adjacency relation A_α, while M_2 is a closed digital hypersurface that k-separates D. Then it turns out that the digital set $M_1 \cup M_2$ that consists of (at least) two connected components, has no k-gap with respect to A_α.

Despite such kind of phenomena, the above definition is adequate for the studies that follow.

[6] Also called an "arithmetic" hyperplane.

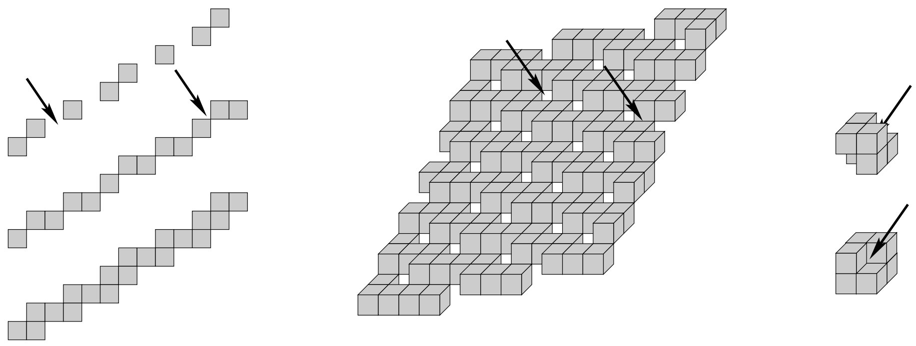

Fig. 3. *Left:* From top to bottom: portions of arithmetic lines defined by $0 \leq 3x-5y < 3$, $0 \leq 3x-5y < 5$ (naive line), and $0 \leq 3x-5y < 8$ (standard line). The first one has 1-gaps (and, therefore, also 0-gaps; a 1-gap is pointed out by an arrow), the second one has 0-gaps (one of them pointed out by an arrow) but no 1-gaps, and the third one is gap-free. *Middle:* Portion of an arithmetic plane defined by $0 \leq 2x+5y+9z < 7$. It has 2-gaps (and, therefore, also 1- and 0-gaps). A 2-gap and a 1-gap are pointed out by arrows. *Right:* Configuration of voxels (in two different orientations) that features a 0-gap (pointed out by an arrow).

C. Dimension

Recall that $A_\alpha(c) = N_\alpha(c) \setminus \{c\}$. A nonempty set $D \subseteq \mathbb{C}_n^{(n)}$ is called *totally α-disconnected* iff $A_\alpha(x) \cap D = \emptyset$ for any $x \in D$ (i.e., there is no pair of cells $c, c' \in D$ such that $c \neq c'$ and $\{c, c'\}$ is α-connected). $D \subseteq \mathbb{C}_n^{(n)}$ is called *linearly α-connected* whenever $|A_\alpha(x) \cap D| \leq 2$ for all $x \in D$ and $|A_\alpha(x) \cap D| > 0$ for at least one $x \in D$.

Let $B_\alpha(c)$ be the union of $N_\alpha(c)$ with all n-cells c' for which there exist $c_1, c_2 \in N_\alpha(c)$ such that a shortest α-path from c_1 to c_2 not passing through c passes through c'. For example, $B_1(c) = B_0(c) = N_0(c)$ for $n = 2$, and $B_2(c) = B_1(c) = N_1(c)$, $B_0(c) = N_0(c)$ for $n = 3$. Denote $B^*_\alpha(c) = B_\alpha(c) \setminus \{c\}$.

Mylopoulos and Pavlidis [39] proposed the following recursive definition of dimension of a (finite or infinite) set of n-cells S with respect to an adjacency relation A_α. Let D be a digital object in $\mathbb{C}_n^{(n)}$ and A_α an adjacency relation on $\mathbb{C}_n^{(n)}$. The *dimension* $\dim_\alpha(D)$ is defined as follows:

(1) $\dim_\alpha(D) = -1$ iff $D = \emptyset$,

(2) $\dim_\alpha(D) = 0$ iff D is a totally α-disconnected nonempty set,

(3) $\dim_\alpha(D) = 1$ if D is linearly α-connected,

(4) $\dim_\alpha(D) = \max_{c \in S} \dim_\alpha(B^*_\alpha(c) \cap D) + 1$ otherwise.

If in the last item of the definition the maximum is reached for an n-cell c, we will also say that D is $\dim_\alpha(D)$-dimensional at c.

Note that in the recursive Step (4), the dimension of the set $B^*_\alpha(c) \cap D = (B_\alpha(c) \setminus \{c\}) \cap D$ can be computed by consecutive applications of the same recursive step.

The next lemma provides a characterization of one-dimensionality in $\mathbb{C}_n^{(n)}$. An *elementary grid triangle* consists of three cells c_1, c_2, c_3, such that any two of them are α-adjacent. An *elementary grid square* consists of four cells c_1, c_2, c_3, c_4 with coordinates $c_1 = (x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)$, $c_2 = (x_1, \ldots, x_i + 1, \ldots, x_j, \ldots, x_n)$, $c_3 = (x_1, \ldots, x_i, \ldots, x_j + 1, \ldots, x_n)$, $c_4 = (x_1, \ldots, x_i + 1, \ldots, x_j + 1, \ldots, x_n)$, for some indices i, j, $1 \leq i, j \leq n$.

Lemma 1: Let $D \subseteq \mathbb{C}_n^{(n)}$ be a non-empty, α-connected set.

(a) If $0 \leq \alpha \leq n-2$, then D is one-dimensional with respect to adjacency A_α iff it does not contain an elementary grid triangle as a *proper* subset.

(b) If $\alpha = (n-1)$, then D is one-dimensional with respect to adjacency A_α iff it does not contain an elementary grid square as a *proper* subset.

Proof (a) Let $\alpha \leq n-2$. Assume at first that D does not contain an elementary grid triangle. Let c be an arbitrary n-cell of D. Assume that D is not linearly connected. Then either there is an n-cell $c' \in D$ with $|A_\alpha(c') \cap D| \leq 2$ or for any $c \in D$ it holds $|A_\alpha(c') \cap D| = 0$. In the latter case, D would clearly be disconnected and thus 0-dimensional. In the former, we have that D is neither empty, nor totally disconnected, nor linearly connected. Then we have

$$\dim_\alpha(D) = \max_{c \in D} \dim(B^*_\alpha(c) \cap D) + 1 \tag{1}$$

Let the maximum in (1) be reached for a point $c_0 \in D$. Since D does not contain any elementary grid triangle, it is easy to deduce that $\dim(A_\alpha(c)) = 0$ and thus $\dim_\alpha(D) = 1$.

Now let $\dim_\alpha(D) = 1$ and assume by contradiction that D contains as a proper subset an elementary grid triangle T. Let c be an arbitrary n-cell of T. We have $|A_\alpha(c) \cap D| \geq 2$. Then $\dim_\alpha A_\alpha(c) \cap D \geq 1$. $A_\alpha(c) \cap D$ is non-empty, not totally disconnected, and not linearly connected. Then its dimension satisfies $\dim_\alpha(A_\alpha(c) \cap D) = \max_{p \in A_\alpha(c) \cap D} \dim(B^*_\alpha(p) \cap D) + 1 \geq 2$ - a contradiction.

(b) Let $\alpha = (n-1)$. Assuming that D does not contain an elementary grid square, one proves that D is one-dimensional by analogous arguments as in part (a). Now let $\dim_\alpha(D) = 1$ and assume by contradiction that D contains as a proper subset an elementary grid square Q. Then there is an n-cell from Q that is $(n-1)$-adjacent to two n-cells from Q and to at least one n-cell of D not belonging to Q. We have $|A_\alpha(c) \cap D| \geq 3$. Hence, $\dim_\alpha A_\alpha(c) \cap D \geq 1$, and $A_\alpha(c) \cap D$ is non-empty, not totally disconnected, and not linearly connected. Then its dimension satisfies $\dim_\alpha(A_\alpha(c) \cap D) = \max_{p \in A_\alpha(c) \cap D} \dim(B^*_\alpha(p) \cap D') + 1$, where $D' = A_\alpha(c) \cap D$. It is easy to see that $\dim(B^*_\alpha(p) \cap D') \geq 1$. Then $\dim_\alpha(A_\alpha(c) \cap D) \geq 2$, which is a contradiction. □

Remark 1: Note that in Case (a) an elementary grid triangle is not two-dimensional, since it is not a proper subset of itself. Similarly, in Case (b) an elementary grid square is not two-dimensional. Actually, the definition of dimension implies that these are one-dimensional.

III. Digital Curves and Hypersurfaces

In what follows, we consider digital analogs of simple closed curves and of hypersurfaces that separate the space $\mathbb{C}_n^{(n)}$. We will consider analogs of either bounded closed separating hypersurfaces, or unbounded hypersurfaces (such as hyperplanes) that separate $\mathbb{R}^n$. (The latter can also be considered as "closed" in the infinite point.) We will not specify whether we consider closed or unbounded hypersurfaces whenever the definitions and results apply to both cases and no confusions arise. We also omit the word "digital" where possible.

The considerations take place in the n-dimensional space $\mathbb{C}_n^{(n)}$. We allow adjacency relations A_α as defined above. We are interested to establish basic definitions for this space that:

- reflect properties which are analogous to the topological connectivity of curves or hypersurfaces in Euclidean topology,
- reflect the one- or $(n-1)$-dimensionality of a curve or a hypersurface, respectively,
- characterize hypersurfaces with respect to gaps.

A digital curve (hypersurface), considered in the context of an adjacency relation A_{α}, will be called an α-curve (α-hypersurface).

A. Digital Curves

A set $\tau \subset \mathbb{C}_n^{(n)}$ is an α-curve iff it is α-connected and one-dimensional with respect to A_{α}. (Note that Urysohn-Menger curves in $\mathbb{R}^n$ are defined to be one-dimensional continua.) Figure 4 presents examples and counterexamples of curves in $\mathbb{C}_2^{(2)}$.

In the rest of this section we define and study digital analogs of simple closed curves (i.e., those that have branching index two at any point). The following lemma provides necessary and sufficient conditions for a set of n-cells to be connected and a simple loop with respect to adjacency relation A_{α}.

Lemma 2: Let $\rho = \{c_1, c_2, \dots, c_l\}$ be a set of n-cells. Then the following conditions are equivalent:

(A1) c_i is α-adjacent to c_j iff $i = j \pm 1 (modulo\ l)$.

(A2) ρ is α-connected and $\forall\ c \in \rho$, $\mathrm{card}(A_{\alpha}(c) \cap \rho) = 2$.

(A3) The α-adjacency graph $G_{\alpha}(\rho, E)$ is a simple loop.

The proof of the above lemma is straightforward. Note that each of conditions (A1) and (A3) implies connectivity of ρ, while α-connectivity of ρ is explicitly required in (A2), otherwise ρ may have more than one connected component.

Lemmas 1 and 2 allow us to give the following general definition, summarizing three equivalent ways for defining a simple α-curve.

Definition 1: A simple α-curve ($0 \leq \alpha \leq n-1$) of length l is a set $\rho = \{c_1, c_2, \dots, c_l\} \subseteq \mathbb{C}_n^{(n)}$, that is one-dimensional with respect to A_{α} adjacency and satisfies a property (Ai) for some $1 \leq i \leq 3$.

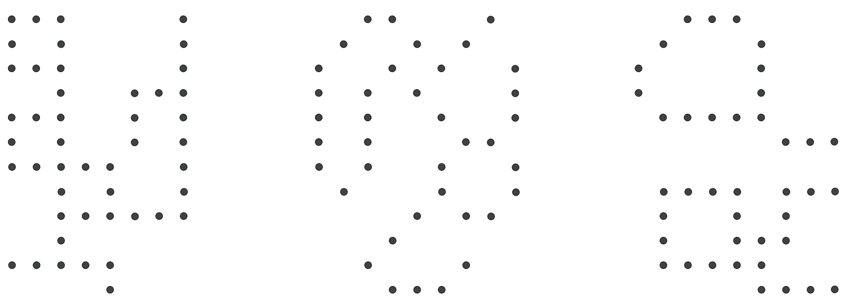

Fig. 4. Examples of a 1-curve (left), 0-curve (middle), and two 0-connected sets in the digital plane that are neither 0- nor 1-curves (right).

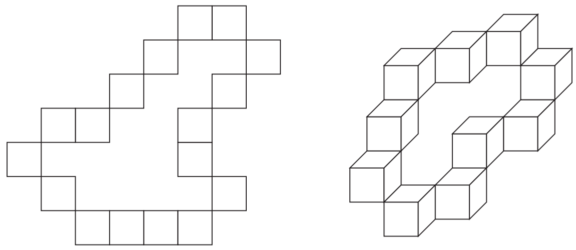

Fig. 5. A proper 0-curve in 2D (left) and an improper 0-curve in 3D (right).

A simple α-curve will also be called a *one-dimensional* α*-manifold*. A simple α-curve ρ $(0 \leq \alpha < n-1)$ is a *proper* α-curve (or a proper one-dimensional α-manifold) if it is not an $(\alpha+1)$-curve.

Example 1: A proper 0-curve in $\mathbb{C}_2^{(2)}$ is a 0-curve which is not a 1-curve (see Figure 5, left). It follows that any closed 0-curve is a proper 0-curve.

A *proper* 0-curve in $\mathbb{C}_3^{(3)}$ is a 0-curve which is not a 1- or 2-curve, and a *proper* 1-curve is a 1-curve which is not a 2-curve.

Any 1-curve is a proper 1-curve. This follows from the facts that a curve is closed and one-dimensional with respect to 1-adjacency. If we assume the opposite, we would obtain that the curve is either an infinite sequence of voxels (e.g., of the form $(0,0,1),(0,0,2),(0,0,3),\ldots$), or it is two-dimensional. However, a closed 0-curve does not need to be proper (see Figure 5, right).

A *simple* α*-arc* σ is an α-connected proper subset of a simple α-curve. It contains exactly two n-cells c, c' such that $\mathrm{card}(A_\alpha(c) \cap \rho) = \mathrm{card} A_\alpha(c') \cap \rho) = 1$.

Remark 2: The difference between Definition 1 and previous definitions of curves is that here we require a curve to be one-dimensional. See again Figure 4, right. This set of points is not a 1-curve since it is not 1-connected although it is 1-dimensional with respect to 1-adjacency. Also, according to Definition 1, it is not a 0-curve since it is 2-dimensional with respect to 0-adjacency. Without the requirement for 1-dimensionality, that set of points is a curve.

Remark 3: With reference to Figure 6 (left) note that according to our definition the two sets of pixels on the left (top and bottom) are not digital curves as, by Lemma 1, they are not one-dimensional. Let us mention that our definition establishes a one-to-one correspondence between the two classes of digital curves or arcs in the plane (0- and 1-curves or arcs) and the two basic classes of digital straight line segments, known as *naive* and *standard*. See Figure 6 (middle, right). The properties of these classes are very well studied over the years (see, e.g., [41]). In particular, it is well-known that both the naive and the standard lines satisfy the criterion of Lemma 1 (respectively parts (a) and (b) restricted to $\mathbb{C}_2^{(2)}$) and thus are one-dimensional. This way our definition supports a uniform classification of digital curves or arcs and digital straight lines and line segments.

We illustrate how early results in digital geometry relate to our definition. As an example, consider the following theorem proved first by A. Rosenfeld [42]:

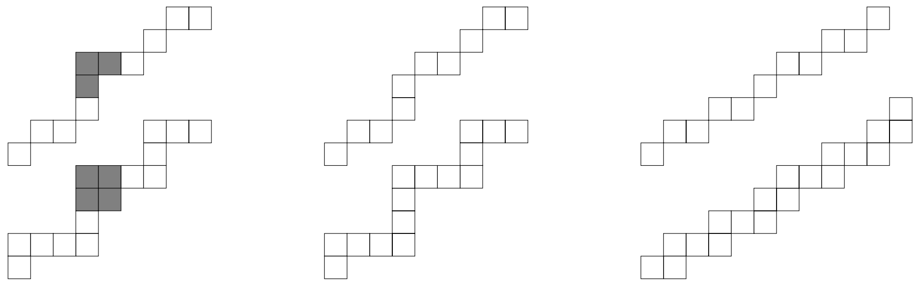

Fig. 6. *Left (top, bottom):* Sets of pixels that are not 1-dimensional with respect to 0- (top) and 1- (bottom) adjacency and therefore are not digital arcs in the sense of Definition 1. *Middle:* 1-dimensional digital 0- (top) and 1- (bottom) arcs. *Right:* Naive (top) and standard (bottom) digital straight line segments, defined by the inequalities $0 \leq 3x - 5y < 5$ and $0 \leq 3x - 5y < 8$, respectively. These are 1-dimensional with respect to 0- (1-) adjacency.

Theorem 1: A simple closed 1-curve (0-curve) γ 0-separates (1-separates) all pixels inside γ from all pixels outside γ. More precisely, we have that a simple closed 1-curve has exactly one 0-hole and a simple closed 0-curve has exactly one 1-hole. A simple closed 1-curve 0-separates its 0-hole from the background and a simple closed 0-curve 1-separates its 1-hole from the background.

In $\mathbb{C}_2^{(2)}$, we have the following characterization of digital curves.

Proposition 1: A finite set of pixels ρ, that is α-separating in $\mathbb{C}_2^{(2)}$ $(\alpha = 0, 1)$, is a simple α-curve in $\mathbb{C}_2^{(2)}$ iff it is $(1 - \alpha)$-minimal in $\mathbb{C}_2^{(2)}$.

Proof Let ρ be a simple 0-curve (1-curve) and p an arbitrary element of ρ. Then from Definition 1 and in view of Lemma 1, we have that $\rho \setminus \{p\}$ is not 0-separating (1-separating) in $\mathbb{C}_2^{(2)}$.

Conversely, let ρ be 1-minimal (0-minimal) in $\mathbb{C}_2^{(2)}$. Then ρ cannot contain an elementary grid triangle (elementary grid square) since otherwise ρ would have a simple point. Hence, by Lemma 1, ρ is a simple curve. □

Note that this last result does not generalize to higher dimensions since a one-dimensional digital object cannot separate $\mathbb{C}_n^{(n)}$ if $n > 2$.

B. *Digital Hypersurfaces*

We consider digital analogs of hole-free hypersurfaces. Accordingly, we are interested in hypersurfaces without $(n-1)$-gaps, although the theory can be extended to cover this case as well. However, in the framework of our approach, a hypersurface with $(n-1)$-gaps can be an $(n-2)$-dimensional set of n-cells (see Figure 7, left), while we want a digital hypersurface to be $(n-1)$-dimensional, in conformity with the continuous case. We give the following recursive definition.

Definition 2: (i) M is an one-dimensional $(n-1)$-manifold in $\mathbb{C}_n^{(n)}$ if it is an $(n-1)$-curve in $\mathbb{C}_n^{(n)}$.

For $2 \leq k \leq n-1$, M is a k-dimensional $(n-1)$-manifold in $\mathbb{C}_n^{(n)}$ if:

(1) M is $(n-1)$-connected (or, equivalently, M consists of a single $(n-1)$-component);

(2) for any $x \in M$, the set $A_0(x) \cap M$ is a $(k-1)$-dimensional $(n-1)$-manifold in $\mathbb{C}_n^{(n)}$.

(ii) M is a 1-dimensional α-manifold $(0 \le \alpha \le n-2)$ in $\mathbb{C}_n^{(n)}$ if M is an α-curve in $\mathbb{C}_n^{(n)}$;
M is a k-dimensional α-manifold $(0 \le k \le n-1, 0 \le \alpha \le n-2)$ in $\mathbb{C}_n^{(n)}$ if:
(1) M is α-connected (or, equivalently, M consists of a single α-component);
(2) for any $x \in M$ the set $A_0(x) \cap M$ is a $(k-1)$-dimensional α-manifold in $\mathbb{C}_n^{(n)}$ but is not a $(k-1)$-dimensional $(\alpha+1)$-manifold in $\mathbb{C}_n^{(n)}$. (Such an α-manifold is also called *proper*.)

In the particular case when M is an $(n-1)$-dimensional α-manifold in $\mathbb{C}_n^{(n)}$ for $\alpha = n-2$ or $n-1$, we say that M is a digital *α-hypersurface*. M is a *proper α-hypersurface* for $\alpha = n-2$ if it is not an $(n-1)$-hypersurface for $\alpha = n-1$. See Figure 7 (middle and right) for illustrations of 1- and 2-surfaces. One can observe that an α-hypersurface M is $(n-1)$-dimensional at any n-cell of M with respect to adjacency relation A_α. It is also clear that any proper one-dimensional α-manifold is an α-curve. Note also that if Condition (1) is missing, then M may have more than one connected component. In such a case Condition (2) implies that any connected component of M is an α-hypersurface. Some other points are clarified by the following remarks.

Remark 4: As defined, a digital hypersurface cannot have "singularities," which may appear, e.g., in case of a digitized 3D "pinched sphere" or "strangled torus," because these would not satisfy our definition of a surface. Nevertheless, in Section V we clarify how the proposed definitions can be extended in a way to cover also such kind of more general cases.

Remark 5: In the definition of an α-hypersurface we use the adjacency set $A_0(x)$ rather than $A_\alpha(x)$ (if $\alpha \neq 0$), since the latter could cause certain incompatibilities. This can be seen in the 3D case: if we use adjacency A_2 to define a 2-surface, $A_2(x) \cap M$ may be a 1-curve rather than a 2-curve. Similarly, if we use adjacency A_1 to define a 1-surface, $A_1(x) \cap M$ may be a 0-curve rather than a 1-curve. This is avoided by using A_0 in all cases.

Remark 6: Indeed, one can consider more general digital hypersurfaces which are not covered by the above definitions. If, for instance, we do not require in Definition 2 the manifold $A_0(x) \cap S$ to be proper, we may obtain a "hypersurface" that has subsets of diverse hypersurface types. More general digital hypersurfaces would be just "mixtures" of patches of hypersurfaces of some of the considered types, and their combinatorial study would lose

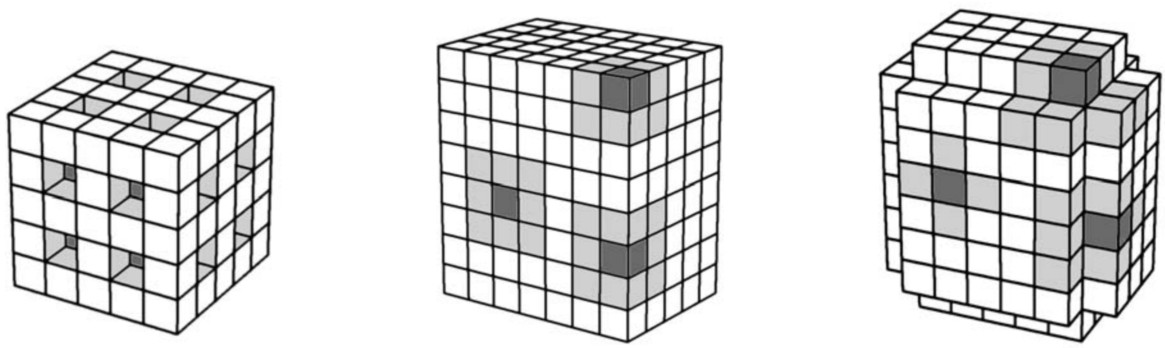

Fig. 7. *Left:* This digital object has 2-gaps and is one-dimensional with respect to 2-adjacency. *Middle* (*Right*): A 1-surface (2-surface) on which three sample voxels are emphasized (in dark gray), together with the 1-curves (2-curves) adjacent to them (in light gray).

its focus. Note that the considered hypersurfaces feature certain combinatorial properties to be studied at the end of this section.

The digital hypersurfaces defined above have the following properties:

Proposition 2: (a) An $(n-2)$-hypersurface S in $\mathbb{C}_n^{(n)}$ is $(n-1)$-gapfree, has $(n-2)$-gaps, and is $(n-1)$-minimal. (b) An $(n-1)$-hypersurface S in $\mathbb{C}_n^{(n)}$ is gapfree and 0-minimal.

Proof We sketch the proof of part (a), the one of part (b) is being similar.

Note that here S is an $(n-2)$-hypersurface defined in the framework of part (ii) of Definition 2. Since the set of n-cells $M_1 = A_0(x) \cap S$, $x \in S$, is a proper $(n-2)$-dimensional $(n-2)$-manifold in $\mathbb{C}_n^{(n)}$, it has $(n-2)$-gaps and no $(n-1)$-gaps. Since this holds for any $x \in S$, it follows that S has $(n-2)$-gaps and no $(n-1)$-gaps, too. The above argument about the manifold M_1 and the recursion of Definition 2 also imply that S is $(n-1)$-separating in $\mathbb{C}_n^{(n)}$.

Now assume that S is not $(n-1)$-minimal in $\mathbb{C}_n^{(n)}$, i.e., there is an n-cell x_0 that is an $(n-1)$-simple point of S. Since S is $(n-2)$-connected, any n-cell $x \in S$ is $(n-2)$-adjacent to at least one n-cell from $S \setminus \{x\}$. Let $y_1 \in S \setminus \{x\}$ be an n-cell that is $(n-2)$-adjacent to x_0. By definition, we have that $M_2 = A_0(y_1) \cap S$ is $(n-2)$-dimensional $(n-2)$-manifold, $x_0 \in A_0(y_1) \cap S$, and there is an n-cell $y_2 \in A_0(y_1) \cap S$ that is $(n-2)$-adjacent to x_0 and $x_0 \in A_0(y_2) \cap S$, where $M_3 = A_0(y_2) \cap S$ is an $(n-3)$-dimensional $(n-2)$-manifold. Continuing this process, after a finite number of steps we obtain that there is an n-cell $y_{n-1} \in A_0(y_{n-2}) \cap S$, such that y_{n-1} is $(n-2)$-adjacent to x_0 and $x_0 \in A_0(y_{n-1}) \cap S$, where $M_{n-1} = A_0(y_{n-1}) \cap S$ is a one-dimensional $(n-2)$-manifold. Note that, since S is an $(n-2)$-hypersurface satisfying Definition 2, all manifolds $M_2, M_3, \ldots, M_{n-1}$ are proper. Keeping this last fact in mind, if we now remove the simple point x_0 from S, it will cause occurrence of an $(n-1)$-gap in the one-dimensional $(n-2)$-manifold (i.e., an $(n-2)$-curve) M_{n-1}, and that gap will propagate over all other manifolds $M_{n-2}, M_{n-3}, \ldots, M_1$, and S. In other words, x_0 is not an $(n-1)$-simple point of S - a contradiction. □

This last proposition suggests the following classification of digital hypersurfaces introduced by Definition 2.

There are two and only two basic types of α-hypersurfaces: one for $\alpha = n-1$ and one for $\alpha = n-2$:

> For $\alpha = n-2$, a hypersurface S has $(n-2)$-gaps which appear on the $(n-2)$-manifolds that build it and, possibly, between adjacent pairs[7] of such $(n-2)$-manifolds.
>
> For $\alpha = n-1$, the hypersurface S is gapfree.

Thus, knowing the type of a given digital surface, one can have correct expectations about the result of tracing the surface by digital rays of a certain type.

Important examples of digital hypersurfaces are the *digital hyperplanes*. These are well-studied from various points of view. In particular, digital hyperplanes admit an analytical description (recall the definition at the end of Section II-A and see Figures 6 (right) and 8 for illustration). We have the following fact:

[7] Actually, two such manifolds, called “adjacent,” may have both adjacent and common n-cells.

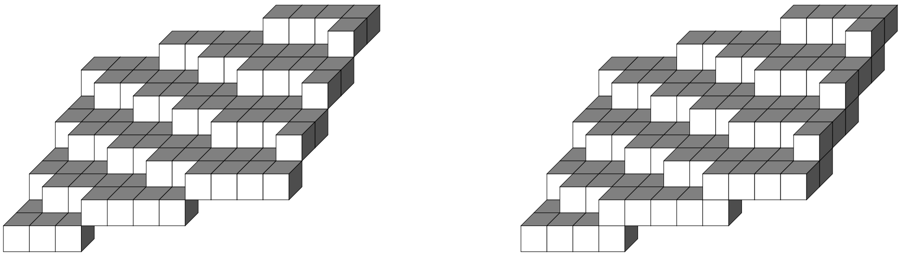

Fig. 8. Portion of a naive arithmetic plane (*left*) and a standard arithmetic plane (*right*).

Proposition 3: A naive digital hyperplane is an $(n-2)$-hypersurface and a standard digital plane is an $(n-1)$-hypersurface.

Proof This follows from the fact (see [3]) that a digital plane with $\omega \geq \sum_{i=1}^{n} |a_i|$ is gapfree and 0-minimal while one with $\omega = |a|_{\max}$ is $(n-1)$-gapfree and $(n-1)$-minimal. □

These studies are related to earlier ones (see, e.g., [5]) on digital hypersurfaces obtained as digitizations of certain continuous hypersurfaces. Here we introduce the following definition.

Definition 3: Let Γ be a closed hypersurface in $\mathbb{R}^n$ and $J^+(\Gamma)$ its outer Jordan digitization. Let $\mathcal{D}_k(\Gamma)$ be the family of all subsets of $J^+(\Gamma)$ that are k-minimal for $k = 0$ or $n-1$. We call a set of n-cells $D_k(\Gamma) \in \mathcal{D}_k(\Gamma)$ a *k-digitization* of Γ if the Hausdorff distance $H_d(\Gamma, V(D_k(\Gamma)))$ is minimal over all elements of $\mathcal{D}_k(\Gamma)$.
Then we have the following fact:

Proposition 4: Any $(n-1)$-hypersurface [$(n-2)$-hypersurface] is an $(n-1)$-digitization [0-digitization] of a certain hypersurface $\Gamma \subset \mathbb{R}^n$.

Proof By Proposition 2 we have that any digital hypersurface S in $\mathbb{C}_n^{(n)}$ is k-minimal for $k = n-1$ if S is an $(n-1)$-hypersurface, or for $k = 0$, if S is an $(n-2)$-hypersurface. Moreover, one can always choose a closed hypersurface Γ such that (i) Γ is completely contained in the polyhedron $P(S)$ obtained as a union of the n-cells of S, and (ii) Γ contains the centers of the n-cells of S. Then S will appear to be a digitization of Γ that satisfies the conditions of Definition 3. □

The digital hyperplanes considered above appear to be hyperplane digitizations (see [6]). The following has been shown in [5]:

Theorem 2: A naive (standard) digital plane $P(b, a_1, a_2, \ldots, a_n, \omega)$ is an $(n-1)$-digitization (0-digitization) of a hyperplane with equation $a_1x_1 + a_2x_2 + \ldots + a_nx_n + b = 0$.

The structure of k-digitizations can also be studied from a combinatorial point of view. Let E be a finite set and $\mathcal{F}$ a family of subsets of E. Recall that $(E, \mathcal{F})$ is a *matroid*[8] if the following axioms are satisfied:

(1) $\emptyset \in \mathcal{F}$,

(2) If $F_2 \in \mathcal{F}$ and $F_1 \subseteq F_2$, then $F_1 \in \mathcal{F}$,

[8]For an introduction to matroid, see the monograph by Welsh [51].

(3) If $F_1, F_2 \in \mathcal{F}$ and $\mathrm{card}(F_1) < \mathrm{card}(F_2)$, then there is an element $x \in F_2$ such that $F_1 \cup \{x\} \in \mathcal{F}$.

We have the following fact:

Proposition 5: Given a closed hypersurface Γ, denote by $G_k^m(\Gamma)$ $(0 \leq k \leq n-1)$ an arbitrary family of k-digitizations of Γ of cardinality m together with all subsets of those digitizations. Then $G_k^m(\Gamma)$ is a matroid that we call the *hypersurface digitization matroid*.

Proof We have (see, e.g., [51]) that an equivalent definition of a matroid is obtained through substituting Condition (3) by the following:

(3′) All maximal (by inclusion) elements of $\mathcal{F}$ have the same cardinality.

These are called the matroid *bases*. Specifically, all k-digitizations in $G_k^m(\Gamma)$ have the same cardinality m and thus satisfy Condition (3). Moreover, $G_k^m(\Gamma)$ contains all subsets of these bases, i.e., conditions (1) and (2) are valid as well. Thus, $G_k^m(\Gamma)$ is a matroid. □

Matroids provide a structural framework for greedy-type algorithms. Thus, the above theorem demonstrates in particular the possibility to generate closed digital surfaces using a greedy approach. It also shows that the greedy approach is potentially applicable to surface digitization, in a sense that, starting from an "empty" digitization, one can always consecutively agglomerate a set of n-cells so that one eventually obtains a digital surface satisfying the proposed definition.

IV. Good Pairs

As already mentioned above, studies on digital (hyper)surfaces interfere with studies on good pairs of adjacency relations. An important motivation for studying good pairs is seen in the possibility that some results of digital topology may hold uniformly for several pairs of adjacency relations. Thus one could obtain a proof which is valid for all of them by proving a statement just for a single good pair of adjacencies.

A. Variations of the Notion "Good Pair"

Different approaches in the literature lead to diverse proposals of good pairs (note: they may be called differently, but address the same basic concept). The rest of this section briefly reviews several possible definitions related to certain fundamental concepts of digital topology.

Good pairs in terms of strictly normal digital picture spaces have been considered in [30]. In that framework, it is shown that adjacencies (1,0) and (0,1) in 2D, and (2,0), (0,2), (2,1) and (1,2) in 3D define strictly normal digital picture spaces, while (1,1) and (0,0) in 2D and (2,2), (1,1), (0,0), (1,0) and (0,1) in 3D do not.

In [26] good pairs have been defined for 2D as follows: (β_1, β_2) is called a *good pair* in the 2D grid iff (for $(i,k) \in \{(1,2),(2,1)\}$) any simple β_i-curve β_k-separates its (at least one) β_k-holes from the background and any totally β_i-disconnected set cannot β_k-separate any β_k-hole from the background. It follows that (1,0) and (0,1) are good pairs, but (1,1) and (0,0) are not. [26] does not generalize this definition to higher dimensions, but suggests the use of (α,β)-separators for the case $n = 3$. ($M \subseteq \mathbb{Z}^3$ is called an (α,β)-*separator* iff M is α-connected, M divides

$\mathbb{Z}^3 \setminus M$ into (exactly) two β-components, and there exists a $p \in M$ such that $\mathbb{Z}^3 \setminus (M \setminus \{p\}) = (\mathbb{Z}^3 \setminus M) \cup \{p\}$ is β-connected.) (α, β)-and (β, α)-separators exist for (α, β) = (0,2),(2,0), (1,2), (2,1), and (1,1). However, there are some difficulties with the case $(\alpha, \beta) = (1, 1)$, as an example from [26] illustrates. Further "strange" examples of separators in $\mathbb{Z}^3$ suggest to refine this notion.

Another approach is based on the following digital variant of the Jordan-Veblen curve theorem (of 2D Euclidean topology) due to A. Rosenfeld [43]: *If C is the set of points of a simple closed 1-curve* (*0-curve*) *and* $\mathrm{card}(C) > 4$ ($\mathrm{card}(C) > 3$), *then $\overline{C}$ has exactly two 0-components* (*1-components*).

This last result defines good pairs of adjacency relations in 2D, as follows. (α, β) *is a* 2D good pair *if for a simple closed α-curve C, $\overline{C}$ has exactly two β-components*. It follows that $(1, 0)$ and $(0, 1)$ are good pairs. It is also easy to see that $(1, 1)$ and $(0, 0)$ are not good pairs.

This above definition can be extended to 3D as follows: (α, β) *is a* 3D good pair *if for a simple closed α-surface S, $\overline{S}$ has exactly two β-components*. We remark that in view of the definition of an α-surface from Section III-B, a 0-digital surface would not be a true surface and should not be called "surface" since it would have 2-gaps. In fact, 3D digital surfaces need to be at least 1-connected. Thus (0,2) would not be a good pair in the sense of allowing a theorem about separating surfaces.

Another approach is based on separation through surfaces (see, e.g., [20], [26]). Relying on Theorem 1, we can give the following definition:

Definition 4: (α, β) is called a *2D good pair* iff any simple closed α-curve β-separates its β-holes from the background.

Clearly, $(1, 0)$ and $(0, 1)$ are good pairs, while (0,0) is not. Note that here also (1,1) is a good pair, as distinct from the case of good pairs defined trough the digital version of the 2D Jordan-Veblen curve theorem.

Let us mention that in a definition from [26] both (α, β) and (β, α) are required to satisfy the conditions of a good pair. To avoid confusion, we suggest to treat this case as a special event: (α, β) *is called a* perfect pair *in 2D if any simple closed α-curve β-separates its β-holes from the background and any simple closed β-curve α-separates its α-holes from the background*.

In what follows, we consider good pairs defined by Definition 4. This allows to obtain a complete characterization of all possible good pairs and perfect pairs.

B. Good Pairs for the Space of n-Cells

We generalize Definition 4 to the nD space of grid cells:

Definition 5: (α, β) is called a *good pair* of adjacency relations in $\mathbb{C}_n^{(n)}$ iff any closed α-hypersurface β-separates its β-holes from the background.

Here, α is a label of the hypersurface type in accordance with our hypersurface classification from Section III-B, while β is an integer representing an adjacency. More precisely, $\alpha = n - 1$ or $\alpha = n - 2$, and $0 \leq \beta \leq n - 1$.

Thus, chosen a good pair (α, β) (e.g., for a certain application), its first component specifies the type of the surfaces to be used further.

In view of the considerations and results from the previous section we have the following theorem:

Theorem 3: There are exactly $n+1$ good pairs in the n-dimensional digital space $\mathbb{C}_n^{(n)}$: $(n-1, i)$ for $0 \leq i \leq n-1$, and $(n-2, n-1)$. The only perfect pairs are $(n-2, n-1)$ and $(n-1, n-1)$.

We illustrate the last theorem for $n = 2$ and $n = 3$. For $n = 2$, the good pairs are $(1, 0)$, $(1, 1)$, and $(0, 1)$. See Figure 9. For $n = 3$, the good pairs are $(2, 0)$, $(2, 1)$, $(2, 2)$, and $(1, 2)$.

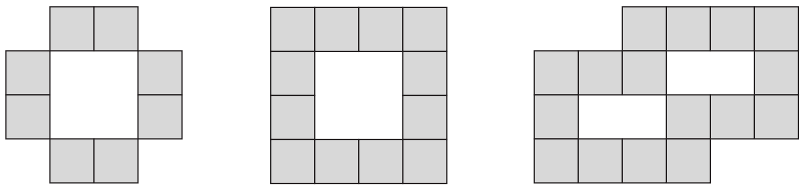

Fig. 9. Illustration to good pairs in 2D: $(0, 1)$ (left), $(1, 1)$ (middle), and $(1, 0)$ (right).

V. Towards Applications

A number of applications will benefit from having mathematically sound definitions of digital manifolds and, respectively, of good pairs. In this section we provide some examples of such problems.

We start with a short discussion on possible extensions of the proposed definitions in a way to incorporate more general notions of curves or surfaces that model the complex geometry of various real objects, sometimes in presence of noise and aliasing.

A. More General Digital Manifolds

Recall that by the Urysohn-Menger definition, a curve $\gamma \subset \mathbb{R}^2$ is known to be a one-dimensional continuum. One can define a *digital continuum* to be any nonempty, finite, and α-connected set of cells in a digital space $\mathbb{C}_2$ (where α is the adopted adjacency relation). Then the above definition would apply to the case of digital curves as follows:

Definition 6: A *digital curve* $\gamma \subset \mathbb{C}_2$ (with respect to a certain adjacency relation) is a one-dimensional digital continuum.

See Figure 10 for illustration. The above definition straightforwardly generalizes for digital curves in a space of arbitrary dimension. It allows to model complicated structures as those considered in the next Section V-B.

Similarly, one can define digital manifolds in more general settings. This would be one more step towards developing a unified topological theory for both continuous and discrete spaces. Clearly, the proposed definition applies not only to a digital surface that appears to be a digitization of the frontier of a simply connected set, but also to digitizations of objects with other genus, e.g. such as the one in Figure 11, left. Note that some studies have considered digitizations of non-smooth surfaces or of those with other peculiar features (see, e.g., [10]).

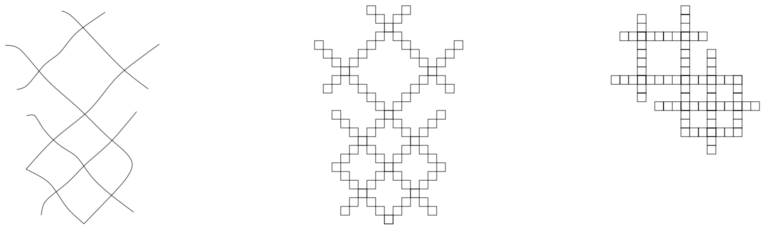

Fig. 10. *Left:* Curve in R^2. *Middle:* 0-curve in $C_2^{(2)}$. *Right:* 1-curve in $C_2^{(2)}$.

In the rest of this subsection we briefly comment on a possible approach to defining digital manifolds that model "pinched" surfaces.[9] Such manifolds may exhibit complex structures that involve singularities. It is not hard to see that the recursive Definition 2 can appropriately be modified in order to cover such situations. To this end, one should describe the conditions satisfied by the point(s) where the digital surface is "pinched." For example, let a digital 1-surface Γ model a continuous surface (e.g., that of a sphere) that is pinched at a single point. Then Part (ii-2) of Definition 2 changes as follows: There is an $x_0 \in M$, such that for any $x \in M$, $x \neq x_0$, the set $A_0(x) \cap M$ is a one-dimensional 1-manifold in $\mathbb{C}_3^{(3)}$ but is not a one-dimensional 2-manifold in $\mathbb{C}_3^{(3)}$, and the set $A_0(x_0) \cap M$ consists of two disjoint one-dimensional 1-manifolds in $\mathbb{C}_3^{(3)}$ that are not one-dimensional 2-manifold in $\mathbb{C}_3^{(3)}$. See Figure 11. The general case of an arbitrary dimension n can be handled similarly.

Fig. 11. *Left:* Digitization of a ball with four holes. *Right:* Illustration to the modified definition of a pinched surface. A digital surface is "pinched" at the dark voxel, which is 1-adjacent to two 1-curves. The dashed lines indicate the rest of the digital surface which contains the two 1-curves.

B. Skeletonization of Digital Objects

The presented theory is particularly relevant to the analysis of curve-like structures in biomedical images. An ongoing research project [24] aims at analyzing confocal microscope images of human brain tissue (which contain

[9]Informally, if a manifold has changed continuously so that two points of it are brought together, then it is called a pinched manifold.

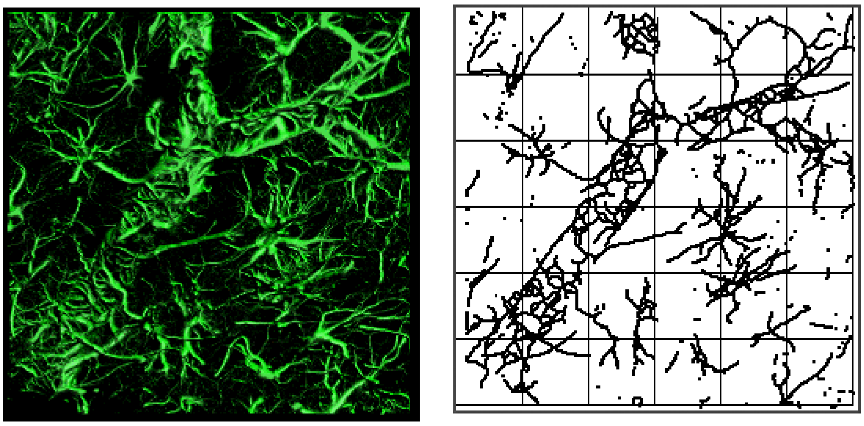

Fig. 12. [24] *Left:* Example of an input data set composed of 42 slices of 256 $\times$ 256 density images generated by confocal microscopy from a sample of human brain tissue. *Right:* A skeleton of the binarized volume shown on the left.

cells called astrocytes, see Figure 12, left). These images have been taken layer by layer and constitute a volume defined on a 3D regular orthogonal grid. The curve-like structures have been obtained by applying a thinning algorithm (see Figure 12, right). [23] proposes a classification of voxels in 3D skeletons of binarized volumes for subsequent structural analysis and length measurements of digital arcs. For the former, a specific graph is associated with the skeleton (see Figure 13). The nodes of the graph, called *junctions*, exhibit certain interesting properties. However, within the proposed model they are considered as singletons that constitute the set of graph vertices. For the purposes of length measurements, the digital curves are segmented into subsequent maximum-length digital straight-line segments, and the total length of those is used to evaluate the length of the curves. For more details we refer to [23].

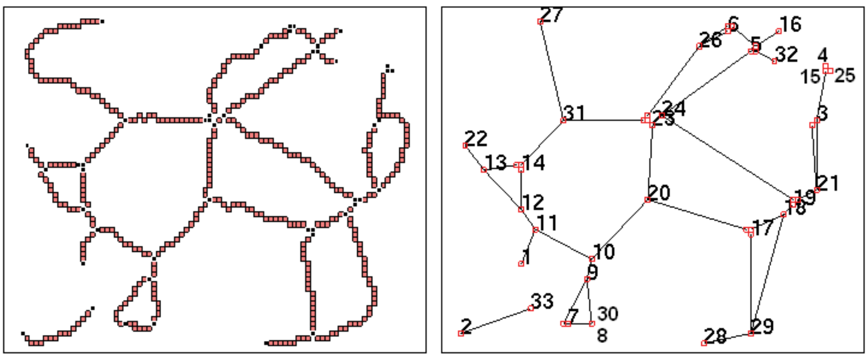

Fig. 13. [23] *Left:* Portion of the skeleton from Figure 12 (junctions are shown as small black squares). *Right:* A graph associated with the skeleton from the left. Nodes are labeled by positive integers).

The interesting point is that, although not explicitly stated as a goal, the arcs of the skeleton form one-dimensional digital curves; as a whole, the skeleton is a digital curve satisfying the definition of the previous section (see Figure

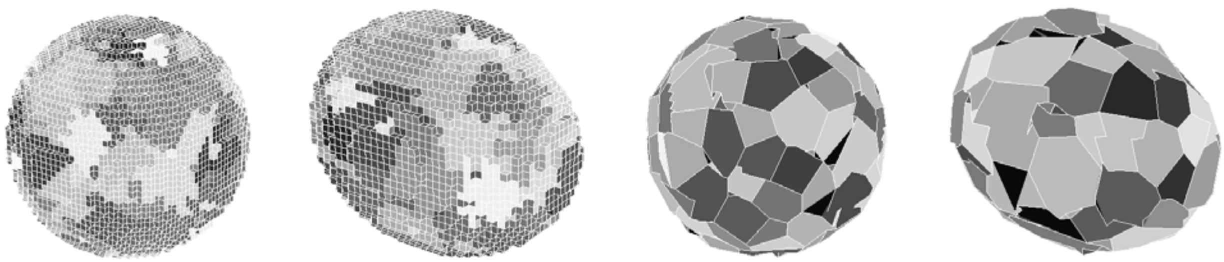

Fig. 14. Illustration to processing steps of an algorithm from [27] (see also [7]). *From left to right:* The first two figures illustrate the agglomeration into digital plane segments (DPS) of the faces of a digitized sphere with radius 20 and an ellipsoid with semi-axes 20, 16, and 12 (grid resolution $h = 40$). The algorithm applies a breadth-first search strategy. Faces of voxels that have the same gray level belong to the same DPS. The respective numbers of faces of the digital surfaces of the sphere and ellipsoid are 7,584 and 4,744, respectively. The numbers of DPSs are 285 and 197; the average sizes of these DPSs are 27 and 24 faces. The third and fourth figures present polyhedrized sphere and ellipsoid.

13, left).[10] These properties support the segmentation process through a number of available efficient algorithms and, in turn, the curve length measurements. Note that curve-like structures appear also in other biomedical images, for example in 3D scans of blood vessels or in 3D ultrasound images.

C. *Determination of Object Boundary*

Another possible application of our definitions and results is seen in designing new algorithms for determining the border of a digital object. Because of its importance, this problem has attracted considerable attention (see, e.g., [15], [32], [35] and the bibliographies in those).

Our hypothesis is that one would benefit from an algorithm that constructs the border as a digital surface as defined above. As already discussed earlier, the reason for this is the knowledge about the gaps in the surface.

If a digital object has been obtained by digitizing a set with a "regular" shape (e.g., featuring convexity), then, in practice, the border voxels indeed constitute a digital surface satisfying our definition. Moreover, for data compression purposes the obtained digital surface can be "linearized" by partitioning it into polygonal portions of digital planes (usually "naive" planes). See Figure 14. As demonstrated in Section III-B, any digital plane satisfies our definition of surface. This explains why in practice the requirement for two-dimensionality supports the minimization of the number of digital plane patches. For more details we refer to [27].

In some cases however it is possible that the border voxels of a digital set do not constitute a digital surface. This usually happens when the digital object has a very complex and irregular structure. An illustration of such a complexity is provided in Figures 15 and 16. They present digitized images of a human brain tissues, studied within the previously mentioned astrocyte project. In such cases, one possibility is to algorithmically "repair" the set of border voxels in order to make it two-dimensional. Theorem 5 suggests that such a digitization always exists.

[10] Except for two locations that exhibit two-dimensionality.

Repairing digital objects in order to achieve desired properties has been already used by some researchers (e.g., [46], [35]).

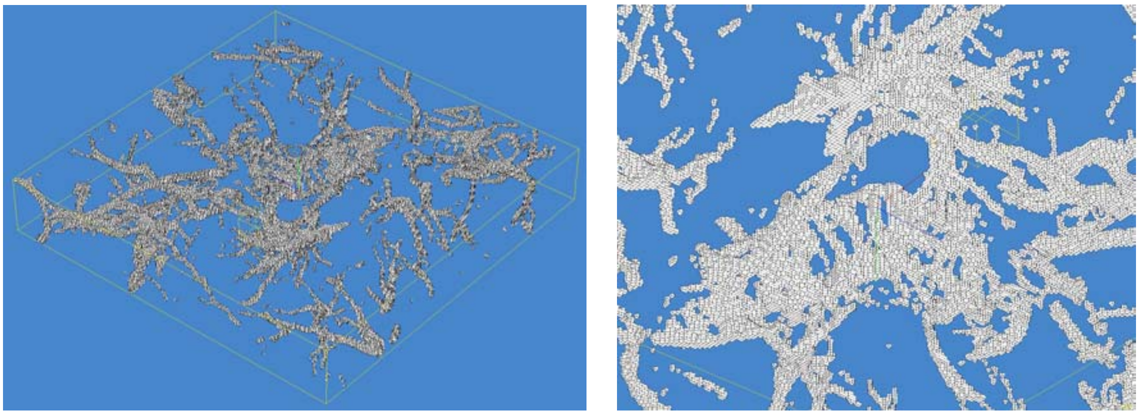

Fig. 15. *Left:* Large view of a sample of human brain tissue, studied within the astrocyte project [24]. The data have been obtained by confocal microscopy and visualized in voxel view mode. *Right:* Enlarged view of a detail of the volume on the left.

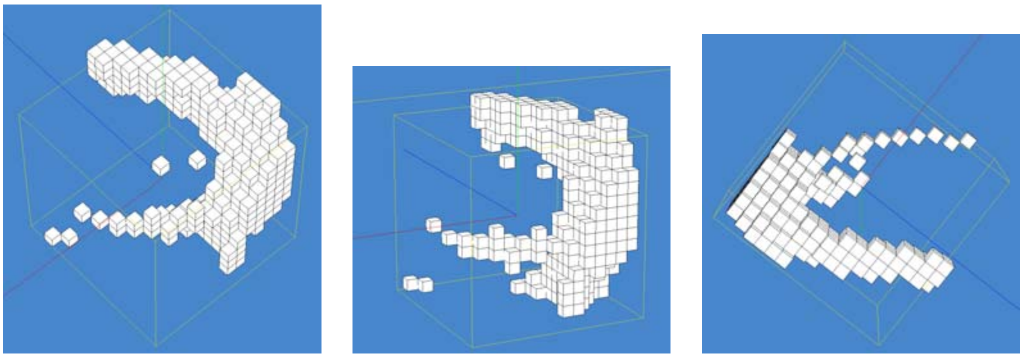

Fig. 16. Further enlargements of subvolumes of the digital image of Figure 15.

VI. Concluding Remarks

In this paper we proposed several equivalent definitions of digital curves and hypersurfaces in arbitrary dimension. The definitions involve properties (such as one-dimensionality of curves and $(n-1)$-dimensionality of hypersurfaces) that characterize them to be digital analogs of definitions for Euclidean spaces. Further research should pursue designing efficient algorithms for recognizing whether a given set of n-cells is a digital curve or a hypersurface.

We also proposed a uniform approach to studying good pairs defined by separation and, in that framework, obtained a classification of good pairs in arbitrary dimension. A future task is seen in extending the obtained results under other reasonable definitions of good pairs.

Some of the presented results have been reported in part at the 10th International Workshop on Combinatorial Image Analysis, Auckland, New Zealand, 2004, [8].

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