B-problem

Akira Nakamura[†], Kenichi Morita[‡] and Katsunobu Imai[‡]

[†]*Hiroshima University*[*]*, Japan*

[‡]*Hiroshima University, Graduate School of Engineering, Japan*

Abstract

Let us consider a 3D image $I = (Z^3, 26, 6, B)$, where B is well-composed and does not contain any cavity or any tunnel. This paper proves that B is SD-equivalent to a single voxel.

1 Introduction

In [8], the first author proposed the following problem of digital topology: Let us consider a 3D image $I = (Z^3, 26, 6, B)$, where B is well-composed and does not contain any cavity or any tunnel. Is B SD-equivalent to a single voxel? This problem was named "B-problem" by the late Prof. A. Rosenfeld. This paper is to give a positive solution of this problem. A key idea of the proof is to use a "magnification technique" that was defined in [6], [8] and [9]. This technique is based on simple deformation (abbreviated to SD). The details will be described in the later sections. Furthermore, another open question called the "animal problem" is fairly well known. Let us call here the animal problem "A-problem". An *animal* — according to Pách — is any topological 3-ball in R^3, consisting of unit cubes. The A-problem is whether every animal can be reduced to a single unit cube by a finite sequence of moves each consisting of either adding or deleting a cube provided the

[*]Professor Emeritus of Hiroshima University

result is an animal at each move. It is well known that this problem is extremely hard. In B-problem, we can use SD (i.e., deformation based on adding or deleting a simple voxel), but in A-problem we cannot use SD. In other words, SD is a deformation based on the meaning of "homotopy", but the deformation used in the A-problem must be "homeomorphic". However, the technique used in B-problem seems to be very useful to solve A-problem. The details will be described in the later sections. It is assumed that readers are familiar with the basic definitions in digital topology that have been given in [2] and [4].

2 Preliminary concepts

Let us consider a 3D image $I = (Z^3, 26, 6, B)$. That is, we consider the set of lattice points in 3D space that are occupied by 1 and 0; and B is a non-empty finite set of 1's, and we use 26-*neighboring relation* for 1's and 6-*neighboring* for 0's. The definition of "well-composedness" of 3D pictures was given in [5]. By making use of the magnification technique, we can SD-deform a general 3D picture to a well-composed one. This was shown in [6]. Therefore, we assume that a 3D picture B in this paper is well-composed. Now, let us consider an image $I = (Z^3, 26, 6, B)$, where B is well-composed and B does not contain any cavity or any tunnel. We here introduce a method of associating a real solid [B] that is a polyhedron of B. We center a "black" unit cube at each 1 of B. Then, the obtained polyhedron is denoted by $[B]$. This polyhedron corresponds to Kong's $\cup I$ of [3]. Let us consider the surface of $[B]$. This surface is a topological 2D sphere. This has been proved by making use of (well-known) classification problem of the 2D real closed surfaces. Note that $[B]$ is in 3D space and it is well-composed and does not have any cavity or any tunnel. We denote this surface by S^2. Hence, $[B]$ has the animal property. (This may need a rigorous proof. But, this follows from the following result that is essentially due to Alexander: On the subdivision of 3-space by a polyhedron, Proc. Nat. Acad. Sci. USA 10 (1924), 6-8.)

The definitions of simple pixels and simple voxels are well-known. (See [2], [6],

[9].) Two images are said to differ by *simple deformation* if one can be obtained from the other by repeatedly changing simple voxel from 1 to 0 or vice versa. For brevity, we will refer to this process as SD. It is obvious that SD is an equivalence relation between two images, so that we use a word "SD-equivalent". We will prove that B is SD-equivalent to a single voxel.

Here, we give a magnification technique that is introduced in [6] and [9]. We have defined this technique for 2D case in [9] and for 3D case in [6]. For the 3D case, it is easily obtained from the 2D case, so that we explain the 2D case. Let the portion of I that contains 1's have n rows, which we number $1, \cdots, n$ starting from the bottom row; the row of 0's below the bottom row is numbered 0. Each column of I consists of alternating runs of 1's (if any) and 0's; a run whose uppermost pixel is in row i will be denoted by r_i . Let $h(i) = i(t-1)$ for each i of $1, \cdots, n$, where t is a larger integer than 1. We dilate each r_i upward by the amount $h(i)$. A run r_i will be dilated upward only after the runs r_j, for all $j > i$, have been dilated upward. For each i, the runs r_i of 1's will all be dilated first, in any order, and then the runs r_i of 0's will be dilated, in any order. In [9] and [6], it has been prove that any such sequence of upward dilations involves only changes in the values of simple pixels or simple voxels. In this case, we say that I is *upward magnified by an amount* t.

We assume that B has been magnified to the z-direction, the x-direction, and the y-direction by a sufficiently large amount. More exactly speaking, we first magnify B to z-direction, and then the new image to the x-direction, and then the new image to the y-direction.

Let V be a set of voxels and $c(V)$ its complement. The faces of the voxels in V that are shared with voxels $c(V)$ are called *surface pixels* of V. (See [10].) It is obvious that the boundary of the 3-ball $[B]$ is a 2D ball S^2. In this case, S^2 consists of surface pixels and it is isothetic. Further, a surface pixel is called a *top pixel* if its upper voxel (i.e., unit cube) is white and its lower voxel (i.e., unit cube) is black, and a surface pixel is called a *roof pixel* if its upper voxel (i.e., unit cube) is black and its lower voxel (i.e., unit cube) is white.

Since S^2 was the surface of $[B]$, it consists of surface pixels. These surface pixels are horizontal or vertical. We are interested in horizontal surface pixels. Two surface pixels are called *edge-adjacent* if they have a common edge. Two surface pixels are called *edge-connected* if there exists a sequence of surface pixels $P = P_0, P_1, \cdots, P_m = Q$ such that P_i is edge-adjacent to $P_{i-1}, 1 \leq i \leq m$.

Let H be the maximal set of horizontal surface pixels of S^2 that satisfies the following conditions:

(h_1) H does not contain any vertical surface pixel,

(h_2) Any two surface pixels of H are edge-connected each other.

Such a H of horizontal surface pixels is called a *horizontal surface component*. In this paper, the horizontal surface components of S^2 plays a very important role. Here, we can assume that there are no horizontal surface components such that they have the same z-coordinate. The reason is as follows: If H_1 and H_2 are two horizontal surface components such that they have the same z-coordinate, then we can lift-up one of them. Since we are considering B that is magnified by a sufficiently large amount, this is possible by SD.

3 Claims and theorem

In this section, we prove the main theorem. To do this, we first explain the exact meaning of cavity and tunnel. Let us consider an $I = (Z^3, 26, 6, B)$, where B is only one 26-component of 1's. The complement $c(B)$ of B may have more than one 6-component of 0's. Only one of the compoenets of 0's can be infinite; this component is called the *background* of B, and the others, if any, are called *cavities* of B. On the other hand, it is quite hard to define a "tunnel". We do not define the tunnel, but define the number of tunnels in I. It is well-known that the *Euler characteristic* can be defined in terms of the number of certain simple patterns of 1's in I. We can then define the *number of tunnels* in I as the number of finite components of 1's and 0's minus the Euler characteristic. In the following theorem, we consider an $I = (Z^3, 26, 6, B)$, where B is only one 26-component and has no cavity and its Euler characteristic is 1. This means that the number of tunnels in I is 0, so that

B has no tunnel.

Theorem 3.1 Let us consider an $I = (Z^3, 26, 6, B)$, where B is well-composed and does not contain any cavity or any tunnel. Then, B is SD-equivalent to a single voxel.

Note that there is a case where we get a single voxel from B by "successive deletions only". This is a easy case. But, this method is not used for Bing's house. (As for Bing's house, see [1].) We propose here a general technique that can be applied to Bing's hous-like B. To prove this theorem, let us consider several claims. A horizontal surface component has some borders. In general, a vertical surface pixel stands upward at an local part of a border L, and another vertical surface pixel falls downward at another local part of L. Such a L is called a *mixed border.* But, we can assume that there are no mixed borders in any horizontal surface component of $[B]$.

Claim 3.1 We can assume that there are no mixed borders in any horizontal surface component of $[B]$.

Proof. First of all, note that the image $I = (Z^3, 26, 6, B)$ has been magnified to the x-direction, the y-direction, and the z-direction by a sufficiently large amount. Assume that L is a mixed border of a horizontal surface component H. We successively add unit cubes to $[B]$ so that H is enlarged to a horizontal surface component H'. This is done by SD. L can be also changed to L' that is an upward border or a downward border. □

Note that the new B' obtained in this claim is also well-composed and does not contain any cavity or any tunnel. Hence, the new B' is also written as B. Here, we define critical components (CCs) of horizontal surface components. That is, let F be a horizontal surface component.

(a.1) If F has exactly two borders such that one is downward and another is upward, then F is called an *(a.1) non-CC.*

(a.2) If F has exactly two borders such that both borders are downward or both borders are upward, then F is called an *(a.2) non-CC.*

An (a.1) non-CC and an (a.2) non-CC are called a *non-CC.*

(b) If F has exactly one downward border, then F is called a *CC of type 2.*

(c) If F has exactly one upward border, then F is called a *CC of type 0.*

(d) If F is not a non-CC such that F is not of type 2 or not of type 0, then F is called a *CC of type 1.*

Fig. 1 shows (a.1), (a.2), (b), (c), (d). (In figures of this paper, curves are actually isothetic.)

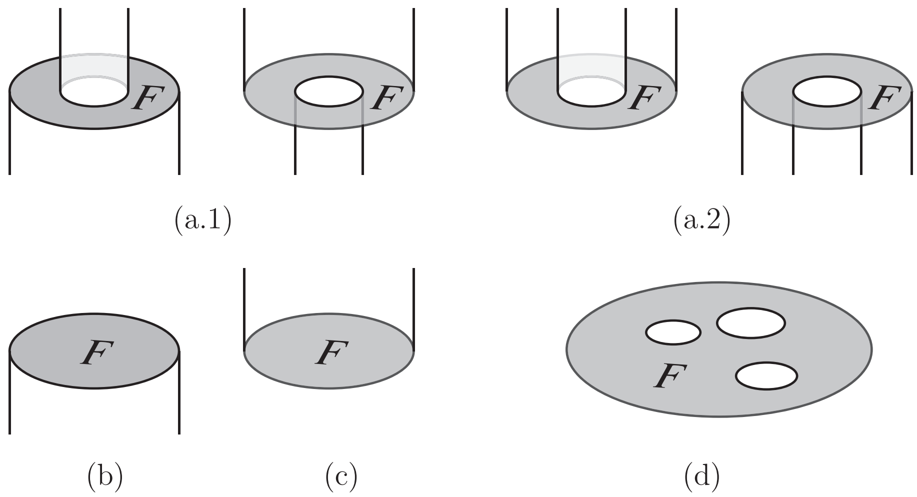


Figure 1: Critical components and non-critical components

Let us consider a border L of an arbitrary horizontal surface component. B was well-composed and $[B]$ was a 3-ball. Hence, it is obvious that L separates S^2 (i.e., the surface of $[B]$) into two parts. Let us denote these two parts by Γ_1 and Γ_2. Obviously, Γ_1 and Γ_2 are 2D balls. $\Gamma_i (i = 1, 2)$ is called a *protuberance from L.*

Let us consider the following travel on S^2 that starts from a (real) point on a horizontal component F_0.

(i) We go horizontally on F_0 and reach a border (say, M_0) of F_0.

(ii) We go upward (or downward) from M_0. Then, we reach a horizontal plane (say, F_1) at its border (i.e., a border of F_1). Let this border be M_1. In this case, F_1 is called the *next horizontal surface* of F_0 .

(iii) From this border M_1, we go horizontally on F_1. Then, we reach another border M_1' (if exists) of F_1.

(iv) We go upward (or downward) from the new border M_1' of (iii). Then, we reach another horizontal plane (say, F_2) at its border (i.e., a border of F_2). In this case, F_2 is also called the next horizontal surface of F_1.

We continue this travel. This travel stops only when we reach a CC of type 0 or type 2. Because a CC of type 0 or 2 has exactly one border. Since there are no mixed borders in S^2, this travel is well-defined. This travel is called a *surface travel from* F_0 . In particular, if all horizontal surface components (i.e., $F_0, F_1, F_2, \cdots$) except the last one have exactly two borders, then the surface travel is "uniquely" determined.

Claim 3.2 There is at least one CC of type 0 or 2 in S^2.

Proof. Assume that every horizontal surface is a non-CC or of type 1. Then, we consider the surface travel from a horizontal surface component on S^2. From this assumption, there must exist an infinite sequence $F_0, F_1, F_2, \cdots$, where F_{i+1} is the next horizontal surface of F_i. Since B is finite, this sequence must contain a cycle. Let us consider the shortest cycle among them. Let $F_i, M_i, F_{i+1}, M_{i+1}, \cdots, F_{i+s}, M_{i+s},\ F_i$ be the shortest one. Here, M_i means a border of F_i as in the above definition. Since M_i is a Jordan curve, it divides S^2 into two region Γ_1 and Γ_2 that intersect only at M_i. F_i is in Γ_1 and F_{i+1} is in Γ_2, and any path on S^2 from F_i to F_{i+1} must intersect M_i. If none of $M_{i+1}, \cdots, F_{i+s}, M_{i+s}$ can intersect M_i, this is a contradiction. If any of them intersected M_i there would be another shorter cycle from F_i to back to F_i. Contradiction. This means that there are no cycles in our surface travel, so that there is at least one CC of type 0 or 2 in S^2. $\square$

By the similar argument, it is known that there is at least one CC of type 0 or 2 in a protuberance from L.

Let us denote the "number of horizontal surface components" in S^2 by $\#(S^2)$. There is no B such that $\#(S^2) = 1$. Because if $\#(S^2) = 1$, we cannot have a 3D image B. Our proof strategy of Theorem 3.1 is to use induction on $\#(S^2)$. In other words, we will show that $\#(S^2)$ is reduced by SD. Therefore, we must first prove the following claim:

Claim 3.3 If $\#(S^2) = 2$, B is SD-equivalent to a single black voxel.

Proof. Let us consider a CC F_0 of type 2 whose existence is guaranteed by Claim 3.2. For the case where F_0 is of type 0, the argument is similar. We start the surface travel starting from F_0. We reach the horizontal surface component F_1 (=the next horizontal surface of F_0) at border L_1. Then, this F_1 must be a CC of type 0 since $\#(S^2) = 2$. Therefore, we can successively SD-remove all voxels of B from the uppermost level. Then, we get eventually a 2D image at the bottom level of B. This 2D image is simply connected. Because B has no cavities and no tunnel. Thus, we have this claim by Proposition 2.5.4 of the book [11]. □

Now, let us consider a CC (say, C_0) of type 0 or 2 in S^2. The existence of C_0 is guaranteed by Claim 3.2. Assume that C_0 is of type 2. For type 0, the argument is similar. We consider the surface travel from C_0. Let C_1 be the next surface component of C_0. Then, there are the following two case (shown in Fig. 2(a) and (b)) of C_1. (Each of Figs. 2(a) and (b) shows a cross-section by a vertical plane. Since there are no mixed borders in any horizontal surface component of S^2, we can explain our argument by making use of such figures.)

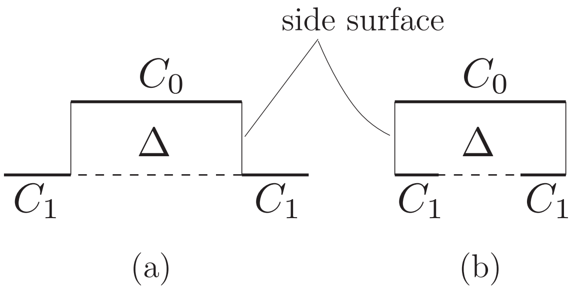


Figure 2: Open type and closed type

Fig. 2(a) shows that the outer border of C_0 is an inner border of C_1. This case is called an *open type*. Fig. 2(b) shows that the outer border of C_0 is the outer border of C_1. This case is called a *closed type*. Let H be a horizontal plane containing C_1. We consider a region (denoted by Δ) that is surrounded by C_0, H, and the vertical side surface between C_0 and C_1. The top of this side surface forms the outer border of C_0. (See Figs. 2(a) and 2(b).)

Claim 3.4 For the open type, we can SD-reduce $\#(S^2)$.

Proof. Assume that Δ consists entirely of white voxels. Then, we can SD-collapse Δ. This is done as follows:

First, we consider the set of 0's on the top level. We can successively SD-remove white voxels from the top level. Then, we have white voxels at the bottom level. Let us this set by Δ'. Since all voxels below the bottom of Δ are white, we can SD-remove the bottom of Δ'. This means that C_0 is absorbed into C_1. Therefore, $\#(S^2)$ is reduced. [For the case where Δ consists entirely black voxels, the argument is similar.] The difficult pattern is a case where Δ contains 1's. (Assume that C_0 consists entirely of "roof pixels".) See Fig. 3(a). (Note that the argument is similar for the case where Δ contains 0's, assuming that C_0 consists entirely of "top pixels".)

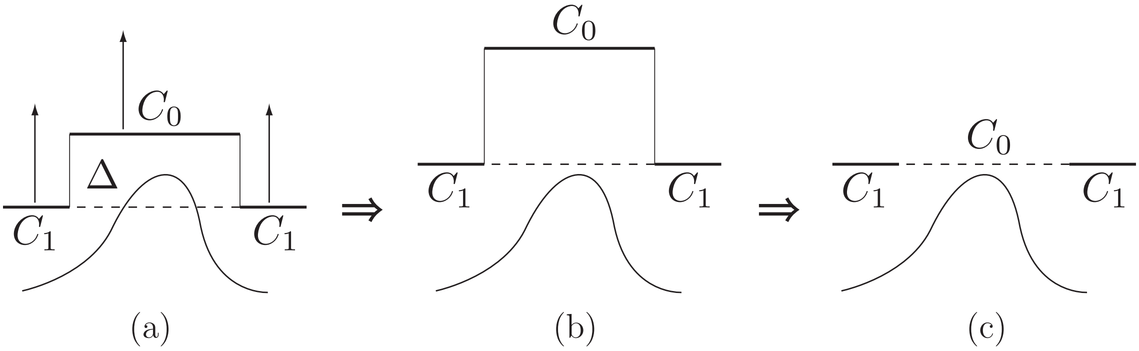


Figure 3: SD-collapsing by making use of partial magnification

For this case, we can use a *partial magnification* explained in Claim 3.5, so that we get Fig. 3(b). Therefore, $\#(S^2)$ is SD-reduced as shown in Fig. 3(c). □

A partial magnification means a magnification for a "subpart" of given picture as described in next Claim 3.5. A concept of a partial magnification is a key idea to prove our main theorem. In Claim 3.5, we consider the 2D case, but it is easily extended the method to the 3D case by considering rectangular parallelepipeds instead of rectangles.

Claim 3.5 Let C be an isothetic rectangle whose leftmost and rightmost columns are both constant. Let A and B be the isothetic rectangular regions just to the left and right of C that contain all the 1's that lie to the left and right of C, and D be the isothetic rectangular region just above A, B, and C that contains all the 1's that lie above A, B, and C. Then, A, B, and D can be magnified upward any desired amount using SD.

(Note that C is not magnified. Thus, we say this Claim 3.5 as a partial magnification method.)

Proof. D can be magnified because SD-magnification works for the rows above any given row that is the top row of A, B, and C. Further A and B can be magnified because the columns of C adjacent to A and B have constant values, and because D has already been magnified. □

Now, let us consider the closed type shown in Fig. 2(b). Before the discussion, let us define a "special protuberance". We have defined a protuberance from a border L (on a horizontal surface component). If this protuberance does not have any CC of type 1, we say this protuberance as a *special protuberance from L*. From this definition, each horizontal surface component of a special protuberance must be of type 0 or 2, or non-CC. Then, there exists at least one special protuberance in S^2. The reason is as follows:

Assume that there is no special protuberance in S^2. Then, all protuberance must be not special, so that each of them has at least one CC of type 1. Then, we can consider again a protuberance (that is different from the old one) from a border of this CC of type 1. We repeat this argument, then the number of protuberances of S^2 must be infinite. Contradiction!

A special protuberance Σ looks like configuration shown in Fig. 4 (in a vertical cross-section). In other words, Σ consists of non-CCs of type (a.1) and/or non-CCs of type (a.2) and/or exactly one CC of type 0 or 2. In this case, Σ must contain exactly one CC of type 0 or 2. This follows from Claim 3.2. Therefore, Σ cannot have an infinitely continuing spiral form as shown in Fig. 5.

We consider the surface travel starting the CC of type 0 or 2. Assume that the CC is of type 2. (For the case where it is of type 0, the argument is similar.) Then, for the closed type, there is one of configurations shown in Fig. 6 (in a vertical

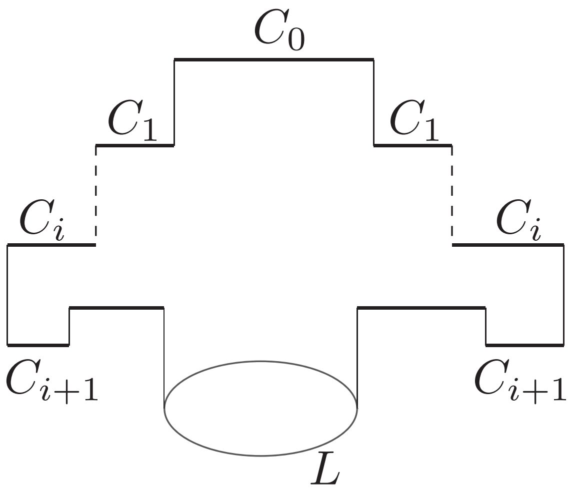


Figure 4: Special protuberance

cross-section) in a special protuberance. (An arrow means the going direction of our surface travel starting from C_0 of type 2.)

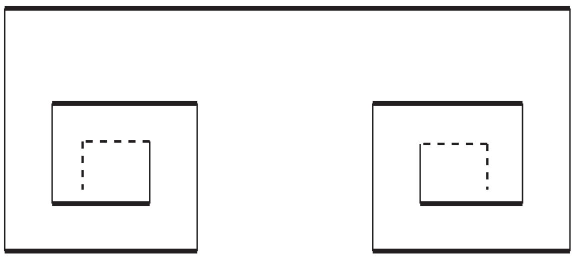


Figure 5: Spiral of horizontal surafces

Δ means a 3D region surrounded by three surfaces of $[B]$ and an open surface shown by a dotted line. The top surface of Δ is an non-CC and the bottom surface of Δ is also non-CC. (If they are not an non-CC, we consider another protuberance from a border on this non-CC. See Fig. 7. But, this argument cannot continue infinitely.)

If this Δ does not contain any 1's (= parts of S^2), Δ is SD-collapsed. Therefore, $\#(S^2)$ is SD-reduced. Even if Δ contains some 1's (that are connected to 1's in the outside of D by passing through the open surface), for (b) and (d) we can SD-turn out these 1's into the outside of Δ by making use of partial magnifications.

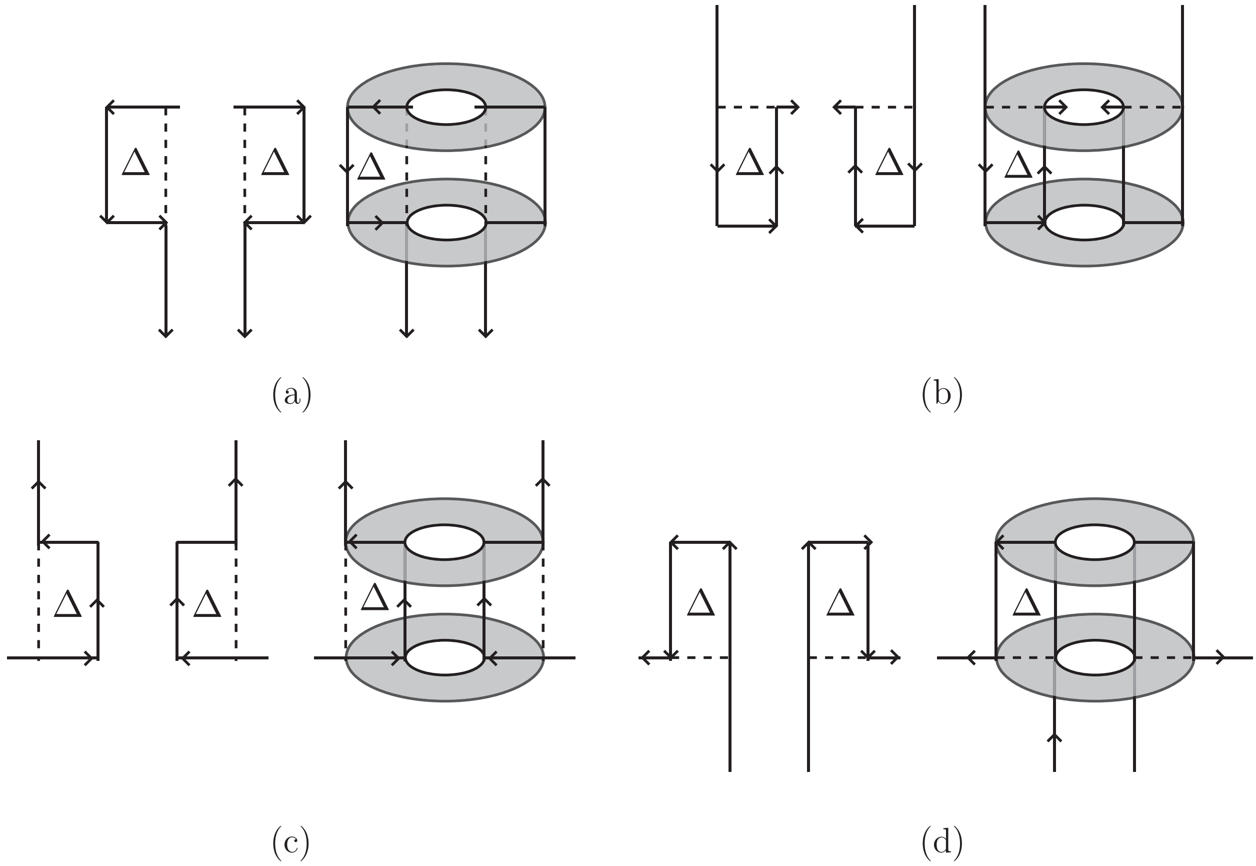


Figure 6: Four cases of surface travel

Therefore, Δ is SD-collapsed, so that $\#(S^2)$ is SD-reduced. Thus, it is enough to give a method how we can SD-turn out these 1's into the outside of Δ for (a) and (c). Here, we introduce a concept of rank of rectangle. The case (a) looks like Fig. 8(a). If we project to a horizontal plane containing C_0 , the situation is shown as Fig. 8(b).

The L is an isothetic simple closed curve, but its shape can be complicated. In general, the interior of L has several convex parts. Let us define such a convex part. First, we go along L clockwise from the left endpoint of the uppermost segment of L If there are two consecutive right turns, L has convex arcs (see Fig. 9). We consider the maximal rectangle G such that G is in the interior of L and G has the convex arcs (=two line segments of L). G is determined successively in such a way that two maximal rectangles do not overlap. This G is called a *convex part of* rank(1) *of L*.

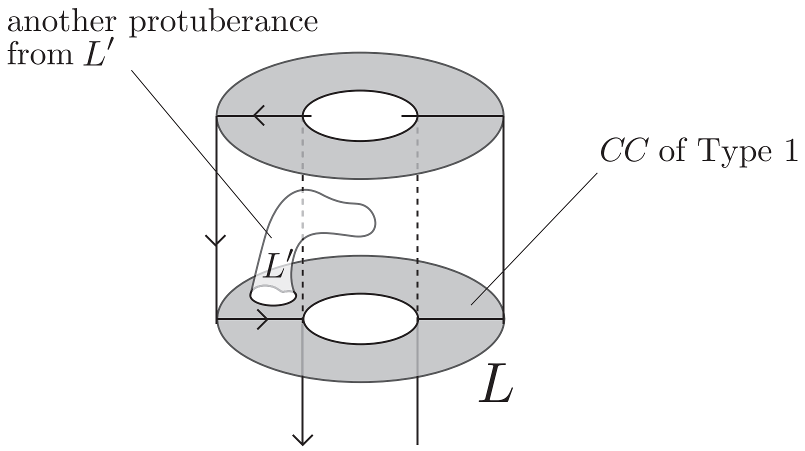


Figure 7: Another protuberance from L'

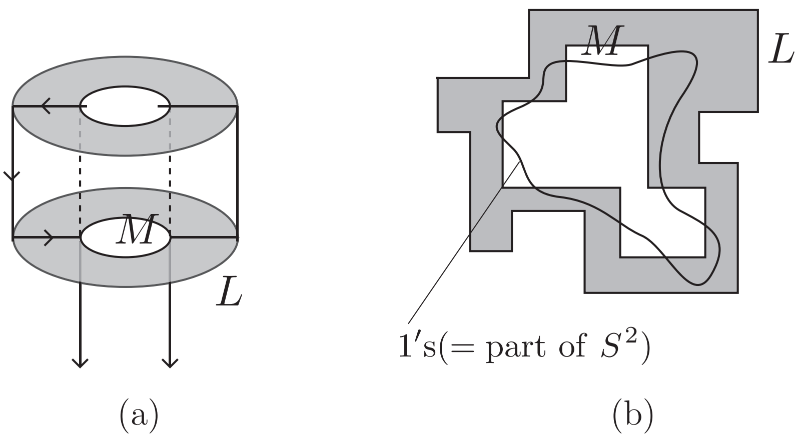


Figure 8: The first case (a) of surface travel and its horizontal projection

Next, we cut all convex parts of rank(1) of L. Then, we get a new isothetic simple closed curve L'. Then, we can define a convex part of rank(1) of L'. This convex part is called a *convex part of* rank(2) *of the original* L. Inductively, we can define a convex part of rank(n) of L. See Fig. 9.

We cut repeatedly all convex parts (of $\text{rank}(1), \text{rank}(2), \cdots, \text{rank}(n)$) of the interior of L. Then, we have a rectangle R (see Fig. 9). Because, if an isothetic simple closed curve is not a rectangle, it has at least one convex part. This R may overlap with the interior of M. But, this is no problem in the following argument. This R is called the *kernel of* L. For the case (a), we have the following claims 3.6

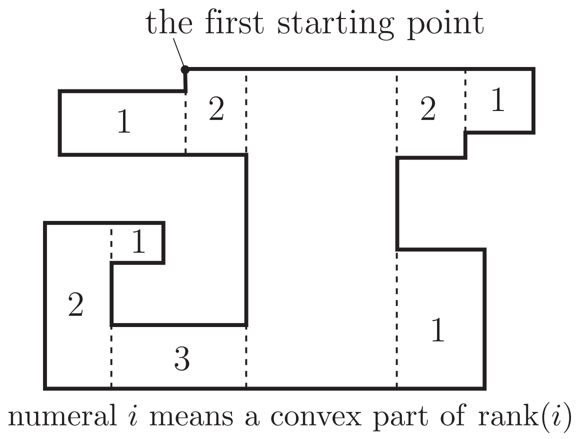


Figure 9: Rank(i)

and 3.7.

Claim 3.6 Assume that the 1's inside Δ do not exist in any convex part. Then, $\#(S^2)$ is SD-reduced.

Proof. The assumption case looks like Fig. 10.

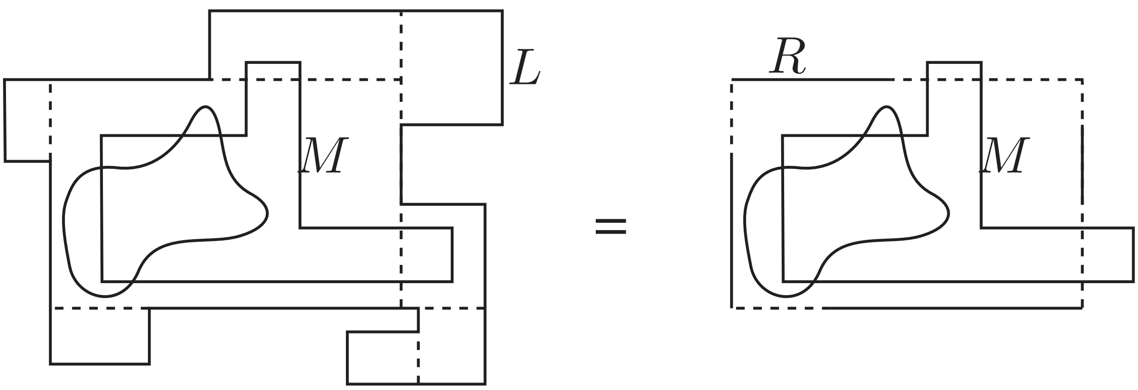


Figure 10: 1's do not exist in any convex part of L

For this case, we deform these 1's into the interior of inner border M by horizontal partial magnifications. We can define the kernel of M by the same way as for L. Let us take a voxel within the kernel of M and we consider this voxel as the "origin". Then, we apply a partial magnification to $+x$ direction and to $-x$ direction., and then to $+y$ directions and to $-y$ direction by a sufficiently large amount in such a way that M is enlarged as shown in Fig. 11. (Note that 1's inside Δ stay where they were.)

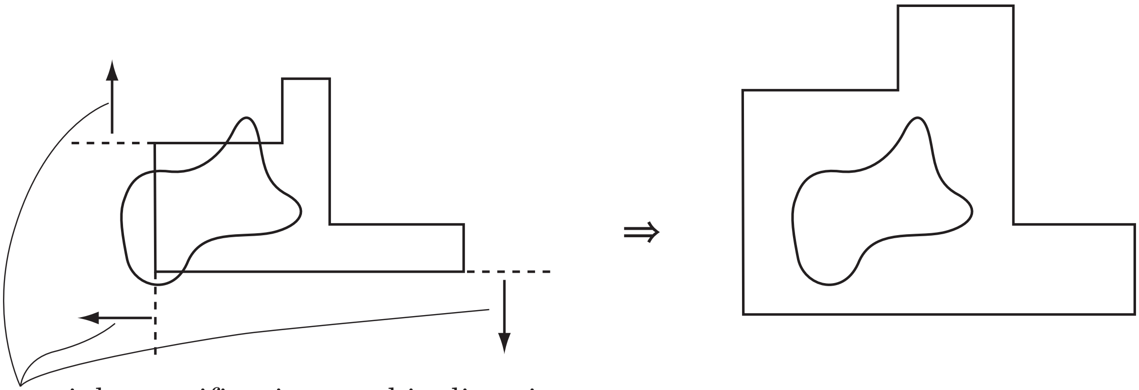


Figure 11: Partial magnification

Therefore, Δ is SD-collapsed, so that C_i disappears. Thus, $\#(S^2)$ is SD-reduced.

□

Note that if the 1's inside Δ exist in some convex part, this procedure does not always work. Because, since these 1's stay where they were, they may interfere this procedure. Hence, it is enough to prove the following claim.

Claim 3.7 Assume that there are 1's in a convex part. Then we deform these 1's into the outside of the convex part by repeated applications of horizontal partial magnifications.

Proof. If the 1's exist in a convex part of rank(1), we can use a partial magnification shown in (i) of Fig. 12. Assume that the 1's exist in a convex part of a rank(k) but there are no 1's in the inside of any convex part of rank($k-1$). Then we can use a horizontal partial magnification shown in (ii) of Fig. 12. Therefore, by induction on rank we can eventually SD-turn out the 1's into the outside of convex part. □

While we are applying a partial magnification, the shape of rectangle R may change, but it preserves the rectangular property.

Next, let us handle the last case (c). Although the discussion of (c) is almost similar to the case (a), there are not many differences. The case (c) looks like Fig. 13.

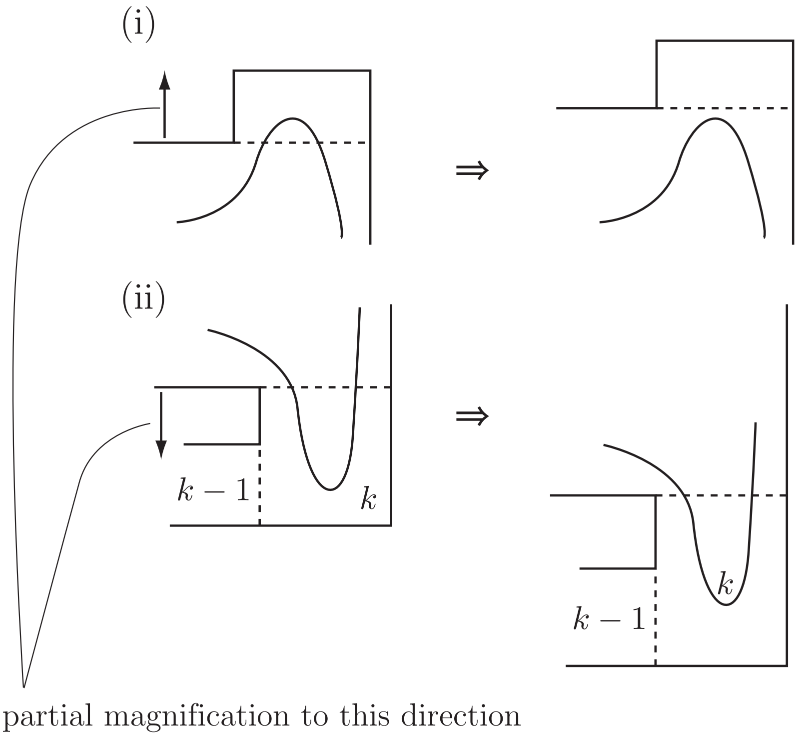


Figure 12: Repeated partial magnifications

Fig. 14 is a projection of Δ to a horizontal plane containing C.

The Δ is a 3D region that is surrounded by the following surfaces.

(1) The top surface C'.

(2) The bottom surface C. C and C' are congruent.

(3) The inner vertical side-surface that below the inner border M of C'.

(4) The outer vertical side-surface that is below the outer border L of C'. This side-surface is open.

If there are no 1's (=parts of S^2) in Δ, Δ is SD-collapsed, so that C' disappears. Hence $\#(S^2)$ is reduced. Therefore, we assume that there are 1's (denoted by T) in the inside of Δ. Note that T is connected to 1's outside Δ by passing through the open outer side-surface. Our purpose is to turn out T into the exterior of L by SD. Let us explain this method. Note that, in (a), the method is to turn out T into the interior of M.) In (a), we have defined a rectangle R from L. But, for (c) we define a convex (in the meaning of Fig. 15(a)) polygon (say, P) from M. The method is

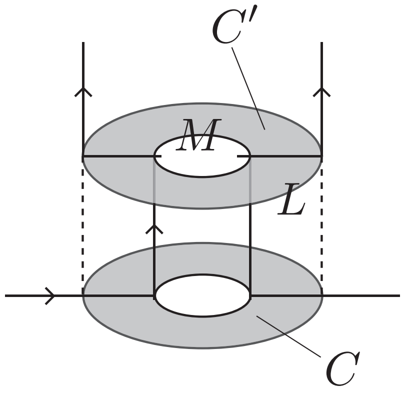


Figure 13: The third case (c) of surface travel

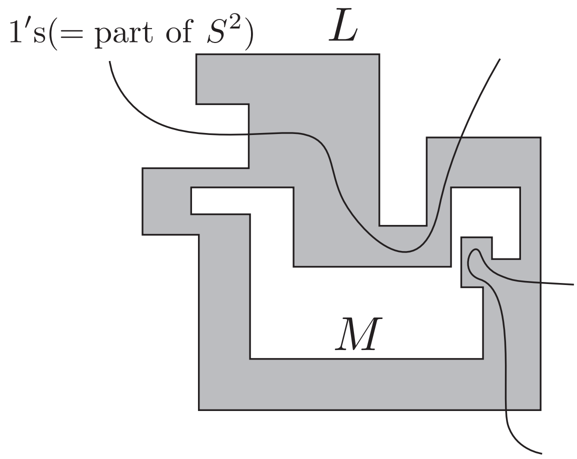
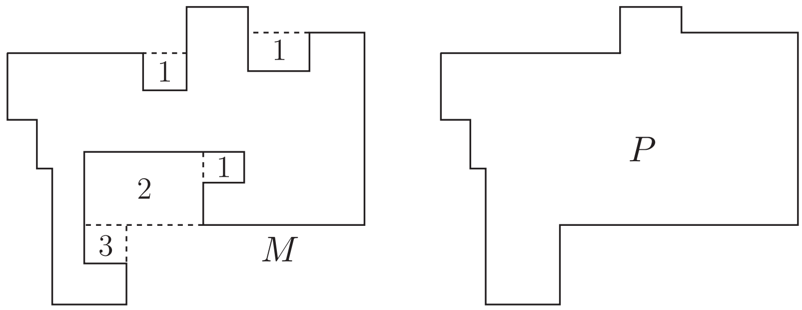


Figure 14: Horizontal projection of the third case (c)

similar to the definition in the case (a). This is as follows: In general, the interior of M has several concave parts. We go along M clockwise from the left endpoint of the uppermost segment of M. If there are two consecutive left turns, M has concave arcs. We consider the maximal rectangle E such that E is in the exterior of M and E has the concave arcs (= two consecutive line segments of M). This E is called a *concave part of* rank(1) *of* M.

By the similar method to case (a), we can inductively define a concave part of rank(n) of M. See Fig. 15(a). We cut repeatedly all concave parts of the interior of M. In other words, we cut successively these concave parts, so that we get



(a) (b)

Figure 15: Rank(i)

an isothetic polygon P. See Fig. 15(b). Note that P has no concave parts. P is considered as an *isothetic convex hull*. P may overlap with the exterior of L, but this is no problem in the following argument. Then, we consider the smallest rectangle R such that R contains P. See Fig. 16.

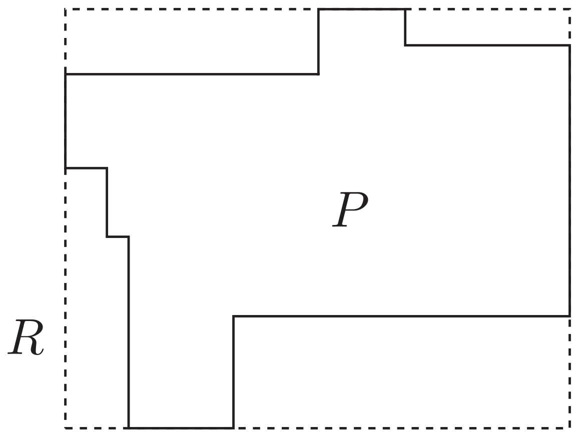


Figure 16: The smallest rectangle R containing P

Now, let us consider a method by which we turn out T inside Δ into the outside P. This is the same as in (a). That is, we repeat horizontal partial magnification as shown in Fig. 17.

Then, we can SD-turn out T to the outside of P. See Fig. 18. In this case, the shape of P changes, but it preserves the convex property.

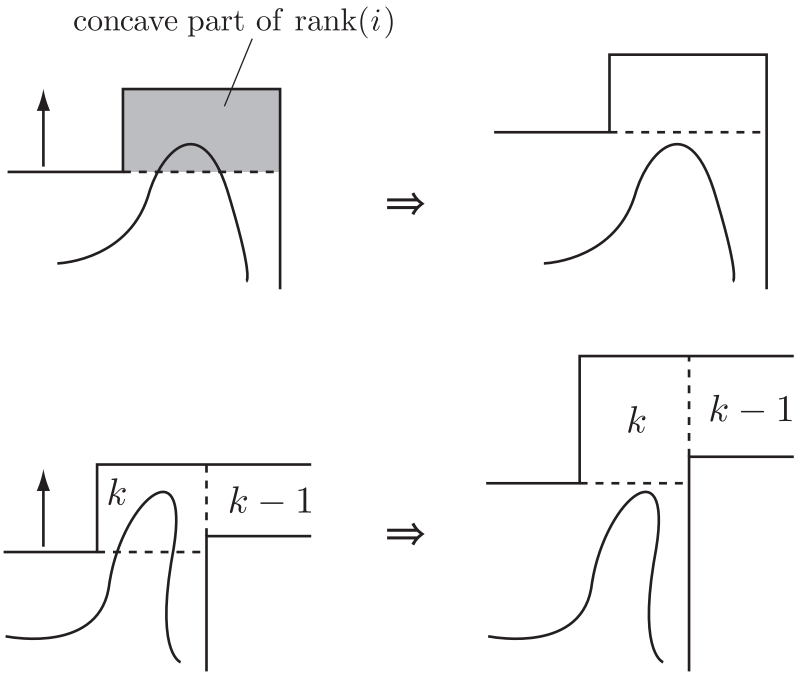


Figure 17: Repeated partial magnifications

Next, we take a pixel $c(o_1, o_2)$ in the interior of P. Here, $c(o_1, o_2)$ means that the coordinate of the center of pixel c (= unit square) is (o_1, o_2). We regard this c as the new origin of the xy-coordinate plane. In this new xy-coordinate plane, we consider the set of pixels $p(x, y)$ such that x is non-negative. This set is denoted by X^+. Similarly, X^-, Y^+, and Y^- are defined. That is, X^- is the set of pixels $q(x, y)$ such that x is non-positive. Then, we apply magnification by a sufficiently large amount in order the following (1), (2), (3), and (4).

(1) The magnification is applied only for X^+ to $+x$ direction.

(2) The magnification is applied only for X^- to $-x$ direction.

(3) The magnification is applied only for Y^+ to $+y$ direction.

(4) The magnification is applied only for Y^- to $-y$ direction.

Then, we have the configuration shown in Fig. 19.

In this time, L, T, and Δ are magnified to L', T', and Δ', respectively. Note that T' is in the exterior of the old L. After that, we partially demagnify Δ' and L', in such a way that L' changes to L, where the magnified T' stays where they were. This

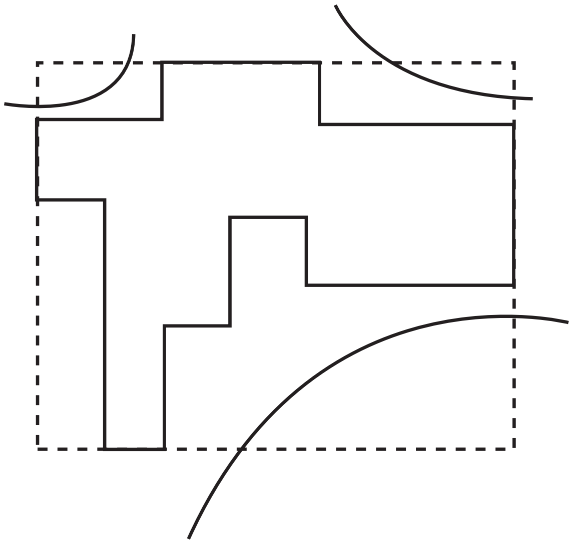


Figure 18: Position of T

is done by SD, because T passes through the open side surface, so that any pixel of T never touch the top surface ($= C'$) and never touch the bottom surface ($= C$) and also T was outside of P. In more detailed explanation, the method is as follows: Let the x-coordinate of top row of the magnified rectangle R' be x_1. We "partially demagnify" a region between o_1 and x_1 for X^+, where the demagnification is not applied for T'. For other X^-, Y^+, and Y^-, we use the similar demagnification. Note that an upward magnification was, row by row, done from the top row, so that its demagnification is, row by row, done from the bottom row. (Of course, we consider the 3D version of this magnification and demaginification.) As the result, T' is in the exterior of the old L. This means that we have SD-turned out T into the outside of Δ.

Claim 3.8 For the case (c), we can SD-turn out 1's inside Δ into the outside of the Δ.

Proof. The method was explained in the explanation before Claim 3.8. □

Therefore, it is known that $\#(S^2)$ is reduced by SD. Thus, we have Theorem 1 by induction on $\#(S^2)$.

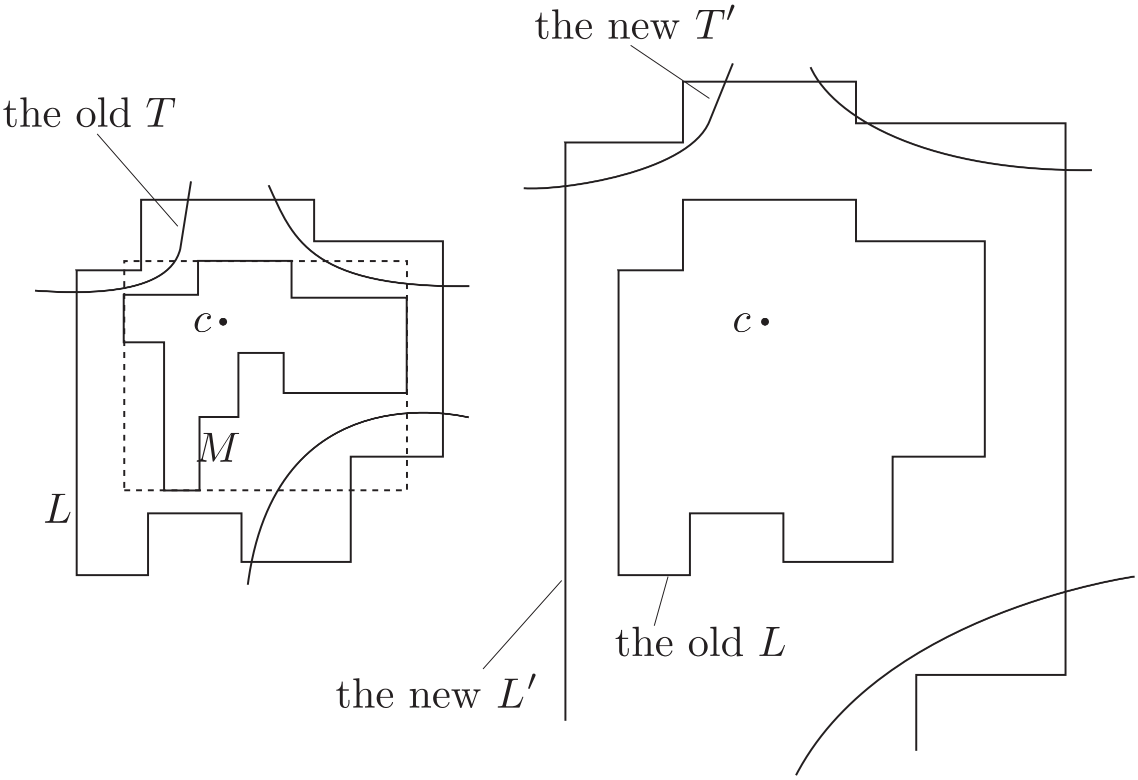


Figure 19: Partial magnification toward each direction and the position of T'

4 Conclusion

We have given a positive solution of the B-problem. As mentioned in the introduction, the A-problem (i.e., animal problem) is still open. In the B-problem, we can use SD so that we can formulate the magnification of a given picture. However, in the A-problem we can use only deformation based on animality-preserving (abbreviated to AD). We do not know whether magnification is possible in AD. At page 274 of [8], we gave a 3D picture in which the upward magnification is impossible. But, the magnification toward x-direction for this example is possible. If AD-magnification (i.e., magnification based on AD) is always possible for an arbitrary picture, the animal problem will be solved. This is a further interesting problem.

Acknowledgements: The first author expresses his appreciation to the late Professor A. Rosenfeld who passed away on 22 Feb. 2004 while he was working with Prof. Rosenfeld. This paper has come from our discussion over a long period.

References

[1] R. H. Bing: Some aspects of the topology of 3-manifolds related to the Poincaré conjecture, *Lectures on Modern Mathematics II*, edited by T.L. Saaty, John Wiley & Sons, Inc., New York (1964), 93-128.

[2] R. Klette and A. Rosenfeld: *Digital Geometry*, Motgan Kaufmann Publ., San Fransisco, (2004).

[3] T.Y. Kong: Topology-preserving deletions of 1's from 2-, 3- and 4-dimensional binary images, *Lecture Notes in Computer Science*, 1347 (1997), 3-18.

[4] T.Y. Kong and A. Rosenfeld: Digital topology: Introduction and survey, *Comput. Vision Graphics Image Process*, 48, (1989), 357-393.

[5] L. Latecki: 3D well-composed sets, *Graphical Models and Image Processing*, 59, (1998), 164-172.

[6] A. Nakamura and A. Rosenfeld: Digital knots, *Pattern Recognition*, 33, (2000), 1541-1553.

[7] A. Nakamura and A. Rosenfeld: Topology-preserving deformations of fuzzy digital pictures, *Fuzzy Techniques in Image Processing* (edited by E.E. Kerre and M. Nachtegael), Physica-Verlag, Heidelberg (2000), 394-404.

[8] A. Nakamura: Magnification in digital topology, *Lecture Notes in Computer Science*, 3322, (2004), 260-275.

[9] A. Rosenfeld, T.Y. Kong, and A. Nakamura: Topology-preserving deformations of two-valued digital pictures, *Graphical Models and Image Processing*, 60, (1998), 24-34.

[10] A. Rosenfeld, T.Y. Kong, and A.Y. Wu: Digital surfaces, *CVGIP: Graphical Models and Image Processing*, 53, (1991), 305-312.

[11] A. Rosenfeld: *Picture Languages*, Academic Press, New York (1979).