PERMUTABLE POLYNOMIALS AND RATIONAL FUNCTIONS

GARRY J. TEE
Department of Mathematics, University of Auckland
Auckland,
New Zealand

2007–1–8

Summary. Many infinite sequences of permutable rational functions and a few infinite sequences of permutable polynomials are constructed, on the basis of elliptic functions and trigonometric functions.

2000 Mathematics Subject Classification (2000). Primary 33E05, 26C05; Secondary 20K99.

Keywords.: Permutable polynomials, Chebyshev polynomials, permutable chains, permutable rational functions, Jacobi elliptic functions, real multiplication, trigonometric functions.

1 Permutable Polynomials

The Chebyshev polynomial of the first type T_d is defined by the initial values

$$T_0(x) \ = \ 1, \qquad T_1(x) \ = \ x, \tag{1}$$

with the 3–term recurrence relation for $n > 1$:

$$T_n(x) \ = \ 2xT_{n-1}(x) - T_{n-2}(x). \tag{2}$$

By induction on n in (2), it follows from (1) that, for all integers $d \geq 0$, $T_d(x)$ is a polynomial in x of degree d with integer coefficients. Moreover, T_d is an even polynomial for even d and an odd polynomial for odd d.

The integer coefficients of T_d are given explicitly by the following formula[1] for $d > 0$ [17, p.79]:

$$T_d(x) \ = \ \sum_{k=0}^{d \div 2} (-1)^k \frac{2^{d-2k-1} d(d-k-1)!}{k!(d-2k)!} \, x^{d-2k} \ =$$

$$2^{d-1}x^d - 2^{d-3}dx^{d-2} + 2^{d-6}d(d-3)x^{d-4} - \frac{2^{d-8}}{3} d(d-4)(d-5)x^{d-6} + \cdots$$

$$\cdots + \begin{cases} (-1)^{d \div 2} & \text{(for even } d\text{)}, \\ (-1)^{(d-1)\div 2}\, dx & \text{(for odd } d\text{)}. \end{cases} \tag{3}$$

[1] The symbol $\div$ denotes integer division, yielding integer quotient. For integers n and $d > 0$, $q = n \div d$, where $n = qd + r$, with remainder $0 \leq r < d$.

For example, in addition to (1),

$$\begin{array}{rclrcl} T_2(x) & = & 2x^2-1, & T_3(x) & = & 4x^3-3x, \\ T_4(x) & = & 8x^4-8x^2+1, & T_5(x) & = & 16x^5-20x^3+5x\ . \end{array} \tag{4}$$

By induction on n in (2), it follows from (1) that for all complex x and integer $n \geq 0$, with $\vartheta \stackrel{\text{def}}{=} \cos^{-1} x$,

$$T_n(x) \;=\; T_n(\cos\vartheta) \;=\; 2\cos\vartheta\ \cos((n-1)\vartheta) - \cos((n-2)\vartheta) \;=\; \cos(n\vartheta). \tag{5}$$

(Any of the infinitely many values of $\cos^{-1} x$ can be used for ϑ.) Therefore, for all complex x and non–negative integers j, k,

$$\begin{aligned} T_j\big(T_k(x)\big) \;&=\; T_j\big(T_k(\cos\vartheta)\big) \;=\; T_j(\cos(k\vartheta)) \;=\; \cos(jk\vartheta) \\ &=\; T_{jk}(\cos\vartheta) =\; T_{jk}(x), \end{aligned} \tag{6}$$

giving the functional identity for complex x and non–negative integers j, k:

$$T_j\big(T_k(x)\big) \;=\; T_{jk}(x). \tag{7}$$

Thus, the Chebyshev polynomials, with the binary operation of function composition, form an infinite Abelian group, with

$$T_j\big(T_k(x)\big) \;=\; T_{jk}(x) \;=\; T_{kj}(x) \;=\; T_k\big(T_j(x)\big). \tag{8}$$

"Two polynomials, p and q, are called *permutable* if $p(q(x)) = q(p(x))$ for all m, x." [20, p.192]. "A sequence of polynomials, each of positive degree, containing at least one of each positive degree and such that every two polynomials in it are permutable is called a *chain*. The Chebyshev polynomials $T_1(x), \ldots, T_n(x), \ldots$ form a chain. So do the powers $\pi_j(x) = x^j$, $j = 1, 2, \ldots$, as is easily verified." [20, p.194]

For $a \neq 0$, the linear transform $y = \lambda(x) = ax + b$ has the inverse transform $x = \lambda^{-1}(y) = (y-b)/a$. A pair of functions $u(x)$ and $v(x)$ are permutable, if and only if the transformed functions $U \stackrel{\text{def}}{=} \lambda^{-1}u\lambda$ and $V \stackrel{\text{def}}{=} \lambda^{-1}v\lambda$ are permutable, since

$$\begin{aligned} U(V(x)) \;&=\; \lambda^{-1}u\Big(\lambda\big(\lambda^{-1}v(\lambda(x))\big)\Big) \;=\; \lambda^{-1}u(v(\lambda(x))) \\ &=\; \lambda^{-1}v(u(\lambda(x))) \;=\; \lambda^{-1}v\Big(\lambda\big(\lambda^{-1}u(\lambda(x))\big)\Big) \;=\; V(U(x))\ . \end{aligned} \tag{9}$$

If $u(x)$ is a polynomial in x of degree n, then so is the transformed function $\lambda^{-1}u\lambda$.

Two chains are called *similar* if there is a linear transformation λ such that each polynomial in one chain is similar (via λ) to the polynomial of the same degree in the other chain. H. David Block and H. P. Thielman [3] and E. Jacobsthal [13] proved the remarkable theorem that "every chain is either similar to $\left\{x^j\right\}$, $j = 1, 2, \ldots$ or to $\{T_j\}$, $j = 1, 2, \ldots$" [20, p.195] [16, p.156] [4, p.34].

1.1 Permutable Even and Odd Functions

The sequence of rational functions $\pi_r(x) = \{x^r\}$ (for integer r) are all permutable, since

$$\begin{aligned} \pi_r\big(\pi_s(x)\big) \;&=\; \pi_r\big(x^s\big) \;=\; \big(x^s\big)^r \;=\; x^{sr} \\ &=\; x^{rs} \;=\; \big(x^r\big)^s \;=\; \pi_s\big(x^r\big) \;=\; \pi_s\big(\pi_r(x)\big). \end{aligned} \tag{10}$$

A function f is called an *even function* iff

$$f(-x) \;=\; f(x) \tag{11}$$

for all x. If u and v are permutable even functions then $-u$ and $-v$ are permutable even functions, since

$$-u(-v(x)) \;=\; -u(v(x)) \;=\; -v(u(x)) \;=\; -v(-u(x)). \tag{12}$$

Hence, if $\{w_r\}$ is a permutable sequence of even functions for integer r, then the negated sequence $\{-w_r\}$ is permutable.

A function f is called an *odd function* iff

$$f(-x) \;=\; -f(x) \tag{13}$$

for all x. If u and v are permutable odd functions then $-u$ and v are permutable odd functions, since

$$-u(v(x)) \;= -v(u(x)) \;=\; v(-u(x)), \tag{14}$$

and similarly $-v$ and u permute. If u and v are permutable odd functions then $-u$ and $-v$ are permutable odd functions, since

$$\begin{aligned} -u(-v(x)) \;&=\; -(-u(v(x))) \;=\; u(v(x)) \\ &=\; v(u(x)) \;=\; -(-v(u(x))) \;=\; -v(-u(x)). \end{aligned} \tag{15}$$

Hence, if w_r are permutable odd functions for integer r, then the reflected sequence $\{w_r,\ -w_r\}$ is permutable. For odd $r = 2i+1$ the odd rational functions x^{2i+1} are permutable, and so is the reflected sequence

$$\{x^{2i+1},\ -x^{2i+1}\} \;=\; \cdots, x^{-3}, -x^{-3}, x^{-1}, -x^{-1}, x, -x, x^3, -x^3, \cdots\ . \tag{16}$$

And the odd Chebyshev polynomials T_{2j+1} are permutable, and hence the reflected sequence of polynomials $\{T_{2j+1},\ -T_{2j+1}\}$ is permutable. For all ϑ, we have the identity [17, p.79]

$$\sin((2j+1)\vartheta) \;=\; (-1)^j T_{2j+1}(\sin\,\vartheta). \tag{17}$$

Denote

$$q_j(z) \;\stackrel{\text{def}}{=}\; (-1)^j T_{2j+1}(z), \tag{18}$$

so that $q_j(\sin\vartheta) = \sin((2j+1)\vartheta)$. Thus the sequence $\{q_j\}$ is a subsequence of the reflected permutable sequence $\{T_{2j+1},\ -T_{2j+1}\}$. Indeed, for all non-negative integers j and k,

$$\begin{aligned} q_j(q_k(\sin\vartheta)) \;&=\; q_j(\sin((2k+1)\vartheta) \\ &=\; \sin((2j+1)(2k+1)\vartheta) \;=\; \sin((4jk+2j+2k+1)\vartheta) \\ &=\; q_{2jk+j+k}(\sin\,\vartheta) \;=\; q_k(q_j(\sin\,\vartheta)), \end{aligned} \tag{19}$$

The linear transform with $\lambda(x) = -x$ converts $u(x)$ to $-u(-x)$. Hence, if u is an odd function then it is unchanged, but if u is an even function then it is converted to $-u$. Thus, the permutable sequence of rational functions $\{x^r\}$ is similar (via $\lambda(x) = -x$) to the permutable sequence

$$\{-(-x)^r\} \;=\; \cdots, -x^{-4}, x^{-3}, -x^{-2}, x^{-1}, -1, x, -x^2, x^3, -x^4, \cdots\ . \tag{20}$$

And the permutable chain of Chebyshev polynomials $\{T_j\}$ is similar (via $\lambda(x) = -x$) to the permutable chain

$$\{-T_j(-x)\} \;=\; -1, x, -T_2(x), T_3(x), -T_4(x), T_5(x), \cdots\ . \tag{21}$$

2 Permutable Rational Functions

For complex $z = x + iy$ and positive integer n.

$$z^n = (x + iy)^n = \sum_{j=0}^{n} \binom{n}{j} i^j x^{n-j} y^j = H_n^{(0)}(x, y) + iH_n^{(1)}(x, y) , \qquad (22)$$

where the real and imaginary parts of z^n are given [17, Appendix 1 §**2.1**] by the harmonic polynomials :

$$H_n^{(0)}(x, y) = \sum_{k=0}^{n\div 2} (-1)^k \binom{n}{2k} x^{n-2k} y^{2k} \qquad (23)$$

$$H_n^{(1)}(x, y) = \sum_{k=0}^{(n-1)\div 2} (-1)^k \binom{n}{2k+1} x^{n-2k-1} y^{2k+1} . \qquad (24)$$

For all $\rho \in \mathbb{C}$, $\log(\cos \rho + i \sin \rho) = i\rho$ (by Cotes's theorem), or

$$\cos \rho + i \sin \rho = e^{i\rho}; \qquad (25)$$

and hence (De Moivre's Theorem) for integer $n \geq 0$

$$\cos(n\rho) + i \sin(n\rho) = e^{in\rho} = (\cos \rho + i \sin \rho)^n. \qquad (26)$$

Therefore

$$\begin{aligned} \cos(n\rho) + i \sin(n\rho) &= (\cos \rho + i \sin \rho)^n \\ &= H_n^{(0)}(\cos \rho, \sin \rho) + iH_n^{(1)}(\cos \rho, \sin \rho) . \end{aligned} \qquad (27)$$

Equating real and imaginary parts (for real ρ), we get that

$$\cos(n\rho) = H_n^{(0)}(\cos \rho, \sin \rho), \quad \sin(n\rho) = H_n^{(1)}(\cos \rho, \sin \rho); \qquad (28)$$

and hence

$$\tan(n\rho) = \frac{\sin(n\rho)}{\cos(n\rho)} = \frac{H_n^{(1)}(\cos \rho, \sin \rho)}{H_n^{(0)}(\cos \rho, \sin \rho)} . \qquad (29)$$

2.1 Tangents of multiple angles

Dividing numerator and denominator in (29) by $\cos^n \rho$, we get [17, p.79] $\tan(n\rho)$ as a rational function of $\tan \rho$, with integer coefficients:

$$\tan(n\rho) = \frac{H_n^{(1)}(1, \tan \rho)}{H_n^{(0)}(1, \tan \rho)} = \frac{\sum_{k=0}^{(n-1)\div 2} \binom{n}{2k+1} (-1)^k \tan^{2k+1} \rho}{\sum_{k=0}^{n\div 2} \binom{n}{2k} (-1)^k \tan^{2k} \rho} . \qquad (30)$$

Denote

$$t \stackrel{\text{def}}{=} \tan \rho, \qquad z \stackrel{\text{def}}{=} -t^2 = -\tan^2 \rho. \qquad (31)$$

Then, it follows from (30) that

$$\tan(n\rho) = r_n(\tan \rho), \qquad (32)$$

where $r_n(t)$ is an odd rational function with positive integer coefficients

$$r_n(t) \; = \; t \, \frac{A_n(z)}{B_n(z)} \; , \tag{33}$$

(except that $A_0(t) = 0$), with :

$$A_n(z) \; = \; \sum_{k=0}^{(n-1)\div 2} \binom{n}{2k+1} z^k, \qquad B_n(z) \; = \; \sum_{k=0}^{n\div 2} \binom{n}{2k} z^k, \tag{34}$$

and $B_n(0) = 1$.

In more detail, for odd $n = 2j+1$:

$$\begin{aligned} r_{2j+1}(t) \; &= \; t \, \frac{A_{2j+1}(z)}{B_{2j+1}(z)} \\ &= \; t \, \frac{2j+1+\binom{2j+1}{3}z+\binom{2j+1}{5}z^2+\cdots+\binom{2j+1}{4}z^{j-2}+\binom{2j+1}{2}z^{j-1}+z^j}{1+\binom{2j+1}{2}z+\binom{2j+1}{4}z^2+\cdots+\binom{2j+1}{5}z^{j-2}+\binom{2j+1}{3}z^{j-1}+(2j+1)z^j} \; , \end{aligned} \tag{35}$$

where the polynomials A_{2j+1} and B_{2j+1} are mutually reciprocal:

$$z^j A_{2j+1}\left(\frac{1}{z}\right) \; = \; B_{2j+1}(z), \qquad z^j B_{2j+1}\left(\frac{1}{z}\right) \; = \; A_{2j+1}(z) \tag{36}$$

for all $z \neq 0$. For even $n = 2j$:

$$\begin{aligned} r_{2j}(t) \; &= \; t \, \frac{A_{2j}(z)}{B_{2j}(z)} \\ &= \; t \, \frac{2j+\binom{2j}{3}z+\binom{2j}{5}z^2+\cdots+\binom{2j}{5}z^{j-3}+\binom{2j}{3}z^{j-2}+2jz^{j-1}}{1+\binom{2j}{2}z+\binom{2j}{4}z^2+\cdots+\binom{2j}{4}z^{j-2}+\binom{2j}{2}z^{j-1}+z^j} \; , \end{aligned} \tag{37}$$

where both A_{2j} and B_{2j} are self–reciprocal polynomials:

$$A_{2j}(z) \; = \; z^{j-1} A_{2j}\left(\frac{1}{z}\right), \qquad B_{2j}(z) \; = \; z^j B_{2j}\left(\frac{1}{z}\right), \tag{38}$$

for all $z \neq 0$.

For example,

$$\begin{aligned} & r_0(t) = t\,\frac{0}{1} = 0 \; , && r_1(t) = t\,\frac{1}{1} = t \; , \\ & r_2(t) = t\,\frac{2}{1+z} \; , && r_3(t) = t\,\frac{3+z}{1+3z} \; , \\ & r_4(t) = t\,\frac{4+4z}{1+6z+z^2} \; , && r_5(t) = t\,\frac{5+10z+z^2}{1+10z+5z^2} \; , \\ & r_6(t) = t\,\frac{6+20z+6z^2}{1+15z+15z^2+z^3} \; , && r_7(t) = t\,\frac{7+35z+21z^2+z^3}{1+21z+35z^2+7z^3} \; , \\ & r_8(t) = t\,\frac{8+56z+56z^2+8z^3}{1+28z+70z^2+28z^3+z^4} \; , \end{aligned} \tag{39}$$

et cetera .

Since $\cot\,\rho = \tan\left(\frac{1}{2}\pi - \rho\right)$, we get that

$$r_n(\cot\,\rho) \;=\; r_n\left(\tan\left(\frac{\pi}{2} - \rho\right)\right) \;=\; \tan\left(\frac{n\pi}{2} - n\rho\right). \tag{40}$$

For odd $n = 2j+1$, this becomes

$$\begin{aligned} r_{2j+1}(\cot\,\rho) \;&=\; \tan\left(j\pi + \tfrac{1}{2}\pi - (2j+1)\rho\right) \\ &=\; \tan\left(\tfrac{1}{2}\pi - (2j+1)\rho\right) \;=\; \cot((2j+1)\rho), \end{aligned} \tag{41}$$

and for even $n = 2j$, this becomes

$$r_{2j}(\cot\,\rho) \;=\; \tan\left(j\pi - 2j\rho\right) \;=\; -\tan(2j\rho). \tag{42}$$

2.2 Permutable rational function $r_j(t)$

For the infinite sequence of odd rational functions $\{r_j\}$, it follows from (32) that, for all complex t and non–negative integers j, k, with $\rho = \tan^{-1} t$,

$$\begin{aligned} r_j\big(r_k(t)\big) \;&=\; r_j\big(r_k(\tan\,\rho)\big) \;=\; r_j(\tan(k\rho)) \;=\; \tan(jk\rho) \\ &=\; r_{jk}(\tan\,\rho) \;=\; r_{jk}(t), \end{aligned} \tag{43}$$

giving the functional identity for complex t and non–negative integers j, k:

$$r_j\big(r_k(t)\big) \;=\; r_{jk}(t). \tag{44}$$

Thus, the rational functions r_n, with the binary operation of function composition, form an infinite Abelian group [4, p.34], with

$$r_{jk}(t) \;=\; r_j\big(r_k(t)\big) \;=\; r_k\big(r_j(t)\big). \tag{45}$$

For example, for all $t \in \mathbb{C}$ and integer $k \geq 0$,

$$r_{4k}(t) \;=\; r_k\left(\frac{4t - 4t^3}{1 - 6t^2 + t^4}\right) \;=\; \frac{4r_k(t) - 4r_k^3(t)}{1 - 6r_k^2(t) + r_k^4(t)}\;. \tag{46}$$

Thus we get for $j = 0, 1, 2, \ldots$ the reflected sequence of permutable odd rational functions $\{r_j(t),\; -r_j(t)\}$

Another infinite sequence of permutable rational functions is given by $\{T_j(x)/1\}$ for $j = 0, 1, 2, \ldots$, which is similar (via $\lambda(x) = -x$) to the permutable rational sequence $\{T_j(-x)/(-1)\}$. And since T_{2i+1} is an odd polynomial, there is also the reflected sequence $\{T_{2i+1}(x)/1,\; T_{2i+1}(x)/(-1)\}$ of permutable rational functions.

2.2.1 Reciprocal rational functions

The bilinear transform $y = \xi(x) = (ax+b)/(cx+d)$, where $ad - bc \neq 0$, has the inverse bilinear transform $x = \xi^{-1}(y) = (dy - b)/(a - yc)$. A function $u(x)$ is rational if and only if the transformed function $\xi^{-1}u(\xi(x))$ is also a rational function of x. Similarly to (9), functions $u(x)$ and $v(x)$ are permutable if and only if the transformed functions $\xi^{-1}u\xi$ and $\xi^{-1}v\xi$ are permutable.

In the important special case where $\xi(x) = 1/x$, any function $v(x)$ gets converted to another function $1/v(1/x)$, which we call the reciprocal of the function v. The reciprocal of an even function is even, and the reciprocal of an odd function is odd. Any pair of rational functions $v(x)$ and $w(x)$ are permutable, if and only if their reciprocal rational functions $f(x) = 1/v(1/x)$ and $g(x) = 1/w(1/x)$ are permutable.

The permutable rational infinite sequence $\{f_n(x) = T_n(x)/1\}$ has the reciprocal permutable sequence $\{s_n(x) = 1/T_n(1/x)\}$, for which

$$f_n(\cos\vartheta) \;=\; \cos(n\vartheta), \qquad s_n(\sec\vartheta) \;=\; \sec(n\vartheta). \tag{47}$$

The sequence $\{s_n(x)\}$ is similar (via $\lambda(x) = -x$) to the permutable rational sequence $\{-s_n(-x)\} = -s_0(x), s_1(x), -s_2(x), s_3(x), -s_4(x), \cdots$. The rational function s_{2i+1} is an odd function, giving the permutable reflected rational sequence $\{s_{2i+1},\ -s_{2i+1}\}$.

The permutable odd rational infinite sequence $\{q_j(x) = T_{2j+1}(x)/(-1)^j\}$ has the reciprocal permutable rational sequence $\{u_j(x) = (-1)^j/T_{2j+1}(1/x)\}$, and in view of (17),

$$q_j(\sin\vartheta) = \sin((2j+1)\vartheta), \qquad u_j(\operatorname{cosec}\vartheta) = \operatorname{cosec}((2j+1)\vartheta). \tag{48}$$

The odd rational function u_j has the reflected permutable sequence $\{u_j,\ -u_j\}$.

The odd rational function $r_n(t)$ has the reciprocal odd rational function $h_n(t) = 1/r_n(1/t)$, for which

$$r_n(\tan\rho) = \tan(n\rho), \qquad h_n(\cot\rho) = \cot(n\rho). \tag{49}$$

In view of (41) and (42), this can be rewritten as

$$h_n(t) \;=\; \begin{cases} r_n(t) & \text{(for odd } n\text{)}, \\[4pt] \dfrac{-1}{r_n(t)} & \text{(for even } n\text{)}. \end{cases} \tag{50}$$

Since $\{r_n\}$ is an odd permutable sequence, we get the odd permutable reflected reciprocal sequence $\{h_n,\ -h_n\}$.

3 Elliptic Functions

An integral of the form $\int R(x,y)\,\mathrm{d}x$, where $R(x,y)$ is a rational function of x and y, and $y^2 = P(x)$ where P is a polynomial of degree 3 or 4, is called an *elliptic integral* [18, §**17.1**].

3.1 Legendre Elliptic Integrals

We shall use Louis Melville Milne-Thomson's notation for Legendre's elliptic integrals and Jacobian elliptic functions [18].

Adrien-Marie Legendre's Incomplete Elliptic Integral of the First Kind, with amplitude φ and parameter m, is defined [18, §**17.2.7**] as

$$F(\varphi \mid m) \;\stackrel{\text{def}}{=}\; \int_0^{\sin\varphi} \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-mt^2)}} \;=\; \int_0^{\varphi} \frac{\mathrm{d}x}{\sqrt{1-m\,\sin^2 x}}\,. \tag{51}$$

The complex parameter m can be reduced to the real case with $0 \le m \le 1$. (Earlier authors often used the modulus k, where $m = k^2$.) $F(\varphi \mid m)$ is often abbreviated to $F(\varphi)$, when the parameter m is to be understood.

Legendre's Complete Elliptic Integral of the First Kind [18, §**17.3.1**] is

$$K(m) \ \stackrel{\text{def}}{=}\ F(\tfrac{1}{2}\pi \mid m) \ = \ \int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-mt^2)}} \ = \ \int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{1-m\,\sin^2 x}}\ . \tag{52}$$

As $m \nearrow 1$, then $K(m) \nearrow \infty$.

Legendre's Complementary Complete Elliptic Integral of the First Kind is defined [18, §**17.3.5**] as

$$K'(m) \ \stackrel{\text{def}}{=}\ K(1-m). \tag{53}$$

$K(m)$ and $K'(m)$ are often abbreviated to K and K', when the parameter m is to be understood.

The elliptic integral $u = F(\varphi \mid m)$ is single-valued for integration along the real interval $[0, \sin\varphi]$, but it has infinitely many values in the case of complex integration.

3.2 Jacobian elliptic functions

In 1827, Niels Henrik Abel inverted elliptic integrals to get elliptic functions [1, p.264], and he shewed that elliptic functions are doubly–periodic single–valued functions.

As functions of the complex variable u, the Jacobian elliptic functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ *et cetera* are doubly–periodic single–valued functions of u.

3.2.1 Jacobian elliptic function sn u

The inverse function of the Legendre elliptic function F is $\varphi = F^{-1}(u)$, and the Jacobian elliptic function $\operatorname{sn}(u \mid m) \ \stackrel{\text{def}}{=}\ \sin\varphi$ is often abbreviated to $\operatorname{sn} u$, with the parameter m implied. The function sn is single–valued for all complex parameters [18, §**16.1.3**], with

$$u \ = \ \int_0^{\operatorname{sn} u} \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-mt^2)}}\ , \tag{54}$$

and $\operatorname{sn} u$ is an odd single–valued function of u.

For real u, the function sn has real period $4K(m)$ and range $[-1, 1]$, with $\operatorname{sn}(0) = 0,\ \operatorname{sn}(K) = 1,\ \operatorname{sn}(2K) = 0,\ \operatorname{sn}(3K) = -1$ and $\operatorname{sn}(4K) = 0$ [18, §**16.2**]. Let $\tau = \operatorname{sn} u$, so that

$$\operatorname{sn}^{-1}\tau \ = \ u \ = \ \int_0^{\tau} \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-mt^2)}} \ = \ F\left(\sin^{-1}\tau \mid m\right). \tag{55}$$

On the real interval $-K \le u \le K$ the function sn increases monotonically from -1 to 1, and so for real $\tau \in [-1, 1]$ the function $\operatorname{sn}^{-1}\tau$ has a single value in the real interval $[-K, K]$.

In addition to the real period $4K$, sn also has the imaginary period $i\,2K'$.

3.2.2 Jacobian elliptic function cn u

The function cn is defined by

$$\mathrm{cn}(u \mid m) \ \stackrel{\mathrm{def}}{=}\ \cos\varphi. \tag{56}$$

It is often abbreviated to $\mathrm{cn}\,u$, with the parameter m implied. Thus,

$$\mathrm{cn}\,u \ = \ \sqrt{1-\mathrm{sn}^2 u}\ , \tag{57}$$

and $\mathrm{cn}\,u$ is an even single–valued function of u. The branch of the square root function in (57) is determined by (56).

For real u, the function cn has real period $4K(m)$ and range $[-1,1]$, with $\mathrm{cn}(0)=1,\ \mathrm{cn}(K)=0,\ \mathrm{cn}(2K)=-1,\ \mathrm{cn}(3K)=0$ and $\mathrm{cn}(4K)=1$ [18, §**16.2**]. On the real interval $0 \le u \le 2K$ the function cn decreases monotonically from 1 to -1, and so for real $r \in [-1,1]$ the function $\mathrm{cn}^{-1} r$ has a single value in the real interval $[0, 2K]$.

In addition to the real period $4K$, cn has the complex period $2K + i\,2K'$.

3.2.3 Jacobian elliptic function dn u

The Jacobian elliptic function dn is defined by

$$\mathrm{dn}(u \mid m) \ \stackrel{\mathrm{def}}{=}\ \sqrt{1-m\,\mathrm{sn}^2 u}\ . \tag{58}$$

It is often abbreviated to $\mathrm{dn}\,u$, with the parameter m implied. Thus, $\mathrm{dn}\,u$ is an even single–valued function of u.

For real u, the function dn has real period $2K(m)$ and range $[\sqrt{m_1}, 1]$, with $\mathrm{dn}(0)=1,\ \mathrm{dn}(K)=\sqrt{m_1}$ and $\mathrm{dn}(2K)=1$ [18, §**16.2**]. On the real interval $0 \le u \le K$ the function dn decreases monotonically from 1 to $\sqrt{m_1}$, and so for real $r \in [\sqrt{m_1}, 1]$ the function $\mathrm{dn}^{-1} r$ has a single value in the real interval $[0, K]$.

In addition to the real period $2K$, dn also has the imaginary period $i\,4K'$.

3.2.4 The 12 Jacobi elliptic functions

In J. W. L. Glaisher's systematic notation, there are 9 other Jacobi elliptic functions [18, §**16.3**]:

$$\begin{array}{lllllllll}
\mathrm{cd}\,u & \stackrel{\mathrm{def}}{=} & \dfrac{\mathrm{cn}\,u}{\mathrm{dn}\,u} & \mathrm{dc}\,u & \stackrel{\mathrm{def}}{=} & \dfrac{\mathrm{dn}\,u}{\mathrm{cn}\,u} & \mathrm{ns}\,u & \stackrel{\mathrm{def}}{=} & \dfrac{1}{\mathrm{sn}\,\mathrm{u}} \\
\mathrm{sd}\,u & \stackrel{\mathrm{def}}{=} & \dfrac{\mathrm{sn}\,\mathrm{u}}{\mathrm{dn}\,\mathrm{u}} & \mathrm{nc}\,u & \stackrel{\mathrm{def}}{=} & \dfrac{1}{\mathrm{cn}\,u} & \mathrm{ds}\,u & \stackrel{\mathrm{def}}{=} & \dfrac{\mathrm{dn}\,u}{\mathrm{sn}\,u} \\
\mathrm{nd}\,u & \stackrel{\mathrm{def}}{=} & \dfrac{1}{\mathrm{dn}\,\mathrm{u}} & \mathrm{sc}\,u & \stackrel{\mathrm{def}}{=} & \dfrac{\mathrm{sn}\,u}{\mathrm{cn}\,u} & \mathrm{cs}\,u & \stackrel{\mathrm{def}}{=} & \dfrac{\mathrm{cn}\,u}{\mathrm{sn}\,u}
\end{array} \tag{59}$$

The 6 functions $\mathrm{sn}\,u$, $\mathrm{sc}\,u$, $\mathrm{sd}\,u$, $\mathrm{ns}\,u$, $\mathrm{cs}\,u$, $\mathrm{ds}\,u$ are odd functions of u, and the other 6 functions are even functions of u.

Milne-Thomson gave clear graphs [18, Figures 16.1, 16.2, 16.3] of the 12 Jacobi elliptic functions for real u (and $m = 0{\cdot}5$). Eugene Jahnke and Fritz Emde gave very effective graphs [14, pp. 92–93] of $\mathrm{sn}\,u$, $\mathrm{cn}\,u$ and $\mathrm{dn}\,u$ for complex u (and $m = 0{\cdot}64$).

3.3 Weierstraß elliptic function $\wp$

Karl Wilhelm Theodor Weierstraß developed his version of elliptic functions on the basis of his elliptic function $\wp$, with complex (or real) parameters g_2 and g_3, which is defined by the differential equation [21, §**18.1.6**]

$$\wp'^2(z) \;=\; 4\wp^3(z) - g_2\wp(z) - g_3 \;. \tag{60}$$

Hence, Weierstraß's function $\wp$ is the inverse of an elliptic integral, with

$$z \;=\; \int_{\wp(z)}^{\infty} \frac{\mathrm{d}w}{\sqrt{4w^3 - g_2 w - g_3}} \tag{61}$$

for complex z.

Many mathematicians (e.g. Eric Temple Bell [2, p.207]) regard Weierstraß's function $\wp$ as being much more difficult than the Jacobi elliptic functions. But some mathematicians seem to have responded to that difficulty as a challenge, and they have chosen to use Weierstraß's version of elliptic functions.

For the parameters $m = 0$ and $m = 1$, the Jacobi elliptic functions reduce respectively [18, §**16.6**] to trigonometric and hyperbolic functions:

$$\begin{array}{lll} \mathrm{sn}(u \mid 0) \;=\; \sin u, & \mathrm{cn}(u \mid 0) \;=\; \cos u, & \mathrm{dn}(u \mid 0) \;=\; 1, \\ \mathrm{sn}(u \mid 1) \;=\; \tanh u, & \mathrm{cn}(u \mid 1) \;=\; \mathrm{sech}\, u, & \mathrm{dn}(u \mid 1) \;=\; \mathrm{sech}\, u, \end{array} \tag{62}$$

et cetera.

For the Weierstraß function, as $u \to 0$,

$$\wp(u) \;=\; \frac{1}{u^2} + O(u^2) \;. \tag{63}$$

The simplest form of $\wp(u)$ occurs as the limit when one period is $i\infty$. Denote the other period as 2ω (with nonzero real part). Then [14, p.105]

$$\wp(u) \;=\; -\frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2 + \left(\frac{\dfrac{\pi}{2\omega}}{\sin\left(\dfrac{\pi u}{2\omega}\right)}\right)^2 \;. \tag{64}$$

In 1956, Harold and Bertha Jeffreys mildly remarked that "Had Abel been alive he would probably have remarked that this property corresponds to taking the fundamental trigonometric function as $\mathrm{cosec}^2 x - \frac{1}{3}$ " [15, p.688].

3.4 Addition Formulæ for Elliptic Functions

For all complex α and β, the Jacobi elliptic functions have the rational addition formulæ: [18, §**16.17**],

$$\begin{aligned} \mathrm{sn}(\alpha+\beta) &= \frac{\mathrm{sn}\,\alpha\cdot\mathrm{cn}\,\beta\cdot\mathrm{dn}\,\beta + \mathrm{sn}\,\beta\cdot\mathrm{cn}\,\alpha\cdot\mathrm{dn}\,\alpha}{1 - m\,\mathrm{sn}^2\alpha\cdot\mathrm{sn}^2\beta} \;, \\ \mathrm{cn}(\alpha+\beta) &= \frac{\mathrm{cn}\,\alpha\cdot\mathrm{cn}\,\beta - \mathrm{sn}\,\alpha\cdot\mathrm{dn}\,\alpha\cdot\mathrm{sn}\,\beta\cdot\mathrm{dn}\,\beta}{1 - m\,\mathrm{sn}^2\alpha\cdot\mathrm{sn}^2\beta} \;, \\ \mathrm{dn}(\alpha+\beta) &= \frac{\mathrm{dn}\,\alpha\cdot\mathrm{dn}\,\beta - m\,\mathrm{sn}\,\alpha\cdot\mathrm{cn}\,\alpha\cdot\mathrm{sn}\,\beta\cdot\mathrm{cn}\,\beta}{1 - m\,\mathrm{sn}^2\alpha\cdot\mathrm{sn}^2\beta} \;. \end{aligned} \tag{65}$$

But the function $\wp$ has the irrational addition formula [21, §**18.4.1**]:

$$\wp(\alpha+\beta) = -\wp(\alpha) - \wp(\beta) + \frac{1}{4}\left[\frac{\sqrt{4\wp(\alpha)^3 - g_2\wp(\alpha) - g_3} - \sqrt{4\wp(\beta)^3 - g_2\wp(\beta) - g_3}}{\wp(\alpha) - \wp(\beta)}\right]^2, \quad (66)$$

for $\wp(\alpha) \neq \wp(\beta)$.

3.5 Elliptic Functions of multiple argument

For brevity, we shall write $s,\ c$ and d for $\operatorname{sn} u$, $\operatorname{cn} u$ and $\operatorname{dn} u$.

Put $\alpha = \beta = u$ in (65), and then it follows that

$$\operatorname{sn} 2u = \frac{2scd}{1-ms^4}, \quad \operatorname{cn} 2u = \frac{1-2s^2+ms^4}{1-ms^4}, \quad \operatorname{dn} 2u = \frac{1-2ms^2+ms^4}{1-ms^4}. \quad (67)$$

For $m = 0$ these reduce to

$$\sin 2u = 2 \sin u \cos u, \qquad \cos 2u = 1 - 2\sin^2 u, \qquad 1 = 1 ; \quad (68)$$

and for $m = 1$ these reduce to

$$\tanh 2u = \frac{2 \tanh u}{1 + \tanh^2 u} \quad (69)$$

and

$$\cosh 2u = 2 \cosh^2 u - 1 \quad \text{(twice)}. \quad (70)$$

For the Weierstraß function, taking the limit of (66) as $\beta \to \alpha$, we get the *rational* formula

$$\begin{aligned}\wp(2\alpha) &= -2\wp(\alpha) + \frac{(3\wp^2(\alpha) - \frac{1}{4}g_2)^2}{4\wp^3(\alpha) - g_2\wp(\alpha) - g_3} \\ &= \frac{16\wp^4(\alpha) + 8g_2\wp^2(\alpha) + 32g_3\wp(\alpha) + g_2^2}{16\big(4\wp^3(\alpha) - g_2\wp(\alpha) - g_3\big)}, \end{aligned} \quad (71)$$

[21, §**18.4.5**]. Induction on (71) shews that for integer $j \geq 0$,

$$\wp\big(2^j\alpha\big) = w_j(\wp(\alpha)), \quad (72)$$

where $w_j(x)$ is a rational function of x with coefficients which are polynomials in g_2 and g_3, with integer coefficients.

But if n is not a power of 2, then the irrational formulæ for $\wp(n\alpha)$ are *very* complicated.

Put $\alpha = 2u$ and $\beta = u$ in (65) and we get rational functions, with denominator and numerators expressed as polynomials in $s = \operatorname{sn} u$ and m;

$$\begin{aligned} D &= 1 - 6ms^4 + 4m[1+m]s^6 - 3m^2s^8, \\ \operatorname{sn} 3u &= s\left(3 - 4[1+m]s^2 + 6ms^4 - m^2s^8\right)/D, \\ \operatorname{cn} 3u &= c\left(1 - 4s^2 + 6ms^4 - 4m^2s^6 + m^2s^8\right)/D, \\ \operatorname{dn} 3u &= d\left(1 - 4ms^2 + 6ms^4 - 4ms^6 + m^2s^8\right)/D. \end{aligned} \quad (73)$$

Hereafter, n is a positive integer.

Those formulæ for addition of Jacobian elliptic functions can be iterated, to get real formulæ for $\mathrm{sn}(nu)$ *et cetera*, in terms of rational functions of $z = \mathrm{sn}^2 u$ [11, pp. 78–87].

For odd $n = 2r + 1$,

$$\mathrm{sn}(nu) \;=\; \frac{s\,A_n(z)}{D_n(z)}\;, \qquad \mathrm{cn}(nu) \;=\; \frac{c\,B_n(z)}{D_n(z)}\;, \qquad \mathrm{dn}(nu) \;=\; \frac{d\,C_n(z)}{D_n(z)}\;; \quad (74)$$

and for even $n = 2r$,

$$\mathrm{sn}(nu) \;=\; \frac{s\,c\,d\,A_n(z)}{D_n(z)}\;, \qquad \mathrm{cn}(nu) \;=\; \frac{B_n(z)}{D_n(z)}\;, \qquad \mathrm{dn}(nu) \;=\; \frac{C_n(z)}{D_n(z)}\;. \quad (75)$$

Each of A_n, B_n, C_n, D_n is a polynomial of degree $n^2 \div 2$ — *except* that A_{2r} has degree $\frac{1}{2}n^2 - 2 = 2r^2 - 2$. In each polynomial, the coefficient of each power of z is a reciprocal polynomial in m of degree less than or equal to $n^2 \div 4$, with integer coefficients [2, p.208].

N.B. Arthur Cayley advised that, when constructing the elliptic functions of $nu = ju + ku$, then $j = k$ should be used for even n and $j = k \pm 1$ should be used for odd n — otherwise the constructed numerators and denominators will have common multinomial factors which need to be divided out to get the rational functions in lowest terms [11, p.79]. The appropriate degrees of $z = s^2$ in the numerators and denominators are given in the previous paragraph. Without that information, it would be necessary to divide numerator and denominator by their gcd, in the form of a polynomial in z whose coefficients are polynomials in m with integer coefficients.

3.6 Transcendental Formulæ for Real Multiplication of Elliptic Functions

Abel inverted some elliptic integrals and invented elliptic functions ϕ, f and F which correspond to sn, cn and dn; and he constructed an explicit formula for $\phi((2r+1)u)$ as a rational function (in factored form) of $\phi(u)$, and similarly for f and F [1, pp.321–323]. That is called *Real Multiplication of Elliptic Functions*.

Thus, the rational functions $s\,A_n(s^2)/D_n(s^2)$ perform real multiplication of the elliptic function sn. And since (5) $T_n(\cos\vartheta) \equiv \cos(n\vartheta)$, the Chebyshev polynomials T_n perform real multiplication of the cosine function; and (33) the rational functions r_n perform real multiplication of the tangent function.

For integers μ and ν, denote

$$Q_{\mu,\nu} \quad \stackrel{\text{def}}{=} \quad 2\mu K(m) + i\,2\nu K'(m)\;. \quad (76)$$

Then $\mathrm{sn}\,Q_{\mu,\nu} = 0$ for all integers μ and ν, and conversely every zero of sn is of that form $Q_{\mu,\nu}$.

Consider an odd positive integer $n = 2r + 1$. Then $\mathrm{sn}(Q_{\mu,\nu}/n)$ has n^2 distinct values for all integers μ and ν, and those are all given by $-r \le \mu \le r$ and $-r \le \nu \le r$. Those n^2 values include $\mathrm{sn}(Q_{0,0}/n) = 0$; so that if we exclude that zero value then there remain $n^2 - 1$ distinct values [11, pp.94–97]. From this, it follows that for odd $n = 2r + 1$, the polynomials $A_n,\ B_n,\ C_n,\ D_n$ can be explicitly factorized, in terms of the transcendental functions sn and K:

$$A_n = (2r+1)\ s \prod_{\mu=0}^{r} \prod_{\nu=-r}^{r}{}' \left(1 - \mathrm{ns}^2\left(\frac{Q_{\mu,\nu}}{n}\right)\ s^2\right), \tag{77}$$

$$B_n = c \prod_{\mu=0}^{r} \prod_{\nu=-r}^{r}{}' \left(1 - \mathrm{ns}^2\left(K(m) - \frac{Q_{\mu,\nu}}{n}\right)\ s^2\right), \tag{78}$$

$$C_n = d \prod_{\mu=0}^{r} \prod_{\nu=-r}^{r}{}' \left(1 - m\,\mathrm{sn}^2\left(K(m) - \frac{Q_{\mu,\nu}}{n}\right)\ s^2\right), \tag{79}$$

$$D_n = \prod_{\mu=0}^{r} \prod_{\nu=-r}^{r}{}' \left(1 - m\,\mathrm{sn}^2\left(\frac{Q_{\mu,\nu}}{n}\right) s^2\right) . \tag{80}$$

Here, $\prod'$ means that for $\nu = 0$, μ ranges from 1 to r; but otherwise ν ranges from $-r$ to r. Thus the double product over μ and ν has $2r(r+1)$ factors.

And similar factorizations apply for even n.

But, repeated application of Abel's addition formulæ for elliptic functions gives the algebraic formulæ (74) (75), with coefficients of powers of z in numerator and denominator as polynomials in m with integer coefficients.

The algebraic complexity of those algebraic formulæ for $\mathrm{sn}(nu)$ *et cetera* increases very rapidly with n. The immensely complicated formulæ for $n = 6$ and $n = 7$ were constructed by Baehr in 1861, and were partially verified (for $m = 0$ and $m = 1$) by Cayley. In 1895, Cayley fitted the formulæ for $\mathrm{sn}(7u)$, $\mathrm{cn}(7u)$ and $\mathrm{dn}(7u)$ onto 2 pages of fine print, only by using heavily abbreviated notation which requires careful attention to detail for interpreting it [11, pp.84–85]. The construction of explicit algebraic formulæ for those coefficients of powers of z for general n was a problem which defied solution for over a century after Abel [6] [7] [8] [9] [12]. In 1847, Cayley sighed that "it seems hopeless to continue this investigation any further" [5, p.298], and in 1889 he declared that "the investigation seems to show that the integration cannot be effected in any tolerably simple form" [10, p.589].

In 1932, Eric Temple Bell explained that "The *transcendental* solution of the problem of real multiplication of elliptic functions has been classic for over a century; the *algebraic* solution has not yet been achieved, although it has engaged the attention of many writers". Bell then gave the first general method (very complicated) for constructing all coefficients of D_n as polynomials in n^2 (and m), for general n [2]. The coefficients in the numerators A_n, B_n, C_n can be constructed from the coefficients of D_n, with different rules for even n and odd n [11, pp. 81–89], [12, pp.196–209].

The low-order terms of D_n are:

$$\begin{aligned} D_n(z) \ = \ & 1 + 0z - \tfrac{1}{12}n^2(n^2-1^2)mz^2 \\ & + \tfrac{1}{90}n^2(n^2-1^2)(n^2-2^2)[1+m]mz^3 - \cdots . \end{aligned} \tag{81}$$

For even $n = 2r$, the high–order terms are:

$$\begin{aligned} D_{2r}(z) \ = \ & \cdots + (-1)^r \tfrac{8}{45} r^2(r^2-1)(4r^2-1)[1+m]m^{r^2-2}z^{2r^2-3} \\ & + (-1)^{r-1}\tfrac{1}{3}r^2(4r^2-1)(mz^2)^{r^2-1} + 0z^{2r^2-1} + (-1)^r(mz^2)^{r^2}; \end{aligned} \tag{82}$$

and for odd $n = 2r + 1$, the high–order terms are:

$$D_{2r+1}(z) \;=\; \cdots + (-1)^{r-1}\tfrac{2}{3}r(r+1)(2r+1)[1+m]m^{r^2+r-1}z^{2r^2+2r-1} \\ +(-1)^r(2r+1)(mz^2)^{r(r+1)}. \quad (83)$$

For odd $n = 2r + 1$,

$$\begin{aligned} A_{2r+1}(z) &= (2r+1) - \tfrac{2}{3}r(r+1)(2r+1)[1+m]z + \cdots \\ &\quad +(-1)^r\tfrac{2}{45}r(r+1)(2r-1)(2r+1)^2(2r+3)\times \\ &\quad \times[1+m]m^{r^2+r-2}z^{2r^2+2r-3} \\ &\quad +(-1)^{r-1}\tfrac{1}{3}r(r+1)(2r+1)^2(mz^2)^{r^2+r-1} \\ &\quad +0z^{2r^2+2r-1} + (-1)^r(mz^2)^{r(r+1)}, \\ B_{2r+1}(z) &= 1 - 2r(r+1)z \\ &\quad +\tfrac{1}{3}r(r+1)[2(r^2+r-2)+(2r+1)^2m]z^2 - \cdots \\ &\quad +\tfrac{1}{3}r(r+1)[(2r+1)^2+2(r^2+r-2)m](mz^2)^{r^2+r-1} \\ &\quad -2r(r+1)m^{r(r+1)}z^{2r^2+2r-1} + (mz^2)^{r(r+1)}, \\ C_{2r+1}(z) &= 1 - 2r(r+1)mz \\ &\quad +\tfrac{1}{3}r(r+1)[(2r+1)^2+2(r^2+r-2)m]mz^2 - \cdots \\ &\quad +\tfrac{1}{3}r(r+1)[2(r^2+r-2)+(2r+1)^2m]m^{r^2+r-2}z^{2(r^2+r-1)} \\ &\quad -2r(r+1)m^{r^2+r-1}z^{2r^2+2r-1} + (mz^2)^{r(r+1)}. \end{aligned} \quad (84)$$

With $m = 0$, these reduce to

$$\sin((2r+1)u) = \frac{(-1)^r T_{2r+1}(\sin u)}{1}\;, \quad \cos((2r+1)u) = \frac{T_{2r+1}(\cos u)}{1}\;. \quad (85)$$

For even $n = 2r$,

$$\begin{aligned} A_{2r}(z) &= 2r - \tfrac{4}{3}r(r^2-1)[1+m]z + \cdots \\ &\quad +(-1)^r\tfrac{4}{3}r(r^2-1)[1+m]m^{r^2-2}z^{2r^2-3} + (-1)^{r-1}2r(mz^2)^{r^2-1}, \\ B_{2r}(z) &= 1 - 2r^2z + \tfrac{1}{3}r^2[2(r^2-1)+(4r^2-1)m]z^2 + \cdots \\ &\quad +\tfrac{1}{3}r^2[2(r^2-1)+(4r^2-1)m]m^{r^2-2}z^{2r^2-2} \\ &\quad -2r^2m^{r^2-1}z^{2r^2-1} + (mz^2)^{r^2}, \\ C_{2r}(z) &= 1 - 2r^2mz + \tfrac{1}{3}r^2[2(r^2-1)+(4r^2-1)m]mz^2 + \cdots \\ &\quad +\tfrac{1}{3}r^2[4r^2-1+2(r^2-1)m]m^{r^2-2}z^{2r^2-2} \\ &\quad -2r^2m^{r^2-1}z^{2r^2-1} + (mz^2)^{r^2}. \end{aligned} \quad (86)$$

With $m = 0$, these reduce to

$$\cos(2ru) \;=\; \frac{T_{2r}(\cos u)}{1}\;. \quad (87)$$

In each expression for the polynomials A_n, B_n, C_n, D_n, some powers of z can appear both in the low–order terms (before the $\cdots$) and in the high–order terms (after the $\cdots$). For instance, in A_{2r+1} with $r = 1$, the term in z^1 does appear before and after the dots, with expressions which both give the term $-4[1+m]z$ for $r = 1$. In any such case, the term is to be used only once (not twice) in evaluating the polynomial.

4 From elliptic functions to permutable rational functions

In 1923, J. F. Ritt made a very general study [19] of permutable polynomials and permutable rational functions "that classifies all rational functions r and s that commute in the sense that $r \circ s = s \circ r$" [4, p.34]. Ritt's major result is the following:

> THEOREM *If the rational functions $\Phi(z)$ and $\Psi(z)$, each of degree greater than unity, are permutable, and if no iterate of $\Phi(z)$ is identical with any iterate of $\Psi(z)$, there exist a periodic meromorphic function $f(z)$, and four numbers a, b, c and d, such that*
>
> $$f(az+b) = \Phi\big[f(z)\big], \qquad f(cz+d) = \Psi\big[f(z)\big].$$
>
> The possibilities for $f(z)$ are: any linear function of e^z, $\cos z$, $\wp z$; in the lemniscatic case ($g_3 = 0$), $\wp^2 z$; in the equianharmonic case ($g_2 = 0$), $\wp' z$ and $\wp^3 z$. These are, essentially, the only periodic meromorphic functions which have rational multiplication theorems.
>
> The multipliers a and c must be such that if ω is any period of $f(z)$, $a\omega$ and $c\omega$ are also periods of $f(z)$.
>
> If p represents the order of $f(z)$, that is, the number of times $f(z)$ assumes any given value in a primitive period strip or in a primitive period parallelogram, the products
>
> $$b\Big(1 - e^{2\pi i/p}\Big), \qquad d\Big(1 - e^{2\pi i/p}\Big)$$
>
> must be periods of $f(z)$.
>
> Finally,
>
> $$(a-1)d - (c-1)b$$
>
> must be a period of $f(z)$. [19, pp.399–400]

Ritt's treatment of this topic is extremely abstract. In 50 pages of abstruse analysis of the Weierstraß elliptic function $\wp(z)$ on Riemann surfaces, he gave only 2 detailed examples of a permutable pair of rational functions. His first example is "the permutable pair z^p and εz^q where p and q are positive or negative integers, and where $\varepsilon^{p-1} = 1$" [19, p.413]. His second example involves functions which are not based on periodic trigonometric or elliptic functions: "Let

$$\phi(z) = \frac{\varepsilon^2 z^2 + 2}{\varepsilon z + 1}, \qquad \psi(z) = \frac{z^2 + 2}{z + 1}, \qquad \sigma(z) = \frac{z^2 - 4}{z - 1},$$

where ε is a primitive third root of unity. We shall see below that $\Phi = \phi\sigma$ and $\Psi = \psi\sigma$ are permutable" [19, p.447].

For the rational function w_j (cf. (72)), for all complex α, g_2 and g_3 and for all non-negative integers j and k,

$$w_j\big(w_k(\wp(\alpha)\big) \;=\; w_j\Big(\wp\big(2^k\alpha\big)\Big) \;=\; \wp\Big(2^j\,\big(2^k\alpha\big)\Big) \;=\; \wp\big(2^{j+k}\alpha\big) \;=\; w_{j+k}(\wp(\alpha)). \tag{88}$$

Hence

$$w_j\big(w_k(\wp(\alpha)\big) \;=\; w_{j+k}(\wp(\alpha)) \;=\; w_k\big(w_j(\wp(\alpha)\big), \tag{89}$$

and thus the sequence of rational functions $\{w_j\}$ is permutable.

The Weierstraß theory of elliptic functions, which Ritt used, is much more complicated than the version using the Jacobi elliptic functions.

For each of the 12 Jacobi elliptic functions, some infinite sequences of permutable rational functions are here constructed explicitly, based on the equations (74) and (75) for elliptic functions of multiple argument. In each case the parameter m is implied.

4.1 Functions $\mathbf{sn}((2r+1)u)$ and $\mathbf{ns}((2r+1)u)$

From (74),

$$\mathrm{sn}((2r+1)u) \;=\; \alpha_{2r+1}(\mathrm{sn}\,u), \tag{90}$$

where α_{2r+1} is the odd rational function:

$$\alpha_{2r+1}(x) \;\stackrel{\mathrm{def}}{=}\; \frac{x\,A_{2r+1}(x^2)}{D_{2r+1}(x^2)}\;. \tag{91}$$

Also, using the factored forms (80) for A_{2r+1} and D_{2r+1}, we get

$$\alpha_{2r+1}(x) \;=\; (2r+1)\;x\;\prod_{g=0}^{r}\;\prod_{h=-r}^{r}{}'\left(\frac{1-\mathrm{ns}^2\left(\dfrac{Q_{g,h}}{2r+1}\right)x^2}{1-\mathrm{sn}^2\left(\dfrac{Q_{g,h}}{2r+1}\right)m\,x^2}\right). \tag{92}$$

With parameter $m=0$, these reduce to $\sin((2r+1)u) = (-1)^r T_{2r+1}(\sin\,u)$.

As with (45), we get the functional identity for complex x and positive odd integers j,k:

$$\alpha_{jk}(x) \;=\; \alpha_k\big(\alpha_j(x)\big) \;=\; \alpha_j\big(\alpha_k(x)\big). \tag{93}$$

Thus, all rational functions in the infinite reflected sequence $\{\alpha_{2r+1},\ -\alpha_{2r+1}\}$ are permutable.

The reciprocal function is

$$\beta_{2r+1}(x) \;\stackrel{\mathrm{def}}{=}\; \frac{x\,D_{2r+1}(1/x^2)}{A_{2r+1}(1/x^2)}\;, \tag{94}$$

with

$$\mathrm{ns}((2r+1)u) \;=\; \beta_{2r+1}(\mathrm{ns}\,u), \tag{95}$$

giving the infinite reflected sequence $\{\beta_{2r+1},\ -\beta_{2r+1}\}$ of permutable rational functions. Also, using the factored form (92) for $\alpha_{2r+1}(x)$, we get

$$\begin{aligned}\beta_{2r+1}(x) &= \frac{1}{\alpha_{2r+1}(1/x)} \\ &= \frac{x}{2r+1} \prod_{g=0}^{r} \prod_{h=-r}^{r}{}' \left(\frac{x^2 - \mathrm{sn}^2\left(\frac{Q_{g,h}}{2r+1}\right) m}{x^2 - \mathrm{ns}^2\left(\frac{Q_{g,h}}{2r+1}\right)} \right) . \qquad (96)\end{aligned}$$

4.2 Functions sd$((2r+1)u)$ and ds$((2r+1)u)$

From (58), we get that

$$\mathrm{sd}^2 u = \frac{\mathrm{sn}^2 u}{\mathrm{dn}^2 u} = \frac{\mathrm{sn}^2 u}{1 - m\,\mathrm{sn}^2 u} , \qquad (97)$$

and hence

$$z = \mathrm{sn}^2 u = \frac{\mathrm{sd}^2 u}{1 + m\,\mathrm{sd}^2 u} . \qquad (98)$$

From (74) we get that for odd $n = 2r+1$,

$$\mathrm{sd}((2r+1)u) = \frac{\mathrm{sd}(u) A_{2r+1}(\mathrm{sn}^2 u)}{C_{2r+1}(\mathrm{sn}^2 u)} . \qquad (99)$$

Therefore

$$\mathrm{sd}((2r+1)u) = \gamma_{2r+1}(\mathrm{sd}\, u), \qquad (100)$$

where γ_{2r+1} is the odd rational function:

$$\gamma_{2r+1}(x) = \frac{x\, A_{2r+1}\left(\frac{x^2}{1+m\,x^2}\right)}{C_{2r+1}\left(\frac{x^2}{1+m\,x^2}\right)} . \qquad (101)$$

Also, using the factored forms (80) for A_{2r+1} and C_{2r+1}, we get

$$\begin{aligned}\gamma_{2r+1}(x) &= (2r+1)\, x \prod_{g=0}^{r} \prod_{h=-r}^{r}{}' \left(\frac{1 - \mathrm{ns}^2\left(\frac{Q_{g,h}}{2r+1}\right) \frac{x^2}{1+mx^2}}{1 - \mathrm{ns}^2\left(K(m) - \frac{Q_{g,h}}{2r+1}\right) \frac{mx^2}{1+mx^2}} \right) \\ &= (2r+1)\, x \prod_{g=0}^{r} \prod_{h=-r}^{r}{}' \left(\frac{1 + \left(m - \mathrm{ns}^2\left(\frac{Q_{g,h}}{2r+1}\right)\right) x^2}{1 + m\left(1 - \mathrm{ns}^2\left(K(m) - \frac{Q_{g,h}}{2r+1}\right)\right) x^2} \right) \\ &= (2r+1)\, x \prod_{g=0}^{r} \prod_{h=-r}^{r}{}' \left(\frac{1 - \mathrm{ds}^2\left(\frac{Q_{g,h}}{2r+1}\right) x^2}{1 - \mathrm{cs}^2\left(K(m) - \frac{Q_{g,h}}{2r+1}\right) mx^2} \right) . \qquad (102)\end{aligned}$$

With parameter $m = 0$ this reduces to $\sin((2r+1)u) = (-1)^r T_{2r+1}(\sin u)$.

As with (45), we get the functional identity for complex x and positive odd positive integers j, k:

$$\gamma_{jk}(x) \;=\; \gamma_j\big(\gamma_k(x)\big) \;=\; \gamma_k\big(\gamma_j(x)\big). \tag{103}$$

Thus, all rational functions in the infinite reflected sequence $\{\gamma_{2r+1},\ -\gamma_{2r+1}\}$ are permutable.

The reciprocal function is

$$\delta_{2r+1}(x) \;\stackrel{\text{def}}{=}\; \frac{1}{\gamma_{2r+1}(1/x)} \;=\; \frac{x\, C_{2r+1}\left(\dfrac{1}{m+x^2}\right)}{A_{2r+1}\left(\dfrac{1}{m+x^2}\right)}\ , \tag{104}$$

with

$$\mathrm{ds}((2r+1)u) \;=\; \delta_{2r+1}(\mathrm{ds}\, u), \tag{105}$$

giving the infinite reflected sequence $\{\delta_{2r+1},\ -\delta_{2r+1}\}$ of permutable rational functions.

Also, using the factored form (102) for $\gamma_{2r+1}(x)$, we get

$$\begin{aligned}\delta_{2r+1}(x) \;&=\; \frac{1}{\gamma_{2r+1}(1/x)}\\ &=\; \frac{x}{2r+1}\prod_{g=0}^{r}\ \prod_{h=-r}^{r}{}'\left(\frac{x^2-\mathrm{cs}^2\left(K(m)-\dfrac{Q_{g,h}}{2r+1}\right)m}{x^2-\mathrm{ds}^2\left(\dfrac{Q_{g,h}}{2r+1}\right)}\right).\end{aligned} \tag{106}$$

4.3 Functions sc$((2p+1)u)$ and cs$((2p+1)u)$

From (58), we get that

$$\mathrm{sc}^2 u \;=\; \frac{\mathrm{sn}^2 u}{\mathrm{cn}^2 u} \;=\; \frac{\mathrm{sn}^2 u}{1-\mathrm{sn}^2 u}\ , \tag{107}$$

and hence

$$z \;=\; \mathrm{sn}^2 u \;=\; \frac{\mathrm{sc}^2 u}{1+\mathrm{sc}^2 u}\ . \tag{108}$$

From (74) we get that for odd $n = 2p+1$,

$$\mathrm{sc}((2p+1)u) \;=\; \frac{\mathrm{sc}(u) A_{2p+1}(\mathrm{sn}^2 u)}{B_{2p+1}(\mathrm{sn}^2 u)}\ . \tag{109}$$

Therefore

$$\mathrm{sc}((2p+1)u) \;=\; \varepsilon_{2p+1}(\mathrm{sc}\, u), \tag{110}$$

where ε_{2p+1} is the odd rational function:

$$\varepsilon_{2p+1}(x) \;=\; \frac{x\, A_{2p+1}\left(\dfrac{x^2}{1+x^2}\right)}{B_{2p+1}\left(\dfrac{x^2}{1+x^2}\right)}\ . \tag{111}$$

With parameter $m=0$, this reduces (cf. (32)) to $\tan((2p+1)\rho) = r_{2p+1}(\tan\rho)$.
Also, using the factored forms (80) for A_{2r+1} and B_{2r+1}, we get

$$\begin{aligned}
\varepsilon_{2p+1}(x) &= (2p+1)\, x \prod_{g=0}^{r} \prod_{h=-r}^{r}{}' \left(\frac{1 - \mathrm{ns}^2\left(\dfrac{Q_{g,h}}{2p+1}\right) \dfrac{x^2}{1+x^2}}{1 - \mathrm{ns}^2\left(K(m) - \dfrac{Q_{g,h}}{2p+1}\right) \dfrac{x^2}{1+x^2}} \right) \\
&= (2p+1)\, x \prod_{g=0}^{r} \prod_{h=-r}^{r}{}' \left(\frac{1 + \left(1 - \mathrm{ns}^2\left(\dfrac{Q_{g,h}}{2p+1}\right)\right) x^2}{1 + \left(1 - \mathrm{ns}^2\left(K(m) - \dfrac{Q_{g,h}}{2p+1}\right)\right) x^2} \right) \\
&= (2p+1)\, x \prod_{g=0}^{r} \prod_{h=-r}^{r}{}' \left(\frac{1 - \mathrm{cs}^2\left(\dfrac{Q_{g,h}}{2p+1}\right) x^2}{1 - \mathrm{cs}^2\left(K(m) - \dfrac{Q_{g,h}}{2p+1}\right) x^2} \right). \qquad (112)
\end{aligned}$$

As with (45), we get the functional identity for complex x and positive odd positive integers j, k:

$$\varepsilon_{jk}(x) = \varepsilon_j\big(\varepsilon_k(x)\big) = \varepsilon_k\big(\varepsilon_j(x)\big). \qquad (113)$$

Thus, all rational functions in the infinite reflected sequence $\{\varepsilon_{2p+1},\ -\varepsilon_{2p+1}\}$ are permutable.

The reciprocal function is

$$\zeta_{2p+1}(x) \stackrel{\text{def}}{=} \frac{1}{\varepsilon_{2p+1}(1/x)} = \frac{x\, B_{2p+1}\left(\dfrac{1}{1+x^2}\right)}{A_{2p+1}\left(\dfrac{1}{1+x^2}\right)}, \qquad (114)$$

with

$$\mathrm{cs}((2p+1)u) = \zeta_{2p+1}(\mathrm{cs}\, u), \qquad (115)$$

giving the infinite reflected sequence $\{\zeta_{2p+1},\ -\zeta_{2p+1}\}$ of permutable rational functions.

Also, using the factored form (112) for ε_{2p+1}, we get

$$\begin{aligned}
\zeta_{2p+1}(x) &= \frac{1}{\varepsilon_{2p+1}(1/x)} \\
&= \frac{x}{2p+1} \prod_{g=0}^{r} \prod_{h=-r}^{r}{}' \left(\frac{x^2 - \mathrm{cs}^2\left(K(m) - \dfrac{Q_{g,h}}{2p+1}\right)}{x^2 - \mathrm{cs}^2\left(\dfrac{Q_{g,h}}{2p+1}\right)} \right). \qquad (116)
\end{aligned}$$

4.4 Functions cn(ku) and nc(ku)

From (56), $z = \mathrm{sn}^2 u = 1 - \mathrm{cn}^2 u$, and from (74) and (75) we get that

$$\mathrm{cn}(ku) = \eta_k(\mathrm{cn}\, u), \qquad (117)$$

where η_k is an odd rational function for odd $k = 2r+1$ and an even rational function for even $k = 2r$:

$$\eta_k(x) \stackrel{\text{def}}{=} \begin{cases} \dfrac{x\, B_{2r+1}(1-x^2)}{D_{2r+1}(1-x^2)} & (k = 2r+1), \\ \dfrac{B_{2r}(1-x^2)}{D_{2r}(1-x^2)} & (k = 2r). \end{cases} \tag{118}$$

With parameter $m = 0$, these reduce to $\cos(ku) = T_k(\cos u)$.

As with (45), we get the functional identity for complex x and positive integers j, k:

$$\eta_{jk}(x) \;=\; \eta_j\big(\eta_k(x)\big) \;=\; \eta_k\big(\eta_j(x)\big). \tag{119}$$

Thus, all rational functions in the infinite sequence $\{\eta_k\}$ are permutable.

This sequence is similar (via $\lambda(x) = -x$) to the permutable rational sequence $\{-\eta_k(-x)\} = -\eta_0(x), \eta_1(x), -\eta_2(x), \eta_3(x), \cdots$.

The odd rational function η_{2r+1} has the reflected permutable rational sequence $\{\eta_{2r+1},\ -\eta_{2r+1}\}$.

The reciprocal function is

$$\theta_k(x) \stackrel{\text{def}}{=} \frac{1}{\eta_k(1/x)}\ , \tag{120}$$

with

$$\mathrm{nc}(ku) \;=\; \theta_k(\mathrm{nc}\, u), \tag{121}$$

giving the sequence of permutable rational functions $\{\theta_k\}$.

This sequence is similar (via $\lambda(x) = -x$) to the permutable rational sequence $\{-\theta_k(-x)\} = -\theta_0(x), \theta_1(x), -\theta_2(x), \theta_3(x), \cdots$.

The odd rational function θ_{2r+1} has the reflected permutable rational sequence $\{\theta_{2r+1},\ -\theta_{2r+1}\}$.

4.5 Functions dn(ku) and nd(ku)

From (58),

$$z \;=\; \mathrm{sn}^2 u \;=\; \frac{1-\mathrm{dn}^2 u}{m}\ , \tag{122}$$

and from (74) and (75) we get that

$$\mathrm{dn}(ku) \;=\; \iota_k(\mathrm{dn}\, u), \tag{123}$$

where ι_k is an odd rational function for odd $k = 2r+1$ and an even rational function for even $k = 2r$:

$$\iota_k(x) \stackrel{\text{def}}{=} \begin{cases} \dfrac{x\, C_{2r+1}\left(\dfrac{1-x^2}{m}\right)}{D_{2r+1}\left(\dfrac{1-x^2}{m}\right)} & (k = 2r+1), \\[2ex] \dfrac{C_{2r}\left(\dfrac{1-x^2}{m}\right)}{D_{2r}\left(\dfrac{1-x^2}{m}\right)} & (k = 2r). \end{cases} \tag{124}$$

As with (45), we get the functional identity for complex x and positive integers j, k:

$$\iota_{jk}(x) \;=\; \iota_j\big(\iota_k(x)\big) \;=\; \iota_k\big(\iota_j(x)\big). \tag{125}$$

Thus, all rational functions in the infinite sequence $\{\iota_k\}$ are permutable.

This sequence is similar (via $\lambda(x) = -x$) to the permutable rational sequence $\{-\iota_k(-x)\} = \; -\iota_0(x), \iota_1(x), -\iota_2(x), \iota_3(x), \cdots$.

The odd rational function ι_{2r+1} has the reflected permutable rational sequence $\{\iota_{2r+1},\; -\iota_{2r+1}\}$.

The reciprocal function is

$$\kappa_k(x) \;\stackrel{\text{def}}{=}\; \frac{1}{\iota_k(1/x)} \;, \tag{126}$$

with

$$\text{nd}(ku) \;=\; \kappa_k(\text{nd}\, u), \tag{127}$$

giving the sequence of permutable rational functions $\{\kappa_k\}$.

This sequence is similar (via $\lambda(x) = -x$) to the permutable rational sequence $\{-\kappa_k(-x)\} = \; -\kappa_0(x), \kappa_1(x), -\kappa_2(x), \kappa_3(x), \cdots$.

The odd rational function κ_{2r+1} has the reflected permutable rational sequence $\{\kappa_{2r+1},\; -\kappa_{2r+1}\}$.

4.5.1 Duality of cn and dn

The rational functions η_k perform (118) real multiplication of the elliptic function cn, and the rational functions ι_k perform (124) real multiplication of the elliptic function dn. Rewrite those functions with m as an explicit parameter, and we get:

$$\eta_k(\text{cn}(u \,|\, m), m) \;=\; \text{cn}(ku \,|\, m), \qquad \iota_k(\text{dn}(u \,|\, m), m) \;=\; \text{dn}(ku \,|\, m). \tag{128}$$

Denote

$$\rho \;\stackrel{\text{def}}{=}\; 1/m, \qquad v \;\stackrel{\text{def}}{=}\; u\sqrt{m} \;. \tag{129}$$

Then, by Jacobi's Real Transformation [18, §**16.11.3** & §**16.11.4**],

$$\text{cn}(u \,|\, m) \;\equiv\; \text{dn}(v \,|\, \rho), \qquad \text{dn}(u \,|\, m) \;\equiv\; \text{cn}(v \,|\, \rho) \;. \tag{130}$$

Substitute this in (128), and we get the identity

$$\begin{aligned} \eta_k(\text{dn}(v \,|\, \rho), m) \;&\equiv\; \eta_k(\text{cn}(u \,|\, m), m) \;\equiv\; \text{cn}(ku \,|\, m) \\ &\equiv\; \text{dn}(kv \,|\, \rho) \;\equiv\; \iota_k(\text{dn}(v \,|\, \rho), \rho) \;. \end{aligned} \tag{131}$$

Write $x = \text{dn}(v \,|\, \rho)$, and we get an identity (in complex x and m, and non-negative integer k) between two families of permutable rational functions:

$$\eta_k(x, m) \;\equiv\; \iota_k(x, 1/m) \;. \tag{132}$$

And we get an identity for their permutable reciprocal functions $\theta_k(x) = 1/\eta_k(1/x)$ (which perform real multiplication of the function nc) and $\kappa_k(x) = 1/\iota_k(1/x)$ (which perform real multiplication of the function nd):

$$\theta_k(x, m) \;\equiv\; \kappa_k(x, 1/m) \;. \tag{133}$$

4.6 Functions cd(ku) and dc(ku)

From (56) and (58), we get that

$$\mathrm{cd}^2 u \;=\; \frac{\mathrm{cn}^2 u}{\mathrm{dn}^2 u} \;=\; \frac{1-\mathrm{sn}^2 u}{1-m\,\mathrm{sn}^2 u}\;, \tag{134}$$

and hence

$$z \;=\; \mathrm{sn}^2 u \;=\; \frac{1-\mathrm{cd}^2 u}{1-m\,\mathrm{cd}^2 u}\;. \tag{135}$$

From (74) and (75) we get that

$$\mathrm{cd}(ku) \;=\; \begin{cases} \dfrac{\mathrm{cd}(u)B_{2r+1}(\mathrm{sn}^2 u)}{C_{2r+1}(\mathrm{sn}^2 u)} & (k=2r+1),\\[2ex] \dfrac{B_{2r}(\mathrm{sn}^2 u)}{C_{2r}(\mathrm{sn}^2 u)} & (k=2r). \end{cases} \tag{136}$$

Therefore

$$\mathrm{cd}(ku) \;=\; \lambda_k(\mathrm{cd}\,u), \tag{137}$$

where λ_k is an odd rational function for odd $k=2r+1$ and an even rational function for even $k=2r$:

$$\lambda_k(x) \;=\; \begin{cases} \dfrac{x\,B_{2r+1}\left(\dfrac{1-x^2}{1-mx^2}\right)}{C_{2r+1}\left(\dfrac{1-x^2}{1-mx^2}\right)} & (k=2r+1),\\[4ex] \dfrac{B_{2r}\left(\dfrac{1-x^2}{1-mx^2}\right)}{C_{2r}\left(\dfrac{1-x^2}{1-mx^2}\right)} & (k=2r). \end{cases} \tag{138}$$

With parameter $m=0$, these reduce to $\cos(ku)=T_k(\cos\,u)$.

As with (45), we get the functional identity for complex x and positive integers j,k:

$$\lambda_{jk}(x) \;=\; \lambda_j\big(\lambda_k(x)\big) \;=\; \lambda_k\big(\lambda_j(x)\big). \tag{139}$$

Thus, all rational functions in the infinite sequence $\{\lambda_k\}$ are permutable.

This sequence is similar to the permutable rational sequence $\{-\lambda_k(-x)\} = -\lambda_0(x), \lambda_1(x), -\lambda_2(x), \lambda_3(x), \cdots$.

The odd rational function λ_{2r+1} has the reflected permutable rational sequence $\{\lambda_{2r+1},\ -\lambda_{2r+1}\}$.

The reciprocal function is

$$\mu_k(x) \;\stackrel{\mathrm{def}}{=}\; \frac{1}{\lambda_k(1/x)}\;, \tag{140}$$

with

$$\mathrm{dc}(ku) \;=\; \mu_k(\mathrm{dc}\,u), \tag{141}$$

giving the sequence of permutable rational functions $\{\mu_k\}$.

This sequence is similar to the permutable rational sequence $\{-\mu_k(-x)\} = -\mu_0(x), \mu_1(x), -\mu_2(x), \mu_3(x), \cdots$.

The odd rational function μ_{2r+1} has the reflected permutable rational sequence $\{\mu_{2r+1},\ -\mu_{2r+1}\}$.

Thus, many infinite sequences of permutable rational functions have been constructed, on the basis of elliptic functions.

References

[1] Niels Henrik Abel, Recherches sur les Fonctions Elliptiques. Journal für die reine und angewandte Mathematik, vol. 2 & 3 (1827–1828).
(Paper 16 in **Œuvres Complètes de Niels Henrik Abel**, 2nd edition edited by L. Sylow & S. Lie, Grondahl & Son, Christiana, 1881, Tome Premier, 263–388).

[2] Eric Temple Bell, On the real multiplication of elliptic functions. Annals of Mathematics vol. 33 (1932), 207–213.

[3] Henry David Block & Henry P. Thielman, Commutative polynomials, Quarterly Journal of Mathematics; Oxford Series 2 vol. 2 (1951), 241–243. MR 13, 552f.

[4] Peter Borwein & Tamás Erdélyi, Polynomials and Polynomial Inequalities, Springer, Berlin, 1995. MR 97e:41001.

[5] Arthur Cayley, On the theory of elliptic functions. Cambridge and Dublin Mathematical Journal vol. 2 (1847); Reprinted as Paper 45 in [10, pp. 290–300].

[6] Arthur Cayley, On the theory of elliptic functions. Cambridge and Dublin Mathematical Journal vol. 3 (1848); Reprinted as Paper 57 in [10, pp. 364–365].

[7] Arthur Cayley, Note sur les fonctions elliptiques. Journal für die reine und angewandte Mathematik vol. 37 (1848); Reprinted as Paper 67 in [10, pp. 402–404].

[8] Arthur Cayley, Note sur quelques formules qui se rapportent à la multiplication des fonctions elliptiques. Journal für die reine und angewandte Mathematik vol. 39 (1850); Reprinted as Paper 93 in [10, pp. 534–539].

[9] Arthur Cayley, Note sur quelques formules qui se rapportent à la multiplication des fonctions elliptiques. Journal für die reine und angewandte Mathematik vol. 41 (1851); Reprinted as Paper 99 in [10, pp. 568–576].

[10] Athur Cayley, The Collected Mathematical Papers of Arthur Cayley, Volume 1. Cambridge University Press, Cambridge, 1889.

[11] Arthur Cayley, An Elementary Treatise on Elliptic Functions (2nd edition). George Bell & Sons, London; and Deighton, Bell & Co., Cambridge, 1895.

[12] Robert Fricke, Die Elliptischen Funktionen und ihre Anwedungen (2 volumes). B. G. Teubner, Leipzig und Berlin, 1916 & 1922.

[13] Ernst Jacobsthal, Über vertauschbare Polynome, Mathematische Zeitschrift 63 (1955), 243–276. MR 17, 574a.

[14] Eugene Jahnke & Fritz Emde, Tables of Functions with Formulæ and Curves (4th edition), Dover, New York, 1945.

[15] Harold Jeffreys & Bertha Swiles Jeffreys, Methods of Mathematical Physics (3rd edition), Cambridge University Press, Cambridge, 1956. MR 8, 447b.

[16] Hans Lausch & Wilfried Nöbauer, Algebra of Polynomials, North-Holland Publishing Company, Amsterdam, 1973. MR 50 #2037.

[17] Lazar' Aronovich Lyusternik, O. A. Chervonenkis & A. R. Yanpol'skii, Handbook for Computing Elementary Functions (translated from Russian by G. J. Tee), Pergamon Press, Oxford, 1965. MR 32 #584.

[18] Louis Melville Milne–Thomson, Jacobian Elliptic Functions and Theta Functions (Chapter 16), and Elliptic Integrals (Chapter 17).
In **Handbook of Mathematical Functions With Formulas, Graphs and Mathematical Tables**, Applied Mathematics Series **55**, edited by Milton Abramowitz & Irene A. Stegun. National Bureau of Standards, Washington, D.C., 1964 (and many later corrected editions). MR 29 #4914.

[19] Joseph Fels Ritt, Permutable rational functions. Transactions of the American Mathematical Society 25 (1923), 399–448.

[20] Theodore J. Rivlin, Chebyshev Polynomials, From Approximation Theory to Algebra and Number Theory (2nd edition), John Wiley & Sons, New York, 1990. MR 92a:41016.

[21] Thomas H. Southard, Weierstrass Elliptic and Related Functions (Chapter 18),
In **Handbook of Mathematical Functions With Formulas, Graphs and Mathematical Tables**, Applied Mathematics Series **55**, edited by Milton Abramowitz & Irene A. Stegun. National Bureau of Standards, Washington, D.C., 1964 (and many later corrected editions). MR 29 #4914.

G. J. Tee
Department of Mathematics
University of Auckland
Private Bag 92019
Auckland
New Zealand
e-mail: tee@math.auckland.ac.nz

2007–1–8