University of Auckland Research Repository, ResearchSpace

Version

This is the Accepted Manuscript version. This version is defined in the NISO recommended practice RP-8-2008 http://www.niso.org/publications/rp/

Suggested Reference

doi: 10.1016/j.topol.2005.11.003

Copyright

Items in ResearchSpace are protected by copyright, with all rights reserved, unless otherwise indicated. Previously published items are made available in accordance with the copyright policy of the publisher.

https://www.elsevier.com/about/company-information/policies/sharing

http://www.sherpa.ac.uk/romeo/issn/0166-8641/

https://researchspace.auckland.ac.nz/docs/uoa-docs/rights.htm
HIGH DISTANCE HEEGAARD SPLITTINGS OF 3-MANIFOLDS

TATIANA EVANS

Abstract. J. Hempel [4] used the curve complex associated to the Heegaard surface of a splitting of a 3-manifold to study its complexity. He introduced the distance of a Heegaard splitting as the distance between two subsets of the curve complex associated to the handlebodies. Inspired by a construction of T. Kobayashi [7], J. Hempel [4] proved the existence of arbitrarily high distance Heegaard splittings.

In this work we explicitly define an infinite sequence of 3-manifolds \( \{ M^n \} \) via their representative Heegaard diagrams by iterating a 2-fold Dehn twist operator. Using purely combinatorial techniques we are able to prove that the distance of the Heegaard splitting of \( M^n \) is at least \( n \).

Moreover, we show that \( \pi_1(M^n) \) surjects onto \( \pi_1(M^{n-1}) \). Hence, if we assume that \( M^n \) has non-trivial boundary then it follows that the first Betti number \( \beta_1(M^n) > 0 \) for all \( n \geq 1 \). Therefore, the sequence \( \{ M^n \} \) consists of Haken 3-manifolds for \( n \geq 1 \) and hyperbolizable 3-manifolds for \( n \geq 3 \).

1. Introduction

A Heegaard splitting \((S; V_1, V_2)\) for a closed 3-manifold \( M \) is a representation \( M = V_1 \cup_S V_2 \) where \( V_1 \) and \( V_2 \) are handlebodies and \( S = \partial V_1 = \partial V_2 = V_1 \cap V_2 \).

The distance of a Heegaard splitting \((S; V_1, V_2)\) is the length of a shortest path in the curve complex of \( S \) which connects the subcomplexes \( K_{V_1} \) and \( K_{V_2} \), where \( K_{V_i} \) is the subcomplex consisting of all vertices that correspond to simple closed curves bounding disks in \( V_i \) for \( i = 1, 2 \).

In this paper we continue to analyze the correlation between subcomplexes of the curve complex and the corresponding Heegaard splittings of 3-manifolds. In particular, we construct a sequence of 3-manifolds (in fact Haken 3-manifolds) which have arbitrarily large distance (see theorem 4.4 for a precise statement).

Theorem 1.1. Let \( S \) be an orientable surface of genus \( g \geq 2 \). Suppose \( X = \{ x_1, x_2, \ldots, x_g \} \) is a collection of standard meridians on \( S \) and \( y \) is a simple closed curve on \( S \). Let \( (S; X, y) \) describe a Heegaard diagram for a 3-manifold. Let \( Y^0 = y \) and then iteratively define \( Y^k = \tau_2^X Y^{k-1}(X), k = 1, \ldots, n \). If the curve \( y \) is sufficiently complicated then \( \text{dist}(K_X, K_{Y^n}) \geq n \).

Here the notation \( \tau_2^X(X) \) means the square of the Dehn twist operator of \( X \) along \( Y^0 \) and \( d(K_X, K_{Y^n}) \) denotes the distance of the Heegaard splitting defined by the Heegaard diagram \((S; X, Y^n)\).

There have been several similar results in the past. J. Hempel [4] showed that the set of distances of Heegaard splittings is unbounded for 3-manifolds obtained by...
using a construction of T. Kobayashi [7]. The proof proceeds by choosing a certain pseudo-Anosov map \( h \) defined on a Heegaard surface corresponding to handlebodies \( V_1 \) and \( V_2 \). For each \( n \) he then considers the manifold obtained by gluing \( V_1 \) to \( V_2 \) by the map \( h^n \). By analyzing the action of \( h \) on the space \( PML(S) \) of projective measured laminations Hempel proves that the set of distances of these Heegaard splittings is unbounded. A. Abrams and S. Schleimer [1] later showed that with the same set up the distance of the splittings grows linearly with \( n \) using the result of H. Masur and Y. Minsky [10] that the curve complex is Gromov hyperbolic.

Whereas the above results are existential our construction is explicit and purely combinatorial.

In contrast to our theorem Schleimer [14] proved that each fixed 3-manifold has a bound on distances of its Heegaard splittings. In particular this implies that our sequence contains infinitely many non-homeomorphic 3-manifolds.

In Section 2, we introduce the necessary definitions and state a few of the main theorems in the field as a form of motivation.

In section 3 we define the Dehn twist operator which is used iteratively to construct a sequence of Heegaard diagrams. We prove that if we start with a manifold with non-trivial boundary then the resulting sequence consists of closed 3-manifolds each containing an incompressible surface.

In Section 4 we continue to analyze the set up introduced in Section 3 by proving the main theorem. From the definition of the distance it follows that the constructed 3-manifolds are irreducible. Since we observed before that they each contain an incompressible surface it follows that they are Haken 3-manifolds.

In section 5 we consider positive Heegaard diagrams of genus 2. It is relatively easy to encode such diagrams in the form of vectors in \( \mathbb{Z}^5 \) and make conclusions about the action of the Dehn twisting operator on the set of those vectors. Finally we show some examples of representative diagrams and make a few steps in constructing the iterating sequence of hyperbolizable 3-manifolds.

2. Preliminaries

Throughout this work we will assume a basic familiarity with common notions in 3-manifold topology, all of which can be found in [5] and [6].

2.1. The curve complex. Let us denote by \( S \) a closed, connected, orientable surface of genus \( g \geq 2 \). The curve complex of \( S \), denoted by \( C(S) \), is a simplicial complex in which vertices are isotopy classes of essential simple closed curves on \( S \), and \( k+1 \) vertices determine a \( k \)-simplex if they are represented by pairwise disjoint simple closed curves.

If we put a hyperbolic metric on \( S \), then each isotopy class contains a unique geodesic. Since two isotopy classes have disjoint representatives if and only if their geodesic representatives are disjoint, we can think about \( C(S) \) as having geodesics as its vertices and the corresponding collections of \( k+1 \) pairwise disjoint simple closed curves as its \( k \)-simplexes, and thus we can think of a \( k \)-simplex as a subset of \( S \).

A principal simplex of \( C(S) \) is a collection of \( 3g-3 \) simple closed curves which splits \( S \) into pairs of pants (thrice punctured 2-spheres). This is the maximum collection of pairwise disjoint, non-isotopic simple closed curves on \( S \) up to homeomorphism. Hence, the maximal dimension of a simplex is \( 3g-4 \). So, \( \text{dim}(C(S)) = 3g-4 \).
2.2. **Heegaard splittings.** In further considerations we will suppress the difference between simple closed curves and their isotopy classes.

**Definition 2.1.** A $k$-simplex $X = (x_0, x_1, \ldots, x_k) \in C(S)$ defines a compression body as follows: start with $S \times [0, 1]$, attach 2-handles to $S \times \{1\}$ along the curves of the collection $X$, and then fill in any resulting 2-sphere boundary components with 3-cells. Denote the resulting space by $V_X = S \times [0, 1] \cup_{S \times \{1\}} 2$-handles. $S \times 0$ is called the outer boundary of $V_X$ and is naturally identified with $S$. The second boundary component $\partial V_X = S \times 0$ is called the inner boundary and may be empty.

**Definition 2.2.** A compression body $V_X$ with an empty inner boundary is called a handlebody.

Define $N_X = \text{normal closure of } \{x_0, x_1, \ldots, x_k\} \text{ in } \pi_1(S)$. Then, $N_X = \ker\{\pi_1(S) \to \pi_1(V_X)\}$ determines $V_X$ up to homeomorphisms which restrict to the identity on $S$.

**Definition 2.3.** A Heegaard splitting of a compact, orientable 3-manifold $M$ is a representation of $M$ as the union of two compression bodies which intersect on their outer boundaries. Thus, a pair $X, Y$ of simplexes of the curve complex $C(S)$ determines a splitting $(S; V_X, V_Y)$ of the 3-manifold $M_{X,Y} = V_X \cup_S V_Y$.

The genus of the splitting is simply the genus $g$ of the splitting surface $S$.

By assuming that the genus of $S$ is $\geq 2$ we are excluding the standard genus zero and genus one Heegaard splittings of $S^3$, Lens spaces, and $S^2 \times S^1$.

Note that a 3-manifold $M$ is closed if and only if both $V_X$ and $V_Y$ are handlebodies in a Heegaard splitting $M_{X,Y} = V_X \cup_S V_Y$.

**Definition 2.4.** For a Heegaard splitting $(S; V_X, V_Y)$ call the pair of simplexes $X, Y$ a Heegaard diagram and denote it by $(S; X, Y)$.

There are many simplexes of $C(S)$ besides $X$ which determine a fixed compression body $V_X$.

**Definition 2.5.** The collection of all simplexes which determine the same compression body defines a subcomplex of the curve complex. The collection of simple closed curves bounding disks in $V_X$ is exactly the collection of vertices of this subcomplex. Denote it by $K_X$. We call $K_X$ the disk system subcomplex associated to the compression body $V_X$.

**Theorem 2.6** (Feng Luo [9]). Two $(3g - 4)$-simplexes $X, X'$ of $C(S)$ determine the same handlebody, $(V_X, S) = (V_{X'}, S)$, if and only if there is a sequence $X = X_0, X_1, \ldots, X_n = X'$ of $(3g - 4)$-simplexes of $C(S)$ such that $X_{i-1} \cap X_i$ is a full $(3g - 5)$-face of each for $i = 1, 2, \ldots, n$.

Thus, the pair $K_X, K_Y$ of subcomplexes of the curve complex describe all the different Heegaard diagrams which determine the same Heegaard splitting.
2.3. Irreducibility of Heegaard splittings. Recall that a closed 3-manifold $M$ is irreducible if every embedded 2-sphere in $M$ bounds a 3-cell in $M$. Otherwise $M$ is reducible. Also, $M$ is toroidal if $M$ contains an incompressible torus. Otherwise, $M$ is called atoroidal. Moreover, a closed, orientable 3-manifold is Haken if it is irreducible and contains a 2-sided incompressible surface.

The geometric intersection number of simple closed curves $\alpha_1, \alpha_2$ on $S$ is

$$i(\alpha_1, \alpha_2) = \min \{ \#(\alpha_i' \cap \alpha_j') \text{ where } \alpha_i' \text{ isotopic to } \alpha_i, i = 1, 2 \}.$$ 

We say that simple closed curves $\alpha, \beta$ meet efficiently if they are in general position and $i(\alpha, \beta) = \#(\alpha \cap \beta)$. This is equivalent to having no disk (or “bigon”) $D$ on $S$ with $D \cap (\alpha \cup \beta) = \partial D = a \cup b$ where $a, b$ are arcs such that $a \subset \alpha$ and $b \subset \beta$.

Definition 2.7. A properly embedded disk $D$ in a 3-manifold $M$ is essential if $\partial D$ does not bound a disk in $\partial M$.

Definition 2.8. For a given Heegaard splitting $(S; V_X, V_Y)$ define the disk system $D_X$ to be the collection of proper isotopy classes of essential disks in $V_X$. The disk system $D_Y$ is defined similarly.

Definition 2.9. A Heegaard splitting $(S; V_X, V_Y)$ is reducible if there are disks $A \in D_X$ and $B \in D_Y$ such that $\partial A = \partial B$. If no such pair exists then the splitting is irreducible.

This is a canonical definition, given the following lemma of Haken:

Lemma 2.10. If a 3-manifold $M$ is reducible then every splitting of $M$ is reducible.

Definition 2.11. A Heegaard splitting $(S; V_X, V_Y)$ is stabilized if there are disks $A \in D_X$ and $B \in D_Y$ which intersect transversely and $\#(\partial A \cap \partial B) = 1$.

Definition 2.12. A Heegaard splitting $(S; V_X, V_Y)$ is weakly reducible if there are disks $A \in D_X$ and $B \in D_Y$ such that $\partial A \cap \partial B = \emptyset$. If no such pair exists then the splitting is strongly irreducible.

The significance of this notion first comes from the following result:

Theorem 2.13 (Casson, Gordon [2]). A weakly reducible Heegaard splitting of a 3-manifold $M$ is either reducible or $M$ contains an incompressible surface.

2.4. Distance.

Definition 2.14. A distance function is defined on the 0-skeleton of $C(S)$ by

$$d(x, y) = \min \{ \text{numbers of 1-simplexes in simplicial path joining } x \text{ to } y \}.$$ 

Hence,

$$d(x, y) \leq 1 \text{ if and only if } x \cap y = \emptyset$$

and

$$d(x, y) \leq 2 \text{ if and only if there is some } z \text{ such that } x \cap z = y \cap z = \emptyset. \text{ In other words, } x \cup y \text{ does not fill } S.$$ 

Theorem 2.15 (H. Masur and Y. Minsky [10]). The curve complex has infinite diameter with respect to $d$.

Definition 2.16. A distance of the splitting is defined by

$$d(K_X, K_Y) = \min \{ d(x, y), \text{ where } x \in K_X \text{ and } y \in K_Y \}.$$
We can restate the above definitions in terms of the distance on \( C(S) \) as follows: Suppose \((S; V_X, V_Y)\) is a splitting of a closed, orientable 3-manifold. Then,
\[
d(K_X, K_Y) = 0 \text{ if and only if the splitting is reducible,}
\]
and
\[
d(K_X, K_Y) \leq 1 \text{ if and only if the splitting is weakly reducible.}
\]
If we are given a Heegaard diagram, there are some computable obstructions that can be read off the diagram that tell us that the corresponding splitting can not be reducible, weakly reducible, or be a distance 2 splitting. Also, there are obstructions for a 3-manifold to be Seifert fibered and contain an essential torus. See [4] for details and proofs.

These conclusions arise from the consideration of a Heegaard diagram using stacks which are unions of “squares” of \( S - X \cap Y \) that share common edges (see section 3.1). The stack intersection matrix provides information about the complexity of the Heegaard splitting.

These ideas were first introduced by Casson and Gordon [2] and extended by Kobayashi [7] to get an obstruction for being a weakly reducible splitting:

**Theorem 2.17** (Casson-Gordon condition [7]). If every \( X \)-stack intersects every \( Y \)-stack for a given Heegaard diagram then the corresponding splitting is not weakly reducible.

### 3. The Dehn twist operator

In this Section we define a Dehn twist operator. Then, we construct a sequence of Heegaard diagrams of 3-manifolds by considering the image of a given Heegaard diagram under iterations of the Dehn twist operator. If the initial diagram corresponds to a 3-manifold with boundary then the resulting sequence consists of diagrams of 3-manifolds which contain incompressible surfaces.

#### 3.1. Definition of a Dehn twist operator

First we define the notion of “stacks” on a surface \( S \) which is in some sense analogous to train tracks.

Suppose \( X, Y \) are simplexes of the curve complex \( C(S) \) such that they fill \( S \). Then, the components of \( S - (X \cup Y) \) are polygonal cells, every point of \( X \cap Y \) is a vertex of order 4 and every face has an even number of edges which lie alternately in \( X \) and \( Y \). Moreover, each polygon is at least a rectangle, since we are assuming that all intersections of \( X \) and \( Y \) are efficient, i.e. there are no “bigons”.

**Observation (J. Hempel [4])** If \( X \) and \( Y \) are simplexes of \( C(S) \) with \( S - (X \cup Y) \) simply connected and having \( n_i \) \( 2i \)-gon components \((i = 1, 2, \ldots)\), then
\[
\chi(S) = \sum (1 - i/2)n_i.
\]

Since \( n_1 = 0 \) and \( \chi(S) < 0 \), the number of polygons with 6 or more edges is bounded by \( |\chi(S)| \). Therefore, in a case of “not very trivial” intersection of \( X \) and \( Y \), most of the complementary polygons will be rectangles with one pair of opposite edges lying in \( X \) and the other in \( Y \).

**Definition 3.1.** An \( X \)-stack is a maximal collection of rectangles which are adjacent along common edges in \( X \). The edges, which lie in large regions with \( \geq 6 \) edges, are called the top and the bottom edges of the \( X \)-stack. The union of all \( Y \)
edges belonging to the $X$-stack defines the *sides* of the stack. There are, obviously, two sides in each $X$-stack which either lie in different curves of $Y$, or possibly in the same curve.

Every stack must have a top edge and bottom edge which do not coincide except for the degenerate case when there is only one edge. The $Y$-stacks are defined by interchanging the roles of $X$ and $Y$.

The *height* of a stack is the number of its rectangles. A stack of height 0 consists of the common edge of two large polygonal regions. 0-height stacks occur rarely and throughout this work we almost always assume that intersection of curves of $X$ and $Y$ are complicated enough to have stacks of height at least 2.

**Definition 3.2.** Suppose $S$ is a genus $g$ orientable surface. Let $X = \{x_1, \ldots, x_g\}$ be a collection of pairwise disjoint simple closed curves on $S$. Call $X = \{x_1, \ldots, x_g\}$ a collection of *standard meridians* on $S$ if $S \setminus X$ is a single planar component.

If we attach a 2-handle along each $x_i$ and glue a 3-ball for each 2-sphere boundary component we obtain the handlebody corresponding to the standard meridians. We will call this handlebody $V_X$.

The following definition is an extension of a notion of a standard Dehn twist along a curve on a surface.

**Definition 3.3.** Suppose $X = \{x_1, \ldots, x_g\}$ and $Y = \{y_1, \ldots, y_s\}$ are collections of simple closed curves such that $x_i \cap x_j = \emptyset$, $y_i \cap y_j = \emptyset$ and $X \cap Y \neq \emptyset$ for all $i, j$ and all intersections of $X$ with $Y$ are efficient. An image of a collection $X$ under the *Dehn twist operator along a collection $Y$, denoted by $\tau_Y(X)$, is the union of images $\{\tau_Y(x_1), \ldots, \tau_Y(x_g)\}$ of $\{x_1, \ldots, x_g\}$ under compositions of standard Dehn twists $\tau_{y_1} \circ \tau_{y_2} \circ \ldots \circ \tau_{y_s}$.

The following describes how to obtain $\tau_Y(X)$. For each $j$ choose an annular neighbourhood $A_j$ of $y_j$ so that $A_i \cap A_j = \emptyset$ for all $i, j$. The image of the collection of $g$ disjoint simple closed curves, $X = \{x_1, x_2, \ldots, x_g\}$, under the homeomorphism $\tau_Y$ is a collection of $g$ disjoint simple closed curves. To obtain the image of some $x_i$ under the Dehn twist operator for each $j = 1, \ldots, s$ replace each arc of $A_j \cap x_i$ by an arc which circles around $A_j$ once and smooth to general position relative to $X$.

Alternatively $\tau_Y(X)$ is the Haken sum (or oriented cut and paste) of a collection $X$ and $k$ copies of a collection $Y$, where $k = i(Y, X)$. That is for each $y_j$ take $k_j$ parallel copies of $y_j$, where $k_j = i(y_j, X)$. Call this collection $Y$. Denote an annular neighbourhood of $y_j$ containing $k_j$ parallel copies by $A_j$. Choose annular neighbourhoods $\{\cup A_j\}$ so that they are pairwise disjoint. Then resolve each point of intersection of $Y$ with $X$ as shown in figure 1.

![Figure 1. Resolution of a point of intersection](image)

Note that the resolution of a point of intersection is independent of the orientation on the curves but is dependent on the orientation of $S$.  


Consider intervals of \( X - N(X \cap \overline{Y}) \). Call an interval \( \text{small} \) if it lies between two parallel copies of some \( y_j \). Call all the other intervals which lie between different components of \( Y \) \( \text{large} \). Then \( \tau_Y(x_i) \) contains almost all of each \( \text{large} \) interval in \( x_i \) except for the smoothed areas. As we continue along \( \tau_Y(x_i) \) and exit a \( \text{large} \) interval of \( x_i \), we enter some annular neighbourhood \( A_j \) containing \( k_j \) parallel copies of some \( y_j \). Now, since we resolved points of intersection of all parallel copies of \( y_j \) with \( X \) we have to follow along the first copy of \( y_j \). As we circle this annulus, each time we encounter \( X \) we switch to the next parallel copy of \( y_j \). By the time we have circled around \( A_j \) one full time we have switched over all \( k_j \) copies of \( y_j \). Therefore, we must exit to the next \( \text{large} \) interval of \( x_i \). See figure 2.

![Figure 2. Construction of Dehn twist operator](image)

Now consider the regions of \( S - (\tau_Y(X) \cup X) \). The regions are of two types. The ‘old’ regions are essentially the regions of \( S - (Y \cup X) \). The ‘new’ regions form \( \text{partial} \) \( X \)-stacks relative to \( \tau_Y(X) \) each of which begins at an old region on one side of some \( A_j \), circles \( A_j \) a total of \((k_j - 1)/k_j\)-times and ends at an ‘old’ region on the other side of \( A_j \). There are \( k_j \) \( \text{partial} \) \( X \)-stacks relative to \( \tau_Y(X) \) in each \( A_j \). Comparing \( X \)-partial stacks relative to \( \tau_Y(X) \) to \( X \)-partial stacks relative to \( \tau_Y(x_k) \) for some \( k \), we note that there are fewer rectangles in \( X \)-partial stacks relative to \( \tau_Y(x_k) \) and consequently there are fewer \( \text{partial} \) \( X \)-stacks relative to \( \tau_Y(x_k) \) in \( A_j \).

\textbf{Remark:} If instead of \( k_j \) parallel copies of \( y_j \) we take \( n \times k_j \) copies and proceed as above, we obtain the image under \( n \)-fold Dehn twist operator, or \( \tau_Y^n(X) \).

3.2. \textbf{Properties of Dehn twist operator}. Let \( X = \{x_1, ..., x_g\} \) be a complete set of standard meridians for a genus \( g \) surface \( S \). Let \( V_X \) be the corresponding
handlebody. Let \( Y = \{y_1, \ldots, y_k\} \) be a collection of essential, pairwise disjoint simple closed curves in \( \partial V_X = S \) such that \( X \cap y_j \neq \emptyset \) for all \( j \) and all intersections of \( Y \) and \( X \) are efficient.

We get a new collection \( Y^1 \) of simple closed curves by taking the image of \( X \) under \( n \)-fold Dehn twist operator along \( Y \), or \( Y^1 = \tau^n_Y(X) \).

**Theorem 3.4.** Assuming the set up from the above let \( M \) be a 3-manifold determined by the Heegaard diagram \((\partial V_X; X, Y)\), possibly with boundary (if \( k < g \)). Let \( M^1 \) be a 3-manifold determined by the Heegaard diagram \((\partial V_X; X, Y^1)\) where \( Y^1 = \tau^n_Y(X) \). Then, \( \text{id}: \pi_1(V_X) \rightarrow \pi_1(V_X^1) \) extends to an epimorphism \( \pi(M^1) \rightarrow \pi(M) \).

**Proof.** Given a Heegaard diagram \((\partial V_X; X, Y)\), we can construct a presentation for \( \pi_1(M) \) as follows: Choose the free basis \( \{X_1, X_2, \ldots, X_g\} \) for the free group \( \pi_1(V_X) \) which is “dual to” \( \{x_1, \ldots, x_g\} \). For \( j = 1, \ldots, k \) let \( r_j \) be a word in \( X_1, X_2, \ldots, X_g \) representing the element of \( \pi_1(V_X) \) determined by \( y_j \). Note that \( r_j \) is unique up to inversion and conjugation. Then, it follows from Van Kampen’s Theorem that \( < X_1, \ldots, X_g : r_1, \ldots, r_k > \) is a presentation for \( \pi_1(M) \). Similarly, \( < X_1, \ldots, X_g : r^1_1, \ldots, r^1_k > \) is a presentation for \( \pi_1(M^1) \) where \( r^1_j \) represents an element of \( \pi_1(V_X^1) \) determined by \( y^1_j = \tau^n_Y(x_j) \).

By construction it follows that \( y^1_j \) is homologous to \( x_i + nk_1y_1 + nk_2y_2 + \ldots + nk_ky_k \) where \( k_i = i(x, y_i) \). Since \( x_i \) is null homotopic it follows that \( y^1_i \) is homotopic to products of conjugations of powers of the \( \{y_j\} \). Denote by \( \psi : \pi_1(V_X) \rightarrow \pi_1(M^1) \) canonical epimorphisms.

Then, \( \text{Ker}(\psi^1) \subset \text{Ker}(\psi) \).

Therefore, the diagram in figure 3 commutes giving the desired conclusion. \( \square \)

\[
\begin{array}{ccc}
\pi(M) & \xrightarrow{\psi} & \pi(M) \\
\downarrow{\psi^1} & & \downarrow{\psi} \\
\pi(V_X) & & \pi(V_X)
\end{array}
\]

**Figure 3.** Commutative diagram

**Corollary 3.5.** If \( M \) has nontrivial boundary then \( M^1 \) is a closed 3-manifold containing an incompressible surface.

**Proof.** The fact that \( M^1 \) is closed follows easily from the observation that the image of the set of \( g \) standard meridians under compositions of homeomorphisms is a collection of exactly \( g \) pairwise disjoint simple closed curves such that \( S = Y^1 \) is a single planar component.

If \( k < g \) then \( \partial M \neq \emptyset \), hence the first Betti number \( \beta_1(M) > 0 \). Since \( \varphi : \pi_1(M^1) \rightarrow \pi_1(M) \) is an epimorphism, it follows that \( \beta_1(M^1) > 0 \). The rest is given by standard facts of 3-manifold topology. See J. Hempel [5] for details. \( \square \)

3.3. Waves.

**Definition 3.6.** Suppose \( X = \{x_i\} \) and \( Y = \{y_j\} \) are collections of simple closed curves on a surface \( S \) determining a Heegaard diagram \((S; X, Y)\). A wave for the
diagram which is relative to $X$ is an arc in $S$ whose endpoints lie in the same component of $X$, whose interior misses $X \cup Y$, which lies on the same side of $X$ near its endpoints, and which can not be isotoped to an arc in $X$.

Throughout this work we will be assuming that for a given Heegaard diagram $(S;X,Y)$ there are no waves relative to $X$ where $X$ is a collection of standard meridians. There is no harm in adding this assumption, since otherwise we can always perform a surgery along a wave and reduce the complexity of the diagram. See J. Hempel [4] for details.

Lemma 3.7. Assume the setup of section 3.2. Suppose $(\partial V_X;X,Y)$ is a Heegaard diagram for some 3-manifold $M$. Let $M^1$ be a 3-manifold determined by the Heegaard diagram $(\partial V_X;X,Y^1)$ where $Y^1 = \tau_Y(X)$. If there are no waves relative to $X$ for the diagram $(\partial V_X;X,Y)$, then there are no waves relative to $X$ and $Y^1$ for the diagram $(\partial V_X;X,Y^1)$.

Proof. Assume there is a wave $w$ relative to $X$ or $Y^1$. Then, interior of $w$ lies in some “old” region of $\partial V_X - (X \cup Y^1)$. Consider the preimage of $w$ under $\tau_Y$. Since “old” regions are unchanged we get a wave $(\tau_Y)\left^{-1}(w)$ for the diagram $(\partial V_X;X,Y)$. Hence, we reach the desired contradiction.

4. Main theorem

In this section we prove the main theorem which heavily relies on the proofs of the following lemmas.

4.1. Lemmas.

Lemma 4.1. Let $(S;X,Y)$ describe a Heegaard diagram for a 3-manifold, where $S$ is a surface of genus $g$, $X = \{x_1, \ldots, x_g\}$ is a collection of standard meridians. Let $V_X$ be the corresponding handlebody bounded by $S$. Assume $Y$ is a collection of pairwise disjoint simple closed curves such that $Y$ intersects $X$ nontrivially and efficiently and there are no waves relative to $X$. Let $\gamma$ be a simple closed curve bounding a disk in $V_X$, i.e. $\gamma \in K_X$. Then $\gamma$ crosses some $Y$-stack.

Proof. Note that a curve crosses a $Y$-stack if it enters the stack through the top (bottom) edge, crosses every rectangular region and exits through the bottom (top) edge. A curve partially crosses a $Y$-stack if it enters the stack through the top (bottom) edge, crosses some (possibly all) of the rectangular regions and exits through the side of the stack, i.e. through an $X$-curve.

We assume that all intersections of $\gamma$ with $X$ and $Y$ curves are efficient. We first suppose that $\gamma \cap X = \emptyset$. If $\gamma \cap Y = \emptyset$ also then we may tube $\gamma$ to some component of $X$ to create a wave. Hence we reach a contradiction. Thus $\gamma \cap Y \neq \emptyset$. Since $\gamma \cap X = \emptyset$, by our observation above $\gamma$ cannot partially cross a $Y$-stack. Therefore $\gamma$ crosses a $Y$-stack.

Let us now consider the case that $\gamma \cap X \neq \emptyset$. Denote by $E$ a disk bounded by $\gamma$ and denote by $D_i$ disks bounded by $x_i$. Consider the arcs of $E \cap \cup D_i$, assuming that those intersections are efficient, i.e. can not be isotoped off $E$. Choose an outermost arc of $E \cap \cup D_i$ on $E$ and call it $e$. The arc $e$ cobounds a disk with a subarc of $\gamma$. Call the subarc $f$. See figure 4. We will show that $f$ satisfies several of the properties required by a wave. Firstly note that the endpoints of $f$ lie on the same component of $X$, say $x_j$. Next observe that the interior of $f$ lies on the same.
side of \( x_j \) near its endpoints. For assume otherwise and consider the homology of \( V_X \) relative its boundary \( S \). Then \( e \cup f \) can be adjusted in a neighborhood of \( x_j \) on \( S \) so that a 1-cycle representing \( e \cup f \) intersects a 2-cycle represented by \( D_{x_j} \) exactly once. Homology intersection number is a topological invariant, therefore \( e \cup f \) can not be null homologous in \( H_1(V_X; S) \). This contradicts the fact that \( e \cup f \) is homotopically trivial in \( V_X \). Lastly observe that since the arc \( e \) intersects the disk \( E \) efficiently, it follows that \( f \) and a subarc of \( x_j \) do not cobound a disk on \( S \). Therefore, the cobounded area must include some component \( x_k \).

We are now ready to show that \( f \) crosses a \( Y \)-stack. Assume otherwise. There are two cases to consider.

The first case is that \( f \cap Y = \emptyset \). By our choice of arc \( f \) we have that the interior of \( f \) is disjoint from \( X \). Together with the properties of \( f \) noted above we conclude that \( f \) is a wave, a contradiction.

The second case to consider is that \( f \cap Y \neq \emptyset \) but every intersection of \( f \) with a \( Y \)-stack is a partial crossing. If \( f \) partially crosses at least three \( Y \)-stacks then
by our initial observation \( f \) has at least three points of intersection with \( X \). In particular this implies that the interior of \( f \) must have a point of intersection with \( X \) contradicting our choice of \( f \). If \( f \) partially crosses a \( Y \)-stack that doesn’t have \( x_j \) as a side then by our initial observation the interior of \( f \) must intersect \( X \). Again this gives a point of intersection of the interior of \( f \) with \( X \), a contradiction. Thus \( f \) partially crosses at most two \( Y \)-stacks each with \( x_j \) as a side; denote these \( Y \)-stacks by \( Y_f \). Note that there are at most two components of \( f \cap Y_f \) and each component contains an endpoint of \( f \). Modify \( f \) by ‘sliding’ each component of \( f \cap Y_f \) off \( Y_f \), keeping the endpoints within the curve \( x_j \). The resulting curve \( f' \) has no intersection with \( Y \) but retains the properties of \( f \) noted above. Thus \( f' \) is a wave, a contradiction.

Figure 5 of the 2-sphere with 2\( g \) disks removed represents a surface \( S \) cut open along a collection of \( g \) simple closed curves \( X = \{x_1, ..., x_g\} \); this demonstrates a typical scenario for the various curves in this lemma.

\[\text{Definition 4.2 (Jason Leasure [8]). Suppose } X = \{x_i\} \text{ is a collection of pairwise disjoint simple closed curves, } y \text{ and } \gamma \text{ are simple closed curves which meet efficiently and nontrivially. Assume } y \text{ intersects each component of } X \text{ efficiently and nontrivially and } \gamma \text{ intersects } X \text{ efficiently. If } y \cap \gamma \text{ where } \gamma \text{ is an arc of } y - X \text{ then we say that } y \text{ is almost contained in } \gamma \text{ relative to } X \text{ and denote this by } y \preceq_X \gamma.\]

This idea is most useful when \( y \preceq_X \gamma \) and there is a curve \( \gamma' \) such that \( \gamma \cap \gamma' = \emptyset \). If this is the case, then \( \gamma' \) can intersect \( y \) in at most one arc of \( y - X \), namely the arc containing \( a \). We say that \( y \) is almost disjoint from \( \gamma' \). See figure 6.

\[\text{Figure 6. “almost contained” relation}\]

\[\text{Lemma 4.3. Let } S \text{ be a genus } g \text{ orientable surface. Suppose } X = \{x_1, ..., x_g\} \text{ is a collection of standard meridians on } S \text{ and } Y = \{y_i\} \text{ is a collection of pairwise disjoint simple closed curves on } S \text{ such that } i(x_i, y_j) \geq 2 \text{ for each } i \text{ and } j. \text{ Suppose } \gamma' \text{ is a simple closed curve on } S \text{ that meets } Y \text{ efficiently. Let } Y^1 = \tau^g(X) \text{ where } \tau \text{ is the Dehn twist operator. Let } \gamma \text{ be a simple closed curve on } S \text{ such that } \gamma \cap \gamma' = \emptyset. \text{ Assume } \gamma \text{ meets } Y \text{ efficiently and nontrivially, and } \gamma \text{ intersects } Y^1 \text{ and } X \text{ efficiently. If there exists a component } y^1_k \text{ of } Y^1 \text{ such that } y^1_k \preceq_X \gamma' \text{ then there exists a component } y_l \text{ of } Y \text{ such that } \gamma \text{ can be isotoped so that } y_l \preceq_X \gamma.}\]
Proof. The image of the collection of $g$ disjoint simple closed curves, $X = \{x_1, x_2, \ldots, x_g\}$, under the homeomorphism $\tau_2^Y$ is the collection of $g$ disjoint simple closed curves $Y^1$. For $1 \leq s \leq g$ let $A_s$ be an annular neighbourhood of the simple closed curve $y_s$ from the collection $Y$. We require that the collection of annuli $\{A_s\}$ are pairwise disjoint. Let $k_s = i(y_s, X)$. To obtain one of the simple closed curves of $Y^1$ from the simple closed curve $x_i$ of $X$, replace each arc of $A_s \cap x_i$ by an arc which circles around $A_s$ twice and smooth to general position relative to $X$.

Figure 7. Arcs of $y^1_k$ inside the annulus $A_j$

From the assumptions we have $y^1_k \cap X \gamma'$, i.e. $y^1_k \subseteq \gamma' \cup a$ where $a$ is some arc of $y^1_k - X$. Therefore, since $\gamma \cap \gamma' = \emptyset$ it follows that $\gamma$ can intersect $y^1_k$ only in the arc $a$. By assumption $\gamma \cap Y \neq \emptyset$. Therefore there exists a component $y_j$ of $Y$ such that $\gamma \cap \partial A_j \neq \emptyset$. Now consider $y^1_k \cap A_j$. Note, from the assumption $i(y_j, x_i) \geq 2$ for any $i, j$ and the observation that $i(y^1_k, x_j) = 2i(Y, x_i) \times i(Y, x_j)$ where $y^1_k$ denotes the image of $x_i$ under $\tau_2^Y$ it follows that $i(y^1_k, x_j) \geq 2$ for all $i, j$. This implies that
there are at least two arcs of $y_1^k \cap A_j$. The following situation represents the worst possible case:

1. there are only two arcs of $y_1^k \cap A_j$ circle twice around $A_j$ (that happens when $i(y_j, x_k) = 2$), and

2. the ends of the two arcs are located in the ‘closest’ possible position i.e. if $y_k$ is the image of $x_k$ under the square of the Dehn twist then $|y_j \cap x_k| = 2$ and these points of intersection occur consecutively along $x_k$. See figure 7.

In this worst possible case there are two $X$-partial stacks rel $y_1^k$ each of which circles around $A_j$ slightly more than once. See section 3 for the detailed description of stacks in $A_j$.

We now analyse an arc of $y_1^k \cap A_j$. In the worst case scenario $y_j$ enters the annulus $A_j$ inside of one of the $X$-partial stacks rel $y_1^k$.

There are two possibilities for $y_j$. Either $y_j$ circles around $A_j$ inside the $X$-partial stack rel $y_1^k$ or $y_j$ intersects $y_1^k$. Note that $\gamma$ can only intersect $y_1^k$ once since $\gamma$ intersects $y_1^k$ in at most one arc $a$ of $y_1^k - X$ (see explanation above). In this latter case $\gamma$ is forced to be inside the other $X$-partial stack rel $y_1^k$ and must circle $A_j$ within that partial stack.

In either case there is a subarc $b$ of $\gamma$ which circles around $A_j$ and comes back to the same rectangle $D$ of $A_j - X$ where it started. Hence, we can isotope $\gamma$ so that $b$ coincides with the core of the annulus everywhere except in the rectangle $D$.

Thus, $y_j = b \cup c$ where $c$ is the subarc of $y_j \cap D$, i.e. $c \subset D$ connects the ends of the arc $b$. Thus $y_j \prec X \gamma$. See figure 8.

Note that the isotopy is supported inside of the annulus $A_j$ and is “perpendicular” to the core of the annulus, i.e. each $x_i$ is fixed as a set. This isotopy simply
moves points of $\gamma$ toward the core. Thus, we can assume that we are not introducing inefficient intersections of $\gamma$ and $X$.

Now we are ready to prove the main theorem.

4.2. Main Theorem.

Theorem 4.4. Let $S$ be an orientable surface of genus $g \geq 2$. Suppose $X = \{x_1, x_2, \ldots, x_g\}$ is a collection of standard meridians on $S$ and $y$ is a simple closed curve on $S$ such that $i(y,x_i) \geq 2$ and each $Y$-stack is of height at least 2. Let $(S;X,y)$ describe a Heegaard diagram for a 3-manifold. Assume there are no waves relative to $X$. For any $n \geq 1$ let:

\[
\begin{align*}
Y^0 &= y \\
Y^1 &= \gamma_{y_0}^2(X) \\
&\vdots \\
Y^n &= \gamma_{y_{n-1}}^2(X).
\end{align*}
\]

Then $\text{dist}(K_X, K_{Y^n}) \geq n$.

Proof. We proceed by contradiction. Suppose $\text{dist}(K_X, K_{Y^n}) = d$ for $d \leq n - 1$, i.e. there exists a sequence of curves $\gamma_0, \gamma_1, \ldots, \gamma_d$ where $\gamma_0 \in K_X$, $\gamma_d \in K_{Y^n}$ and $\gamma_{i-1} \cap \gamma_i = \emptyset$ for all $i$.

From the assumption that there are no waves relative to $X$ and from the properties of the Dehn twist operator (see Lemma 3.7) it follows that there are no waves for the diagram $(S;X,Y^n)$ relative to $Y^n$.

By Lemma 4.1 $\gamma_d \in K_{Y^n}$ either does not intersect $Y^n$ or has a subarc $\alpha$ which is based on some component $y^n_{i_0}$ of $Y^n$ and is not isotopic into $y^n_{i_0}$ relative to its base points. In any case some subarc of $\gamma$ crosses some $X$-stack relative to $Y^n$. Since $\alpha$ is not isotopic into $y^n_{i_0}$, there exists a component $y^n_{i_0}$ of $Y^n$ such that $\alpha \cap \gamma_{i_0} = \emptyset$ and $\alpha$ crosses an $X$-stack relative to $Y^n$ with $y^n_{i_0}$ as a side of this stack.

Let us consider a different picture introduced in Lemma 4.3 where we look at the collection of pairwise disjoint annuli $\{A_1^{n-1}, A_2^{n-1}, \ldots, A_g^{n-1}\}$ on the surface $S$ and partial $X$-stacks circling around those annuli. Recall that $A_i^{n-1}$ corresponds to an annular neighbourhood of $y_{i_0}^{n-1}$. Since we assume $i(y,x_i) \geq 2$ it follows that any partial $X$-stack relative to $Y^n$ circles around any annulus $A_i^{n-1}$ at least once. Since $\alpha$ crosses the $X$-stack relative to $Y^n$ then $\alpha$ has to intersect a large interval of some $x_i$ and then follow the next rectangle which enters a partial $X$-stack relative to $Y^n$ which is inside of some annulus, say $A_i^{n-1}$. See figure 9.

Note that $\alpha$ can not intersect $Y^n$, therefore $\alpha$ has to stay inside of that partial $X$-stack relative to $Y^n$. That means $\alpha$ has to circle around the annulus $A_i^{n-1}$ and come back to the same rectangle $D$ of $A_i^{n-1} - X$. Thus, we can isotope the subarc $b$ equal to $\alpha \cap (A_i^{n-1} - D)$ to coincide with the core of the annulus which is $y_{i_0}^{n-1}$. That is possible to do everywhere except in the rectangle $D$. Then, connect the ends of the isotoped $b$ by an arc $b' \subset y_{i_0}^{n-1}$ that lies in the rectangle $D$. Therefore, after ambient isotopy we have $y_{i_0}^{n-1} \prec_X \gamma_d$. We will isotope the curves $\{\gamma_{d-1}, \ldots, \gamma_0\}$ by the same ambient isotopy. Continue to call the resulting curves $\{\gamma_{d-1}, \gamma_{d-1}, \ldots, \gamma_0\}$.

Thus, the property $\gamma_i \cap \gamma_{i-1} = \emptyset$ is preserved. Note, this isotopy is supported inside of the annulus $A_i^{n-1}$ and can be chosen so that each $x_i$ is fixed as a set. Therefore, we are not introducing any inefficient intersections of $X$ and $\{\gamma_{d-1}, \ldots, \gamma_0\}$. Also, we
can choose this isotopy so that $\gamma_{d-1}$ intersects efficiently $Y^{n-j},... Y^{0}$ for $1 \leq i \leq d$ and $2 \leq j \leq n$.

Applying Lemma 4.3 to $y^{n-1}_{j}$ and inducting using the set $(\gamma_{d}, \gamma_{d-1},... \gamma_{0})$ we conclude that $\gamma_{0}$ can be isotoped so that

$$y^{n-(d+1)}_{i,k} \prec_{X} \gamma_{0} \text{ where } \gamma_{0} \in K_{X} \quad (*)$$

Note: In order to apply Lemma 2 we need to assume that $d_{k} \not \in Y_{n}(k+1)$; for $k=1,..., d$ (**).

Let us consider it later as a special case and for now let us assume that (** holds.

From the assumptions and properties of the Dehn twist operator (see Lemma 3.7) it follows that there are no waves relative to $X$ for the diagram $(S; X, Y^{n-(d+1)})$.

By lemma 4.1 either $\gamma_{0} \cap X = \emptyset$ or there exists an outermost subarc $c$ of $\gamma_{0} - X$ such that $c$ is based on the same component $x_{i}$. In either case there is a subarc $c$ which crosses some $Y^{n-(d+1)}$-stack. It follows from the assumption on $y$ that $y^{n-(d+1)}_{i,k+1}$ has at least two arcs in that stack. Therefore, the subarc $c$ must cross at least two arcs of $y^{n-(d+1)}_{i,k+1} - X$. It follows from (*) that only one arc of $y^{n-(d+1)}_{i,k+1} - X$ does not lie in $\gamma_{0}$. Therefore $\gamma_{0}$ must be singular. Hence, we have reached a contradiction.

Let us show that for each inductive step (** holds). That is, if we have found a component $y_{n-k}^{n-k}$ of $Y^{n-k}$ with $y_{n-k}^{n-k} \prec_{X} \gamma_{d-k+1}$ then $\gamma_{d-k} \cap Y^{n-(k+1)} \not= \emptyset$. Now suppose (** does not hold, i.e. $\gamma_{d-k} \cap Y^{n-(k+1)} = \emptyset$. If $d-k > 1$, then $\gamma_{d-k}$ meets every component $x_{i}$ of $X$. Otherwise $x_{i}, \gamma_{d-k}, \gamma_{d-k+1},... , \gamma_{d}$ is a shorter path connecting $K_{X}$ with $K_{Y^{n}}$. That contradicts our assumptions. In fact $\gamma_{d-k}$ meets each $x_{i}$ in at least two arcs of $x_{i} - Y^{n-(k+1)}$, since it must go around an $X$-stack relative to $Y^{n-(k+1)}$ and every $X$-stack contains at least two arcs of $x_{i} - Y^{n-(k+1)}$. 

**Figure 9.** Arc $\alpha$ inside of annulus $A^{n-1}_{j}$
Since \( y_{n-k}^i = \tau_{Y^{n-k+1}}^k(x_i) \), it follows that \( \gamma_{d-k} \) crosses \( y_{n-k}^i \) in at least two arcs of \( y_{n-k}^i - X \). Since \( y_{n-k}^i \cap X \gamma_{d-k+1} \), we conclude that \( \gamma_{d-k} \cap \gamma_{d-k+1} \neq \emptyset \). This contradicts our assumptions. The last case to consider is when \( d = 1 \). So \( \gamma_1 \cap Y^{n-d} = \emptyset \). By assumption \( d < n \), so \( n - d > 1 \). But then \( \gamma_1 \) bounds a disk in \( V_{Y^{n-d}} \). Therefore, the distance of \( (S; V_X, V_{Y^{n-d}}) \) is \( \leq 1 \), i.e. the splitting is weakly reducible. However, in this diagram every \( X \)-stack meets every \( Y^{n-d} \)-stack. This is the Casson-Gordon condition that the splitting is not weakly reducible. Hence, we have reached the desired contradiction. \( \square \)

5. Genus two Heegaard diagrams and examples

5.1. Positive Heegaard diagrams of genus two.

Definition 5.1. An oriented Heegaard diagram \((S; X, Y)\) is a Heegaard diagram where \( X \) and \( Y \) are given specific orientations.

Let \( <, > : H_1(S) \times H_1(S) \to \mathbb{Z} \) denote the algebraic intersection number on a surface \( S \). So for oriented simple closed curves \( x, y \) on \( S \) meeting efficiently, \( <x, y> = i(x, y) \) means that the algebraic intersection number is +1 at each point of \( x \cap y \).

Definition 5.2. A positive Heegaard diagram \((S; X, Y)\) is an oriented Heegaard diagram where the algebraic intersection number \( <X, Y>_p \) of \( X \) with \( Y \) is +1 at each point \( p \in X \cap Y \).

Every compact, oriented 3-manifold with no 2-sphere boundary components can be represented by a positive diagram; see Hempel [5]. In this section we will be focusing on genus two positive Heegaard diagrams.

For a given positive Heegaard diagram \((S; X, Y)\) we can construct a picture by cutting \( S \) open along \( X \). The result will be a 2-manifold \( S_1 \) whose boundary contains disjoint copies \( X^+ \) and \( X^- \) of \( X \) together with a map \( f : S_1 \to S \) which maps \( S_1 - X^+ \cup X^- \) homeomorphically onto \( M - X \) and maps each of \( X^+ \) and \( X^- \) homeomorphically onto \( X \).

If the genus of \( S \) is two and \( X \) contains exactly two components \( x_1 \) and \( x_2 \), then \( S_1 \) is a four times punctured 2-sphere with the boundary components \( x_1^1, x_1^2, x_2^1, x_2^2 \). The components of \( Y \) will be strands connecting \( x_1^1, x_1^2, x_2^1, x_2^2 \). See figure 10.

![Figure 10. S cut open along X](image_url)
Since we are assuming that the diagram is positive, it follows that the diagram 
will be in a shape of a “square”, i.e. there are no strands connecting $x_i^+$ with $x_j^+$ 
and $x_i^-$ with $x_j^-$. 

Similarly we can construct an analogous picture by cutting $S$ open along $Y$. In 
this case we will call it a $Y$-side of the diagram.

Given such a picture, we need specific instructions how to recover the original 
diagram. For that we need to describe how to glue back $x_i^+$ with $x_i^-$ and $x_j^+$ with 
$x_j^-$. 

**Definition 5.3.** We define as the *twist number* from $x_i^+$ to $x_i^-$ for $i = 1, 2$ the 
amount of twist used in gluing $x_i^+$ back to $x_i^-$ to reconstruct the original diagram.

**Definition 5.4.** For a positive Heegaard diagram $(S; X, Y)$ define a five-tuple vector 
$(p, q, r, n, m)$ by specifying the following:

$$
\begin{align*}
p &= \text{number of } Y \text{ strands from } x_1^+ \to x_2^+ \\
q &= \text{number of } Y \text{ strands from } x_1^- \to x_2^- \\
r &= \text{number of } Y \text{ strands from } x_1^+ \to x_2^- \\
n &= \text{twist number from } x_1^- \to x_1^- \\
m &= \text{twist number from } x_1^+ \to x_2^-
\end{align*}
$$

Thus, given this vector $(p, q, r, n, m)$ we can draw the cut-open diagram for this 
splitting. The values $p, q, r$ allow us to draw the strands between each of the (cut-
open) components of $X$. We can then number the intersection points on $x_i^-$ and 
$x_i^+$ consecutively following the orientation, starting at an arbitrary point on each.

The twist numbers $n$ and $m$ then tell us how to label the points on $x_i^-$ and $x_i^+$.

In our example in figure 10 the corresponding vector is $(2, 3, 6, 3, 3)$ and represents 
two disjoint simple closed $Y$-curves.

The following proposition follows immediately from the definition of the vector 
$(p, q, r, n, m)$.

**Proposition 5.5.** Suppose the vectors $v(y_1) = (p_1, q_1, r_1, n_1, m_1)$ and $v(y_2) = 
(p_2, q_2, r_2, n_2, m_2)$ represent pairwise disjoint simple closed curves $y_1$ and $y_2$ 
respectively. Then their union $y = y_1 \cup y_2$ is represented by the vector $v(y) = 
(p_1 + p_2, q_1 + q_2, r_1 + r_2, n_1 + n_2, m_1 + m_2)$.

Next we will attempt to consider the action of Dehn twisting operator on five-
tuple vectors.

Let $X = \{x_1, x_2\}$ be a set of oriented meridians for an oriented genus two 
handlebody bounded by $S$ and let $Y = \{y_i\}$, $i \leq 2$ be a collection of oriented 
pairwise disjoint curves which meet $X$ positively. Thus $y_i$ can be represented by a 
vector $v(y_i) = (p_i, q_i, r_i, n_i, m_i)$.

For $a_1, a_2 \in \mathbb{Z}_+$ let $\tau = \tau_{a_1y_1} \circ \tau_{a_2y_2}$ be the $a_1$-fold Dehn twist 
along $y_1$ together with the $a_2$-fold Dehn twist along $y_2$. Let $l_{ij} = < x_j, y_i >$ and 
$l_i = l_{1i} + l_{2i} = < X, y_i >$.

**Proposition 5.6.** $v(\tau(x_j)) = a_1l_{1j}v(y_1) + a_2l_{2j}v(y_2) + \epsilon_j$ where

$$
\epsilon = \begin{cases} 
(0, 0, 0, 1, 0) & j=1 \\
(0, 0, 0, 1, 1) & j=2
\end{cases}
$$

**Proof.** For a detailed description of the image of $X$ under the Dehn twist operator 
along the collection of $a_1$ parallel copies of $y_1$ and $a_2$ parallel copies of $y_2$ see section 
3.
So, there are $a_1 l_1 + a_2 l_2$ strands of $\tau(x_j) - X$ parallel to each strand of $Y - X$. This establishes the first three coordinates of the proposition. Fix a homological basis $(x_1, x_2, X_1, X_2)$ where $X_1$ is a longitude meeting $x_i$ in a single point for $i = 1, 2$ so that $\langle x_i, X_i \rangle = +1$. Then $n_i = \langle y_i, X_1 \rangle$ and $m_i = \langle y_i, X_2 \rangle$.

Observe that $\tau(x_j)$ is homologous to $x_j + a_1 < x_j, y_1 > y_1 + a_2 < x_j, y_2 > y_2 = x_j + a_1 l_1 y_1 + a_2 l_2 y_2$. Thus $\langle \tau(x_j), X_1 \rangle = \delta_{1j} + l_1 y_1 + l_2 n_2$ and $\langle \tau(x_j), X_2 \rangle = \delta_{2j} + l_1 m_1 + l_2 m_2$

where $\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$

\[\Box\]

**Corollary 5.7.** $v(\tau(X)) = a_1 l_1 v(y_1) + a_2 l_2 v(y_2) + (0, 0, 0, 1, 1)$

**Proposition 5.8.** Let $M_0, M$ be the 3-manifolds represented by positive Heegaard diagrams $(S; X, Y)$ and $(S; X, \tau(X))$ respectively. Then $H_1(M)$ is presented by the matrix

\[
\begin{pmatrix}
p_1 + q_1 & p_2 + q_2 \\
p_1 + r_1 & p_2 + r_2
\end{pmatrix}
\begin{pmatrix}
a_1 & 0 \\
0 & a_2
\end{pmatrix}
\begin{pmatrix}
p_1 + q_1 & p_1 + r_1 \\
p_2 + q_2 & p_2 + r_2
\end{pmatrix}
\]

where

\[
\begin{pmatrix}
p_1 + q_1 & p_2 + q_2 \\
0 & a_2
\end{pmatrix}
\]

presents $H_1(M_0)$.

**Proof.**

\[
\begin{pmatrix}
< x_1, \tau(x_1) > & < x_1, \tau(x_2) > \\
< x_2, \tau(x_1) > & < x_2, \tau(x_2) >
\end{pmatrix}
\]

presents $H_1(M)$.

Also, $< x_1, \tau(x_j) > = p(\tau(x_j)) + q(\tau(x_j))$ and $< x_2, \tau(x_j) > = p(\tau(x_j)) + r(\tau(x_j))$.

Similarly

\[
\begin{pmatrix}
< x_1, y_1 > & < x_1, y_2 > \\
< x_2, y_1 > & < x_2, y_2 >
\end{pmatrix}
\]

presents $H_1(M_0)$ where $< x_1, y_j > = p_j + q_j$ and $< x_2, y_j > = p_j + r_j$. Since

\[
l_{ij} = < x_j, y_i > = \begin{cases} p_i + q_i & \text{if } j = 1 \\ p_i + r_i & \text{if } j = 2 \end{cases}
\]

the result of the claim follows from direct calculation using the equalities:

\[
\begin{align*}
p(\tau(x_j)) &= a_1 l_1 p_1 + a_2 l_2 p_2 \\
q(\tau(x_j)) &= a_1 l_1 q_1 + a_2 l_2 q_2 \\
r(\tau(x_j)) &= a_1 l_1 r_1 + a_2 l_2 r_2
\end{align*}
\]

\[\Box\]

**Corollary 5.9.** If $Y$ has a single component (i.e. the 3-manifold $M_0$ has non-trivial boundary) then $H_1(M)$ is infinite.

**Proof.** Let $P$ denote a representation matrix of $H_1(M)$. We can assume that $a_2 = 0$. Then by proposition 5.8 $\det(P) = 0$.

\[\Box\]

Note, this corollary follows from Theorem 3.4 as well.

**Corollary 5.10.** Suppose $a_1 a_2 \neq 0$, i.e. $H_1(M_0)$ is finite and $M_0$ is necessarily closed then $\alpha(H_1(M)) = a_1 a_2 \times \alpha(H_1(M_0))^2$.
5.2. Examples. Suppose $Y$ has a single component $y$ represented by the vector $v(y) = (2, 2, 1, 2)$ on a genus two surface $S$ which bounds a handlebody determined by standard meridians $X = \{x_1, x_2\}$. Let $Y_0 = y$ and $Y^n = \tau_{y, -1}(X)$ for $n \geq 1$. Then $l_1 = \langle y, X, y \rangle = 8$. Let $a_1 = 2$.

By proposition 5.6 the image of $X$ under 2-fold Dehn twisting operator is represented by $v(Y^1) = (32, 32, 32, 17, 33)$. By Theorem 4.4 it follows that the 3-manifold $M^1$ determined by the Heegaard diagram $(S; X, Y^1)$ is closed, irreducible, Haken 3-manifold and the distance of this splitting is $\geq 1$.

The next step in the iteration gives $v(Y^2) = (4096, 4096, 4096, 2177, 4225)$. The 3-manifold $M^2$ defined by the Heegaard diagram $(S; X, Y^2)$ is again a closed, irreducible, Haken 3-manifold and the distance of this splitting is $\geq 2$.

After iterating one more time we get $v(Y^3) = (67108864, 67108864, 67108864, 35667969, 69222401)$. The 3-manifold $M^3$ determined by the Heegaard diagram $(S, X, Y^3)$ is closed, irreducible, Haken and atoroidal since the distance of this splitting is $\geq 3$ (see Hempel [5]). Thus by Thurston’s hyperbolisation theorem $M^3$ admits a hyperbolic metric.

If we keep iterating we get an infinite sequence of hyperbolizable 3-manifolds with arbitrarily large distance.

References