PERMUTABLE POLYNOMIALS AND RATIONAL FUNCTIONS

GARRY J. TEE
Department of Mathematics, University of Auckland
Auckland, New Zealand
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Summary. Many infinite sequences of permutable rational functions and a few infinite sequences of permutable polynomials are constructed, on the basis of elliptic functions and trigonometric functions.

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1 Permutable Polynomials

The Chebyshev polynomial of the first type $T_d$ is defined by the initial values

$$T_0(x) = 1, \quad T_1(x) = x,$$

with the 3–term recurrence relation for $n > 1$:

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x).$$

By induction on $n$ in (2), it follows from (1) that, for all integers $d \geq 0$, $T_d(x)$ is a polynomial in $x$ of degree $d$ with integer coefficients. Moreover, $T_d$ is an even polynomial for even $d$ and an odd polynomial for odd $d$.

The integer coefficients of $T_d$ are given explicitly by the following formula\(^1\) for $d > 0$ [17, p.79]:

$$T_d(x) = \sum_{k=0}^{d+2} (-1)^k \frac{2^{d-2k-1}d(d-1)!}{k!(d-2k)!} x^{d-2k} = 2^{d-1}x^d - 2^{d-3}dx^{d-2} + 2^{d-6}d(d-3)x^{d-4} - \frac{2^{d-8}}{3}d(d-4)(d-5)x^{d-6} + \cdots$$

\[\cdots + \begin{cases} (-1)^{d+2} & \text{(for even } d), \\ (-1)^{(d-1)+2} dx & \text{(for odd } d). \end{cases}\]

\(^1\)The symbol $\div$ denotes integer division, yielding integer quotient. For integers $n$ and $d > 0$, $q = n \div d$, where $n = qd + r$, with remainder $0 \leq r < d$. 

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For example, in addition to (1),
\begin{align*}
T_2(x) &= 2x^2 - 1, \\
T_3(x) &= 4x^3 - 3x,
\end{align*}

(4)

By induction on \( n \) in (2), it follows from (1) that for all complex \( x \) and integer \( n \geq 0 \), with \( \vartheta \overset{\text{def}}{=} \cos^{-1} x \),
\[
T_n(x) = T_n(\cos \vartheta) = 2 \cos \vartheta \cos((n-1)\vartheta) - \cos((n-2)\vartheta) = \cos(n\vartheta). \tag{5}
\]
(Any of the infinitely many values of \( \cos^{-1} x \) can be used for \( \vartheta \).) Therefore, for all complex \( x \) and non-negative integers \( j,k \),
\[
T_j(T_k(x)) = T_j(T_k(\cos \vartheta)) = T_j(\cos(k\vartheta)) = \cos(jk\vartheta) = T_{jk}(\cos \vartheta) = T_{jk}(x), \tag{6}
\]
giving the functional identity for complex \( x \) and non-negative integers \( j,k \):
\[
T_j(T_k(x)) = T_{jk}(x). \tag{7}
\]

Thus, the Chebyshev polynomials, with the binary operation of function composition, form an infinite Abelian group, with
\[
T_j(T_k(x)) = T_{jk}(x) = T_{kj}(x) = T_k(T_j(x)). \tag{8}
\]

“Two polynomials, \( p \) and \( q \), are called permutable if \( p(q(x)) = q(p(x)) \) for all \( m,x \).” [20, p.192]. “A sequence of polynomials, each of positive degree, containing at least one of each positive degree and such that every two polynomials in it are permutable is called a chain. The Chebyshev polynomials \( T_1(x), \ldots, T_n(x), \ldots \) form a chain. So do the powers \( \pi_j(x) = x^j, j = 1, 2, \ldots \), as is easily verified.” [20, p.194]

For \( a \neq 0 \), the linear transform \( y = \lambda(x) = ax + b \) has the inverse transform \( x = \lambda^{-1}(y) = (y-b)/a \). A pair of functions \( u(x) \) and \( v(x) \) are permutable, if and only if the transformed functions \( U \overset{\text{def}}{=} \lambda^{-1}u\lambda \) and \( V \overset{\text{def}}{=} \lambda^{-1}v\lambda \) are permutable, since
\[
U(V(x)) = \lambda^{-1}u\lambda(\lambda^{-1}v\lambda(x)) = \lambda^{-1}u(v\lambda(x))
\]
\[
= \lambda^{-1}v(u\lambda(x)) = \lambda^{-1}v(\lambda^{-1}u\lambda(x)) = V(U(x)). \tag{9}
\]

If \( u(x) \) is a polynomial in \( x \) of degree \( n \), then so is the transformed function \( \lambda^{-1}u\lambda \).

Two chains are called similar if there is a linear transformation \( \lambda \) such that each polynomial in one chain is similar (via \( \lambda \)) to the polynomial of the same degree in the other chain. H. David Block and H. P. Thielman [3] and E. Jacobsthal [13] proved the remarkable theorem that “every chain is either similar to \( \{x^j\}, j = 1, 2, \ldots \) or to \( \{T_j\}, j = 1, 2, \ldots \)” [20, p.195] [16, p.156] [4, p.34].

1.1 Permutable Even and Odd Functions

The sequence of rational functions \( \pi_r(x) = \{x^r\} \) (for integer \( r \)) are all permutable, since
\[
\pi_s(\pi_r(x)) = \pi_r(x^s) = (x^s)^r = x^{sr} = x^r s = \pi_s(x^r) = \pi_s(\pi_r(x)). \tag{10}
\]
A function $f$ is called an **even function** iff

$$f(-x) = f(x)$$ \hfill (11)

for all $x$. If $u$ and $v$ are permutable even functions then $-u$ and $-v$ are permutable even functions, since

$$-u(-v(x)) = -u(v(x)) = v(-u(x)) = -v(-u(x)).$$ \hfill (12)

Hence, if $\{w_r\}$ is a permutable sequence of even functions for integer $r$, then the negated sequence $\{-w_r\}$ is permutable.

A function $f$ is called an **odd function** iff

$$f(-x) = -f(x)$$ \hfill (13)

for all $x$. If $u$ and $v$ are permutable odd functions then $-u$ and $v$ are permutable odd functions, since

$$-u(v(x)) = -v(u(x)) = v(-u(x)), \quad \text{and similarly } -v \text{ and } u \text{ permute.}$$ \hfill (14)

If $u$ and $v$ are permutable odd functions then $-u$ and $v$ are permutable odd functions, since

$$-u(-v(x)) = -(-u(v(x))) = u(v(x)) = v(u(x)) = -v(-u(x)) = -v(-u(x)).$$ \hfill (15)

Hence, if $w_r$ are permutable odd functions for integer $r$, then the reflected sequence $\{w_r, -w_r\}$ is permutable. For odd $r = 2i+1$ the odd rational functions $x^{2i+1}$ are permutable, and so is the reflected sequence

$$\{x^{2i+1}, -x^{2i+1}\} = \cdots, -x^3, x^{-1}, -x^1, x, -x, x^3, -x^3, \cdots.$$ \hfill (16)

And the odd Chebyshev polynomials $T_{2j+1}$ are permutable, and hence the reflected sequence of polynomials $\{T_{2j+1}, -T_{2j+1}\}$ is permutable. For all $\vartheta$, we have the identity [17, p.79]

$$\sin((2j+1)\vartheta) = (-1)^j T_{2j+1}(\sin \vartheta).$$ \hfill (17)

Denote

$$q_j(z) \overset{\text{def}}{=} (-1)^j T_{2j+1}(z),$$ \hfill (18)

so that $q_j(\sin \vartheta) = \sin((2j+1)\vartheta)$. Thus the sequence $\{q_j\}$ is a subsequence of the reflected permutable sequence $\{T_{2j+1}, -T_{2j+1}\}$. Indeed, for all non-negative integers $j$ and $k$,

$$q_j(q_k(\sin \vartheta)) = q_j(\sin((2k+1)\vartheta)) = \sin((2j+1)(2k+1)\vartheta) = \sin((4jk+2j+2k+1)\vartheta) = q_{2jk+j+k}(\sin \vartheta) = q_k(q_j(\sin \vartheta)), $$ \hfill (19)

The linear transform with $\lambda(x) = -x$ converts $u(x)$ to $-u(-x)$. Hence, if $u$ is an odd function then it is unchanged, but if $u$ is an even function then it is converted to $-u$. Thus, the permutable sequence of rational functions $\{x^r\}$ is similar (via $\lambda(x) = -x$) to the permutable sequence

$$\{-(-x)^r\} = \cdots, -x^{-4}, x^{-3}, -x^{-2}, x^{-1}, -1, x, -x^2, x^3, -x^4, \cdots.$$ \hfill (20)

And the permutable chain of Chebyshev polynomials $\{T_j\}$ is similar (via $\lambda(x) = -x$) to the permutable chain

$$\{-T_j(-x)\} = -1, x, -T_2(x), T_3(x), -T_4(x), T_5(x), \cdots.$$ \hfill (21)
2 Permutable Rational Functions

For complex \( z = x + iy \) and positive integer \( n \).

\[
z^n = (x + iy)^n = \sum_{j=0}^{n} \binom{n}{j} i^j x^{n-j} y^j = H_n^{(0)}(x, y) + iH_n^{(1)}(x, y), \tag{22}
\]

where the real and imaginary parts of \( z^n \) are given [17, Appendix 1 §2.1] by the harmonic polynomials:

\[
H_n^{(0)}(x, y) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k}, \tag{23}
\]

\[
H_n^{(1)}(x, y) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1}. \tag{24}
\]

For all \( \rho \in \mathbb{C} \), \( \log(\cos \rho + i \sin \rho) = i\rho \) (by Cotes’s theorem), or

\[
\cos \rho + i \sin \rho = e^{i\rho}; \tag{25}
\]

and hence (De Moivre’s Theorem) for integer \( n \geq 0 \)

\[
\cos(n\rho) + i \sin(n\rho) = e^{in\rho} = (\cos \rho + i \sin \rho)^n. \tag{26}
\]

Therefore

\[
\cos(n\rho) + i \sin(n\rho) = (\cos \rho + i \sin \rho)^n
= H_n^{(0)}(\cos \rho, \sin \rho) + iH_n^{(1)}(\cos \rho, \sin \rho). \tag{27}
\]

Equating real and imaginary parts (for real \( \rho \)), we get that

\[
\cos(n\rho) = H_n^{(0)}(\cos \rho, \sin \rho), \quad \sin(n\rho) = H_n^{(1)}(\cos \rho, \sin \rho); \tag{28}
\]

and hence

\[
\tan(n\rho) = \frac{\sin(n\rho)}{\cos(n\rho)} = \frac{H_n^{(1)}(\cos \rho, \sin \rho)}{H_n^{(0)}(\cos \rho, \sin \rho)}. \tag{29}
\]

2.1 Tangents of multiple angles

Dividing numerator and denominator in (29) by \( \cos^n \rho \), we get [17, p.79] \( \tan(n\rho) \) as a rational function of \( \tan \rho \), with integer coefficients:

\[
\tan(n\rho) = \frac{H_n^{(1)}(1, \tan \rho)}{H_n^{(0)}(1, \tan \rho)} = \frac{\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2k+1} (-1)^k \tan^{2k+1} \rho}{\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (-1)^k \tan^{2k} \rho}. \tag{30}
\]

Denote

\[
t \overset{\text{def}}{=} \tan \rho, \quad z \overset{\text{def}}{=} -t^2 = -\tan^2 \rho. \tag{31}
\]

Then, it follows from (30) that

\[
\tan(n\rho) = r_n(\tan \rho), \tag{32}
\]
where \( r_n(t) \) is an odd rational function with positive integer coefficients

\[
r_n(t) = t \frac{A_n(z)}{B_n(z)},
\]

(except that \( A_0(t) = 0 \), with :

\[
A_n(z) = \sum_{k=0}^{(n-1)/2} \binom{n}{2k+1} z^k, \quad B_n(z) = \sum_{k=0}^{n/2} \binom{n}{2k} z^k,
\]

and \( B_n(0) = 1 \).

In more detail, for odd \( n = 2j + 1 \):

\[
r_{2j+1}(t) = t \frac{A_{2j+1}(z)}{B_{2j+1}(z)} = t \frac{2j + 1 + (2j+1) z + (2j+1)^2 z^2 + \cdots + (2j+1)^j z^{j-2} + (2j+1)^j z^{j-1} + z^j}{1 + \frac{(2j+1)^2}{2} z + \frac{(2j+1)^3}{3} z^2 + \cdots + \frac{(2j+1)^j}{j} z^{j-2} + \frac{(2j+1)^j}{j} z^{j-1} + z^j},
\]

where the polynomials \( A_{2j+1} \) and \( B_{2j+1} \) are mutually reciprocal:

\[
z^j A_{2j+1} \left( \frac{1}{z} \right) = B_{2j+1}(z), \quad z^j B_{2j+1} \left( \frac{1}{z} \right) = A_{2j+1}(z)
\]

for all \( z \neq 0 \). For even \( n = 2j \):

\[
r_{2j}(t) = t \frac{A_{2j}(z)}{B_{2j}(z)} = t \frac{2j + \left( \frac{2j}{3} \right) z + \left( \frac{2j}{5} \right) z^2 + \cdots + \left( \frac{2j}{3} \right) z^{j-3} + \left( \frac{2j}{5} \right) z^{j-2} + 2j z^{j-1}}{1 + \left( \frac{2j}{3} \right) z + \left( \frac{2j}{5} \right) z^2 + \cdots + \left( \frac{2j}{3} \right) z^{j-2} + \left( \frac{2j}{5} \right) z^{j-1} + z^j},
\]

where both \( A_{2j} \) and \( B_{2j} \) are self–reciprocal polynomials:

\[
A_{2j}(z) = z^{j-1} A_{2j} \left( \frac{1}{z} \right), \quad B_{2j}(z) = z^j B_{2j} \left( \frac{1}{z} \right),
\]

for all \( z \neq 0 \).

For example,

\[
\begin{align*}
r_0(t) &= t^0 = 0, & r_1(t) &= t^1 = t, \\
r_2(t) &= t^2 \frac{2}{1 + z}, & r_3(t) &= t^3 \frac{3 + z}{1 + 3z}, \\
r_4(t) &= t^4 \frac{4 + 4z}{1 + 6z + z^2}, & r_5(t) &= t^5 \frac{5 + 10z + z^2}{1 + 10z + 5z^2}, \\
r_6(t) &= t^6 \frac{6 + 20z + 6z^2}{1 + 15z + 15z^2 + z^3}, & r_7(t) &= t^7 \frac{7 + 35z + 21z^2 + z^3}{1 + 21z + 35z^2 + 7z^3}, \\
r_8(t) &= t^8 \frac{8 + 56z + 56z^2 + 8z^3}{1 + 28z + 70z^2 + 28z^3 + z^4},
\end{align*}
\]

(39)
et cetera.

Since \( \cot \rho = \tan \left( \frac{1}{2} \pi - \rho \right) \), we get that

\[
 r_n(\cot \rho) = r_n \left( \tan \left( \frac{\pi}{2} - \rho \right) \right) = \tan \left( \frac{n\pi}{2} - n\rho \right).
\] (40)

For odd \( n = 2j + 1 \), this becomes

\[
r_{2j+1}(\cot \rho) = \tan \left( j\pi + \frac{1}{2} \pi - (2j + 1)\rho \right) = \tan \left( \frac{3}{2} \pi - (2j + 1)\rho \right) = \cot((2j + 1)\rho),
\] (41)

and for even \( n = 2j \), this becomes

\[
r_{2j}(\cot \rho) = \tan (j\pi - 2j\rho) = -\tan(2j\rho).
\] (42)

2.2 Permutable rational function \( r_j(t) \)

For the infinite sequence of odd rational functions \( \{r_j\} \), it follows from (32) that, for all complex \( t \) and non-negative integers \( j, k \), with \( \rho = \tan^{-1} t \),

\[
r_j(r_k(t)) = r_j(r_k(\tan \rho)) = r_j(\tan(k\rho)) = \tan(jk\rho) = r_{jk}(\tan \rho) = r_{jk}(t),
\] (43)

giving the functional identity for complex \( t \) and non-negative integers \( j, k \):

\[
r_j(r_k(t)) = r_{jk}(t).
\] (44)

Thus, the rational functions \( r_n \), with the binary operation of function composition, form an infinite Abelian group [4, p.34], with

\[
r_{jk}(t) = r_j(r_k(t)) = r_k(r_j(t)).
\] (45)

For example, for all \( t \in \mathbb{C} \) and integer \( k \geq 0 \),

\[
r_{4k}(t) = r_k \left( \frac{4t - 4t^3}{1 - 6t^2 + t^4} \right) = \frac{4r_k(t) - 4r_k^3(t)}{1 - 6r_k^2(t) + r_k^4(t)}.
\] (46)

Thus we get for \( j = 0, 1, 2, \ldots \) the reflected sequence of permutable odd rational functions \( \{r_j(t), -r_j(t)\} \).

Another infinite sequence of permutable rational functions is given by \( \{T_j(x)/1\} \) for \( j = 0, 1, 2, \ldots \), which is similar (via \( \lambda(x) = -x \)) to the permutable rational sequence \( \{T_j(-x)/(-1)\} \). And since \( T_{2i+1} \) is an odd polynomial, there is also the reflected sequence \( \{T_{2i+1}(x)/1, T_{2i+1}(x)/(-1)\} \) of permutable rational functions.

2.2.1 Reciprocal rational functions

The bilinear transform \( y = \xi(x) = (ax + b)/(cx + d) \), where \( ad - bc \neq 0 \), has the inverse bilinear transform \( x = \xi^{-1}(y) = (dy - b)/(a - yc) \). A function \( u(x) \) is rational if and only if the transformed function \( \xi^{-1}u(\xi(x)) \) is also a rational function of \( x \). Similarly to (9), functions \( u(x) \) and \( v(x) \) are permutable if and only if the transformed functions \( \xi^{-1}u\xi \) and \( \xi^{-1}v\xi \) are permutable.
In the important special case where \( \xi(x) = 1/x \), any function \( v(x) \) gets converted to another function \( 1/v(1/x) \), which we call the reciprocal of the function \( v \). The reciprocal of an even function is even, and the reciprocal of an odd function is odd. Any pair of rational functions \( v(x) \) and \( w(x) \) are permutable, if and only if their reciprocal rational functions \( f(x) = 1/v(1/x) \) and \( g(x) = 1/w(1/x) \) are permutable.

The permutable rational infinite sequence \( \{f_n(x) = T_n(x)/1\} \) has the reciprocal permutable sequence \( \{s_n(x) = 1/T_n(1/x)\} \), for which

\[
f_n(\cos \vartheta) = \cos(n\vartheta), \quad s_n(\sec \vartheta) = \sec(n\vartheta).
\] (47)

The sequence \( \{s_n(x)\} \) is similar (via \( \lambda(x) = -x \)) to the permutable rational sequence \( \{-s_n(-x) = -s_0(x), s_1(x), -s_2(x), s_3(x), -s_4(x), \cdots \} \). The rational function \( s_{2i+1} \) is an odd function, giving the permutable reflected rational sequence \( \{s_{2i+1}, -s_{2i+1}\} \).

The permutable odd rational infinite sequence \( \{q_j(x) = T_{2j+1}(x)/(-1)^j\} \) has the reciprocal permutable rational sequence \( \{u_j(x) = (-1)^j/T_{2j+1}(1/x)\} \), and in view of (17),

\[
q_j(\sin \vartheta) = \sin((2j+1)\vartheta), \quad u_j(\csc \vartheta) = \csc((2j+1)\vartheta).
\] (48)

The odd rational function \( u_j \) has the reflected permutable sequence \( \{u_j, -u_j\} \).

The odd rational function \( r_n(t) \) has the reciprocal odd rational function \( h_n(t) = 1/r_n(1/t) \), for which

\[
r_n(\tan \rho) = \tan(n\rho), \quad h_n(\cot \rho) = \cot(n\rho).
\] (49)

In view of (41) and (42), this can be rewritten as

\[
h_n(t) = \begin{cases} 
  r_n(t) & \text{for odd } n, \\
  -1/r_n(t) & \text{for even } n.
\end{cases}
\] (50)

Since \( \{r_n\} \) is an odd permutable sequence, we get the odd permutable reflected reciprocal sequence \( \{h_n, -h_n\} \).

3 Elliptic Functions

An integral of the form \( \int R(x, y) \, dx \), where \( R(x, y) \) is a rational function of \( x \) and \( y \), and \( y^2 = P(x) \) where \( P \) is a polynomial of degree 3 or 4, is called an elliptic integral [18, §17.1].

3.1 Legendre Elliptic Integrals

We shall use Louis Melville Milne-Thomson’s notation for Legendre’s elliptic integrals and Jacobian elliptic functions [18].

Adrien-Marie Legendre’s Incomplete Elliptic Integral of the First Kind, with amplitude \( \varphi \) and parameter \( m \), is defined [18, §17.2.7] as

\[
F(\varphi \mid m) \overset{\text{def}}{=} \int_0^{\sin \varphi} \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}} = \int_0^\varphi \frac{dx}{\sqrt{1 - m \sin^2 x}}.
\] (51)
The complex parameter \( m \) can be reduced to the real case with \( 0 \leq m \leq 1 \). Earlier authors often used the modulus \( k \), where \( m = k^2 \). \( F(\varphi \mid m) \) is often abbreviated to \( F(\varphi) \), when the parameter \( m \) is to be understood.

Legendre’s Complete Elliptic Integral of the First Kind [18, §17.3.1] is

\[
K(m) \overset{\text{def}}{=} F(\frac{1}{2} \pi \mid m) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}} = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - m \sin^2 x}}.
\]

As \( m \to 1 \), then \( K(m) \to \infty \).

Legendre’s Complementary Complete Elliptic Integral of the First Kind is defined [18, §17.3.5] as

\[
K'(m) \overset{\text{def}}{=} K(1 - m).
\]

\( K(m) \) and \( K'(m) \) are often abbreviated to \( K \) and \( K' \), when the parameter \( m \) is to be understood.

The elliptic integral \( u = F(\varphi \mid m) \) is single-valued for integration along the real interval \([0, \sin \varphi]\), but it has infinitely many values in the case of complex integration.

### 3.2 Jacobian elliptic functions

In 1827, Niels Henrik Abel inverted elliptic integrals to get elliptic functions [1, p.264], and he shewed that elliptic functions are doubly-periodic single-valued functions.

As functions of the complex variable \( u \), the Jacobian elliptic functions \( \text{sn} u \), \( \text{cn} u \), \( \text{dn} u \) \( \text{et cetera} \) are doubly-periodic single-valued functions of \( u \).

#### 3.2.1 Jacobian elliptic function \( \text{sn} u \)

The inverse function of the Legendre elliptic function \( F \) is \( \varphi = F^{-1}(u) \), and the Jacobian elliptic function \( \text{sn}(u \mid m) \overset{\text{def}}{=} \sin \varphi \) is often abbreviated to \( \text{sn} u \), with the parameter \( m \) implied. The function \( \text{sn} \) is single-valued for all complex parameters [18, §16.1.3], with

\[
u = \int_{\text{sn} u}^0 \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}} ,
\]

and \( \text{sn} u \) is an odd single-valued function of \( u \).

For real \( u \), the function \( \text{sn} \) has real period \( 4K(m) \) and range \([-1, 1] \), with \( \text{sn}(0) = 0, \text{sn}(K) = 1, \text{sn}(2K) = 0, \text{sn}(3K) = -1 \) and \( \text{sn}(4K) = 0 \) [18, §16.2]. Let \( \tau = \text{sn} u \), so that

\[
\text{sn}^{-1} \tau = u = \int_0^\tau \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}} = F(\sin^{-1} \tau \mid m) .
\]

On the real interval \(-K \leq u \leq K\) the function \( \text{sn} \) increases monotonically from \(-1\) to \(1\), and so for real \( \tau \in [-1, 1] \) the function \( \text{sn}^{-1} \tau \) has a single value in the real interval \([-K, K]\).

In addition to the real period \( 4K \), \( \text{sn} \) also has the imaginary period \( i2K' \).
3.2.2 Jacobian elliptic function \( cn_u \)

The function \( cn \) is defined by

\[
\text{cn}(u | m) \overset{\text{def}}{=} \cos \varphi.
\] (56)

It is often abbreviated to \( cn_u \), with the parameter \( m \) implied. Thus,

\[
\text{cn} u = \sqrt{1 - \text{sn}^2 u},
\] (57)

and \( cn_u \) is an even single–valued function of \( u \). The branch of the square root function in (57) is determined by (56).

For real \( u \), the function \( cn \) has real period \( 4K(m) \) and range \([-1, 1] \), with \( cn(0) = 1, \ cn(K) = 0, \ cn(2K) = -1, \ cn(3K) = 0 \) and \( cn(4K) = 1 \) [18, §16.2]. On the real interval \( 0 \leq u \leq 2K \) the function \( cn \) decreases monotonically from 1 to -1, and so for real \( r \in [-1, 1] \) the function \( cn^{-1}r \) has a single value in the real interval \([0, 2K]\).

In addition to the real period \( 4K \), \( cn \) has the complex period \( 2K + i2K' \).

3.2.3 Jacobian elliptic function \( dn_u \)

The Jacobian elliptic function \( dn \) is defined by

\[
\text{dn}(u | m) \overset{\text{def}}{=} \sqrt{1 - m \text{sn}^2 u}.
\] (58)

It is often abbreviated to \( dn_u \), with the parameter \( m \) implied. Thus, \( dn_u \) is an even single–valued function of \( u \).

For real \( u \), the function \( dn \) has real period \( 2K(m) \) and range \( [\sqrt{m_1}, 1] \), with \( dn(0) = 1, \ dn(K) = \sqrt{m_1} \) and \( dn(2K) = 1 \) [18, §16.2]. On the real interval \( 0 \leq u \leq K \) the function \( dn \) decreases monotonically from 1 to \( \sqrt{m_1} \), and so for real \( r \in [\sqrt{m_1}, 1] \) the function \( dn^{-1}r \) has a single value in the real interval \([0, K]\).

In addition to the real period \( 2K \), \( dn \) also has the imaginary period \( i4K' \).

3.2.4 The 12 Jacobi elliptic functions

In J. W. L. Glaisher’s systematic notation, there are 9 other Jacobi elliptic functions [18, §16.3]:

- \( cd \)  \( \overset{\text{def}}{=} \frac{cn u}{dn u} \)
- \( dc \)  \( \overset{\text{def}}{=} \frac{dn u}{cn u} \)
- \( ds \)  \( \overset{\text{def}}{=} \frac{1}{\text{sn} u} \)
- \( ns \)  \( \overset{\text{def}}{=} \frac{1}{\text{cn} u} \)
- \( sc \)  \( \overset{\text{def}}{=} \frac{sn u}{dn u} \)
- \( cs \)  \( \overset{\text{def}}{=} \frac{cn u}{sn u} \)
- \( nd \)  \( \overset{\text{def}}{=} \frac{1}{dn u} \)
- \( sn \)  \( \overset{\text{def}}{=} \frac{sn u}{cn u} \)
- \( sd \)  \( \overset{\text{def}}{=} \frac{sn u}{dn u} \)
- \( sc \)  \( \overset{\text{def}}{=} \frac{sn u}{cn u} \)
- \( cs \)  \( \overset{\text{def}}{=} \frac{cn u}{sn u} \)
- \( nd \)  \( \overset{\text{def}}{=} \frac{1}{dn u} \)

The 6 functions \( sn \), \( sc \), \( sd \), \( ns \), \( cs \), \( ds \) are odd functions of \( u \), and the other 6 functions are even functions of \( u \).

Milne-Thomson gave clear graphs [18, Figures 16.1, 16.2, 16.3] of the 12 Jacobi elliptic functions for real \( u \) (and \( m = 0.5 \)). Eugene Jahnke and Fritz Emde gave very effective graphs [14, pp. 92–93] of \( sn \), \( cn \) and \( dn \) for complex \( u \) (and \( m = 0.64 \)).
3.3 Weierstraß elliptic function \( \wp \)

Karl Wilhelm Theodor Weierstraß developed his version of elliptic functions on the basis of his elliptic function \( \wp \), with complex (or real) parameters \( g_2 \) and \( g_3 \), which is defined by the differential equation \[21, \S 18.1.6\]

\[ \wp'(z)^2 = 4 \wp^3(z) - g_2 \wp(z) - g_3. \] (60)

Hence, Weierstraß’s function \( \wp \) is the inverse of an elliptic integral, with

\[ z = \int_{\infty}^{\wp(z)} \frac{dw}{\sqrt{4w^3 - g_2w - g_3}} \] (61)

for complex \( z \).

Many mathematicians (e.g. Eric Temple Bell [2, p.207]) regard Weierstraß’s function \( \wp \) as being much more difficult than the Jacobi elliptic functions. But some mathematicians seem to have responded to that difficulty as a challenge, and they have chosen to use Weierstraß’s version of elliptic functions.

For the parameters \( m = 0 \) and \( m = 1 \), the Jacobi elliptic functions reduce respectively \[18, \S 16.6\] to trigonometric and hyperbolic functions:

\[ \begin{align*}
\text{sn}(u \mid 0) &= \sin u, & \text{cn}(u \mid 0) &= \cos u, & \text{dn}(u \mid 0) &= 1, \\
\text{sn}(u \mid 1) &= \tanh u, & \text{cn}(u \mid 1) &= \text{sech} u, & \text{dn}(u \mid 1) &= \text{sech} u,
\end{align*} \]

et cetera.

For the Weierstraß function, as \( u \to 0 \),

\[ \wp(u) = \frac{1}{u^2} + O(u^2). \] (63)

The simplest form of \( \wp(u) \) occurs as the limit when one period is \( i\infty \). Denote the other period as \( 2\omega \) (with nonzero real part). Then \[14, \text{p.105}\]

\[ \wp(u) = -\frac{1}{3} \left( \frac{\pi}{2\omega} \right)^2 \left( \frac{\pi}{\sin \left( \frac{\pi u}{2\omega} \right)} \right)^2. \] (64)

In 1956, Harold and Bertha Jeffreys mildly remarked that “Had Abel been alive he would probably have remarked that this property corresponds to taking the fundamental trigonometric function as \( \csc^2 x - \frac{1}{3} \)” [15, p.688].

3.4 Addition Formulæ for Elliptic Functions

For all complex \( \alpha \) and \( \beta \), the Jacobi elliptic functions have the rational addition formulæ: \[18, \S 16.17\],

\[ \begin{align*}
\text{sn}(\alpha + \beta) &= \frac{\text{sn} \alpha \cdot \text{cn} \beta \cdot \text{dn} \beta + \text{sn} \beta \cdot \text{cn} \alpha \cdot \text{dn} \alpha}{1 - m \text{sn}^2 \alpha \cdot \text{sn}^2 \beta}, \\
\text{cn}(\alpha + \beta) &= \frac{\text{cn} \alpha \cdot \text{cn} \beta - \text{sn} \alpha \cdot \text{dn} \alpha \cdot \text{sn} \beta \cdot \text{dn} \beta}{1 - m \text{sn}^2 \alpha \cdot \text{sn}^2 \beta}, \\
\text{dn}(\alpha + \beta) &= \frac{\text{dn} \alpha \cdot \text{dn} \beta - m \text{sn} \alpha \cdot \text{cn} \alpha \cdot \text{sn} \beta \cdot \text{cn} \beta}{1 - m \text{sn}^2 \alpha \cdot \text{sn}^2 \beta}. \end{align*} \] (65)
But the function $\wp$ has the irrational addition formula [21, §18.4.1]:

$$\wp(\alpha + \beta) = -\wp(\alpha) - \wp(\beta) + \frac{1}{4} \left[ \sqrt[4]{4\wp(\alpha)^3 - g_2\wp(\alpha) - g_3} - \sqrt[4]{4\wp(\beta)^3 - g_2\wp(\beta) - g_3} / \wp(\alpha) - \wp(\beta) \right]^2,$$  \hspace{1cm} (66)

for $\wp(\alpha) \neq \wp(\beta)$.

### 3.5 Elliptic Functions of multiple argument

For brevity, we shall write $s$, $c$ and $d$ for $sn u$, $cn u$ and $dn u$.

Put $\alpha = \beta = u$ in (65), and then it follows that

$$sn 2u = \frac{2scd}{1-ms^4}, \quad cn 2u = \frac{1-2s^2+ms^4}{1-ms^4}, \quad dn 2u = \frac{1-2ms^2+ms^4}{1-ms^4}. \hspace{1cm} (67)$$

For $m = 0$ these reduce to

$$\sin 2u = 2 \sin u \cos u, \quad \cos 2u = 1 - 2\sin^2 u, \quad 1 = 1; \hspace{1cm} (68)$$

and for $m = 1$ these reduce to

$$\tanh 2u = \frac{2 \tanh u}{1+\tanh^2 u} \hspace{1cm} (69)$$

and

$$\cosh 2u = 2 \cosh^2 u - 1 \quad \text{(twice)}. \hspace{1cm} (70)$$

For the Weierstraß function, taking the limit of (66) as $\beta \to \alpha$, we get the rational formula

$$\wp(2\alpha) = -2\wp(\alpha) + \frac{(3\wp^2(\alpha) - \frac{1}{4}g_2)^2}{4\wp^3(\alpha) - g_2\wp(\alpha) - g_3} = \frac{16\wp^4(\alpha) + 8g_2\wp^2(\alpha) + 32g_4\wp(\alpha) + g_2^2}{16(4\wp^3(\alpha) - g_2\wp(\alpha) - g_3)}, \hspace{1cm} (71)$$

[21, §18.4.5]. Induction on (71) shews that for integer $j \geq 0$,

$$\wp(2^j \alpha) = w_j(\wp(\alpha)), \hspace{1cm} (72)$$

where $w_j(x)$ is a rational function of $x$ with coefficients which are polynomials in $g_2$ and $g_3$, with integer coefficients.

But if $n$ is not a power of 2, then the irrational formulæ for $\wp(n\alpha)$ are very complicated.

Put $\alpha = 2u$ and $\beta = u$ in (65) and we get rational functions, with denominator and numerators expressed as polynomials in $s = sn u$ and $m$;

$$D = 1 - 6ms^4 + 4m[1 + m]s^6 - 3m^2s^8, \hspace{1cm} (73)$$
$$sn 3u = s\left(3 - 4[1 + m]s^2 + 6ms^4 - m^2s^8 \big/ D, \right.$$  
$$cn 3u = c\left(1 - 4s^2 + 6ms^4 - 4m^2s^6 + m^2s^8 \big/ D, \right.$$  
$$dn 3u = d\left(1 - 4ms^2 + 6ms^4 - 4ms^6 + m^2s^8 \big/ D. \right.$$
Hereafter, \( n \) is a positive integer.

Those formulæ for addition of Jacobian elliptic functions can be iterated, to get real formulæ for \( \text{sn}(nu) \), \( \text{cn}(nu) \) et cetera, in terms of rational functions of \( z = \text{sn}^2 u \) [11, pp. 78–87].

For odd \( n = 2r + 1 \),

\[
\text{sn}(nu) = \frac{sA_n(z)}{D_n(z)}, \quad \text{cn}(nu) = \frac{cB_n(z)}{D_n(z)}, \quad \text{dn}(nu) = \frac{dC_n(z)}{D_n(z)}; \quad (74)
\]

and for even \( n = 2r \),

\[
\text{sn}(nu) = \frac{scdA_n(z)}{D_n(z)}, \quad \text{cn}(nu) = \frac{B_n(z)}{D_n(z)}, \quad \text{dn}(nu) = \frac{C_n(z)}{D_n(z)}. \quad (75)
\]

Each of \( A_n, B_n, C_n, D_n \) is a polynomial of degree \( n^2 - 2 \) except that \( A_{2r} \) has degree \( 1 \) and \( n^2 - 2 = 2r^2 - 2 \). In each polynomial, the coefficient of each power of \( z \) is a reciprocal polynomial in \( m \) of degree less than or equal to \( n^2 / 4 \), with integer coefficients [2, p.208].

\text{N.B.} \ Arthur Cayley advised that, when constructing the elliptic functions of \( nu = ju + ku \), then \( j = k \) should be used for even \( n \) and \( j = k \pm 1 \) should be used for odd \( n \) — otherwise the constructed numerators and denominators will have common multinomial factors which need to be divided out to get the rational functions in lowest terms [11, p.79]. The appropriate degrees of \( z = s^2 \) in the numerators and denominators are given in the previous paragraph. Without that information, it would be necessary to divide numerator and denominator by their \( \gcd \), in the form of a polynomial in \( z \) whose coefficients are polynomials in \( m \) with integer coefficients.

3.6 Transcendental Formulae for Real Multiplication of Elliptic Functions

Abel inverted some elliptic integrals and invented elliptic functions \( \phi, f \) and \( F \) which correspond to \( \text{sn}, \text{cn} \) and \( \text{dn} \); and he constructed an explicit formula for \( \phi((2r+1)u) \) as a rational function (in factored form) of \( \phi(u) \), and similarly for \( f \) and \( F \) [1, pp.321–323]. That is called Real Multiplication of Elliptic Functions.

Thus, the rational functions \( sA_n(s^2)/D_n(s^2) \) perform real multiplication of the elliptic function \( \text{sn} \). And since (5) \( T_n(\cos \vartheta) = \cos(n\vartheta) \), the Chebyshev polynomials \( T_n \) perform real multiplication of the cosine function; and (33) the rational functions \( r_n \) perform real multiplication of the tangent function.

For integers \( \mu \) and \( \nu \), denote

\[
Q_{\mu,\nu} \overset{\text{def}}{=} 2\mu K(m) + i 2\nu K'(m). \quad (76)
\]

Then \( \text{sn}Q_{\mu,\nu} = 0 \) for all integers \( \mu \) and \( \nu \), and conversely every zero of \( \text{sn} \) is of that form \( Q_{\mu,\nu} \).

Consider an odd positive integer \( n = 2r + 1 \). Then \( \text{sn}(Q_{\mu,\nu}/n) \) has \( n^2 \) distinct values for all integers \( \mu \) and \( \nu \), and those are all given by \( -r \leq \mu \leq r \) and \( -r \leq \nu \leq r \). Those \( n^2 \) values include \( \text{sn}(Q_{0,0}/n) = 0 \); so that if we exclude that zero value then there remain \( n^2 - 1 \) distinct values [11, pp.94–97]. From this, it follows that for odd \( n = 2r + 1 \), the polynomials \( A_n, B_n, C_n, D_n \) can
be explicitly factorized, in terms of the transcendental functions \(sn\) and \(K\):

\[
A_n = (2r + 1) s \prod_{\mu=0}^{r} \prod_{\nu=-r}^{r} \left(1 - n s^2 \left(\frac{Q_{\mu,\nu}}{n}\right) s^2\right),
\]

\[
B_n = c \prod_{\mu=0}^{r} \prod_{\nu=-r}^{r} \left(1 - n s^2 \left(K(m) - \frac{Q_{\mu,\nu}}{n}\right) s^2\right),
\]

\[
C_n = d \prod_{\mu=0}^{r} \prod_{\nu=-r}^{r} \left(1 - m s^2 \left(K(m) - \frac{Q_{\mu,\nu}}{n}\right) s^2\right),
\]

\[
D_n = \prod_{\mu=0}^{r} \prod_{\nu=-r}^{r} \left(1 - m s^2 \left(\frac{Q_{\mu,\nu}}{n}\right) s^2\right).
\]  

Here, \(\prod'\) means that for \(\nu = 0\), \(\mu\) ranges from 1 to \(r\); but otherwise \(\nu\) ranges from \(-r\) to \(r\). Thus the double product over \(\mu\) and \(\nu\) has \(2r(r + 1)\) factors.

And similar factorizations apply for even \(n\).

But, repeated application of Abel’s addition formulæ for elliptic functions gives the algebraic formulæ (74) (75), with coefficients of powers of \(z\) in numerator and denominator as polynomials in \(m\) with integer coefficients.

The algebraic complexity of those algebraic formulæ for \(sn(nu)\ et\ et\ et\ etc\) increases very rapidly with \(n\). The immensely complicated formulæ for \(n = 6\) and \(n = 7\) were constructed by Baehr in 1861, and were partially verified (for \(m = 0\) and \(m = 1\)) by Cayley. In 1895, Cayley fitted the formulæ for \(sn(7u)\) and \(dn(7u)\) onto 2 pages of fine print, only by using heavily abbreviated notation which requires care to detail for interpreting it [11, pp.84–85]. The construction of explicit algebraic formulæ for those coefficients of powers of \(z\) for general \(n\) was a problem which defied solution for over a century after Abel [6] [7] [8] [9] [12]. In 1847, Cayley sighed that “it seems hopeless to continue this investigation any further” [5, p.298], and in 1889 he declared that “the investigation seems to show that the integration cannot be effected in any tolerably simple form” [10, p.589].

In 1932, Eric Temple Bell explained that “The transcendental solution of the problem of real multiplication of elliptic functions has been classic for over a century; the algebraic solution has not yet been achieved, although it has engaged the attention of many writers”. Bell then gave the first general method (very complicated) for constructing all coefficients of \(D_n\) as polynomials in \(n^2\) (and \(m\)), for general \(n\) [2]. The coefficients in the numerators \(A_n, B_n, C_n\) can be constructed from the coefficients of \(D_n\), with different rules for even \(n\) and odd \(n\) [11, pp. 81–89], [12, pp.196–209].

The low-order terms of \(D_n\) are:

\[
D_n(z) = 1 + 0z - \frac{1}{12} n^2(n^2 - 1)^2 m z^2 + \frac{1}{96} n^2(n^2 - 1)(n^2 - 2^2)(1 + m) m z^3 - \cdots.
\]  

For even \(n = 2r\), the high–order terms are:

\[
D_{2r}(z) = \cdots + (-1)^r \frac{8}{15} r^2 (r^2 - 1) (4r^2 - 1)(1 + m) m \ z^{2r^2 - 3} + (-1)^r - 1 \frac{4}{3} r^2 (4r^2 - 1)(mz)^{r^2 - 1} + 0 z^{2r^2 - 1} + (-1)^r (mz)^{r^2};
\]  

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and for odd \( n = 2r + 1 \), the high–order terms are:

\[
D_{2r+1}(z) = \cdots + (-1)^r \frac{2}{3} r (r + 1)(2r + 1) [1 + m] r^{2r+2} z^{2r+2r-1} + (-1)^r (2r + 1)(m z^2)^{r(r+1)}.
\]

For odd \( n = 2r + 1 \),

\[
A_{2r+1}(z) = (2r + 1) - \frac{2}{3} r (r + 1)(2r + 1) [1 + m] z + \cdots
+ (-1)^r \frac{2}{3} r (r + 1)(2r - 1)(2r + 1)^2 (2r + 3) \times [1 + m] m r^{2r+2} z^{2r+2r-3} + (-1)^r \frac{1}{3} r (r + 1)(2r + 1)^2 (m z^2)^{r+1} + 0 z^{2r+2r-1} + (-1)^r (m z^2)^{r(r+1)},
\]

\[
B_{2r+1}(z) = 1 - 2r(r + 1) z + \frac{1}{3} r (r + 1)[2(r^2 + r - 2) + (2r + 1)^2 m] z^2 - \cdots
+ \frac{1}{3} r (r + 1)[(2r + 1)^2 + 2(r^2 + r - 2) m] (m z^2)^{r+1} + 2r(r + 1)m(r+1) z^{2r+2r-1} + (m z^2)^{r(r+1)}.
\]

\[
C_{2r+1}(z) = 1 - 2r(r + 1) m z + \frac{1}{3} r (r + 1)[(2r + 1)^2 + 2(r^2 + r - 2) m] m z^2 - \cdots
+ \frac{1}{3} r (r + 1)[2(r^2 + r - 2) + (2r + 1)^2 m] m z^{2r+2r-1} + 2r(r + 1)m(r+1) z^{2r+2r-1} + (m z^2)^{r(r+1)}.
\]

With \( m = 0 \), these reduce to

\[
\sin((2r + 1)u) = \frac{(-1)^r T_{2r+1}(\sin u)}{1}, \quad \cos((2r + 1)u) = \frac{T_{2r+1}(\cos u)}{1}.
\]

For even \( n = 2r \),

\[
A_{2r}(z) = 2r - \frac{3}{4} r (r^2 - 1) [1 + m] z + \cdots
+ (-1)^r \frac{4}{3} r (r^2 - 1) [1 + m] m r^{2r^2-2} z^{2r^2-3} + (-1)^r 2r(r+1) m z^{2r^2-2},
\]

\[
B_{2r}(z) = 1 - 2r^2 z + \frac{1}{3} r^2 [2(r^2 - 1) + (4r^2 - 1) m] z^2 + \cdots
+ \frac{1}{3} r^2 [2(r^2 - 1) + (4r^2 - 1) m] m r^{2r^2-2} z^{2r^2-2} + 2r^2 m r^{2r^2-2} z^{2r^2-2} + (m z^2)^{r^2},
\]

\[
C_{2r}(z) = 1 - 2r^2 m z + \frac{1}{3} r^2 [2(r^2 - 1) + (4r^2 - 1) m] m z^2 + \cdots
+ \frac{1}{3} r^2 [4r^2 - 1 + 2(r^2 - 1) m] m r^{2r^2-2} z^{2r^2-2} + 2r^2 m r^{2r^2-2} z^{2r^2-2} + (m z^2)^{r^2}.
\]

With \( m = 0 \), these reduce to

\[
\cos(2ru) = \frac{T_{2r}(\cos u)}{1}.
\]

In each expression for the polynomials \( A_n, B_n, C_n, D_n \), some powers of \( z \) can appear both in the low–order terms (before the \( \cdots \)) and in the high–order terms.
(after the · · ·). For instance, in $A_{2r+1}$ with $r = 1$, the term in $z^1$ does appear before and after the dots, with expressions which both give the term $-4[1+m]z$ for $r = 1$. In any such case, the term is to be used only once (not twice) in evaluating the polynomial.

4 From elliptic functions to permutable rational functions

In 1923, J. F. Ritt made a very general study [19] of permutable polynomials and permutable rational functions “that classifies all rational functions $r$ and $s$ that commute in the sense that $r \circ s = s \circ r$” [4, p.34]. Ritt’s major result is the following:

Theorem If the rational functions $\Phi(z)$ and $\Psi(z)$, each of degree greater than unity, are permutable, and if no iterate of $\Phi(z)$ is identical with any iterate of $\Psi(z)$, there exist a periodic meromorphic function $f(z)$, and four numbers $a$, $b$, $c$ and $d$, such that

\[ f(az + b) = \Phi[f(z)], \quad f(cz + d) = \Psi[f(z)]. \]

The possibilities for $f(z)$ are: any linear function of $e^z$, $\cos z$, $\wp z$; in the lemniscatic case ($g_3 = 0$), $\wp^2 z$; in the equianharmonic case ($g_2 = 0$), $\wp' z$ and $\wp^3 z$. These are, essentially, the only periodic meromorphic functions which have rational multiplication theorems.

The multipliers $a$ and $c$ must be such that if $\omega$ is any period of $f(z)$, $a\omega$ and $c\omega$ are also periods of $f(z)$.

If $p$ represents the order of $f(z)$, that is, the number of times $f(z)$ assumes any given value in a primitive period strip or in a primitive period parallelogram, the products

\[ b\left(1 - e^{2\pi i/p}\right), \quad d\left(1 - e^{2\pi i/p}\right) \]

must be periods of $f(z)$.

Finally,

\[ (a - 1)d - (c - 1)b \]

must be a period of $f(z)$. [19, pp.399–400]

Ritt’s treatment of this topic is extremely abstract. In 50 pages of abstruse analysis of the Weierstraß elliptic function $\wp(z)$ on Riemann surfaces, he gave only 2 detailed examples of a permutable pair of rational functions. His first example is “the permutable pair $z^p$ and $\varepsilon z^q$ where $p$ and $q$ are positive or negative integers, and where $\varepsilon^{p-1} = 1$” [19, p.413]. His second example involves functions which are not based on periodic trigonometric or elliptic functions: “Let

\[ \phi(z) = \frac{\varepsilon z^2 + 2}{\varepsilon z + 1}, \quad \psi(z) = \frac{z^2 + 2}{z + 1}, \quad \sigma(z) = \frac{z^2 - 4}{z - 1}, \]

where $\varepsilon$ is a primitive third root of unity. We shall see below that $\Phi = \phi\sigma$ and $\Psi = \psi\sigma$ are permutable” [19, p.447].
For the rational function $w_j$ (cf. (72)), for all complex $\alpha$, $g_2$ and $g_3$ and for all non-negative integers $j$ and $k$,

$$w_j(w_k(\wp(\alpha))) = \wp(2^j(2^k\alpha)) = \wp(2^{j+k}\alpha) = w_{j+k}(\wp(\alpha)).$$

Hence

$$w_j(w_k(\wp(\alpha))) = w_{j+k}(\wp(\wp(\alpha))) = \wp(2^j(2^k\alpha)) = w_{j+k}(\wp(\alpha)),
$$

and thus the sequence of rational functions $\{w_j\}$ is permutable.

The Weierstraß theory of elliptic functions, which Ritt used, is much more complicated than the version using the Jacobi elliptic functions.

For each of the 12 Jacobi elliptic functions, some infinite sequences of permutable rational functions are here constructed explicitly, based on the equations (74) and (75) for elliptic functions of multiple argument. In each case the parameter $m$ is implied.

4.1 Functions $\text{sn}((2r+1)u)$ and $\text{ns}((2r+1)u)$

From (74),

$$\text{sn}((2r+1)u) = \alpha_{2r+1}(\text{sn} u),$$

where $\alpha_{2r+1}$ is the odd rational function:

$$\alpha_{2r+1}(x) \overset{\text{def}}{=} \frac{x A_{2r+1}(x^2)}{D_{2r+1}(x^2)}.$$

Also, using the factored forms (80) for $A_{2r+1}$ and $D_{2r+1}$, we get

$$\alpha_{2r+1}(x) = (2r+1)x \prod_{g=0}^{r} \prod_{h=-r}^{r} \frac{1 - \text{sn}^2 \left( \frac{Q_{g,h}}{2r+1} \right) x^2}{1 - \text{sn}^2 \left( \frac{Q_{g,h}}{2r+1} \right) m x^2}.$$

With parameter $m = 0$, these reduce to $\text{sn}((2r+1)u) = (-1)^r T_{2r+1}(\sin u)$.

As with (45), we get the functional identity for complex $x$ and positive odd integers $j, k$:

$$\alpha_{jk}(x) = \alpha_k(\alpha_j(x)) = \alpha_j(\alpha_k(x)).$$

Thus, all rational functions in the infinite reflected sequence $\{\alpha_{2r+1}, -\alpha_{2r+1}\}$ are permutable.

The reciprocal function is

$$\beta_{2r+1}(x) \overset{\text{def}}{=} \frac{x D_{2r+1}(1/x^2)}{A_{2r+1}(1/x^2)},$$

with

$$\text{ns}((2r+1)u) = \beta_{2r+1}(\text{ns} u).$$
giving the infinite reflected sequence \( \{ \beta_{2r+1}, -\beta_{2r+1} \} \) of permutable rational functions. Also, using the factored form (92) for \( \alpha_{2r+1}(x) \), we get

\[
\beta_{2r+1}(x) = \frac{1}{\alpha_{2r+1}(1/x)} = \frac{x}{2r+1} \prod_{g=0}^{r} \prod_{h=-r}^{r} \left( \frac{x^2 - \text{sn}^2 \left( \frac{Q_{g,h}}{2r+1} \right) m}{x^2 - \text{ns}^2 \left( \frac{Q_{g,h}}{2r+1} \right)} \right). \tag{96}
\]

### 4.2 Functions \( \text{sd}((2r+1)u) \) and \( \text{ds}((2r+1)u) \)

From (58), we get that

\[
\text{sd}^2 u = \frac{\text{sn}^2 u}{\text{dn}^2 u} = \frac{\text{sn}^2 u}{1 - m \text{sn}^2 u}, \tag{97}
\]

and hence

\[
z = \text{sn}^2 u = \frac{\text{sd}^2 u}{1 + m \text{sd}^2 u}. \tag{98}
\]

From (74) we get that for odd \( n = 2r + 1 \),

\[
\text{sd}((2r + 1)u) = \frac{\text{sd}(u) A_{2r+1}(\text{sn}^2 u)}{C_{2r+1}(\text{sn}^2 u)}. \tag{99}
\]

Therefore

\[
\text{sd}((2r + 1)u) = \gamma_{2r+1}(\text{sd} u), \tag{100}
\]

where \( \gamma_{2r+1} \) is the odd rational function:

\[
\gamma_{2r+1}(x) = \frac{x A_{2r+1} \left( \frac{x^2}{1 + mx^2} \right)}{C_{2r+1} \left( \frac{x^2}{1 + mx^2} \right)}. \tag{101}
\]

Also, using the factored forms (80) for \( A_{2r+1} \) and \( C_{2r+1} \), we get

\[
\gamma_{2r+1}(x) = (2r + 1) x \prod_{g=0}^{r} \prod_{h=-r}^{r} \left( \frac{1 - \text{ns}^2 \left( \frac{Q_{g,h}}{2r+1} \right) x^2}{1 - \text{ns}^2 \left( K(m) - \frac{Q_{g,h}}{2r+1} \right) x^2} \right) \left( \frac{1 + \left( m - \text{ns}^2 \left( \frac{Q_{g,h}}{2r+1} \right) \right) x^2}{1 + m \left( 1 - \text{ns}^2 \left( K(m) - \frac{Q_{g,h}}{2r+1} \right) \right) x^2} \right) \left( \frac{1 - \text{ds}^2 \left( \frac{Q_{g,h}}{2r+1} \right) x^2}{1 + \text{cs}^2 \left( K(m) - \frac{Q_{g,h}}{2r+1} \right) x^2} \right) \left( \frac{1 - \text{cs}^2 \left( K(m) - \frac{Q_{g,h}}{2r+1} \right) m x^2}{1 - \text{cs}^2 \left( K(m) - \frac{Q_{g,h}}{2r+1} \right) m x^2} \right). \tag{102}
\]
With parameter $m = 0$ this reduces to $\sin((2r+1)u) = (-1)^r T_{2r+1}(\sin u)$.
As with (45), we get the functional identity for complex $x$ and positive odd positive integers $j,k$:
\[
\gamma_{jk}(x) = \gamma_j(\gamma_k(x)) = \gamma_k(\gamma_j(x)). \tag{103}
\]
Thus, all rational functions in the infinite reflected sequence $\{\gamma_{2r+1}, -\gamma_{2r+1}\}$ are permutable.

The reciprocal function is
\[
\delta_{2r+1}(x) \overset{\text{def}}{=} \frac{1}{\gamma_{2r+1}(1/x)} = \frac{x C_{2r+1} \left( \frac{1}{m+x^2} \right)}{A_{2r+1} \left( \frac{1}{m+x^2} \right)}, \tag{104}
\]
with
\[
ds((2r+1)u) = \delta_{2r+1}(ds u), \tag{105}
\]
giving the infinite reflected sequence $\{\delta_{2r+1}, -\delta_{2r+1}\}$ of permutable rational functions.

Also, using the factored form (102) for $\gamma_{2r+1}(x)$, we get
\[
\delta_{2r+1}(x) = \frac{1}{\gamma_{2r+1}(1/x)} = \frac{x}{2r+1} \prod_{g=0}^{r} \prod_{h=-r}^{r} \left( \frac{x^2 - \text{cs}^2 \left( \frac{K(m) - Q_{g,h}}{2r+1} \right) m}{x^2 - \text{ds}^2 \left( \frac{Q_{g,h}}{2r+1} \right)} \right). \tag{106}
\]

### 4.3 Functions $sc((2p+1)u)$ and $cs((2p+1)u)$

From (58), we get that
\[
sc^2 u = \frac{\text{sn}^2 u}{\text{cn}^2 u} = \frac{\text{sn}^2 u}{1 - \text{sn}^2 u}, \tag{107}
\]
and hence
\[
z = \text{sn}^2 u = \frac{sc^2 u}{1 + sc^2 u}. \tag{108}
\]
From (74) we get that for odd $n = 2p+1$,
\[
sc((2p+1)u) = \frac{sc(u) A_{2p+1}(\text{sn}^2 u)}{B_{2p+1}(\text{sn}^2 u)}. \tag{109}
\]
Therefore
\[
sc((2p+1)u) = \varepsilon_{2p+1}(sc u), \tag{110}
\]
where $\varepsilon_{2p+1}$ is the odd rational function:
\[
\varepsilon_{2p+1}(x) = \frac{x A_{2p+1} \left( \frac{x^2}{1+x^2} \right)}{B_{2p+1} \left( \frac{x^2}{1+x^2} \right)}. \tag{111}
\]
With parameter $m = 0$, this reduces (cf. (32)) to $\tan((2p+1)\rho) = r_{2p+1}(\tan \rho)$.

Also, using the factored forms (80) for $A_{2r+1}$ and $B_{2r+1}$, we get

$$
\varepsilon_{2p+1}(x) = (2p+1) x \prod_{g=0}^{r} \prod_{h=-r}^{r} \left( \frac{1 - \eta^2 \left( \frac{Q_{g,h}}{2p+1} \right) x^2}{1 - \eta^2 \left( \frac{1}{2p+1} \right) x^2} \right).
$$

$$
= (2p+1) x \prod_{g=0}^{r} \prod_{h=-r}^{r} \left( \frac{1 + \left( 1 - \eta^2 \left( \frac{Q_{g,h}}{2p+1} \right) x^2 \right)}{1 + \left( 1 - \eta^2 \left( \frac{1}{2p+1} \right) x^2 \right)} \right).
$$

$$
= (2p+1) x \prod_{g=0}^{r} \prod_{h=-r}^{r} \left( \frac{1 - \xi^2 \left( \frac{Q_{g,h}}{2p+1} \right) x^2}{1 - \xi^2 \left( \frac{1}{2p+1} \right) x^2} \right). \quad (112)
$$

As with (45), we get the functional identity for complex $x$ and positive odd positive integers $j, k$:

$$
\varepsilon_{jk}(x) = \varepsilon_j(\varepsilon_k(x)) = \varepsilon_k(\varepsilon_j(x)). \quad (113)
$$

Thus, all rational functions in the infinite reflected sequence $\{\varepsilon_{2p+1}, -\varepsilon_{2p+1}\}$ are permutable.

The reciprocal function is

$$
\zeta_{2p+1}(x) \overset{\text{def}}{=} \frac{1}{\varepsilon_{2p+1}(1/x)} = \frac{x B_{2p+1} \left( \frac{1}{1 + x^2} \right)}{A_{2p+1} \left( \frac{1}{1 + x^2} \right)}, \quad (114)
$$

with

$$
cs((2p+1)u) = \zeta_{2p+1}(\xi u), \quad (115)
$$
giving the infinite reflected sequence $\{\zeta_{2p+1}, -\zeta_{2p+1}\}$ of permutable rational functions.

Also, using the factored form (112) for $\varepsilon_{2p+1}$, we get

$$
\zeta_{2p+1}(x) = \frac{1}{\varepsilon_{2p+1}(1/x)}
\quad = \frac{x}{2p+1} \prod_{g=0}^{r} \prod_{h=-r}^{r} \left( \frac{x^2 - cs^2 \left( \frac{K(m) - Q_{g,h}}{2p+1} \right) x^2}{x^2 - cs^2 \left( \frac{Q_{g,h}}{2p+1} \right) x^2} \right). \quad (116)
$$

**4.4 Functions cn(ku) and nc(ku)**

From (56), $z = sn^2 u = 1 - cn^2 u$, and from (74) and (75) we get that

$$
\text{cn}(ku) = \eta_k(\text{cn} u), \quad (117)
$$
where $\eta_k$ is an odd rational function for odd $k = 2r + 1$ and an even rational function for even $k = 2r$:

$$
\eta_k(x) \equiv \begin{cases} 
  \frac{x B_{2r+1}(1-x^2)}{D_{2r+1}(1-x^2)} & (k = 2r + 1), \\
  \frac{B_{2r}(1-x^2)}{D_{2r}(1-x^2)} & (k = 2r).
\end{cases} \tag{118}
$$

With parameter $m = 0$, these reduce to $\cos(ku) = T_k(\cos u)$.

As with (45), we get the functional identity for complex $x$ and positive integers $j, k$:

$$
\eta_{jk}(x) = \eta_j(\eta_k(x)) = \eta_k(\eta_j(x)). \tag{119}
$$

Thus, all rational functions in the infinite sequence $\{\eta_k\}$ are permutable.

This sequence is similar (via $\lambda(x) = -x$) to the permutable rational sequence

$$
\{ -\eta_k(-x) \} = -\eta_0(x), -\eta_1(x), -\eta_2(x), -\eta_3(x), \cdots.
$$

The odd rational function $\eta_{2r+1}$ has the reflected permutable rational sequence $\{ \eta_{2r+1}, -\eta_{2r+1} \}$.

The reciprocal function is

$$
\theta_k(x) \equiv \frac{1}{\eta_k(1/x)}, \tag{120}
$$

with

$$
\text{nc}(ku) = \theta_k(\text{nc} u), \tag{121}
$$

giving the sequence of permutable rational functions $\{\theta_k\}$.

This sequence is similar (via $\lambda(x) = -x$) to the permutable rational sequence

$$
\{ -\theta_k(-x) \} = -\theta_0(x), -\theta_1(x), -\theta_2(x), -\theta_3(x), \cdots.
$$

The odd rational function $\theta_{2r+1}$ has the reflected permutable rational sequence $\{ \theta_{2r+1}, -\theta_{2r+1} \}$.

### 4.5 Functions $dn(ku)$ and $nd(ku)$

From (58),

$$
z = \sin^2 u = \frac{1 - \text{dn}^2 u}{m}, \tag{122}
$$

and from (74) and (75) we get that

$$
\text{dn}(ku) = \iota_k(\text{dn} u), \tag{123}
$$

where $\iota_k$ is an odd rational function for odd $k = 2r + 1$ and an even rational function for even $k = 2r$:

$$
\iota_k(x) \equiv \begin{cases} 
  \frac{x C_{2r+1} \left( \frac{1-x^2}{m} \right)}{D_{2r+1} \left( \frac{1-x^2}{m} \right)} & (k = 2r + 1), \\
  \frac{C_{2r} \left( \frac{1-x^2}{m} \right)}{D_{2r} \left( \frac{1-x^2}{m} \right)} & (k = 2r).
\end{cases} \tag{124}
$$
As with (45), we get the functional identity for complex \( x \) and positive integers \( j, k \):

\[
\iota_{jk}(x) = \iota_j(\iota_k(x)) = \iota_k(\iota_j(x)).
\]

(125)

Thus, all rational functions in the infinite sequence \( \{\iota_k\} \) are permutable.

This sequence is similar (via \( \lambda(x) = -x \)) to the permutable rational sequence \( \{-\iota_k(-x)\} = -\iota_0(x), \iota_1(x), -\iota_2(x), \iota_3(x), \cdots \).

The odd rational function \( \iota_{2r+1} \) has the reflected permutable rational sequence \( \{\iota_{2r+1}, -\iota_{2r+1}\} \).

The reciprocal function is

\[
\kappa_k(x) \overset{\text{def}}{=} \frac{1}{\iota_k(1/x)},
\]

(126)

with

\[
\text{nd}(ku) = \kappa_k(\text{nd} u),
\]

(127)

giving the sequence of permutable rational functions \( \{\kappa_k\} \).

This sequence is similar (via \( \lambda(x) = -x \)) to the permutable rational sequence \( \{-\kappa_k(-x)\} = -\kappa_0(x), \kappa_1(x), -\kappa_2(x), \kappa_3(x), \cdots \).

The odd rational function \( \kappa_{2r+1} \) has the reflected permutable rational sequence \( \{\kappa_{2r+1}, -\kappa_{2r+1}\} \).

### 4.5.1 Duality of \( \eta \) and \( \kappa \)

The rational functions \( \eta_k \) perform (118) real multiplication of the elliptic function \( \text{cn} \), and the rational functions \( \iota_k \) perform (124) real multiplication of the elliptic function \( \text{dn} \). Rewrite those functions with \( m \) as an explicit parameter, and we get:

\[
\eta_k(\text{cn}(u \mid m), m) = \text{cn}(ku \mid m), \quad \iota_k(\text{dn}(u \mid m), m) = \text{dn}(ku \mid m).
\]

(128)

Denote

\[
\rho \overset{\text{def}}{=} 1/m, \quad v \overset{\text{def}}{=} u\sqrt{m}.
\]

(129)

Then, by Jacobi’s Real Transformation [18, §16.11.3 & §16.11.4],

\[
\text{cn}(u \mid m) \equiv \text{dn}(v \mid \rho), \quad \text{dn}(u \mid m) \equiv \text{cn}(v \mid \rho).
\]

(130)

Substitute this in (128), and we get the identity

\[
\eta_k(\text{dn}(v \mid \rho), m) \equiv \eta_k(\text{cn}(u \mid m), m) \equiv \text{cn}(ku \mid m)
\equiv \text{dn}(kv \mid \rho) \equiv \iota_k(\text{dn}(v \mid \rho), \rho).
\]

(131)

Write \( x = \text{dn}(v \mid \rho) \), and we get an identity (in complex \( x \) and \( m \), and non-negative integer \( k \)) between two families of permutable rational functions:

\[
\eta_k(x, m) \equiv \iota_k(x, 1/m).
\]

(132)

And we get an identity for their permutable reciprocal functions

\[
\theta_k(x) = 1/\eta_k(1/x) \text{ (which perform real multiplication of the function nc) and }
\kappa_k(x) = 1/\iota_k(1/x) \text{ (which perform real multiplication of the function nd):}
\]

\[
\theta_k(x, m) \equiv \kappa_k(x, 1/m).
\]

(133)
4.6 Functions \( cd(ku) \) and \( dc(ku) \)

From (56) and (58), we get that
\[
\frac{cd^2}{dn^2} u = \frac{cn^2}{1 - mn^2} u ,
\]
and hence
\[
z = sn^2 u = \frac{1 - cd^2}{1 - m cd^2} u .
\]

From (74) and (75) we get that
\[
\frac{cd}{ku} = \left\{ \begin{array}{ll}
\frac{cd(u)B_{2r+1}(sn^2 u)}{C_{2r+1}(sn^2 u)} & (k = 2r + 1), \\
\frac{B_{2r}(sn^2 u)}{C_{2r}(sn^2 u)} & (k = 2r).
\end{array} \right.
\]

Therefore
\[
\frac{cd}{ku} = \lambda_k(cd u),
\]
where \( \lambda_k \) is an odd rational function for odd \( k = 2r + 1 \) and an even rational function for even \( k = 2r \):
\[
\lambda_k(x) = \left\{ \begin{array}{ll}
x B_{2r+1} \left( \frac{1 - x^2}{1 - mx^2} \right) & (k = 2r + 1), \\
B_{2r} \left( \frac{1 - x^2}{1 - mx^2} \right) & (k = 2r).
\end{array} \right.
\]

With parameter \( m = 0 \), these reduce to \( \cos(ku) = T_k(\cos u) \).

As with (45), we get the functional identity for complex \( x \) and positive integers \( j, k \):
\[
\lambda_{jk}(x) = \lambda_j(\lambda_k(x)) = \lambda_k(\lambda_j(x)).
\]

Thus, all rational functions in the infinite sequence \( \{\lambda_k\} \) are permutable.

This sequence is similar to the permutable rational sequence \( \{-\lambda_k(-x)\} = -\lambda_0(x), \lambda_1(x), -\lambda_2(x), \lambda_3(x), \ldots \).

The odd rational function \( \lambda_{2r+1} \) has the reflected permutable rational sequence \( \{\lambda_{2r+1}, -\lambda_{2r+1}\} \).

The reciprocal function is
\[
\mu_k(x) \overset{\text{def}}{=} \frac{1}{\lambda_k(1/x)} ,
\]
with
\[
\frac{dc}{ku} = \mu_k(dc u),
\]
giving the sequence of permutable rational functions \( \{\mu_k\} \).
This sequence is similar to the permutable rational sequence \( \{-\mu_k(-x)\} = -\mu_0(x), \mu_1(x), -\mu_2(x), \mu_3(x), \cdots \).
The odd rational function \( \mu_{2r+1} \) has the reflected permutable rational sequence \( \{\mu_{2r+1}, -\mu_{2r+1}\} \).

Thus, many infinite sequences of permutable rational functions have been constructed, on the basis of elliptic functions.

References


G. J. Tee
Department of Mathematics
University of Auckland
Private Bag 92019
Auckland
New Zealand
e-mail: tee@math.auckland.ac.nz

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