Digital Manifolds of Higher Dimensions and Good Pairs of Adjacency Relations

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Abstract. In this paper we define and study digital manifolds of arbitrary dimension, and provide (in particular) a general theoretical basis for curve or surface tracing in picture analysis. The studies involve properties such as one-dimensionality of digital curves and $(n-1)$-dimensionality of digital hypersurfaces that makes them discrete analogs of corresponding notions in topology. The presented approach is fully based on good pairs of adjacency relations and complements the concept of dimension as common in combinatorial topology. This work appears to be the first one on digital manifolds based on a graph-theoretical definition of dimension. In particular, a digital hypersurface in nD is an $(n-1)$-dimensional object, as it is in the case of continuous hypersurfaces. Relying on the obtained properties of digital hypersurfaces, we propose a uniform approach for studying good pairs defined by separations and obtain a classification of good pairs in arbitrary dimension.
Keywords: *digital geometry, digital topology, discrete dimension, digital manifold, digital curve, digital hypersurface, good pair*

1 Introduction

Combinatorial topology (see, e.g., [2]) introduced the concept of dimension for topologies defined on discrete sets (e.g., poset topologies). We will show that the concept of dimension is also a very useful concept for picture analysis in general.

A regular orthogonal grid subdivides $\mathbb{R}^n$ into n-dimensional hypercubes (e.g., unit squares for $n = 2$ or unit cubes for $n = 3$, also called *n-cells* for short) defining a class $\mathbb{C}_n^{(n)}$. Let $\mathbb{C}_n^{(k)}$ be the class of all k-dimensional facets of n-dimensional hypercubes, for $0 \leq k < n$. The grid-cell space $\mathbb{C}_n$ is the union of all these classes $\mathbb{C}_n^{(k)}$, for $0 \leq k \leq n$. The *grid cell topology* (see [17] for reviewing material) is defined on $\mathbb{C}_n$; open or closed sets of n-cells correspond to $(n-1)$- or 0-connected regions in the graph-theoretical model of adjacencies between n-cells; see [16]. (This is a general justification for the alternative use of 4- or 8-adjacency in binary pictures, as proposed in [10].)

The grid cell topology (in particular the alternative use of $(n-1)$- or 0-adjacency for binary pictures) supports "topologically sound" picture analysis, and it is an example of a poset topology. This allows to apply the notion of

dimension within n-dimensional grid cell topology, as known from combinatorial topology. However, this is *not* equivalent to dimensions as considered in this article. For example, a simple $(n-1)$-path of n-cells would be an n-dimensional set in the grid cell topology, but it will be a one-dimensional continuum with respect to the notion of dimension as used in this article.

Digital 2D or 3D pictures can be considered to be defined on $\mathbb{C}_n^{(n)}$, for $n = 2$ or $n = 3$, using either graph-theoretical concepts such as neighborhoods or adjacencies, or a particular digital topology (typically the grid cell topology). Picture analysis benefits from the grid cell topology (using a maximum-label rule for multi-level or multi-channel pictures), and more generally (as we will show), from a theory of *good pairs*[1] because this allows to specify *separation theorems* which form a theoretical justification for any border or surface tracing algorithm. Such separation theorems have been often studied with respect to particular digital topologies or graph-theoretical concepts (see, e.g., [17]).

1.1 Good Pairs

In this paper we study digital manifolds, in particular digital curves and hypersurfaces. On this basis, we define good pairs of adjacency relations in grid-cell spaces $\mathbb{C}_n^{(n)}$ $(n \geq 2)$, equipped with adjacencies A_α (e.g., $\alpha = 0, 1$ for $n = 2$, and $\alpha = 0, 1, 2$ for $n = 3$)[2]. Informally speaking, a good pair combines two adjacency relations on $\mathbb{C}_n^{(n)}$ which appear to be "suitable for image analysis". The reason for suggesting the first good pairs (α, β) in [10], with (α, β) equal to (1,0) or (0,1), were observations in [31]. (A_α is the adjacency relation for $1s$, which are the "object" pixels with value 1, and A_β is the adjacency relation for $0s$.) The benefit of two alternative adjacencies was then formally shown in [28]: (1,0) or (0,1) define region adjacency graphs for binary pictures which form a rooted tree. This simplifies topological studies of binary pictures because it allows to specify a separation theorem (typically formulated in terms of 4- or 8-paths, and 4- or 8-holes). A good pair of adjacency relations can also be "combined" into s-adjacency (see [16, 17]) which can be used for "topologically sound" multi-level or multi-channel picture analysis.

Our study of good pairs is directed on the understanding of separability properties: which kind of sets, defined by one type of adjacency, allow to separate sets defined by another type of adjacency. These separating sets can be defined in the form of digital curves in 2D, or as digital surfaces in 3D. In this way, studies of good pairs and of (separating) surfaces are directly related to one-another.

Several works on digital surfaces are available in the literature. [15] defines digital surfaces in $\mathbb{Z}^3$ based on adjacencies of 3-cells. For obtaining α-surfaces by digitization of surfaces in $\mathbb{R}^3$, see [8]. It is proved in [24] that there is no

[1] The name was created for the oral presentation of [18]. Note that the same term has been used already with different meaning in topology.

[2] In 2D, 0- and 1-adjacency correspond to 8- and 4-adjacency, respectively, while in 3D, 0-, 1-, and 2-adjacency correspond to 26-, 18- and 6-adjacency, respectively. The latter are traditionally used within the grid-point model on $\mathbb{Z}^n$.

local characterization of a 26-connected subset S of $\mathbb{Z}^3$ such that its complement $\overline{S}$ consists of two 6-components and every voxel of S is adjacent to both of these components. [24] defines a class of 18-connected surfaces in $\mathbb{Z}^3$, proves a surface separation theorem for those surfaces, and studies their relationship to the surfaces defined in [25]. [4] introduces a class of "strong" surfaces and proves that both the 26-connected surfaces of [25] and the 18-connected surfaces of [24] are strong. For further studies on 6-surfaces, see [7]. Digital surfaces in the context of arithmetic geometry are studied in [5].

1.2 Digital Topologies

A *digital topology* on $\mathbb{C}_n$ is defined by a family of open subsets that satisfy a number of axioms (see, e.g., Section 6.2 in [17]). For all $n \geq 2$ we have at least two digital topologies, known as *grid cell topology* (first time formulated in 1935 within an exercise in [2] for 2D; in general, the nD case is an example of a poset topology) and as *grid point topology* (as specified in 1970 [36] for nD). Digital topologies on $\mathbb{C}_n$ can be mapped by isomorphisms into digital topologies on $\mathbb{C}_n^{(n)}$; for example, the grid cell topology is isomorphic to the topology of incidence grids, also isomorphic to the Khalimsky topology, and a special case of a topology of Euclidean, or of abstract complexes (see, e.g., [17] for reviewing material).

For example, the topology of incidence grids is one possible approach for considering 2D or 3D picture analysis: frontiers of closed sets of n-cells define hypersurfaces, consisting of $(n-1)$-cells, which separate interior from exterior.

A separation theorem for the Khalimsky topology is proved in [21]. For discrete combinatorial surfaces, see [12]. The approximation of n-dimensional manifolds by graphs is studied in [32, 33], with a special focus on topological properties of such graphs defined by homotopy, and on homology or cohomology groups. Approximation of boundaries of finite sets of grid points (in n dimensions) based on "continuous analogs" is proposed and studied in [23]. [14] discusses local topological configurations (stars) for surfaces in incidence grids.

Frontiers in cell complexes (and related topological concepts such as components and fundamental groups) are studied in [1]. For characterizations of, and algorithms for curves and surfaces in frontier grids, see [13, 22, 30, 34]. [9] defines curves in incidence grids.

[11] shows that there are two digital topologies on $\mathbb{C}_2^{(2)}$, five on $\mathbb{C}_3^{(3)}$, and [19] shows that there are 24 on $\mathbb{C}_4^{(4)}$ (all up to homeomorphisms). The product of all nD digital topologies with the 1D alternating topology of [36] is an $(n+1)$D digital topology, and we also have always the $(n+1)$D grid cell topology. This gives at least $n+20$ digital topologies for all $n \geq 4$. However, many of those digital topologies may have no relevance for being applied in computer analysis of regularly sampled nD data. In applied picture analysis there seems to be a general preference for adjacency-based algorithms compared to topology-based algorithms.

1.3 Digital Topologies versus Good Pairs

Good pairs may induce a digital topology on $\mathbb{C}_n$ (and not vice-versa in general). For example, from [17] we know that the good pair (1,0) is equivalent to regarding 1-components of $1s$ as open regions and 0-components of $0s$ as closed regions in $\mathbb{C}_2$ [or vice versa, for the good pair (0,1)]. According to [16] this can be generalized to arbitrary $n \geq 2$: the good pairs $(0, n-1)$ or $(n-1, 0)$ are equivalent to the grid cell topology in $\mathbb{C}_n$, if both models are using identical total orders of values of n-cells.

The present paper provides a complete characterization of good pairs, showing that there are exactly $n+1$ good pairs on $\mathbb{C}_n^{(n)}$. Together with the lower bound for numbers of digital topologies, as given in the previous subsection for $n \leq 4$, we thus know that there are (for $n \geq 3$) more digital topologies than good pairs. As already mentioned, some of the known digital topologies (for $n = 3$ and $n = 4$) seem to be irrelevant for practical use. On the other hand, all the defined good pairs can be considered to be of practical relevance, also covering the grid cell topology as stated above.

1.4 Results and Structure of the Paper

In this paper we present alternative definitions of digital hypersurfaces, partially following ideas already published in some of the references cited above. In short, a digital α-hypersurface is composed by (closed) α-curves; two such curves are either disjoint and non-adjacent, or disjoint but adjacent, or they have overlapping portions. The main contributions of the paper are as follows ($n \geq 2$):

- We define digital manifolds in arbitrary dimensions, as the definitions involve the notion of *dimension of a digital object* [26]. Thus a digital curve is a one-dimensional digital manifold, while a digital hypersurface in nD is an $(n-1)$-dimensional manifold, in conformity to topology (see, e.g., the topological definitions of curves by Urysohn and Menger, as discussed in [17]). To our knowledge of the available literature, this is the first work involving dimensionality in defining these notions in digital geometry.
- We show that there are two and only two basic types of α-hypersurfaces, one for $\alpha = n-2$ and one for $\alpha = n-1$. For $\alpha = n-2$, a hypersurface S has $(n-2)$-gaps which appear on $(n-2)$-manifolds that build S and, possibly, between adjacent/overlapping pairs of such $(n-2)$-manifolds. Moreover, S is $(n-1)$-gapfree[3] and $(n-1)$-minimal. For $\alpha = n-1$, the hypersurface S is 0-gapfree and 0-minimal.

[3] This was also called "tunnel-free" in earlier publications (e.g., in [3, 27]). The Betti number β_1 defines the number of tunnels in topology. Informally speaking, the location of a tunnel cannot be uniquely identified in general; there is only a unique way to count the number of tunnels. Locations of gaps are identified by defining sets. There are sets (e.g., knots) which have a tunnel (i.e., $\beta_1 > 0$) but no gap (in the sense of [3, 27]).

– We investigate combinatorial properties of digital hypersurfaces, showing that a digital hypersurface can define a matroid.
– Relying on the obtained properties of digital hypersurfaces, we study good pairs of adjacency relations in arbitrary dimension. We define nD good pairs through separation by digital hypersurfaces and show that there are exactly $n+1$ such good pairs. We also provide a short review and comments on some other approaches for defining good pairs which have been communicated elsewhere.

The paper is organized as follows. In the next Section 2 we recall some basic definitions and facts. We also prove a lemma that characterizes one-dimensionality of digital sets and that is used in the further sections. In Section 3 we presents our basic results about digital manifolds. In Section 4 we provide a characterization of good pairs of adjacency relations defined by separation by digital surfaces. We conclude with some remarks in Section 5.

2 Preliminaries

We start with recalling basic definitions; notations follow [17]. In particular, the grid point space $\mathbb{Z}^n$ allows a refined representation by an incidence grid defined on the cellular space $\mathbb{C}_n$ introduced above.

2.1 Some Basic Definitions

Elements in $\mathbb{C}_n^{(k)}$ are k-cells, for $0 \leq k \leq n$. An m-dimensional facet of a k-cell is an m-cell, for $0 \leq m \leq k-1$. Two k-cells are called *m-adjacent* if they share an m-cell. Two k-cells are *properly m-adjacent* if they are m-adjacent but not $(m+1)$-adjacent.

A digital object D is a finite set of n-cells. In dimension two these are usually called *pixels* and in dimension three *voxels*. An *m-path* in D is a sequence of n-cells from D such that every two consecutive n-cells are m-adjacent. The *length* of a path is the number of n-cells it contains. A *proper m-path* is an m-path in which at least two consecutive n-cells are not $(m+1)$-adjacent. Two n-cells of a digital object D are *m-connected* (in D) iff there is an m-path in D between them. A digital object D is *m-connected* iff there is an m-path connecting any two n-cells of D. D is *properly m-connected* iff it contains two n-cells such that all m-paths between them are proper. An *m-component* of D is a maximal by inclusion (i.e., non-extendable) m-connected subset of D.

Let M be a subset of a digital object D. If $D \setminus M$ is not m-connected then the set M is said to be *m-separating* in D. (In particular, the empty set m-separates any set D which is not m-connected.)

Let M be an m-separating digital object in D such that $D \setminus M$ has exactly two m-components. An *m-simple cell* (or *m-simple point*) of M (with respect to D) is an n-cell c such that $M \setminus \{c\}$ is still m-separating in D. An m-separating digital object in D is *m-minimal* (or *m-irreducible*) if it does not contain any m-simple cell (with respect to D).

For a set of n-cells D, by $\overline{D}$ we denote the complement of D to the whole digital space $\mathbb{C}_n^{(n)}$, and by $card(D)$ its cardinality.

$J^+(A)$ is the outer Jordan digitization (also called *supercover*) of a set $A \subseteq \mathbb{R}^n$, which consists of all n-cells intersected by A.

By $N_\alpha(c)$ we denote the *unit α-ball* (also called the *α-neighborhood* of c) with center c consisting of all α-neighbors of c. Furthermore, let $A_\alpha(c) = N_\alpha(c) \setminus \{c\}$ be the *α-adjacency set* of c.

For a given set $D = \{c_1, c_2, \ldots, c_m\} \subseteq \mathbb{C}_n^{(n)}$, we define its *$\alpha$-adjacency graph* $G_\alpha(D, E)$ with D as a set of vertices and a set of edges $E = \{(c_i, c_j) : c_i \text{ and } c_j \text{ are } \alpha\text{−adjacent}\}$.

Finally, we recall the definition of Hausdorff distance of two sets. Given a metric space E with a metric d, let $\mathcal{E}$ be a familly of closed nonempty subsets of E. For every $x \in E$ and every $A \in \mathcal{E}$, let $d(x, A) = \inf\{d(x, y) : y \in A\}$. Then, given two sets $A, B \in \mathcal{E}$, $H_d(A, B) = \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(A, b) : b \in B\}\}$ is called the *Hausdorff distance* between A and B.

2.2 Gaps

An important notion in discrete geometry and topology is the one of a *gap*. Usually, gaps are defined through separability as follows: Let a digital object M be m-separating but not $(m-1)$-separating in a digital object D. Then M is said to have *k-gaps* for any $k < m$. A digital object without m-gaps is called *m-gapfree*.

Although the above definition has been used in a number of papers by different authors, one can reasonably argue that it requires further refinement. Consider, for instance, the following example. Let M_1 and M_2 be two digital objects that are subsets of a superset D, and assume that $M_1 \cap M_2 = \emptyset$ (we may think that M_1 and M_2 are "far away" from each other). In addition, assume that M_1 has a k-gap with respect to an adjacency relation A_α, while M_2 is a closed digital hypersurface that k-separates D. Then it turns out that the digital set $M_1 \cup M_2$ that consists of (at least) two connected components, has no k-gap with respect to A_α.

Despite such kind of phenomena, the above definition is adequate for the studies that follow.

2.3 Dimension

Mylopoulos and Pavlidis [26] proposed definition of dimension of a (finite or infinite) set of n-cells S with respect to an adjacency relation A_α (for its use see also [17]).

Let $B_\alpha(c)$ be the union of $N_\alpha(c)$ with all n-cells c' for which there exist $c_1, c_2 \in N_\alpha(c)$ such that a shortest α-path from c_1 to c_2 not passing through c passes through c'. For example, $B_1(c) = B_0(c) = N_0(c)$ for $n = 2$, and $B_2(c) = B_1(c) = N_1(c)$, $B_0(c) = N_0(c)$ for $n = 3$.

Recall that $A_\alpha(c) = N_\alpha(c) \setminus \{c\}$. Denote $B_\alpha^*(c) = B_\alpha(c) \setminus \{c\}$.

A nonempty set $D \subseteq \mathbb{C}_n^{(n)}$ is called *totally α-disconnected* iff $A_\alpha(x) \cap D = \emptyset$ for any $x \in D$ (i.e., there is no pair of cells $c, c' \in D$ such that $c \neq c'$ and $\{c, c'\}$ is α-connected).

$D \subseteq \mathbb{C}_n^{(n)}$ is called *linearly α-connected* whenever $|A_\alpha(x) \cap D| \leq 2$ for all $x \in D$ and $|A_\alpha(x) \cap D| > 0$ for at least one $x \in D$.

In what follows we will use the definition of dimension from [26]. Let D be a digital object in $\mathbb{C}_n^{(n)}$ and A_α an adjacency relation on $\mathbb{C}_n^{(n)}$. The *dimension* $\dim_\alpha(D)$ is defined as follows:

(1) $\dim_\alpha(D) = -1$ iff $D = \emptyset$,
(2) $\dim_\alpha(D) = 0$ iff D is a totally α-disconnected nonempty set,
(3) $\dim_\alpha(D) = 1$ if D is linearly α-connected,
(4) $\dim_\alpha(D) = \max\limits_{c \in S} \dim_\alpha(B_\alpha^*(c) \cap D) + 1$ otherwise.

If in the last item of the definition the maximum is reached for an n-cell c, we will also say that D is $\dim_\alpha(D)$-dimensional at c.

The next lemma provides characterization of one-dimensionality in $\mathbb{C}_n^{(n)}$ to be used in the sequel.

Lemma 1. *Let $D \subseteq \mathbb{C}_n^{(n)}$ be a non-empty, α-connected set.*

(a) If $0 \leq \alpha \leq n-2$, then D is one-dimensional with respect to adjacency A_α iff it does not contain as a proper subset an elementary grid triangle *consisting of three cells c_1, c_2, c_3, such that any two of them are α-adjacent.*

(b) If $\alpha = (n-1)$, then D is one-dimensional with respect to adjacency A_α iff it does not contain as a proper subset an elementary grid square *consisting of four cells c_1, c_2, c_3, c_4 with coordinates $c_1 = (i, i, \ldots, i, i)$, $c_2 = (i+1, i, \ldots, i, i)$, $c_3 = (i+1, i+1, \ldots, i, i)$, $c_4 = (i, i+1, \ldots, i, i)$, for some $i \in \mathbb{Z}$.*

Proof (a) Let $\alpha \leq n-2$. Assume first that D does not contain an elementary grid triangle. Let c be an arbitrary n-cell of D. Assume that D is not linearly connected. Then either there is an n-cell $c' \in D$ with $|A_\alpha(c') \cap D| \leq 2$ or for any $c \in D$ it holds $|A_\alpha(c') \cap D| = 0$. In the latter case D would clearly be disconnected and thus 0-dimensional. In the former, we have that D is neither empty, nor totally disconnected, nor linearly connected. Then we have

$$\dim_\alpha(D) = \max_{c \in D} \dim(B_\alpha^*(c) \cap D) + 1 \tag{1}$$

Let the maximum in (1) is reached for a point $c_0 \in D$. Since D does not contain any elementary grid triangle, it is easy to deduce that $\dim(A_\alpha(c)) = 0$ and thus $\dim_\alpha(D) = 1$.

Now let $\dim_\alpha(D) = 1$ and assume by contradiction that D contains as a proper subset an elementary grid triangle T. Let c be an arbitrary n-cell of T. We have $|A_\alpha(c) \cap D| \geq 2$. Then $\dim_\alpha A_\alpha(c) \cap D \geq 1$. $A_\alpha(c) \cap D$ is non-empty, non totally disconnected, and non linearly connected. Then its dimension satisfies $\dim_\alpha(A_\alpha(c) \cap D) = \max_{p \in A_\alpha(c) \cap D} \dim(B_\alpha^*(p) \cap D) + 1 \geq 2$ - a contradiction.

(b) Let $\alpha = (n-1)$. Assumming that D does not contain an elementary grid square, one proves that D is one-dimensional by analogous arguments as in part

(a). Now let $\dim_\alpha(D) = 1$ and assume by contradiction that D contains as a proper subset an elementary grid square Q. Then there is an n-cell from Q that is $(n-1)$-adjacent to two n-cells from Q and to at least one n-cell of D not belonging to Q. We have $|A_\alpha(c) \cap D| \geq 3$. Hence, $\dim_\alpha A_\alpha(c) \cap D \geq 1$, and $A_\alpha(c) \cap D$ is non-empty, non totally disconnected, and non linearly connected. Then its dimension satisfies $\dim_\alpha(A_\alpha(c) \cap D) = \max_{p \in A_\alpha(c) \cap D} \dim(B^*_\alpha(p) \cap D') + 1$, where $D' = A_\alpha(c) \cap D$. It is easy to see that $\dim(B^*_\alpha(p) \cap D') \geq 1$. Then $\dim_\alpha(A_\alpha(c) \cap D) \geq 2$, which is a contradiction. □

3 Digital Curves and Hypersurfaces

In what follows we consider digital analogs of simple closed curves and of hypersurfaces that separate the space $\mathbb{C}_n^{(n)}$. We will consider analogs of either bounded closed separating hypersurfaces, or unbounded hypersurfaces (such as hyperplanes) that separate $\mathbb{R}^n$. (The latter can also be considered as "closed" in the infinite point.) We will not specify whether we consider closed or unbounded hypersurfaces whenever the definitions and results apply to both cases and no confusions arise. We also omit the word "digital" where possible.

The considerations take place in the n-dimensional space $\mathbb{C}_n^{(n)}$. We allow adjacency relations A_α as defined above. We are interested to establish basic definitions for this space that:

- reflect properties which are analogous to the topological connectivity of curves or hypersurfaces in Euclidean topology,
- reflect the one- or $(n-1)$-dimensionality of a curve or a hypersurface, respectively,
- characterize hypersurfaces with respect to gaps.

A digital curve (hypersurface), considered in the context of an adjacency relation A_α, will be called an α-curve (α-hypersurface).

3.1 Digital Curves

A set $\tau \subset \mathbb{C}_n^{(n)}$ is an α-curve iff it is α-connected and one-dimensional with respect to A_α. (Note that Urysohn-Menger curves in $\mathbb{R}^n$ are defined to be one-dimensional continua.) Figure 1 presents examples and counterexamples of curves in $\mathbb{C}_2^{(2)}$.

In the rest of this section we define and study digital analogs of simple closed curves (i.e., those that have branching index two at any point). The following lemma provides necessary and sufficient conditions for a set of n-cells to be connected and a loop with respect to adjacency relation A_α.

Lemma 2. *Let $\rho = \{c_1, c_2, \ldots, c_l\}$ be a set of n-cells. Then the following are equivalent:*

(A1) *c_i is α-adjacent to c_j iff $i = j \pm 1$(modulo l).*
(A2) *ρ is α-connected and $\forall\ c \in \rho$, $\mathrm{card}(A_\alpha(c) \cap \rho) = 2$.*
(A3) *The α-adjacency graph $G_\alpha(\rho, E)$ is a simple loop.*

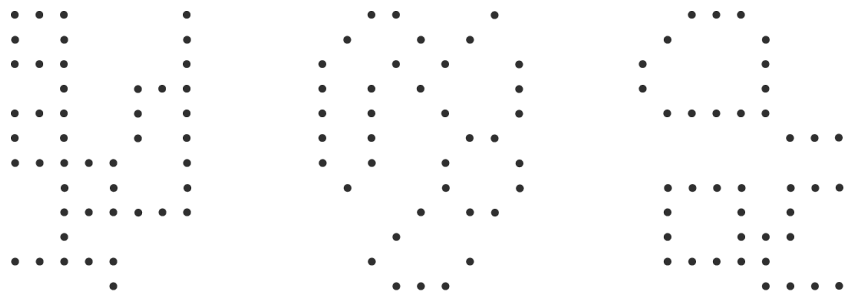

Fig. 1. Examples of a 1-curve (left), 0-curve (middle), and two 0-connected sets in the digital plane that are neither 0- nor 1-curves (right).

The proof of the above lemma is straighforward. Note that each of conditions (A1) and (A3) implies connectivity of ρ, while α-connectivity of ρ is explicitly required in (A2), otherwise ρ may have more than one connected component.

Lemmas 1 and 2 allow us to give the following general definition, summarizing three equivalent ways for defining a simple α-curve.

Definition 1. *A* simple α-curve $(0 \leq \alpha \leq n-1)$ *of length* l *is a set* $\rho = \{c_1, c_2, \ldots, c_l\} \subseteq \mathbb{C}_n^{(n)}$*, that is one-dimensional with respect to* A_α *adjacency and satisfies a property* (Ai) *for some* $1 \leq i \leq 3$.

A simple α-curve will also be called a *one-dimensional* α*-manifold*. A simple α-curve ρ $(0 \leq \alpha < n-1)$ is a *proper* α-curve (or a proper one-dimensional α-manifold) if it is not an $(\alpha+1)$-curve.

Example 1. A proper 0-curve in $\mathbb{C}_2^{(2)}$ is a 0-curve which is not a 1-curve (see Figure 2, left). It follows that any closed 0-curve is a proper 0-curve.

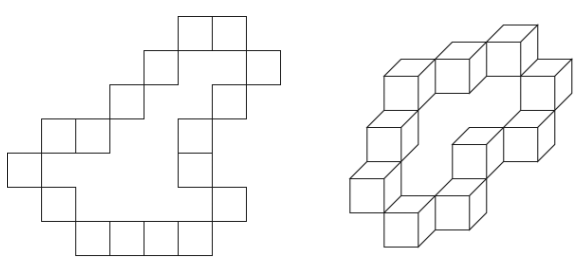

Fig. 2. A proper 0-curve in 2D (left) and an improper 0-curve in 3D (right).

A *proper* 0-curve in $\mathbb{C}_3^{(3)}$ is a 0-curve which is not a 1- or 2-curve, and a *proper* 1-curve is a 1-curve which is not a 2-curve.

Any 1-curve is a proper 1-curve. This follows from the facts that a curve is closed and one-dimensional with respect to 1-adjacency. If we assume the opposite, we would obtain that the curve is either an infinite sequence of voxels (e.g., of the form $(0,0,1),(0,0,2),(0,0,3),\ldots$) or is two-dimensional. However, a closed 0-curve does not need to be proper (see Figure 2, right).

A *simple α-arc* σ is an α-connected proper subset of a simple α-curve. It contains exactly two n-cells c, c' such that $\mathrm{card}(A_\alpha(c) \cap \rho) = \mathrm{card} A_\alpha(c') \cap \rho) = 1$.

Remark 1. The difference between Definition 1 and previous definitions of curves is that here we require a curve to be one-dimensional. See again Figure 1c. It is not a 1-curve since it is not 1-connected although it is 1-dimensional with respect to 1-adjacency. Also, according to Definition 1, it is not a 0-curve since it is 2-dimensional with respect to A_0-adjacency. Without the requirement for 1-dimensionality, that set of points is a curve. Some classical results hold also in the framework of our definition, the latter being a refinement of the earlier. As an example we list for future references the following theorem proved first by A. Rosenfeld [28]:

Theorem 1. *A simple closed 1-curve (0-curve) γ 0-separates (1-separates) all pixels inside γ from all pixels outside γ. More precisely, we have that a simple closed 1-curve has exactly one 0-hole and a simple closed 0-curve has exactly one 1-hole. A simple closed 1-curve 0-separates its 0-hole from the background and a simple closed 0-curve 1-separates its 1-hole from the background.*

In $\mathbb{C}_2^{(2)}$, we have the following characterization of digital curves.

Proposition 1. *A finite set of pixels ρ, that is α-separating in $\mathbb{C}_2^{(2)}$ $(\alpha = 0, 1)$, is a simple α-curve in $\mathbb{C}_2^{(2)}$ iff it is $(1-\alpha)$-minimal in $\mathbb{C}_2^{(2)}$.*

Proof Let ρ be a simple 0-curve (resp. 1-curve) and p an arbitrary element of ρ. Then from Definition 1 and in view of Lemma 1, we have that $\rho \setminus \{p\}$ is not 0-separating (1-separating) in $\mathbb{C}_2^{(2)}$.

Conversely, let ρ be 1-minimal (resp. 0-minimal) in $\mathbb{C}_2^{(2)}$. Then ρ cannot contain an elementary grid triandle (resp. elementary grid square) since otherwise ρ would have a simple point. Hence, by Lemma 1, ρ is a simple curve. $\square$

Note that this last result does not generalize to higher dimensions since a one-dimensional digital object cannot separate $\mathbb{C}_n^{(n)}$ if $n > 2$.

3.2 Digital Hypersurfaces

We consider digital analogs of hole-free hypersurfaces. Accordingly, we are interested in hypersurfaces without $(n-1)$-gaps, although the theory can be extended to cover this case as well. However, in the framework of our approach, a hypersurface with $(n-1)$-gaps can be an $(n-2)$-dimensional set of n-cells, while we want a digital hypersurface to be $(n-1)$-dimensional, in conformity with the continuous case. We give the following recursive definition.

Definition 2. **(i)** *M is a 1-dimensional* $(n-1)$-manifold *in* $\mathbb{C}_n^{(n)}$ *if it is an* $(n-1)$*-curve in* $\mathbb{C}_n^{(n)}$.
M is a k-dimensional $(2 \leq k \leq n-1)$ $(n-1)$-manifold *in* $\mathbb{C}_n^{(n)}$ *if:*
(1) M is $(n-1)$-connected (or, equivalently, M consists of a single $(n-1)$-component);
(2) for any $x \in M$ the set $A_0(x) \cap M$ is a $(k-1)$-dimensional $(n-1)$-manifold in $\mathbb{C}_n^{(n)}$.

(ii) *M is a 1-dimensional* α-manifold $(0 \leq \alpha \leq n-2)$ *in* $\mathbb{C}_n^{(n)}$ *if M is an α-curve in* $\mathbb{C}_n^{(n)}$*;*
M is a k-dimensional α-manifold $(0 \leq k \leq n-1, 0 \leq \alpha \leq n-2)$ *in* $\mathbb{C}_n^{(n)}$ *if:*
(1) M is α-connected (or, equivalently, M consists of a single α-component);
(2) for any $x \in M$ the set $A_0(x) \cap M$ is a $(k-1)$-dimensional α-manifold in $\mathbb{C}_n^{(n)}$ but is not a $(k-1)$-dimensional $(\alpha+1)$-manifold in $\mathbb{C}_n^{(n)}$. (Such an α-manifold will also be called proper.)

In the particular case when M is an $(n-1)$-dimensional α-manifold in $\mathbb{C}_n^{(n)}$ for $\alpha = n-2$ or $n-1$, we say that M is a digital *α-hypersurface*. M is a *proper α-hypersurface* for $\alpha = n-2$ if it is not an $(n-1)$-hypersurface for $\alpha = n-1$. One can observe that an α-hypersurface M is $(n-1)$-dimensional at any n-cell of M with respect to adjacency relation A_α. It is also clear that any proper one-dimensional α-manifold is an α-curve. Note also that if Condition (1) is missing, then M may have more than one connected component. In such a case Condition (2) implies that any connected component of M is an α-hypersurface. Some other points are clarified by the following remarks.

Remark 2. A digital hypersurface cannot have "singularities," which may appear, e.g., in case of a 3D "pinched sphere" or a "strangled torus." In fact, surfaces of that kind would either be non-simple or three-dimensional or both, so they would not satisfy our definition of a surface.

Remark 3. In the definition of an α-hypersurface we use the adjacency set $A_0(x)$ rather than $A_\alpha(x)$ (if $\alpha \neq 0$), since the latter could cause certain incompatibilities. This can be seen in the 3D case: if we use adjacency A_2 to define a 2-surface, $A_2(x) \cap M$ may be a 1-curve rather than a 2-curve. Similarly, if we use adjacency A_1 to define a 1-surface, $A_1(x) \cap M$ may be a 0-curve rather than a 1-curve. This is avoided by using A_0 in all cases.

Remark 4. Indeed, one can consider more general digital hypersurfaces which are not covered by the above definitions. If, for instance, we do not require in Definition 2 the manifold $A_0(x) \cap S$ be proper, we may obtain a "hypersurface" that has subsets of diverse hypersurface types. More general digital hypersurfaces would be just "mixtures" of patches of hypersurfaces of some of the considered types, and their combinatorial study would lose its focus. Note that the considered hypersurfaces feature certain combinatorial properties to be studied at the end of this section.

The digital hypersurfaces defined above have the following properties.

Proposition 2. *(a) An $(n-2)$-hypersurface S in $\mathbb{C}_n^{(n)}$ is $(n-1)$-gapfree, has $(n-2)$-gaps, and is $(n-1)$-minimal.*

(b) An $(n-1)$-hypersurface S in $\mathbb{C}_n^{(n)}$ is gapfree and 0-minimal.

Proof We sketch the proof of part (a), the one of part (b) being similar.

Note that here S is an $(n-2)$-hypersurface defined in the framework of part (ii) of Definition 2. Since the set of n-cells $M_1 = A_0(x) \cap S$, $x \in S$, is a proper $(n-2)$-dimensional $(n-2)$-manifold in $\mathbb{C}_n^{(n)}$, it has $(n-2)$-gaps and no $(n-1)$-gaps. Since this holds for any $x \in S$, it follows that S has $(n-2)$-gaps and no $(n-1)$-gaps too. The above argument about the manifold M_1 and the recursion of Definition 2 also imply that S is $(n-1)$-separating in $\mathbb{C}_n^{(n)}$.

Now assume that S is not $(n-1)$-minimal in $\mathbb{C}_n^{(n)}$, i.e., there is an n-cell x_0 that is an $(n-1)$-simple point of S. Since S is $(n-2)$-connected, any n-cell $x \in S$ is $(n-2)$-adjacent to at least one n-cell from $S \setminus \{x\}$. Let $y_1 \in S \setminus \{x\}$ be an n-cell that is $(n-2)$-adjacent to x_0. By definition, we have that $M_2 = A_0(y_1) \cap S$ is $(n-2)$-dimensional $(n-2)$-manifold, $x_0 \in A_0(y_1) \cap S$, and there is an n-cell $y_2 \in A_0(y_1) \cap S$ that is $(n-2)$-adjacent to x_0 and $x_0 \in A_0(y_2) \cap S$, where $M_3 = A_0(y_2) \cap S$ is an $(n-3)$-dimensional $(n-2)$-manifold. Continuing this process, after a finite number of steps we obtain that there is an n-cell $y_{n-1} \in A_0(y_{n-2}) \cap S$, such that y_{n-1} is $(n-2)$-adjacent to x_0 and $x_0 \in A_0(y_{n-1}) \cap S$, where $M_{n-1} = A_0(y_{n-1}) \cap S$ is a one-dimensional $(n-2)$-manifold. Note that, since S is an $(n-2)$-hypersurface satisfying Definition 2, all manifolds $M_2, M_3, \ldots, M_{n-1}$ are proper. Keeping this last fact in mind, if we now remove the simple point x_0 from S, it will cause occurrence of an $(n-1)$-gap in the one-dimensional $(n-2)$-manifold (i.e., an $(n-2)$-curve) M_{n-1}, and that gap will propagate over all other manifolds $M_{n-2}, M_{n-3}, \ldots, M_1$, and S. In other words, x_0 is not an $(n-1)$-simple point of S - a contradiction. □

This last proposition suggests the following classification of digital surfaces introduced by Definition 2.

There are two and only two basic types of α-hypersurfaces: one for $\alpha = n-1$ and one for $\alpha = n-2$:

For $\alpha = n-2$, a hypersurface S has $(n-2)$-gaps which appear on the $(n-2)$-manifolds that build it and, possibly, between adjacent pairs[4] of such $(n-2)$-manifolds.

For $\alpha = n-1$, the hypersurface S is gapfree.

Important examples of digital surfaces are the *digital hyperplanes*. These are well-studied from various points of view. In particular, digital hyperplanes admit an analytical description. Specifically, a set $P(b, a_1, a_2, \ldots, a_n, \omega) = \{x \in \mathbb{Z}^n | -\frac{\omega}{2} \leq b + \sum_{i=1}^n a_i x_i < \frac{\omega}{2}\}$ is a digital hyperplane with coefficients $a_1, a_2, \ldots, a_n$, and b and *thickness* ω. A digital hyperplane with a thickness $\omega = a_{\max} = \max\{a_1, a_2, \ldots, a_n\}$ is called *naive*, and one with a thickness $\omega = \sum_{i=1}^n a_i$ is called *standard*. We have the following fact.

[4] Actually, two such manifolds, called "adjacent," may have both adjacent and common n-cells.

Proposition 3. *A naive digital plane is an $(n-2)$-surface and a standard digital plane is an $(n-1)$-surface.*

Proof Follows from the well-known fact (see [3]) that a digital plane with $\omega \geq \sum_{i=1}^{n} a_i$ is gapfree and 0-minimal while one with $\omega = a_{\max}$ is $(n-1)$-gapfree and $(n-1)$-minimal. □

These studies are related to earlier ones (see, e.g., [5]) on digital hypersurfaces obtained as digitizations of certain continuous surfaces. Here we introduce the following definition.

Definition 3. *Let Γ be a closed surface in $\mathbb{R}^n$ and $J^+(\Gamma)$ its outer Jordan digitization. Let $\mathcal{D}_k(\Gamma)$ be the family of all subsets of $J^+(\Gamma)$ that are k-minimal for $k = 0$ or $n-1$. We call a set of n-cells $D_k(\Gamma) \in \mathcal{D}_k(\Gamma)$ a* k-digitization *of Γ if the Hausdorff distance $H_d(\Gamma, V(D_k(\Gamma)))$ is minimal over all elements of $\mathcal{D}_k(\Gamma)$.*

We have the following fact.

Proposition 4. *Any $(n-1)$-hypersurface (resp. $(n-2)$-hypersurface) is an $(n-1)$-digitization (resp. 0-digitization) of certain hypersurface $\Gamma \subset \mathbb{R}^n$.*

Proof By Proposition 2 we have that any digital hypersurface S in $\mathbb{C}_n^{(n)}$ is k-minimal for $k = n-1$ if S is an $(n-1)$-hypersurface, or for $k = 0$ if S is an $(n-2)$-hypersurface. Moreover, one can always choose a closed surface Γ such that (i) Γ is completely contained in the polyhedron $P(S)$ obtained as a union of the n-cells of S, and (ii) Γ contains the centers of the n-cells of S. Then S will appear to be a digitization of Γ that satisfies the conditions of Definition 3. □

The digital hyperplanes considered above appear to be hyperplane digitizations (see [6]). It has been shown in [5] that the following holds.

Theorem 2. *A naive (resp. standard) digital plane $P(b, a_1, a_2, \ldots, a_n, \omega)$ is an $(n-1)$-digitization (resp. 0-digitization) of a hyperplane with equation $a_1x_1 + a_2x_2 + \ldots + a_nx_n + b = 0$.*

The structure of k-digitizations can also be studied from a combinatorial point of view. Let E be a finite set and $\mathcal{F}$ a family of subsets of E. Recall that $(E, \mathcal{F})$ is a *matroid*[5] if the following axioms are satisfied:

(1) $\emptyset \in \mathcal{F}$,

(2) If $F_2 \in \mathcal{F}$ and $F_1 \subseteq F_2$, then $F_1 \in \mathcal{F}$,

(3) If $F_1, F_2 \in \mathcal{F}$ and $\mathrm{card}(F_1) < \mathrm{card}(F_2)$, then there is an element $x \in F_2$ such that $F_1 \cup \{x\} \in \mathcal{F}$.

We have the following fact.

[5] For getting acquainted with matroid theory the reader is referred to the monograph by Welsh [35].

Proposition 5. *Given a closed hypersurface Γ, denote by $G_k^m(\Gamma)$ ($0 \leq k \leq n-1$) an arbitrary familly of k-digitizations of Γ of cardinality m together with all subsets of those digitizations. Then $G_k^m(\Gamma)$ is a matroid that we call the* hypersurface digitization matroid.

Proof It is well-known (see, e.g., [35]) that an equivalent definition of a matroid is obtained through substituting Condition (3) by the following:

(3′) All maximal (by inclusion) elements of $\mathcal{F}$ have the same cardinality.

These are called the matroid *bases*. Specifically, all k-digitizations in $G_k^m(\Gamma)$ have the same cardinality m and thus satisfy Condition (3). Moreover, $G_k^m(\Gamma)$ contains all subsets of these bases, i.e., conditions (1) and (2) hold as well. Hence, $G_k^m(\Gamma)$ is a matroid. □

It is well-known that the matroids provide a structural framework for greedy-type algorithms. Thus the above theorem in particular demonstrates the possibility to generate closed digital surfaces using a greedy approach.

4 Good Pairs

As already mentioned, studies on digital surfaces naturally interfere with studies on good pairs of adjacency relations. An important motivation for studying good pairs is seen in the possibility that some results of digital topology may hold uniformly for several pairs of adjacency relations. Thus one could obtain a proof which is valid for all of them by proving a statement just for a single good pair of adjacencies.

4.1 Variations of the Notion "Good Pair"

Different approaches in the literature lead to diverse proposals of good pairs (note: they may be called differently, but address the same basic concept). It seems to be unrealistic to define good pairs in a way to cover all previous studies. Therefore, instead of looking for a universal definition, it might be more reasonable and useful to propose and study a number of definitions related to the fundamental concepts of digital topology. The rest of this section reviews several possible approaches.

Good pairs in terms of strictly normal digital picture spaces have been considered in [20]. In that framework, it is shown that adjacencies (1,0) and (0,1) in 2D, and (2,0), (0,2), (2,1) and (1,2) in 3D define strictly normal digital picture spaces, while (1,1) and (0,0) in 2D and (2,2), (1,1), (0,0), (1,0) and (0,1) in 3D do not.

In [17] good pairs have been defined for 2D as follows: (β_1, β_2) is called a *good pair* in the 2D grid iff (for $(i,k) \in \{(1,2),(2,1)\}$) any simple β_i-curve β_k-separates its (at least one) β_k-holes from the background and any totally β_i-disconnected set cannot β_k-separate any β_k-hole from the background. It follows that (1,0) and (0,1) are good pairs, but (1,1) and (0,0) are not. [17] does not

generalize this definition to higher dimensions, but suggests the use of (α, β)-separators for the case $n = 3$. ($M \subseteq \mathbb{Z}^3$ is called an (α, β)-*separator* iff M is α-connected, M divides $\mathbb{Z}^3 \setminus M$ into (exactly) two β-components, and there exists a $p \in M$ such that $\mathbb{Z}^3 \setminus (M \setminus \{p\}) = (\mathbb{Z}^3 \setminus M) \cup \{p\}$ is β-connected.) (α, β)-and (β, α)-separators exist for $(\alpha, \beta) =$ (0,2),(2,0), (1,2), (2,1), and (1,1). However, there are some difficulties with the case $(\alpha, \beta) = (1, 1)$, as an example from [17] illustrates. Further "strange" examples of separators in $\mathbb{Z}^3$ suggest to refine this notion.

Another approach is based on the following digital variant of the Jordan-Veblen curve theorem (of 2D Euclidean topology) due to A. Rosenfeld [29].

Theorem 3. *If C is the set of points of a simple closed 1-curve (0-curve) and* $\mathrm{card}(C) > 4$ $(\mathrm{card}(C) > 3)$*, then $\overline{C}$ has exactly two 0-components (1-components).*

This theorem defines good pairs of adjacency relations in 2D, as follows. (α, β) *is a* 2D good pair *if for a simple closed α-curve C, $\overline{C}$ has exactly two β-components.* It follows that $(1, 0)$ and $(0, 1)$ are good pairs. It is also easy to see that $(1, 1)$ and $(0, 0)$ are not good pairs.

This above definition can be extended to 3D, as follows: (α, β) *is a* 3D good pair *if for a simple closed α-surface S, $\overline{S}$ has exactly two β-components.* We remark that in view of the definition of an α-surface from Section 3.2, a 0-digital surface would not be a true surface and should not be called "surface" since it would have 2-gaps. In fact, 3D digital surfaces need to be at least 1-connected. Thus (0,2) would not be a good pair in the sense of allowing a theorem about separating surfaces.

Another approach is based on separation through surfaces (see, e.g., [13, 17]). Relying on Theorem 1, we can give the following definition: (α, β) *is called a* 2D good pair *if any simple closed α-curve β-separates its β-holes from the background.*

Clearly, $(1, 0)$ and $(0, 1)$ are good pairs, while (0,0) is not. Note that here also (1,1) is a good pair, as distinct from the case of good pairs defined trough the digital version of the 2D Jordan-Veblen curve theorem.

Let us mention that in a definition from [17] both (α, β) and (β, α) are required to satisfy the conditions of a good pair. To avoid confusion, we suggest to treat this case as a special event: (α, β) *is called a* perfect pair *in 2D if any simple closed α-curve β-separates its β-holes from the background and any simple closed β-curve α-separates its α-holes from the background.*

In what follows we consider good pairs defined by this last approach that seems the most reasonable to the authors.

4.2 Good Pairs for the Space of n-Cells

As already mentioned, we adopt the following definition of a good pair.

Definition 4. (α, β) *is called a* good pair *of adjacency relations in $\mathbb{C}_n^{(n)}$ if any closed α-hypersurface β-separates its β-holes from the background.*

Here α is a label of the hypersurface type in accordance with our hypersurface classification from Section 3.2, while β is an integer representing an adjacency. More precisely, $\alpha = n-1$ or $\alpha = n-2$, and $0 \leq \beta \leq n-1$.

In view of the considerations and results from the previous section we have the following theorem.

Theorem 4. *There are exactly $n+1$ good pairs in the n-dimensional digital space $\mathbb{C}_n^{(n)}$: $(n-1, i)$ for $0 \leq i \leq n-1$, and $(n-2, n-1)$. The only perfect pairs are $(n-2, n-1)$ and $(n-1, n-1)$.*

We illustrate the last theorem for $n = 2$ and $n = 3$. For $n = 2$, the good pairs are $(1, 0)$, $(1, 1)$, and $(0, 1)$. See Figure 3. For $n = 3$, the good pairs are $(2, 0)$, $(2, 1)$, $(2, 2)$, and $(1, 2)$.

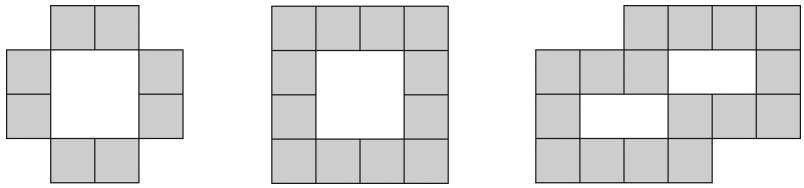

Fig. 3. Illustration to good pairs in 2D: $(0, 1)$ (left), $(1, 1)$ (middle), and $(1, 0)$ (right).

5 Concluding Remarks

In this paper we proposed several equivalent definitions of digital curves and hypersurfaces in arbitrary dimension. The definitions involve properties (such as one-dimensionality of curves and $(n-1)$-dimensionality of hypersurfaces) that characterize them to be digital analogs of definitions for Euclidean spaces. Further research may pursue designing efficient algorithms for recognizing whether a given set of n-cells is a digital curve or hypersurface.

We also proposed a uniform approach to studying good pairs defined by separation and, in that framework, obtained a classification of good pairs in arbitrary dimension. A future task is seen in extending the obtained results under other reasonable definitions of good pairs.

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