

Discrete Volume Polyhedrization is Strongly NP-Hard

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Abstract. Given a set $M \subset \mathbb{Z}^3$, an enclosing polyhedron for M is any polyhedron P such that the set of integer points contained in P is precisely M . Representing a discrete volume by enclosing polyhedron is a fundamental problem in visualization. In this paper we propose the first proof of the long-standing conjecture that the problem of finding an enclosing polyhedron with a minimal number of 2-facets is strongly NP-hard and provide a lower bound for that number.

Keywords: *discrete volume polyhedrization, polyhedron decomposition, strong NP-hardness, polyhedral complexity*

1 Introduction

This paper deals with a problem usually known as “discrete volume polyhedrization” and further abbreviated DVP. In its most general form, it is the following:

Given a set $M \subset \mathbb{Z}^3$, we look for a (possibly, non-convex) polyhedron P , such that the set $P_{\mathbb{Z}}$ of integer points contained in P is precisely M .¹ Such a polyhedron will be called *enclosing* for M . Usually, the 2-dimensional facets of P (2-facets, for short) are required to be convex polygons, as two adjacent polygons may be co-planar. Their number $f_2(P)$ is desired to be as small as possible. Define *polyhedral complexity* of M as $pc(M) = \min_P \{f_2(P) : P \text{ is enclosing for } M\}$. If $f_2(P) = pc(M)$, the polyhedron P will be called *minimally enclosing* for M .

Our goal is to study the computational complexity of finding minimally enclosing polyhedron for a given integer set M , as well as to estimate the polyhedral complexity of such a set.

1.1 Motivation

The main motivation for the above problem comes from medical imaging and other visualization problems, where discrete volumes of voxels result from scanning and MRI techniques. Since digital medical images involve a huge number of points, it is quite problematic to apply traditional rendering or texture algorithms to obtain satisfactory visualization. Moreover, one can face difficulties

¹ In practice, M is often obtained through “digitization” of some (usually unknown) set $S \subset \mathbb{R}^3$ of full dimension.

in storing or transmitting data of that size. There are multiple sources of data being transmitted for many diverse uses, e.g., telemedicine, tele-maintenance, mine detection, ATR, visual display, cueing, and others. In all these applications the coding compression methodology used is paramount. For this, one can try to transform a discrete data set to a polyhedron P , such that the number of its 2-facets is as small as possible. Such polyhedrizations are also searched for the purposes of geometric approximation of surfaces and surface and area estimation.

1.2 Related Work and Open Questions

In recent years DVP is attracting an increasing interest, mainly driven by its rich set of applications. A substantial body of literature impossible to report here has been developed on the subject considered from diverse points of view. For instance, a lot of related work is available in any volume of ACM SIGGRAPH series. The most widely used algorithm is the Marching cube method [13] that generates a triangulated polyhedral surface in which small triangles model local configurations of voxels. The shortcoming here is that the number of triangular facets in the surface may be comparable with the number of points in M . Moreover, the obtained polyhedron is not always hole-free. For other related studies see, e.g., [2, 3, 6, 7, 9–11, 14–20]. The intrinsic complexity of the problem explains the overwhelming usage of greedy algorithms, that, as a rule, are not accompanied with rigorous performance estimation. More importantly, the question about the computational complexity of DVP is still open. The lack of rigorous theoretical results on the problem is somewhat surprising to us, in view of its forcible practical importance.

DVP is germane with some well-known problems in theory of lattice polytopes. For instance, tight bounds exist for the number of the vertices, edges, and facets of a lattice polytope obtained as a convex hull of the integer points in a large ball (or convex body satisfying certain conditions) [1]. We rely on this and other results in obtaining ours. It is not hard to realize (see Section 4) that if in the DVP formulation M is an arbitrary set then a bound $PC(M) = \Theta(|M|)$ holds where $|M|$ is the cardinality of M . The problem becomes nontrivial when $M = S_{\mathbb{Z}}$ for some convex body S , that is the case for which we consider it.

1.3 Results

The main results of the paper are the following.

1. The first proof of the long-standing conjecture that the optimization version of DVP is strongly NP-hard (Section 3). The demonstration is based on the concept of “well-sized lattice polygons” that feature certain optimal properties. This theory, briefly sketched in Section 2, may be of certain independent interest.
2. A lower bound on the polyhedral complexity of a given set $M = S_{\mathbb{Z}} \subset \mathbb{Z}^n$ where S is a large (hyper)ball (Section 4). This result implies a bound on the performance of the convex hull algorithm applied to DVP.

1.4 Some Notations

Let S be a *body* in \mathbb{R}^n , i.e., a subset of \mathbb{R}^n of full dimension $\dim(S) = n$. By $S_{\mathbb{Z}} = S \cap \mathbb{Z}^n$ we denote its *Gaus discretization* defined as the set of all integer points that belong to S . For a set $A \subseteq \mathbb{R}^n$, by $\text{diam}(A)$ we denote its *diameter* defined as $\text{diam}(A) = \max_{x,y \in A} \|x - y\|$, where $\|\cdot\|$ is the Euclidean norm. By kA we denote its homothetic image under a homothety with center $\mathbf{0}$ and a constant of proportionality $k \in \mathbb{R}_+$. Given two sets $A, B \subset \mathbb{R}^n$, by $\rho(A, B)$ we denote the Euclidean distance between them. By $\text{conv}(A)$ we denote the convex hull of A . Given a polyhedron $P \subset \mathbb{R}^n$, the number of its i -facets is denoted by $f_i(P)$, $0 \leq i \leq n$.

2 Well-Sized Lattice Polygons

Let $P \subset \mathbb{R}^2$ be a polygon. By $\nu(P)$ we denote the minimal number of convex polygons into which P can be decomposed. We now define *decomposition complexity* of a set $M \subset \mathbb{Z}^2$ as $dc(M) = \min_P \{\nu(P) : P \text{ is enclosing for } M\}$. We will say that P is *minimally decomposable* with respect to $M = P_{\mathbb{Z}}$ if $\nu(P) = dc(M)$.

Now let $P \subset \mathbb{R}^2$ be a lattice polygon. We call P *irreducible* if there is no real number $\alpha < 1$ for which αP is still a lattice polygon. Otherwise P is *reducible*.

Informally, by *size* of P we will mean its diameter.

Definition 1. A lattice polygon P is well-sized if the following conditions are met:

- (i) P is minimally enclosing for $P_{\mathbb{Z}}$,
- (ii) P is minimally decomposable with respect to $P_{\mathbb{Z}}$.

We call P a *least well-sized* polygon if there is no $\alpha < 1$ for which αP is well-sized. If a well-sized polygon is an irreducible lattice polygon, then clearly it is also a least well-sized polygon. Note that a least well-sized polygon may be reducible (see Fig. 2a).

Obviously, an arbitrary lattice polygon P is enclosing for the set $P_{\mathbb{Z}}$ but not necessarily well-sized, as one or both of conditions (i), (ii) may not be satisfied, both for irreducible (Fig. 1) or reducible lattice polygons (Fig. 2).

Finding $\nu(P)$ is a strongly NP-hard problem [12], so testing if a polygon is well-sized or not is hard, in general. For our pseudopolynomial reduction in the next section, however, we will need an arbitrary lattice polygon P that appears to be well-sized. To this end, we will show that for any lattice polygon P there is a number $\delta_P \in \mathbb{Z}_+$, so that $\delta_P P$ is well-sized. δ_P is called *sizing factor* for P . The procedure of computing δ_P and $\delta_P P$ is called *sizing* of P and is described next.

Let P be a lattice polygon. In what follows, w.l.o.g. we assume that P contains the origin $\mathbf{0}$. Let $W = \{v^1, v^2, \dots, v^m\}$ be the set of vertices of P , where $v^i = (v_1^i, v_2^i) \in \mathbb{Z}^2$ for $1 \leq i \leq m$.

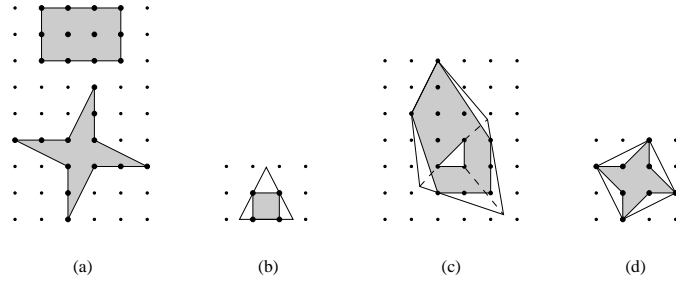


Fig. 1. Irreducible lattice polygons (in gray): a) two minimally enclosing minimally decomposable polygons, b) non-minimally enclosing minimally decomposable, c) minimally enclosing non-minimally decomposable, d) non-minimally enclosing non-minimally decomposable.

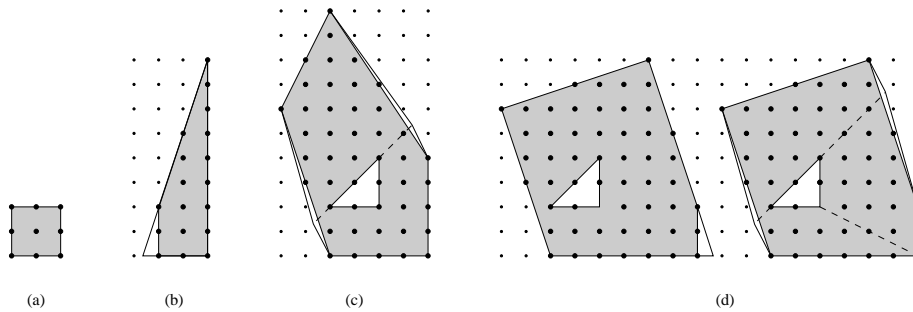


Fig. 2. Reducible lattice polygons (in gray): a) two minimally enclosing minimally decomposable polygons, b) non-minimally enclosing minimally decomposable, c) minimally enclosing non-minimally decomposable, d) non-minimally enclosing non-minimally decomposable.

Let $\phi(P) = \max_{v^j, v^k \in W} \{3, \|v^j - v^k\|\}$, $1 \leq j, k \leq m$. For any $v^i \in W$, we define $\psi_i(P) = \max_{j, k \neq i} \left\{ \frac{3}{\rho(v^i, \overline{v^j v^k})} \right\}$, where $\overline{v^j v^k}$ is the straight line through v^j and v^k . Then we set $\psi(P) = \max_i \psi_i(P)$. Finally, let

$$\delta_P = \max\{\phi(P), \psi(P)\}. \quad (1)$$

We have the following plain fact.

Lemma 1. *Given a lattice polygon P with $d = \text{diam}(P)$ and $\mathbf{0} \in P$, then:*

1. *There is a polynomial $p(x)$ such that $\delta_P = O(p(d))$.*
2. *δ_P can be computed in time $q(m, \log d)$, where $q(x, y)$ is some polynomial in two variables.*

Next we list a series of lemmas that reveal certain properties of well-sized polygons. These will be used later in the proof of OptDVP's strong NP-hardness, but may be of independent interest too.

Let P be a lattice polygon.

Lemma 2. *For any integer $k \geq 2\delta_P^2$, the polygon kP is minimally enclosing for $(kP)_{\mathbb{Z}}$.*

Lemma 3. *For any integer $k \geq 2\delta_P^2$, $\nu(kP) \leq \nu(P')$ for any polygon P' (not necessarily a lattice one) that is enclosing for $(kP)_{\mathbb{Z}}$.*

Corollary 1. *For any integer $k \geq 2\delta_P^2$, the polygon kP is well-sized.*

Lemma 4. *Let $k \geq 2\delta_P^2$. By Lemma 2, kP is minimally enclosing for $P_{\mathbb{Z}}$. Let P' be another minimally enclosing polygon for $P_{\mathbb{Z}}$. Then there is a one-to-one correspondence between the vertices of kP and P' , such that vertex v of kP is corresponding to vertex v' of P' if and only if $\|v - v'\| \leq c\delta_P$ for some positive constant c .*

Lemma 5. *Let kP and P' be as in Lemma 4. Let u, v, w be vertices of P and u', v', w' their corresponding vertices in P' . Then w is in the left/right half-plane determined by the line \overline{uv} iff w' is in the left/right half-plane determined by the line $\overline{u'v'}$.*

We call a *line set* any set v^1, v^2, \dots, v^l , $l \geq 3$ of (at least three) vertices of P that belong to the straight line segment $\overline{u^{(1)}u^{(l)}} \subset P$.

Lemma 6. *Let kP and P' be as in Lemma 4. Let one of the following conditions is met:*

1. *Both kP and P' do not contain line sets.*
2. *kP contains a line set v^1, v^2, \dots, v^l if and only if P' contains a line set $\bar{v}^1, \bar{v}^2, \dots, \bar{v}^l$, where vertex v^i of kP is corresponding to vertex \bar{v}^i of P' for $i = 1, 2, \dots, l$.*

Then $\nu(kP) = \nu(P')$.

Lemma 7. *Let kP and P' be as in Lemma 4 and let the following conditions be met:*

1. *If $\bar{v}^1, \bar{v}^2, \dots, \bar{v}^l$ is a line set in P' , then the set of their corresponding vertices v^1, v^2, \dots, v^l in kP form a line set in kP .*
2. *There exists a line set v^1, v^2, \dots, v^l in kP such that the corresponding set of vertices $\bar{v}^1, \bar{v}^2, \dots, \bar{v}^l$ in P' is not a line set in P' .*

Then $\nu(kP) \leq \nu(P')$.

The proofs of the above lemmas are based on combinatorial-geometric reasoning. They are more or less straightforward but somewhat lengthy and therefore omitted here and left for an exercise.

3 OptDVP is Strongly NP-Hard

Consider the optimization version of DVP.

Optimization Discrete Surface Polyhedrization (OptDVP):

Instance: A set $M \subset \mathbb{Z}^3$ and a bound $\beta \in \mathbb{Z}_+$, $\beta \leq |M|$.

Problem: Decide if there exists a polyhedron P with $M = P_{\mathbb{Z}}$ with no more than β facets that are all convex polygons, some of which may be co-planar.

We prove that OptDVP is strongly NP-hard by exhibiting a pseudopolynomial reduction to it from the following problem that is known to be strongly NP-hard [12].

Minimal Number Convex Partition (MNCP):

Instance: A simple polygon P (given by a sequence of pairs of integer-coordinate points in the plane) with non-rectilinear holes and a bound $\alpha \in \mathbb{Z}_+$.

Problem: Decide if P can be decomposed into no more than α convex polygons with vertices among the vertices of P .

Note that MNCP is in P if holes are not allowed [4].

For getting acquainted with the notion of strong NP-hardness, pseudopolynomial reduction, and related matters the reader is referred to [8]. Here we only recall some basic points. Let $\Pi = (D_{\Pi}, Y_{\Pi})$ and $\Pi' = (D_{\Pi'}, Y_{\Pi'})$ be decision problems with instance sets D_{Π} and $D_{\Pi'}$, respectively, and sets of instances with answer “yes” Y_{Π} and $Y_{\Pi'}$, respectively. Denote by $Max[I]$, $Length[I]$, $Max'[I']$, $Length'[I']$ the maximal number and the input length of an instance $I \in D_{\Pi}$ and $I' \in D_{\Pi'}$, respectively. A *pseudopolynomial reduction* from Π to Π' is a function $f : D_{\Pi} \rightarrow D_{\Pi'}$ such that:

- (a) for all $I \in D_{\Pi}$, $I \in Y_{\Pi}$ iff $f(I) \in Y_{\Pi'}$,
- (b) f can be computed in time polynomial in two variables: $Max[I]$ and $Length[I]$,
- (c) there exists a polynomial q_1 such that, for all $I \in D_{\Pi}$, $q_1(Length'[f(I)]) \geq Length[I]$,

- (d) there exists a two-variable polynomial q_2 such that, for all $I \in D_{II}$, $Max'[f(I)] \leq q_2(Max[I], Length[I])$.

It is proved in [8] that if II is strongly NP-hard and there is a pseudopolynomial reduction from II to II' , then II' is strongly NP-hard.

We now sketch a proof of the following theorem.

Theorem 1. *OptDVP is strongly NP-hard.*

Sketch of Proof It is not hard to see that the pseudopolynomial reduction conditions (b),(c), and (d) are all satisfied by Lemma 1. We show next that condition (a) is met as well, i.e., given an instance $I = (X, \alpha)$ of MNCP, it is possible to construct an instance $I' = (M, \beta)$ of DVP, such that the solution of I' is “yes” if and only if the solution of I is “yes.”

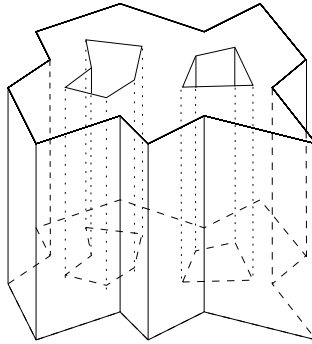


Fig. 3. Illustration to the proof of Theorem 1: the polytope P .

We construct a polytope $P = X \times \tau \subset \mathbb{R}^2 \times \mathbb{R}^1$, i.e., $P \subset \mathbb{R}^3$ is the Cartesian product of X and the interval $\tau = [0, t]$, where t is an integer constant greater than two (see Fig. 3). Then let $M = P \cap \mathbb{Z}^n$. Finally, we set $\beta = 2\alpha + k$.

The so-constructed instance $I' = (P, 2\alpha + k)$ of DVP has solution “yes” iff the instance $I = (X, \alpha)$ of MNCP has solution “yes.” In the one direction the proof of the above is trivial: if the solution for I is “yes,” then obviously so is the solution for I' , since P can be decomposed into no more than $2\alpha + k$ convex polygons.

To demonstrate the other direction, we have to make sure that if the set M can be represented as $M = P' \cap \mathbb{Z}^n$ for some polyhedron P' with no more than $2\alpha + k$ facets, then X can be partitioned into no more than α convex polygons. For this, it is enough to show that any such polyhedron $P' \neq P$ cannot have smaller number of convex facets than P . It is not hard to see that this is implied by Lemmas 2 through 7. \square

4 Lower Polyhedral Complexity Bound

If in the DVP formulation S (and thus also M) is an arbitrary set, then a bound $f_i(P) = \Theta(|M|)$ holds, where $|M|$ denotes the cardinality of set M (see Fig. 4 for an example in dimension two). In what follows we assume that $M = S_{\mathbb{Z}}$, where

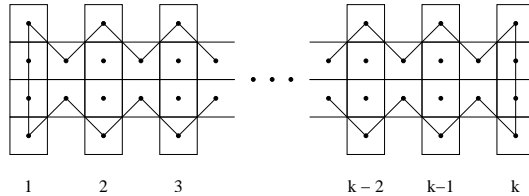


Fig. 4. Example in 2D with $|M| = 6k - 2$ and $f_i(P) = 4k - 2$ for any $k \in \mathbb{Z}_+$.

S is a ball with a sufficiently large diameter and $\mathbf{0} \in S$.² Our considerations hold for an arbitrary dimension n .

Let $P = \text{conv}(S_{\mathbb{Z}})$. Clearly, $P_{\mathbb{Z}} = S_{\mathbb{Z}}$ and $\text{conv}(P_{\mathbb{Z}}) = \text{conv}(S_{\mathbb{Z}}) = P$. Let P^* be a polytope with a minimal number of $(n-1)$ -facets, such that $P_{\mathbb{Z}}^* = P_{\mathbb{Z}} = S_{\mathbb{Z}}$. Then, $\text{conv}(P_{\mathbb{Z}}^*) = P$. Denote $d_P = \text{diam}(P)$, $d_{P^*} = \text{diam}(P^*)$, and $d_S = \text{diam}(S)$. Clearly, $\text{diam}(P) \leq \text{diam}(P^*)$ and $\text{diam}(P) \leq \text{diam}(S)$. It is easy to see that any of the relations $\text{diam}(P^*) < \text{diam}(S)$, $\text{diam}(P^*) = \text{diam}(S)$, and $\text{diam}(P^*) > \text{diam}(S)$ is possible. For S being a sphere, we have

$$d_P \leq d_{P^*} \leq 2d_P, d_P \leq d_S \leq 2d_P, d_S \leq 2d_{P^*}, d_{P^*} \leq 2d_S. \quad (2)$$

In what follows, we will suppose that these diameters are sufficiently large.

An upper bound $O(d_S^{\frac{n(n-1)}{(n+1)\lceil n/2 \rceil}})$ on the number of facets of P^* follows from [1] (see Lemma 8 below). A lower bound is provided by the following theorem.

Theorem 2. *For every $n \geq 2$ there is a constant $c(n)$ depending only on n such that*

$$f_{n-1}(P^*) \geq c(n) \frac{d_S^{\frac{n(n-1)}{(n+1)\lceil n/2 \rceil}}}{\log^{\lceil n/2 \rceil} d_S}.$$

Proof We use the following well-known results.

Lemma 8. [1] *Let $K \in \mathcal{C}(D)$ and $\bar{K} = \text{conv}(K \cap \mathbb{Z}^n)$. Then for every $n \geq 2$ there are constants $c_1(n)$ and $c_2(n)$ such that for all $k \in \{0, 1, \dots, n-1\}$,*

$$c_1(n)d^n \frac{n-1}{n+1} \leq f_k(\bar{K}) \leq c_2(n)d^n \frac{n-1}{n+1}, \quad (3)$$

where $d = \text{diam}(K)$ is sufficiently large.

² The bound follows also if $S \in \mathcal{C}(D)$, where $\mathcal{C}(D)$ is the family of convex bodies with C^2 boundary and radius of curvature at every point and every direction between $1/D$ and D , $D \geq 1$.

Lemma 9. [5] *Let $Q = \{x \in \mathbb{R}^n : Ax \leq \mathbf{b}, A \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m\}$. Consider the polytope $\text{conv}(Q_{\mathbb{Z}})$ and denote by $V(A, \mathbf{b})$ the set of its vertices. Then*

$$V(A, \mathbf{b}) \leq c(n)m^{\lfloor n/2 \rfloor} \log^{n-1}(1+d), \quad (4)$$

where $d = \max_{x,y \in Q} \max_{j=1,2,\dots,n} |x_j - y_j|$.

Clearly, Lemma 9 holds also if d is defined by the Euclidean metric, i.e., if $d = \text{diam}(Q) = \max_{x,y \in Q} (\sum_{j=1}^n (x_j - y_j)^2)^{1/2}$.

From (3) we have that there exist $c_1(n)$ and $c_2(n)$ such that $c_1(n)d^{\frac{n-1}{n+1}} \leq f_k(P) \leq c_2(n)d^{\frac{n-1}{n+1}}$. Denote for short $m = f_{n-1}(P)$, $m_* = f_{n-1}(P^*)$, $v = f_0(P)$, and $v_* = f_0(P^*)$. Then we have

$$m_* \leq m \leq c_2(n)d_S^{\frac{n-1}{n+1}}. \quad (5)$$

Multiplying by $\frac{c_1(n)}{c_2(n)}$ both sides of the last inequality and using once again Lemma 8, we get $\frac{c_1(n)}{c_2(n)}m \leq c_1(n)d_S^{\frac{n-1}{n+1}} \leq v$. Now, from (4) we get $v = v_* \leq c'(n)m_*^{\lfloor n/2 \rfloor} \log^{n-1}(1+d_{P^*})$ and $c_1(n)d_S^{\frac{n-1}{n+1}} \leq c'(n)m_*^{\lfloor n/2 \rfloor} \log^{n-1}(1+d_{P^*})$, where $c'(n)$ is a constant depending only on n . From here, keeping (2) in mind, we obtain consecutively $m_*^{\lfloor n/2 \rfloor} \geq c''(n) \frac{d_S^{\frac{n-1}{n+1}}}{\log^{n-1}(1+d_S)}$ and $m_* = f_{n-1}(P^*) \geq c(n) \frac{d_S^{\frac{n-1}{n+1}}}{\log^{\lfloor n/2 \rfloor} d_S}$ for some constants $c''(n)$ and $c(n)$, respectively. \square

Corollary 2. *Let S be a circle in \mathbb{R}^2 with a radius r . Then there is a constant $\beta_1 \in \mathbb{Z}_+$ such that $f_1(P^*) \geq \beta_1 \frac{r^{2/3}}{\log r}$.*

Let S be a sphere in \mathbb{R}^3 with a radius r . Then there is a constant $\beta_2 \in \mathbb{Z}_+$ such that $f_2(P^) \geq \beta_2 \frac{r^{3/2}}{\log r}$.*

Theorem 2 implies that there exists a constant $\alpha(n)$ depending only on n , such that $\frac{f_{n-1}(P)}{f_{n-1}(P^*)} \leq \Delta(d_S, n) = \alpha(n)d_P^{(1-\frac{1}{\lfloor n/2 \rfloor})n\frac{n-1}{n+1}} \log^{\frac{n-1}{\lfloor n/2 \rfloor}}(1+d_S)$, i.e., the number of facets of the solution provided by the convex hull algorithm is within a factor $\Delta(d_S, n)$ from the optimal one. In particular, for $n = 2$ and $n = 3$ we have $\frac{f_1(P)}{f_1(P^*)} \leq \alpha_1 \log d_S$ and $\frac{f_2(P)}{f_2(P^*)} \leq \alpha_2 \log^2 d_S$ for some constants $\alpha_1, \alpha_2 \in \mathbb{Z}_+$.

5 Concluding Remarks

In this paper we showed that the general discrete volume polyhedrization problem is strongly NP-hard. We also gave a lower bound on its polyhedral complexity. Here are some questions that are still open and are subject of future interest.

1. Is OptDSP in NP?

2. Is OptDSP NP-hard if non-convex polygonal facets are allowed?
3. Is OptDSP NP-hard when S is a convex body in \mathbb{R}^3 ? Is OptDSP NP-hard when S is a sphere?
4. Find a tight bound on $PC(M)$ when M is the set of integer points in a sphere or a circle.

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