HIGH DISTANCE HEEGAARD SPLITTINGS OF 3-MANIFOLDS

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Abstract. J. Hempel [4] used the curve complex associated to the Heegaard surface of a splitting of a 3-manifold to study its complexity. He introduced the distance of a Heegaard splitting as the distance between two subsets of the curve complex associated to the handlebodies. Inspired by a construction of T. Kobayashi [7], J. Hempel [4] proved the existence of arbitrarily high distance Heegaard splittings.

In this work we explicitly define an infinite sequence of 3-manifolds $\{M^n\}$ via their representative Heegaard diagrams by iterating a 2-fold Dehn twist operator. Using purely combinatorial techniques we are able to prove that the distance of the Heegaard splitting of M^n is at least n.

Moreover, we show that $\pi_1(M^n)$ surjects onto $\pi_1(M^{n-1})$. Hence, if we assume that M^0 has non-trivial boundary then it follows that the first Betti number $\beta_1(M^n) > 0$ for all $n \geq 1$. Therefore, the sequence $\{M^n\}$ consists of Haken 3-manifolds for $n \geq 1$ and hyperbolizable 3-manifolds for $n \geq 3$.

1. Introduction

A Heegaard splitting $(S; V_1, V_2)$ for a closed 3-manifold M is a representation $M = V_1 \cup_S V_2$ where V_1 and V_2 are handlebodies and $S = \partial V_1 = \partial V_2 = V_1 \cap V_2$. The distance of a Heegaard splitting $(S; V_1, V_2)$ is the length of a shortest path in the curve complex of S which connects the subcomplexes K_{V_1} and K_{V_2}, where K_{V_i} is the subcomplex consisting of all vertices that correspond to simple closed curves bounding disks in V_i for $i = 1, 2$.

In this paper we continue to analyze the correlation between subcomplexes of the curve complex and the corresponding Heegaard splittings of 3-manifolds. In particular, we construct a sequence of 3-manifolds (in fact Haken 3-manifolds) which have arbitrarily large distance (see theorem 4.4 for a precise statement).

Theorem 1.1. *Let S be an orientable surface of genus $g \geq 2$. Suppose $X = \{x_1, x_2, ..., x_g\}$ is a collection of standard meridians on S and y is a simple closed curve on S. Let $(S; X, y)$ describe a Heegaard diagram for a 3-manifold. Let $Y^0 = y$ and then iteratively define $Y^k = \tau^2_{Y^{k-1}}(X), k = 1, ..., n$. If the curve y is sufficiently complicated then $dist(K_X, K_{Y^n}) \geq n$.*

Here the notation $\tau^2_{Y^0}(X)$ means the square of the Dehn twist operator of X along Y^0 and $d(K_X, K_{Y^n})$ denotes the distance of the Heegaard splitting defined by the Heegaard diagram $(S; X, Y^n)$.

There have been several similar results in the past. J. Hempel [4] showed that the set of distances of Heegaard splittings is unbounded for 3-manifolds obtained by

Date: June 25, 2005.
1991 *Mathematics Subject Classification.* Primary 57N10.
Key words and phrases. Heegaard splittings, Curve Complex.
The results in this paper represent a portion of my Ph.D. thesis completed at Rice University.

using a construction of T. Kobayashi [7]. The proof proceeds by choosing a certain pseudo-Anosov map h defined on a Heegaard surface corresponding to handlebodies V_1 and V_2. For each n he then considers the manifold obtained by gluing V_1 to V_2 by the map h^n. By analyzing the action of h on the space *PML(S)* of projective measured laminations Hempel proves that the set of distances of these Heegaard splittings is unbounded. A. Abrams and S. Schleimer [1] later showed that with the same set up the distance of the splittings grows linearly with n using the result of H. Masur and Y. Minsky [10] that the curve complex is Gromov hyperbolic.

Whereas the above results are existential our construction is explicit and purely combinatorial.

In contrast to our theorem Schleimer [14] proved that each fixed 3-manifold has a bound on distances of its Heegaard splittings. In particular this implies that our sequence contains infinitely many non-homeomorphic 3-manifolds.

In Section 2, we introduce the necessary definitions and state a few of the main theorems in the field as a form of motivation.

In section 3 we define the Dehn twist operator which is used iteratively to construct a sequence of Heegaard diagrams. We prove that if we start with a manifold with non-trivial boundary then the resulting sequence consists of closed 3-manifolds each containing an incompressible surface.

In Section 4 we continue to analyze the set up introduced in Section 3 by proving the main theorem. From the definition of the distance it follows that the constructed 3-manifolds are irreducible. Since we observed before that they each contain an incompressible surface it follows that they are Haken 3-manifolds.

In section 5 we consider positive Heegaard diagrams of genus 2. It is relatively easy to encode such diagrams in the form of vectors in $\mathbb{Z}^5$ and make conclusions about the action of the Dehn twisting operator on the set of those vectors. Finally we show some examples of representative diagrams and make a few steps in constructing the iterating sequence of hyperbolizable 3-manifolds.

2. Preliminaries

Throughout this work we will assume a basic familiarity with common notions in 3-manifold topology, all of which can be found in [5] and [6].

2.1. **The curve complex.** Let us denote by S a closed, connected, orientable surface of genus $g \geq 2$. The *curve complex* of S, denoted by $C(S)$, is a simplicial complex in which vertices are isotopy classes of essential simple closed curves on S, and $k+1$ vertices determine a k-simplex if they are represented by pairwise disjoint simple closed curves.

If we put a hyperbolic metric on S, then each isotopy class contains a unique geodesic. Since two isotopy classes have disjoint representatives if and only if their geodesic representatives are disjoint, we can think about $C(S)$ as having geodesics as its vertices and the corresponding collections of $k+1$ pairwise disjoint simple closed curves as its k-simplexes, and thus we can think of a k-simplex as a subset of S.

A *principal* simplex of $C(S)$ is a collection of $3g-3$ simple closed curves which splits S into *pairs of pants* (thrice punctured 2-spheres). This is the maximum collection of pairwise disjoint, non-isotopic simple closed curves on S up to homeomorphism. Hence, the maximal dimension of a simplex is $3g-4$. So, $dim C(S) = 3g-4$.

2.2. **Heegaard splittings.** In further considerations we will suppress the difference between simple closed curves and their isotopy classes.

Definition 2.1. A k-simplex $X = (x_0, x_1, ..., x_k) \in C(S)$ defines a *compression body* as follows: start with $S \times [0, 1]$, attach 2-handles to $S \times \{1\}$ along the curves of the collection X, and then fill in any resulting 2-sphere boundary components with 3-cells. Denote the resulting space by

$V_X = S \times [0, 1] \cup_{X \times 1} 2\text{-handles} \cup_{S^2} 3\text{-handles}$.

$S \times 0$ is called the *outer boundary* of V_X and is naturally identified with S. The second boundary component $\partial V_X - S \times 0$ is called the *inner boundary* and may be empty.

Definition 2.2. A compression body V_X with an empty inner boundary is called a *handlebody.*

Define

N_X = normal closure of $\{x_0, x_1, ..., x_k\}$ in $\pi_1(S)$.

Then, $N_X = ker\{\pi_1(S) \to \pi_1(V_X)\}$ determines V_X up to homeomorphisms which restrict to the identity on S.

Definition 2.3. A *Heegaard splitting* of a compact, orientable 3-manifold M is a representation of M as the union of two compression bodies which intersect on their outer boundaries. Thus, a pair X, Y of simplexes of the curve complex $C(S)$ determines a splitting $(S; V_X, V_Y)$ of the 3-manifold

$$M_{X,Y} = V_X \cup_S V_Y$$

The *genus* of the splitting is simply the genus g of the splitting surface S.

By assuming that the genus of S is ≥ 2 we are excluding the standard genus zero and genus one Heegaard splittings of S^3, Lens spaces, and $S^2 \times S^1$.

Note that a 3-manifold M is closed if and only if both V_X and V_Y are handlebodies in a Heegaard splitting $M_{X,Y} = V_X \cup_S V_Y$.

Definition 2.4. For a Heegaard splitting $(S; V_X, V_Y)$ call the pair of simplexes X, Y a *Heegaard diagram* and denote it by $(S; X, Y)$.

There are many simplexes of $C(S)$ besides X which determine a fixed compression body V_X.

Definition 2.5. The collection of all simplexes which determine the same compression body defines a subcomplex of the curve complex. The collection of simple closed curves bounding disks in V_X is exactly the collection of vertices of this subcomplex. Denote it by K_X. We call K_X the *disk system subcomplex* associated to the compression body V_X.

Theorem 2.6 (Feng Luo [9])**.** *Two $(3g-4)$-simplexes X, X' of $C(S)$ determine the same handlebody, $(V_X, S) = (V_{X'}, S)$, if and only if there is a sequence $X = X_0, X_1, ..., X_n = X'$ of $(3g-4)$-simplexes of $C(S)$ such that $X_{i-1} \cap X_i$ is a full $(3g-5)$-face of each for $i = 1, 2, ..., n$.*

Thus, the pair K_X, K_Y of subcomplexes of the curve complex describe all the different Heegaard diagrams which determine the same Heegaard splitting.

2.3. **Irreducibility of Heegaard splittings.** Recall that a closed 3-manifold M is irreducible if every embedded 2-sphere in M bounds a 3-cell in M. Otherwise M is reducible. Also, M is *toroidal* if M contains an incompressible torus. Otherwise, M is called *atoroidal*. Moreover, a closed, orientable 3-manifold is *Haken* if it is irreducible and contains a 2-sided incompressible surface.

The *geometric intersection number* of simple closed curves α_1, α_2 on S is

$$i(\alpha_1, \alpha_2) = min\{\#({\alpha'}_1 \cap {\alpha'}_2) \text{ where } {\alpha'}_i \text{ isotopic to } \alpha_i, i = 1, 2\}.$$

We say that simple closed curves α, β meet *efficiently* if they are in general position and $i(\alpha, \beta) = \#(\alpha \cap \beta)$. This is equivalent to having no disk (or "bigon") D on S with $D \cap (\alpha \cup \beta) = \partial D = a \cup b$ where a, b are arcs such that $a \subset \alpha$ and $b \subset \beta$.

Definition 2.7. A properly embedded disk D in a 3-manifold M is essential if ∂D does not bound a disk in ∂M.

Definition 2.8. For a given Heegaard splitting $(S; V_X, V_Y)$ define the *disk system* D_X to be the collection of proper isotopy classes of essential disks in V_X. The *disk system* D_Y is defined similarly.

Definition 2.9. A Heegaard splitting $(S; V_X, V_Y)$ is *reducible* if there are disks $A \in D_X$ and $B \in D_Y$ such that $\partial A = \partial B$. If no such pair exists then the splitting is *irreducible*.

This is a canonical definition, given the following lemma of Haken:

Lemma 2.10. *If a* 3*-manifold M is reducible then every splitting of M is reducible.*

Definition 2.11. A Heegaard splitting $(S; V_X, V_Y)$ is *stabilized* if there are disks $A \in D_X$ and $B \in D_Y$ which intersect transversely and $\sharp(\partial A \cap \partial B) = 1$.

Definition 2.12. A Heegaard splitting $(S; V_X, V_Y)$ is *weakly reducible* if there are disks $A \in D_X$ and $B \in D_y$ such that $\partial A \cap \partial B = \emptyset$. If no such pair exists then the splitting is *strongly irreducible*.

The significance of this notion first comes from the following result:

Theorem 2.13 (Casson, Gordon [2]). *A weakly reducible Heegaard splitting of a* 3*-manifold M is either reducible or M contains an incompressible surface.*

2.4. **Distance.**

Definition 2.14. A *distance function* is defined on the 0-skeleton of $C(S)$ by

$$d(x, y) = \min\{\text{numbers of 1-simplexes in simplicial path joining } x \text{ to } y\}$$

Hence,
$d(x, y) \leq 1$ if and only if $x \cap y = \emptyset$
and
$d(x, y) \leq 2$ if and only if there is some z such that $x \cap z = y \cap z = \emptyset$. In other words, $x \cup y$ does not fill S.

Theorem 2.15 (H. Masur and Y. Minsky [10]). *The curve complex has infinite diameter with respect to d.*

Definition 2.16. A *distance of the splitting* is defined by

$$d(K_X, K_Y) = min\{d(x, y), \text{ where } x \in K_X \text{ and } y \in K_Y\}$$

We can restate the above definitions in terms of the distance on $C(S)$ as follows: Suppose $(S; V_X, V_Y)$ is a splitting of a closed, orientable 3-manifold.
Then,
$d(K_X, K_Y) = 0$ if and only if the splitting is reducible,
and
$d(K_X, K_Y) \leq 1$ if and only if the splitting is weakly reducible.

If we are given a Heegaard diagram, there are some computable obstructions that can be read off the diagram that tell us that the corresponding splitting can not be reducible, weakly reducible, or be a distance 2 splitting. Also, there are obstructions for a 3-manifold to be Seifert fibered and contain an essential torus. See [4] for details and proofs.

These conclusions arise from the consideration of a Heegaard diagram using *stacks* which are unions of "squares" of $S - X \cap Y$ that share common edges (see section 3.1). The *stack intersection matrix* provides information about the complexity of the Heegaard splitting.

These ideas were first introduced by Casson and Gordon [2] and extended by Kobayashi [7] to get an obstruction for being a weakly reducible splitting:

Theorem 2.17 (Casson-Gordon condition [7])**.** *If every X-stack intersects every Y-stack for a given Heegaard diagram then the corresponding splitting is not weakly reducible.*

3. The Dehn twist operator

In this Section we define a Dehn twist operator. Then, we construct a sequence of Heegaard diagrams of 3-manifolds by considering the image of a given Heegaard diagram under iterations of the Dehn twist operator. If the initial diagram corresponds to a 3-manifold with boundary then the resulting sequence consists of diagrams of 3-manifolds which contain incompressible surfaces.

3.1. **Definition of a Dehn twist operator.** First we define the notion of "stacks" on a surface S which is in some sense analogous to train tracks.

Suppose X, Y are simplexes of the curve complex $C(S)$ such that they fill S. Then, the components of $S - (X \cup Y)$ are polygonal cells, every point of $X \cap Y$ is a vertex of order 4 and every face has an even number of edges which lie alternately in X and Y. Moreover, each polygon is at least a rectangle, since we are assuming that all intersections of X and Y are efficient, i.e. there are no "bigons".

Observation (J.Hempel [4]*)* If X and Y are simplexes of $C(S)$ with $S - (X \cup Y)$ simply connected and having n_i $2i$-gon components ($i = 1, 2, ...$), then

$$\chi(S) = \sum (1 - i/2) n_i$$

.

Since $n_1 = 0$ and $\chi(S) < 0$, the number of polygons with 6 or more edges is bounded by $|\chi(S)|$. Therefore, in a case of "not very trivial" intersection of X and Y, most of the complementary polygons will be rectangles with one pair of opposite edges lying in X and the other in Y.

Definition 3.1. An *X-stack* is a maximal collection of rectangles which are adjacent along common edges in X. The edges, which lie in *large* regions with ≥ 6 edges, are called the *top* and the *bottom* edges of the X-stack. The union of all Y

edges belonging to the X-stack defines the *sides* of the stack. There are, obviously, two sides in each X-stack which either lie in different curves of Y, or possibly in the same curve.

Every stack must have a top edge and bottom edge which do not coincide except for the degenerate case when there is only one edge. The Y-stacks are defined by interchanging the roles of X and Y.

The *height* of a stack is the number of its rectangles. A stack of height 0 consists of the common edge of two large polygonal regions. 0-height stacks occur rarely and throughout this work we almost always assume that intersection of curves of X and Y are complicated enough to have stacks of height at least 2.

Definition 3.2. Suppose S is a genus g orientable surface. Let $X = \{x_1, ..., x_g\}$ be a collection of pairwise disjoint simple closed curves on S. Call $X = \{x_1, ..., x_g\}$ a collection of *standard meridians* on S if $S - X$ is a sigle planar component.

If we attach a 2-handle along each x_i and glue a 3-ball for each 2-sphere boundary component we obtain the handlebody corresponding to the standard meridians. We will call this handlebody V_X.

The following definition is an extension of a notion of a standard Dehn twist along a curve on a surface.

Definition 3.3. Suppose $X = \{x_1, ..., x_g\}$ and $Y = \{y_1, ..., y_s\}$ are collections of simple closed curves such that $x_i \cap x_j = \emptyset$, $y_i \cap y_j = \emptyset$ and $X \cap y_j \neq \emptyset$ for all i, j and all intersections of X with Y are efficient. An image of a collection X under the *Dehn twist operator along a collection* Y, denoted by $\tau_Y(X)$, is the union of images $\{\tau_Y(x_1), ..., \tau_Y(x_g)\}$ of $\{x_1, ..., x_g\}$ under compositions of standard Dehn twists $\tau_{y_1} \circ \tau_{y_2} \circ ... \circ \tau_{y_s}$.

The following describes how to obtain $\tau_Y(X)$. For each j choose an annular neighbourhood A_j of y_j so that $A_i \cap A_j = \emptyset$ for all i, j. The image of the collection of g disjoint simple closed curves, $X = \{x_1, x_2, ...x_g\}$, under the homeomorphism τ_Y is a collection of g disjoint simple closed curves. To obtain the image of some x_i under the Dehn twist operator for each $j = 1, ..., s$ replace each arc of $A_j \cap x_i$ by an arc which circles around A_j once and smooth to general position relative to X.

Alternatively $\tau_Y(X)$ is the Haken sum (or oriented cut and paste) of a collection X and k copies of a collection Y, where $k = i(Y, X)$. That is for each y_j take k_j parallel copies of y_j, where $k_j = i(y_j, X)$. Call this collection $\bar{Y}$. Denote an annular neighbourhood of y_j containing k_j parallel copies by A_j. Choose annular neighbourhoods $\{\cup A_j\}$ so that they are pairwise disjoint. Then resolve each point of intersection of $\bar{Y}$ with X as shown in figure 1.

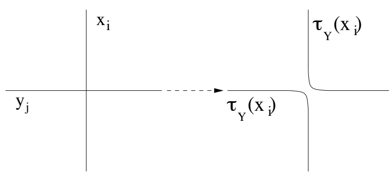

FIGURE 1. Resolution of a point of intersection

Note that the resolution of a point of intersection is independent of the orientation on the curves but is dependent on the orientation of S.

Consider intervals of $X - N(X \cap \bar{Y})$. Call an interval *small* if it lies between two parallel copies of some y_j. Call all the other intervals which lie between different components of Y *large*. Then $\tau_Y(x_i)$ contains almost all of each *large* interval in x_i except for the smoothed areas. As we continue along $\tau_Y(x_i)$ and exit a *large* interval of x_i, we enter some annular neighbourhood A_j containing k_j parallel copies of some y_j. Now, since we resolved points of intersection of all parallel copies of y_j with X we have to follow along the first copy of y_j. As we circle this annulus, each time we encounter X we switch to the next parallel copy of y_j. By the time we have circled around A_j one full time we have switched over all k_j copies of y_j. Therefore, we must exit to the next *large* interval of x_i. See figure 2.

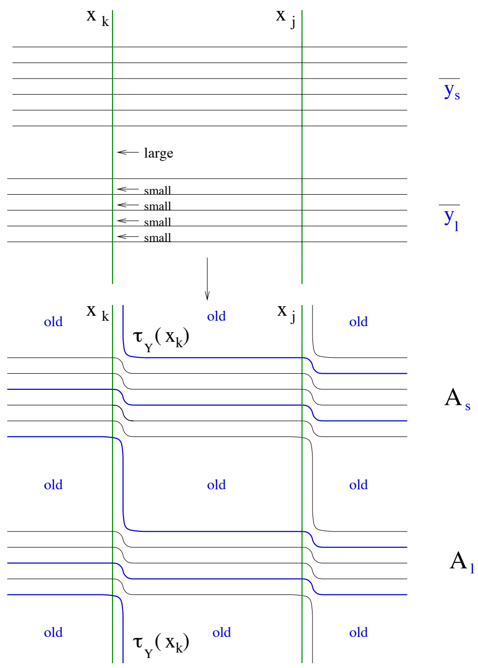

FIGURE 2. Construction of Dehn twist operator

Now consider the regions of $S - (\tau_Y(X) \cup X)$. The regions are of two types. The 'old' regions are essentially the regions of $S - (Y \cup X)$. The 'new' regions form *partial* X-stacks relative to $\tau_Y(X)$ each of which begins at an old region on one side of some A_j, circles A_j a total of $(k_j - 1)/k_j$-times and ends at an 'old' region on the other side of A_j. There are k_j partial X-stacks relative to $\tau_Y(X)$ in each A_j. Comparing X-partial stacks relative to $\tau_Y(X)$ to X-partial stacks relative to $\tau_Y(x_k)$ for some k, we note that there are fewer rectangles in X-partial stacks relative to $\tau_Y(x_k)$ and consequently there are fewer partial X-stacks relative to $\tau_Y(x_k)$ in A_j.

Remark: If instead of k_j parallel copies of y_j we take $n \times k_j$ copies and proceed as above, we obtain the image under n-fold Dehn twist operator, or $\tau_Y^n(X)$.

3.2. **Properties of Dehn twist operator.** Let $X = \{x_1, ..., x_g\}$ be a complete set of standard meridians for a genus g surface S. Let V_X be the corresponding handlebody. Let $Y = \{y_1, ..., y_k\}$ be a collection of essential, pairwise disjoint simple closed curves in $\partial V_X = S$ such that $X \cap y_j \neq \emptyset$ for all j and all intersections of Y and X are efficient.

We get a new collection Y^1 of simple closed curves by taking the image of X under n-fold Dehn twist operator along Y, or $Y^1 = \tau_Y^n(X)$.

Theorem 3.4. *Assuming the set up from the above let M be a 3-manifold determined by the Heegaard diagram $(\partial V_X; X, Y)$, possibly with boundary (if $k < g$). Let M^1 be a 3-manifold determined by the Heegaard diagram $(\partial V_X; X, Y^1)$ where $Y^1 = \tau_Y^n(X)$. Then, id: $\pi_1(V_X) \to \pi_1(V_X)$ extends to an epimorphism $\pi(M^1) \to \pi(M)$.*

Proof. Given a Heegaard diagram $(\partial V_X; X, Y)$, we can construct a presentation for $\pi_1(M)$ as follows: Choose the free basis $\{X_1, X_2, ..., X_g\}$ for the free group $\pi_1(V_X)$ which is "dual to" $\{x_1, ..., x_g\}$. For $j = 1, ..., k$ let r_j be a word in $X_1, X_2, ..., X_g$ representing the element of $\pi_1(V_X)$ determined by y_j. Note that r_j is unique up to inversion and conjugation. Then, it follows from Van Kampen's Theorem that $< X_1, ..., X_g : r_1, ..., r_k >$ is a presentation for $\pi_1(M)$. Similarly, $< X_1, ..., X_g : r_1^1, ..., r_k^1 >$ is a presentation for $\pi_1(M^1)$ where r_j^1 represents an element of $\pi_1(V_X)$ determined by $y_j^1 = \tau_Y^n(x_j)$.

By construction it follows that y_i^1 is homologous to $x_i + nk_1y_1 + nk_2y_2 + ... + nk_ky_k$ where $k_i = i(X, y_i)$. Since x_i is null homotopic it follows that y_i^1 is homotopic to products of conjugations of powers of the $\{y_j\}$. Denote by $\psi : \pi_1(V_X) \to \pi_1(M)$ and $\psi^1 : \pi_1(V_X) \to \pi_1(M^1)$ canonical epimorphisms.

Then, $Ker(\psi^1) \subset Ker(\psi)$.

Therefore, the diagram in figure 3 commutes giving the desired conclusion. □

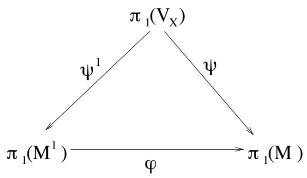

FIGURE 3. Commutative diagram

Corollary 3.5. *If M has nontrivial boundary then M^1 is a closed 3-manifold containing an incompressible surface.*

Proof. The fact that M^1 is closed follows easily from the observation that the image of the set of g standard meridians under compositions of homeomorphisms is a collection of exactly g pairwise disjoint simple closed curves such that $S - Y^1$ is a single planar component.

If $k < g$ then $\partial M \neq \emptyset$, hence the first Betti number $\beta_1(M) > 0$. Since $\varphi : \pi_1(M^1) \to \pi_1(M)$ is an epimorphism, it follows that $\beta_1(M^1) > 0$. The rest is given by standard facts of 3-manifold topology. See J. Hempel [5] for details. □

3.3. **Waves.**

Definition 3.6. Suppose $X = \{x_i\}$ and $Y = \{y_j\}$ are collections of simple closed curves on a surface S determining a Heegaard diagram $(S; X, Y)$. A *wave* for the diagram which is relative to X is an arc in S whose endpoints lie in the same component of X, whose interior misses $X \cup Y$, which lies on the same side of X near its endpoints, and which can not be isotoped to an arc in X.

Throughout this work we will be assuming that for a given Heegaard diagram $(S; X, Y)$ there are no waves relative to X where X is a collection of standard meridians. There is no harm in adding this assumption, since otherwise we can always perform a surgery along a wave and reduce the complexity of the diagram. See J. Hempel [4] for details.

Lemma 3.7. *Assume the setup of section 3.2. Suppose $(\partial V_X; X, Y)$ is a Heegaard diagram for some 3-manifold M. Let M^1 be a 3-manifold determined by the Heegaard diagram $(\partial V_X; X, Y^1)$ where $Y^1 = \tau_Y^n(X)$. If there are no waves relative to X for the diagram $(\partial V_X; X, Y)$, then there are no waves relative to X and Y^1 for the diagram $(\partial V_X; X, Y^1)$.*

Proof. Assume there is a wave w relative to X or Y^1. Then, interior of w lies in some "old" region of $\partial V_X - (X \cup Y^1)$. Consider the preimage of w under τ_Y^n. Since "old"regions are unchanged we get a wave $(\tau_Y^n)^{-1}(w)$ for the diagram $(\partial V_X; X, Y)$. Hence, we reach the desired contradiction. □

4. Main theorem

In this section we prove the main theorem which heavily relies on the proofs of the following lemmas.

4.1. **Lemmas.**

Lemma 4.1. *Let $(S; X, Y)$ describe a Heegaard diagram for a 3-manifold, where S is a surface of genus g, $X = \{x_1, ..., x_g\}$ is a collection of standard meridians. Let V_X be the corresponding handlebody bounded by S. Assume Y is a collection of pairwise disjoint simple closed curves such that Y intersects X nontrivially and efficiently and there are no waves relative to X. Let γ be a simple closed curve bounding a disk in V_X, i.e. $\gamma \in K_X$. Then γ crosses some Y-stack.*

Proof. Note that a curve *crosses a Y-stack* if it enters the stack through the top (bottom) edge, crosses every rectangular region and exits through the bottom (top) edge. A curve *partially crosses a Y-stack* if it enters the stack through the top (bottom) edge, crosses some (possibly all) of the rectangular regions and exits through the side of the stack, i.e. through an X-curve.

We assume that all intersections of γ with X and Y curves are efficient. We first suppose that $\gamma \cap X = \emptyset$. If $\gamma \cap Y = \emptyset$ also then we may tube γ to some component of X to create a wave. Hence we reach a contradiction. Thus $\gamma \cap Y \neq \emptyset$. Since $\gamma \cap X = \emptyset$, by our observation above γ cannot partially cross a Y-stack. Therefore γ crosses a Y-stack.

Let us now consider the case that $\gamma \cap X \neq \emptyset$. Denote by E a disk bounded by γ and denote by D_{x_i} disks bounded by x_i. Consider the arcs of $E \cap \cup D_{x_i}$ assuming that those intersections are efficient, i.e. can not be isotoped off E. Choose an outermost arc of $E \cap \cup D_{x_i}$ on E and call it e. The arc e cobounds a disk with a subarc of γ. Call the subarc f. See figure 4. We will show that f satisfies several of the properties required by a wave. Firstly note that the endpoints of f lie on the same component of X, say x_j. Next observe that the interior of f lies on the same side of x_j near its endpoints. For assume otherwise and consider the homology of V_X relative its boundary S. Then $e \cup f$ can be adjusted in a neighborhood of x_j on S so that a 1-cycle representing $e \cup f$ intersects a 2-cycle represented by D_{x_j} exactly once. Homology intersection number is a topological invariant, therefore $e \cup f$ can not be null homologous in $H_1(V_X; S)$. This contradicts the fact that $e \cup f$ is homotopically trivial in V_X. Lastly observe that since the arc e intersects the disk E efficiently, it follows that f and a subarc of x_j do not cobound a disk on S. Therefore, the cobounded area must include some component x_k.

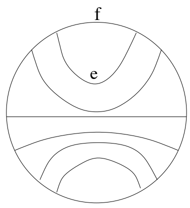

FIGURE 4. Disk E bounded by γ

We are now ready to show that f crosses a Y-stack. Assume otherwise. There are two cases to consider.

The first case is that $f \cap Y = \emptyset$. By our choice of arc f we have that the interior of f is disjoint from X. Together with the properties of f noted above we conclude that f is a wave, a contradiction.

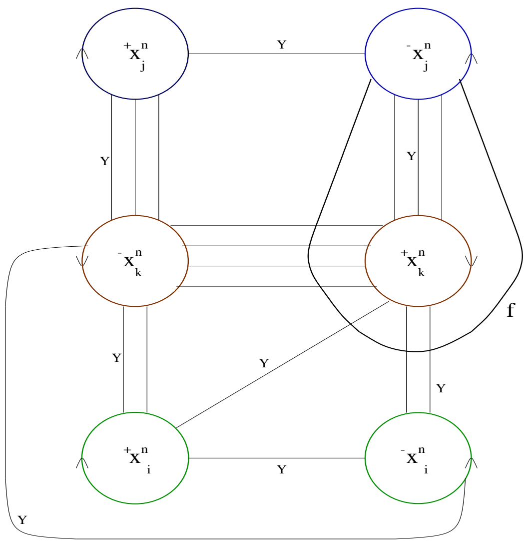

FIGURE 5. S cut open along the collection X and $\gamma \cap X \neq \emptyset$

The second case to consider is that $f \cap Y \neq \emptyset$ but every intersection of f with a Y-stack is a partial crossing. If f partially crosses at least three Y-stacks then by our initial observation f has at least three points of intersection with X. In particular this implies that the interior of f must have a point of intersection with X contradicting our choice of f. If f partially crosses a Y-stack that doesn't have x_j as a side then by our initial observation the interior of f must intersect X. Again this gives a point of intersection of the interior of f with X, a contradiction. Thus f partially crosses at most two Y-stacks each with x_j as a side; denote these Y-stacks by Y_f. Note that there are at most two components of $f \cap Y_f$ and each component contains an endpoint of f. Modify f by 'sliding' each component of $f \cap Y_f$ off Y_f, keeping the endpoints within the curve x_j. The resulting curve f' has no intersection with Y but retains the properties of f noted above. Thus f' is a wave, a contradiction.

Figure 5 of the 2-sphere with $2g$ disks removed represents a surface S cut open along a collection of g simple closed curves $X = \{x_1, ..., x_g\}$; this demonstrates a typical scenario for the various curves in this lemma. □

Definition 4.2 (Jason Leasure [8])**.** Suppose $X = \{x_i\}$ is a collection of pairwise disjoint simple closed curves, y and γ are simple closed curves which meet efficiently and nontrivially. Assume y intersects each component of X efficiently and nontrivially and γ intersects X efficiently. If $y \subset \gamma \cup a$ where a is an arc of $y - X$ then we say that y is almost contained in γ relative to X and denote this by $y \prec_X \gamma$.

This idea is most useful when $y \prec_X \gamma$ and there is a curve γ' such that $\gamma \cap \gamma' = \emptyset$. If this is the case, then γ' can intersect y in at most one arc of $y - X$, namely the arc containing a. We say that y is almost disjoint from γ'. See figure 6.

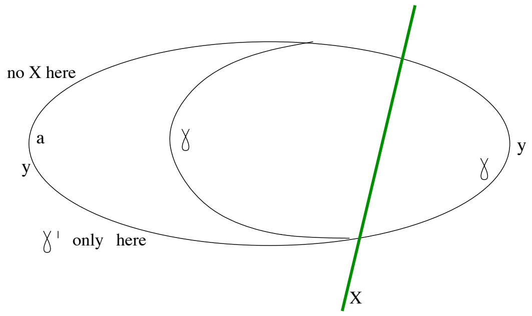

FIGURE 6. "almost contained" relation

Lemma 4.3. *Let S be a genus g orientable surface. Suppose $X = \{x_1, ..., x_g\}$ is a collection of standard meridians on S and $Y = \{\cup y_i\}$ is a collection of pairwise disjoint simple closed curves on S such that $i(x_i, y_j) \geq 2$ for each i and j. Suppose γ' is a simple closed curve on S that meets Y efficiently. Let*

$$Y^1 = \tau_Y^2(X) \text{ where } \tau \text{ is the Dehn twist operator.}$$

Let γ be a simple closed curve on S such that $\gamma \cap \gamma' = \emptyset$. Assume γ meets Y efficiently and nontrivially, and γ intersects Y^1 and X efficiently. If there exists a component y_k^1 of Y^1 such that $y_k^1 \prec_X \gamma'$ then there exists a component y_l of Y such that γ can be isotoped so that $y_l \prec_X \gamma$.

Proof. The image of the collection of g disjoint simple closed curves, $X = \{x_1, x_2, ...x_g\}$, under the homeomorphism τ_Y^2 is the collection of g disjoint simple closed curves Y^1. For $1 \leq s \leq g$ let A_s be an annular neighbourhood of the simple closed curve y_s from the collection Y. We require that the collection of annuli $\{A_s\}$ are pairwise disjoint. Let $k_s = i(y_s, X)$. To obtain one of the simple closed curves of Y^1 from the simple closed curve x_i of X, replace each arc of $A_s \cap x_i$ by an arc which circles around A_s twice and smooth to general position relative to X.

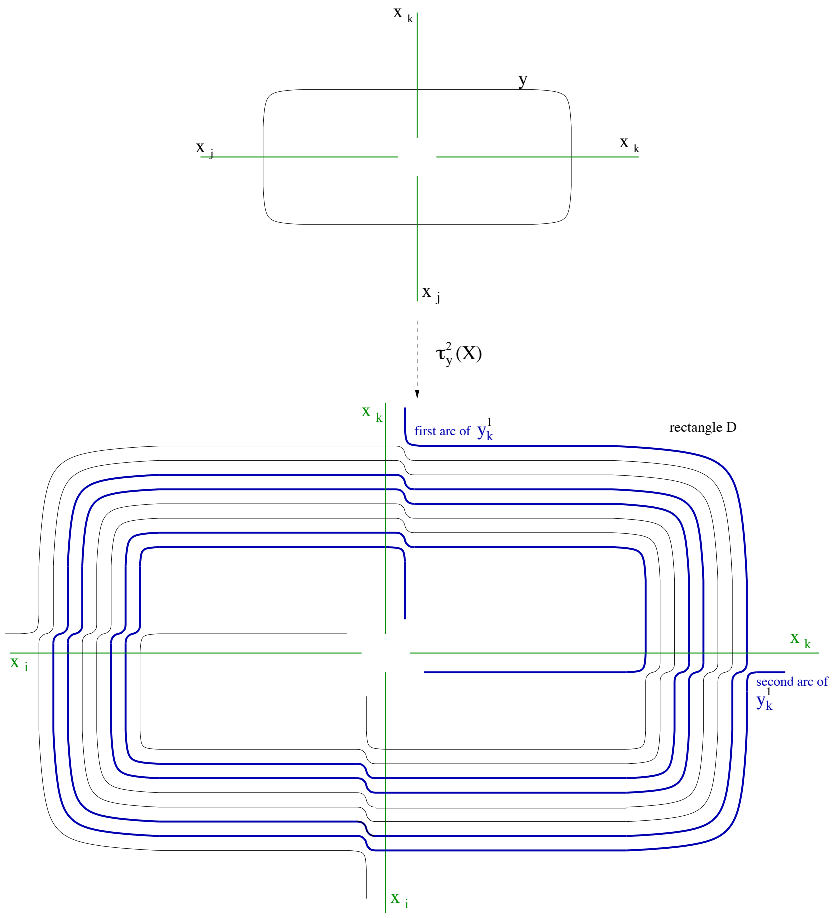

FIGURE 7. Arcs of y_k^1 inside the annulus A_j

From the assumptions we have $y_k^1 \prec_X \gamma'$, i.e. $y_k^1 \subset \gamma' \cup a$ where a is some arc of $y_k^1 - X$. Therefore, since $\gamma \cap \gamma' = \emptyset$ it follows that γ can intersect y_k^1 only in the arc a. By assumption $\gamma \cap Y \neq \emptyset$. Therefore there exists a component y_j of Y such that $\gamma \cap \partial A_j \neq \emptyset$. Now consider $y_k^1 \cap A_j$. Note, from the assumption $i(y_j, x_i) \geq 2$ for any i, j and the observation that $i(y_i^1, x_j) = 2i(Y, x_i) \times i(Y, x_j)$ where y_i^1 denotes the image of x_i under τ_Y^2 it follows that $i(y_i^1, x_j) \geq 2$ for all i, j. This implies that there are at least two arcs of $y_k^1 \cap A_j$. The following situation represents the worst possible case:

(1) there are only two arcs of $y_k^1 \cap A_j$ circle twice around A_j (that happens when $i(y_j, x_k) = 2$), and
(2) the ends of the two arcs are located in the 'closest' possible position i.e. if y_k is the image of x_k under the square of the Dehn twist then $|y_j \cap x_k| = 2$ and these points of intersection occur consecutively along x_k. See figure 7.

In this worst possible case there are two X-partial stacks rel to y_k^1 each of which circles around A_j slightly more than once. See section 3 for the detailed description of stacks in A_j.

We now analyse an arc of $\gamma \cap A_j$. In the worst case scenario γ enters the annulus A_j inside of one of the X-partial stacks rel y_k^1.

There are two possibilities for γ. Either γ circles around A_j inside the X-partial stack rel y_k^1 or γ intersects y_k^1. Note that γ can only intersect y_k^1 once since γ intersects y_k^1 in at most one arc a of $y_k^1 - X$ (see explanation above). In this latter case γ is forced to be inside the other X-partial stack rel y_k^1 and must circle A_j within that partial stack.

In either case there is a subarc b of γ which circles around A_j and comes back to the same rectangle D of $A_j - X$ where it started . Hence, we can isotope γ so that b coincides with the core of the annulus everywhere except in the rectangle D.

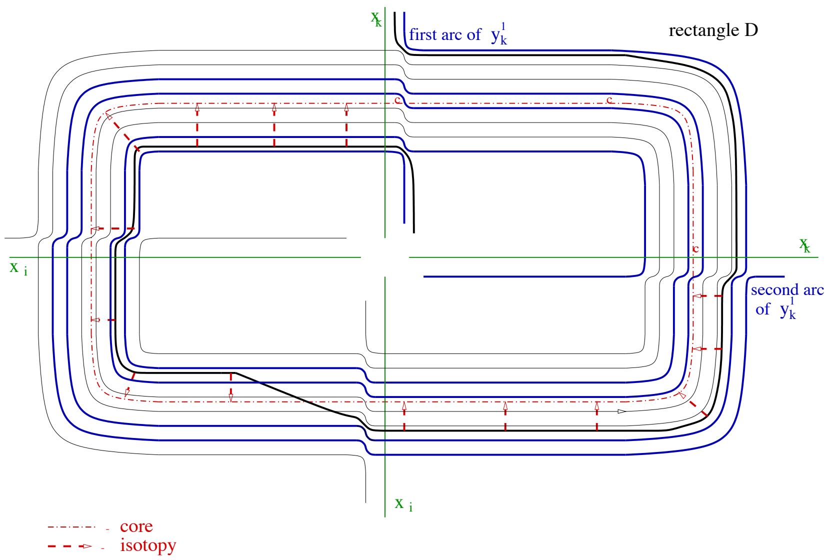

FIGURE 8. y_j is almost contained in γ after isotopy

Thus, $y_j = b \cup c$ where c is the subarc of $y_j \cap D$, i.e. $c \subset D$ connects the ends of the arc b. Thus $y_j \prec_X \gamma$. See figure 8.

Note that the isotopy is supported inside of the annulus A_j and is "perpendicular" to the core of the annulus, i.e. each x_i is fixed as a set. This isotopy simply moves points of γ toward the core. Thus, we can assume that we are not introducing inefficient intersections of γ and X. □

Now we are ready to prove the main theorem.

4.2. **Main Theorem.**

Theorem 4.4. *Let S be an orientable surface of genus $g \geq 2$. Suppose $X = \{x_1, x_2, ..., x_g\}$ is a collection of standard meridians on S and y is a simple closed curve on S such that $i(y, x_i) \geq 2$ and each Y-stack is of height at least 2. Let $(S; X, y)$ describe a Heegaard diagram for a 3-manifold. Assume there are no waves relative to X. For any $n \geq 1$ let*

$$\begin{aligned} Y^0 &= y \\ Y^1 &= \tau^2_{Y^0}(X) \\ &\vdots \\ Y^n &= \tau^2_{Y^{n-1}}(X). \end{aligned}$$

Then $dist(K_X, K_{Y^n}) \geq n$.

Proof. We proceed by contradiction. Suppose $dist(K_X, K_{Y^n}) = d$ for $d \leq n-1$, i.e. there exists a sequence of curves $\gamma_0, \gamma_1, ..., \gamma_d$ where $\gamma_0 \in K_X$, $\gamma_d \in K_{Y^n}$ and $\gamma_{i-1} \cap \gamma_i = \emptyset$ for all i.

From the assumption that there are no waves relative to X and from the properties of the Dehn twist operator (see Lemma 3.7) it follows that there are no waves for the diagram $(S; X, Y^n)$ relative to Y^n.

By Lemma 4.1 $\gamma_d \in K_{Y^n}$ either does not intersect Y^n or has a subarc α which is based on some component $y^n_{i_0}$ of Y^n and is not isotopic into $y^n_{i_0}$ relative to its base points. In any case some subarc of γ crosses some X-stack relative to Y^n. Since α is not isotopic into $y^n_{i_0}$, there exists a component $y^n_{j_0}$ of Y^n such that $\alpha \cap y^n_{j_0} = \emptyset$ and α crosses an X-stack relative to Y^n with $y^n_{j_0}$ as a side of this stack.

Let us consider a different picture introduced in Lemma 4.3 where we look at the collection of pairwise disjoint annuli $\{A^{n-1}_1, A^{n-1}_2, ..., A^{n-1}_g\}$ on the surface S and partial X-stacks circling around those annuli. Recall that A^{n-1}_i corresponds to an annular neighbourhood of y^{n-1}_i. Since we assume $i(y, x_i) \geq 2$ it follows that any partial X-stack relative to Y^n circles around any annulus A^{n-1}_i at least once. Since α crosses the X-stack relative to Y^n then α has to intersect a large interval of some x_i and then follow the next rectangle which enters a partial X-stack relative to Y^n which is inside of some annulus, say A^{n-1}_l. See figure 9.

Note that α can not intersect Y^n, therefore α has to stay inside of that partial X-stack relative to Y^n. That means α has to circle around the annulus A^{n-1}_l and come back to the same rectangle D of $A^{n-1}_l - X$. Thus, we can isotope the subarc b equal to $\alpha \cap (A^{n-1}_l - D)$ to coincide with the core of the annulus which is $y^{n-1}_{l_1}$. That is possible to do everywhere except in the rectangle D. Then, connect the ends of the isotoped b by an arc $b' \subset y^{n-1}_l$ that lies in the rectangle D. Therefore, after ambient isotopy we have $y^{n-1}_{l_1} \prec_X \gamma_d$. We will isotope the curves $\{\gamma_{d-1}, ..., \gamma_0\}$ by the same ambient isotopy. Continue to call the resulting curves $\{\gamma_d, \gamma_{d-1}, ..., \gamma_0\}$. Thus, the property $\gamma_i \cap \gamma_{i-1} = \emptyset$ is preserved. Note, this isotopy is supported inside of the annulus A^{n-1}_l and can be chosen so that each x_i is fixed as a set. Therefore, we are not introducing any inefficient intersections of X and $\{\gamma_{d-1}, ..., \gamma_0\}$. Also, we

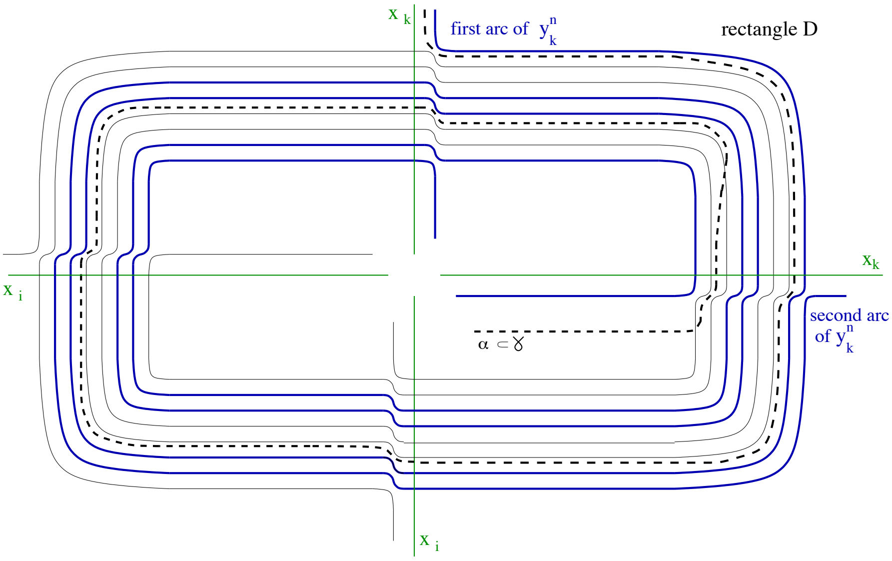

FIGURE 9. Arc α inside of annulus A_j^{n-1}

can choose this isotopy so that γ_{d-i} intersects efficiently $Y^{n-j}, ..., Y^0$ for $1 \le i \le d$ and $2 \le j \le n$.

Applying Lemma 4.3 to $y_{l_1}^{n-1}$ and inducting using the set $(\gamma_d, \gamma_{d-1}, ..., \gamma_0)$ we conclude that γ_0 can be isotoped so that

$$y_{i_{d+1}}^{n-(d+1)} \prec_X \gamma_0 \text{ where } \gamma_0 \in K_X \qquad (*)$$

Note: In order to apply Lemma 2 we need to assume that

$$\gamma_{d-k} \cap Y^{n-(k+1)} \neq \emptyset \text{ for } k = 1, ..., d \qquad (**)$$

Let us consider it later as a special case and for now let us assume that (**) holds.

From the assumptions and properties of the Dehn twist operator (see Lemma 3.7) it follows that there are no waves relative to X for the diagram $(S; X, Y^{n-(d+1)})$. By lemma 4.1 either $\gamma_0 \cap X = \emptyset$ or there exists an outermost subarc c of $\gamma_0 - X$ such that c is based on the same component x_i. In either case there is a subarc c which crosses some $Y^{n-(d+1)}$-stack. It follows from the assumption on y that $y_{i_{d+1}}^{n-(d+1)}$ has at least two arcs in that stack. Therefore, the subarc c must cross at least two arcs of $y_{i_{d+1}}^{n-(d+1)} - X$. It follows from (*) that only one arc of $y_{i_{d+1}}^{n-(d+1)} - X$ does not lie in γ_0. Therefore γ_0 must be singular. Hence, we have reached a contradiction.

Let us show that for each inductive step (**) holds. That is, if we have found a component $y_{i_k}^{n-k}$ of Y^{n-k} with $y_{i_k}^{n-k} \prec_X \gamma_{d-k+1}$ then $\gamma_{d-k} \cap Y^{n-(k+1)} \neq \emptyset$. Now suppose (**) does not hold, i.e. $\gamma_{d-k} \cap Y^{n-(k+1)} = \emptyset$. If $d - k > 1$, then γ_{d-k} meets every component x_i of X. Otherwise $x_i, \gamma_{d-k}, \gamma_{d-k+1}, ..., \gamma_d$ is a shorter path connecting K_X with K_{Y^n}. That contradicts our assumptions. In fact γ_{d-k} meets each x_i in at least two arcs of $x_i - Y^{n-(k+1)}$, since it must go around an X-stack relative to $Y^{n-(k+1)}$ and every X-stack contains at least two arcs of $x_i - Y^{n-(k+1)}$.

Since $y_{i_k}^{n-k} = \tau^2_{Y^{n-(k+1)}}(x_i)$, it follows that γ_{d-k} crosses $y_{i_k}^{n-k}$ in at least two arcs of $y_{i_k}^{n-k} - X$. Since $y_{i_k}^{n-k} \prec_X \gamma_{d-k+1}$, we conclude that $\gamma_{d-k} \cap \gamma_{d-k+1} \neq \emptyset$. This contradicts our assumptions. The last case to consider is when $d-k=1$. So $\gamma_1 \cap Y^{n-d} = \emptyset$. By assumption $d < n$, so $n-d \geq 1$. But then γ_1 bounds a disk in $V_{Y^{n-d}}$. Therefore, the distance of $(S; V_X, V_{Y^{n-d}})$ is ≤ 1, i.e. the splitting is weakly reducible. However, in this diagram every X-stack meets every Y^{n-d}-stack. This is the Casson-Gordon condition that the splitting is not weakly reducible. Hence, we have reached the desired contradiction. □

5. Genus two Heegaard diagrams and examples

5.1. **Positive Heegaard diagrams of genus two.**

Definition 5.1. An *oriented Heegaard diagram* $(S; X, Y)$ is a Heegaard diagram where X and Y are given specific orientations.

Let $<,>$: $H_1(S) \times H_1(S) \to \mathbb{Z}$ denote the algebraic intersection number on a surface S. So for oriented simple closed curves x, y on S meeting efficiently, $< x, y >= i(x, y)$ means that the algebraic intersection number is $+1$ at each point of $x \cap y$.

Definition 5.2. A *positive Heegaard diagram* $(S; X, Y)$ is an oriented Heegaard diagram where the algebraic intersection number $< X, Y >_p$ of X with Y is $+1$ at each point $p \in X \cap Y$.

Every compact, oriented 3-manifold with no 2-sphere boundary components can be represented by a positive diagram; see Hempel [5]. In this section we will be focusing on genus two positive Heegaard diagrams.

For a given positive Heegaard diagram $(S; X, Y)$ we can construct a picture by cutting S open along X. The result will be a 2-manifold S_1 whose boundary contains disjoint copies X^+ and X^- of X together with a map $f : S_1 \to S$ which maps $S_1 - X^+ \cup X^-$ homeomorphically onto $M - X$ and maps each of X^+ and X^- homeomorphically onto X.

If the genus of S is two and X contains exactly two components x_1 and x_2, then S_1 is a four times punctured 2-sphere with the boundary components $x_1^+, x_1^-, x_2^+, x_2^-$. The components of Y will be strands connecting $x_1^+, x_1^-, x_2^+, x_2^-$. See figure 10.

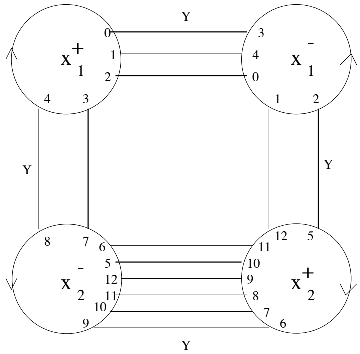

FIGURE 10. S cut open along X

Since we are assuming that the diagram is positive, it follows that the diagram will be in a shape of a "square", i. e. there are no strands connecting x_i^+ with x_j^+ and x_i^- with x_j^-.

Similarly we can construct an analogous picture by cutting S open along Y. In this case we will call it a Y-side of the diagram.

Given such a picture, we need specific instructions how to recover the original diagram. For that we need to describe how to glue back x_1^+ with x_1^- and x_2^+ with x_2^-.

Definition 5.3. We define as the *twist number* from x_i^+ to x_i^- for $i = 1, 2$ the amount of twist used in gluing x_i^+ back to x_i^- to reconstruct the original diagram.

Definition 5.4. For a positive Heegaard diagram $(S; X, Y)$ define a five-tuple vector (p, q, r, n, m) by specifying the following:

$$\begin{array}{rcl} p & = & \text{number of } Y \text{ strands from } x_1^+ \text{ to } x_2^- \\ q & = & \text{number of } Y \text{ strands from } x_1^+ \text{ to } x_1^- \\ r & = & \text{number of } Y \text{ strands from } x_2^+ \text{ to } x_2^- \\ n & = & \text{twist number from } x_1^+ \text{ to } x_1^- \\ m & = & \text{twist number from } x_2^+ \text{ to } x_2^- \end{array}$$

Thus, given this vector (p, q, r, n, m) we can draw the cut-open diagram for this splitting. The values p, q, r allow us to draw the strands between each of the (cut-open) components of X. We can then number the intersection points on x_1^- and x_2^+ consecutively following the orientation, starting at an arbitrary point on each. The twist numbers n and m then tell us how to label the points on x_1^- and x_2^-. In our example in figure 10 the corresponding vector is $(2, 3, 6, 3, 3)$ and represents two disjoint simple closed Y-curves.

The following proposition follows immediately from the definition of the vector (p, q, r, n, m).

Proposition 5.5. *Suppose the vectors* $v(y_1) = (p_1, q_1, r_1, n_1, m_1)$ *and* $v(y_2) = (p_2, q_2, r_2, n_2, m_2)$ *represent pairwise disjoint simple closed curves* y_1 *and* y_2 *respectively. Then their union* $y = y_1 \cup y_2$ *is represented by the vector* $v(y) = (p_1 + p_2, q_1 + q_2, r_1 + r_2, n_1 + n_2, m_1 + m_2)$.

Next we will attempt to consider the action of Dehn twisting operator on five-tuple vectors.

Let $X = \{x_1, x_2\}$ be a set of oriented meridians for an oriented genus two handlebody bounded by S and let $Y = \{y_i\}$, $i \leq 2$ be a collection of oriented pairwise disjoint curves which meet X positively. Thus y_i can be represented by a vector $v(y_i) = (p_i, q_i, r_i, n_i, m_i)$.

For $a_1, a_2 \in \mathbb{Z}_+$ let $\tau = \tau_{a_1 y_1 + a_2 y_2} = \tau_{y_1}^{a_1} \circ \tau_{y_2}^{a_2}$ be a the a_1-fold Dehn twist along y_1 together with the a_2-fold Dehn twist along y_2. Let $l_{ij} =< x_j, y_i >$ and $l_i = l_{i1} + l_{i2} =< X, y_i >$.

Proposition 5.6. $v(\tau(x_j)) = a_1 l_{1j} v(y_1) + a_2 l_{2j} v(y_2) + \epsilon_j$ *where*

$$\epsilon = \begin{cases} (0,0,0,1,0) & j{=}1 \\ (0,0,0,0,1) & j{=}2 \end{cases}$$

Proof. For a detailed description of the image of X under the Dehn twist operator along the collection of a_1 parallel copies of y_1 and a_2 parallel copies of y_2 see section 3.

So, there are $a_1 l_1 + a_2 l_2$ strands of $\tau(x_j) - X$ parallel to each strand of $Y - X$. This establishes the first three coordinates of the proposition. Fix a homological basis (x_1, x_2, X_1, X_2) where X_i is a longitude meeting x_i in a single point for $i = 1, 2$ so that $< x_i, X_i >= +1$. Then $n_i =< y_i, X_1 >$ and $m_i =< y_i, X_2 >$.

Observe that $\tau(x_j)$ is homologous to $x_j + a_1 < x_j, y_1 > y_1 + a_2 < x_i, y_2 > y_2 = x_j + a_1 l_{1j} y_1 + a_2 l_{2j} y_2$. Thus $< \tau(x_j), X_1 >= \delta_{1j} + l_{1j} n_1 + l_{2j} n_2$ and $< \tau(x_j), X_2 >= \delta_{2j} + l_{1j} m_1 + l_{2j} m_2$

$$where \quad \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

□

Corollary 5.7. $v(\tau(X)) = a_1 l_1 v(y_1) + a_2 l_2 v(y_2) + (0, 0, 0, 1, 1)$

Proposition 5.8. *Let M_0, M be the 3-manifolds represented by positive Heegaard diagrams $(S; X, Y)$ and $(S; X, \tau(X))$ respectively. Then $H_1(M)$ is presented by the matrix*

$$\begin{pmatrix} p_1 + q_1 & p_2 + q_2 \\ p_1 + r_1 & p_2 + r_2 \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} p_1 + q_1 & p_1 + r_1 \\ p_2 + q_2 & p_2 + r_2 \end{pmatrix}$$

where

$$\begin{pmatrix} p_1 + q_1 & p_2 + q_2 \\ p_1 + r_1 & p_2 + r_2 \end{pmatrix}$$

presents $H_1(M_0)$.

Proof.

$$\begin{pmatrix} < x_1, \tau(x_1) > & < x_1, \tau(x_2) > \\ < x_2, \tau(x_1) > & < x_2, \tau(x_2) > \end{pmatrix}$$

presents $H_1(M)$.

Also, $< x_1, \tau(x_j) >= p(\tau(x_j) + q(\tau(x_j))$ and $< x_2, \tau(x_j) >= p(\tau(x_j) + r(\tau(x_j))$.

Similarly

$$\begin{pmatrix} < x_1, y_1 > & < x_1, y_2 > \\ < x_2, y_1 > & < x_2, y_2 > \end{pmatrix}$$

presents $H_1(M_0)$ where $< x_1, y_j >= p_j + q_j$ and $< x_2, y_j >= p_j + r_j$. Since

$$l_{ij} =< x_j, y_i >= \begin{cases} p_i + q_i & \text{if j=1} \\ p_i + r_i & \text{if j=2} \end{cases}$$

the result of the claim follows from direct calculation using the equalities:

$$\begin{aligned} p(\tau(x_j)) &= a_1 l_{1j} p_1 + a_2 l_{2j} p_2 \\ q(\tau(x_j)) &= a_1 l_{1j} q_1 + a_2 l_{2j} q_2 \\ r(\tau(x_j)) &= a_1 l_{1j} r_1 + a_2 l_{2j} r_2 \end{aligned}$$

□

Corollary 5.9. *If Y has a single component (i.e. the 3-manifold M_0 has non-trivial boundary) then $H_1(M)$ is infinite.*

Proof. Let P denote a representation matrix of $H_1(M)$. We can assume that $a_2 = 0$. Then by proposition 5.8 $det(P) = 0$. □

Note, this corollary follows from Theorem 3.4 as well.

Corollary 5.10. *Suppose $a_1 a_2 \neq 0$, i.e. $H_1(M_0)$ is finite and M_0 is necessarily closed then $o(H_1(M)) = a_1 a_2 \times o(H_1(M_0))^2$.*

5.2. **Examples.** Suppose Y has a single component y represented by the vector $v(y) = (2,2,2,1,2)$ on a genus two surface S which bounds a handlebody determined by standard meridians $X = \{x_1, x_2\}$. Let $Y^0 = y$ and $Y^n = \tau^2_{Y^{n-1}}(X)$ for $n \geq 1$. Then $l_1 =< y, X >= 8$. Let $a_1 = 2$.

By proposition 5.6 the image of X under 2-fold Dehn twisting operator is represented by $v(Y^1) = (32, 32, 32, 17, 33)$. By Theorem 4.4 it follows that the 3-manifold M^1 determined by the Heegaard diagram $(S; X, Y^1)$ is closed, irreducible, Haken 3-manifold and the distance of this splitting is ≥ 1.

The next step in the iteration gives $v(Y^2) = (4096, 4096, 4096, 2177, 4225)$. The 3-manifold M^2 defined by the Heegaard diagram $(S; X, Y^2)$ is again a closed, irreducible, Haken 3-manifold and the distance of this splitting is ≥ 2.

After iterating one more time we get

$$v(Y^3) = (67108864, 67108864, 67108864, 35667969, 69222401)$$

The 3-manifold M^3 determined by the Heegaard diagram (S, X, Y^3) is closed, irreducible, Haken and atoroidal since the distance of this splitting is ≥ 3 (see Hempel [5]). Thus by Thurston's hyperbolisation theorem M^3 admits a hyperbolic metric.

If we keep iterating we get an infinite sequence of hyperbolizable 3-manifolds with arbitrarily large distance.

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