Minimum-Length Polygon of a Simple Cube-Curve in 3D Space

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Abstract. We consider simple cube-curves in the orthogonal 3D grid of cells. The union of all cells contained in such a curve (also called the tube of this curve) is a polyhedrally bounded set. The curve's length is defined to be that of the minimum-length polygonal curve (MLP) fully contained and complete in the tube of the curve. So far, only a "rubber-band algorithm" is known to compute such a curve approximately. We provide an alternative iterative algorithm for the approximative calculation of the MLP for curves contained in a special class of simple cube-curves (for which we prove the correctness of our alternative algorithm), and the obtained results coincide with those calculated by the rubber-band algorithm.

1 Introduction

The analysis of cube-curves is related to 3D image data analysis. A cube-curve is, for example, the result of a digitization process which maps a curve-like object into a union S of face-connected closed cubes. The computation of the length of a cube-curve was the subject in [3], and the suggested local method has its limitations if studied with respect to multigrid convergence. [1] presents a rubber-band algorithm for an approximative calculation of a minimum-length polygonal curve (MLP) in S. So far it was still an open problem to prove whether results of the rubber-band algorithm always converge to the exact MLP or not. In this paper we provide a non-trivial example where the rubber-band algorithm is converging against the MLP. So far, MLPs could only be tested manually for "simple" examples.

This paper also presents an algorithm for the computation of approximate MLPs for a special class of simple cube-curves. (The example for the rubber-band algorithm is from this class.)

Following [1], a grid point $(i, j, k) \in \mathbb{Z}^3$ is assumed to be the center point of a *grid cube* with *faces* parallel to the coordinate planes, with *edges* of length 1, and *vertices* as its corners. *Cells* are either cubes, faces, edges, or vertices. The intersection of two cells is either empty or a joint *side* of both cells. A *cube-curve* is an alternating sequence $g = (f_0, c_0, f_1, c_1, \ldots, f_n, c_n)$ of faces f_i and cubes c_i, for $0 \leq i \leq n$, such that faces f_i and f_{i+1} are sides of cube c_i, for $0 \leq i \leq n$ and $f_{n+1} = f_0$. It is *simple* iff $n \geq 4$ and for any two cubes $c_i, c_k \in g$ with $|i - k| \geq 2$

(mod $n+1$), if $c_i \bigcap c_k \neq \phi$ then either $|i-k| \geq 2$ (mod $n+1$) and $c_i \bigcap c_k$ is an edge, or $|i-k| \geq 3$ (mod $n+1$) and $c_i \bigcap c_k$ is is a vertex.

A *tube* **g** is the union of all cubes contained in a cube-curve g. A tube is a compact set in $\mathbb{R}^3$, its frontier defines a polyhedron, and it is homeomorphic with a torus in case of a simple cube-curve. A curve in $\mathbb{R}^3$ is *complete* in **g** iff it has a nonempty intersection with every cube contained in g. Following [4, 5], we define:

Definition 1. *A* minimum-length polygon *(MLP) of a simple cube-curve g is a shortest simple curve P which is contained and complete in tube* **g**. *The length of a simple cube-curve g is defined to be the length $l(P)$ of an MLP of g.*

It turns out that such a shortest simple curve P is always a polygonal curve, and it is uniquely defined if the cube-curve is not only contained in a single layer of cubes of the 3D grid (see [4, 5]). If contained in one layer, then the MLP is uniquely defined up to a translation orthogonal to that layer. We speak about *the* MLP of a simple cube-curve.

A *critical edge* of a cube-curve g is such a grid edge which is incident with exactly three different cubes contained in g. Figure 1 shows all the critical edges of a simple cube-curve.

Definition 2. *If e is a critical edge of g and l is a straight line such that $e \subset l$, then l is called a* critical line of e in g *or* critical line *for short.*

Definition 3. *Assume a simple cube-curve g and a triple of consecutive critical edges e_1, e_2, and e_3 such that $e_i \perp e_j$, for all $i, j = 1, 2, 3$ with $i \neq j$. If the x-coordinates (y-coordinates, or z-coordinates) of two vertices (i.e., end points) of e_1 and e_3 are equal when e_2 is parallel to the x-axis (y-axis, or z-axis), we say that e_1, e_2 and e_3 form an* end angle, *and g* has an end angle, *denoted by $\angle(e_1, e_2, e_3)$; otherwise we say that e_1, e_2 and e_3 form a* middle angle, *and g* has a middle angle.

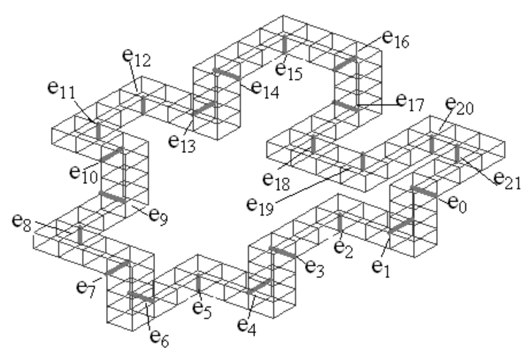

Fig. 1. Example of a first-class simple cube-curve which has middle and end angles.

Figure 1 shows a simple cube-curve which has 5 end angles $\angle(e_{21}, e_0, e_1)$, $\angle(e_4, e_5, e_6)$, $\angle(e_6, e_7, e_8)$, $\angle(e_{14}, e_{15}, e_{16}))$, $\angle(e_{16}, e_{17}, e_{18})$, and many middle angles (e.g., $\angle(e_0, e_1, e_2)$, $\angle(e_1, e_2, e_3)$, and $\angle(e_2, e_3, e_4)$).

Definition 4. *A simple cube-curve g is called* first class *iff each critical edge of g contains exactly one vertex of the MLP of g.*

This paper focuses on first-class simple cube-curves which have at least one end angle (as the one in Figure 1).

Definition 5. *Let $S \subseteq \mathbb{R}^3$. The set $\{(x, y, 0) : \exists z(z \in \mathbb{R} \wedge (x, y, z) \in S)\}$ is the* xy-projection *of S, or* projection *of S for short. Analogously we define the yz- or xz-*projection *of S.*

The paper is organized as follows: Section 2 describes theoretical fundamentals for the length calculation of first-class simple cube-curves. Section 3 presents our algorithm for length computation. Section 4 gives experimental results of an example and a discussion of results obtained by the rubber-band algorithm for this particular input. Section 5 gives the conclusions.

2 Basics

We provide mathematical fundamentals used in our algorithm for computing the MLP of a first-class simple cube-curve. We start with citing a basic theorem from [1]:

Theorem 1. *Let g be a simple cube-curve. Critical edges are the only possible locations of vertices of the MLP of g.*

This theorem is of fundamental importance for both the rubber-band algorithm and our algorithm (to be defined later in this paper). Let $d_e(p, q)$ be the Euclidean distance between points p and q.

Let e_1, e_2, and e_3 be three (not necessarily consecutive) critical edges in a simple cube-curve, and let l_1, l_2, and l_3 be the corresponding three critical lines. We express a point $p_2(t_2) = (x_2 + k_{x_2}t_2, y_2 + k_{y_2}t_2, z_2 + k_{z_2}t_2)$ on l_2 in general form, with $t_2 \in \mathbb{R}$. Analogously, let $p_1(t_1)$, $p_3(t_3)$ be points on l_1, l_3, respectively.

Lemma 1. *Let $d_2(t_1, t_2, t_3) = d_e(p_1, p_2) + d_e(p_2, p_3)$. It follows that $\frac{\partial^2 d_2}{\partial {t_2}^2} > 0$.*

Proof. Let the coordinates of p_i be $(x_i + k_{x_i}t_i, y_i + k_{y_i}t_i, z_i + k_{z_i}t_i)$, where i equals 1 or 3. Since $p_i \in e_i \subset l_i$, and e_i is a critical edge which is an edge of an orthogonal grid, only one of the values k_{x_i}, k_{y_i} and k_{z_i} can be 1 and the other two must be zero. Let us look at one of these cases where the coordinates of p_1 be $(x_1 + t_1, y_1, z_1)$, the coordinates of p_2 be $(x_2, y_2 + t_2, z_2)$, and the coordinates of p_3 be $(x_3, y_3, z_3 + t_3)$. Then $d_2 = d_e(p_1, p_2) + d_e(p_2, p_3) =$

$$= \sqrt{(t_2 - (y_1 - y_2))^2 + (x_1 + t_1 - x_2)^2 + (z_1 - z_2)^2} + \sqrt{(t_2 - (y_3 - y_2))^2 + (x_3 - x_2)^2 + (z_3 + t_3 - z_2)^2}$$

This can be written as $d_2 = \sqrt{(t_2-a_1)^2+b_1^2} + \sqrt{(t_2-a_2)^2+b_2^2}$, where b_1 and b_2 are functions of t_1 and t_3. Then we have

$$\frac{\partial d_2}{\partial t_2} = \frac{t_2-a_1}{\sqrt{(t_2-a_1)^2+b_1^2}} + \frac{t_2-a_2}{\sqrt{(t_2-a_2)^2+b_2^2}} \tag{1}$$

and

$$\begin{aligned}\frac{\partial^2 d_2}{\partial {t_2}^2} = & \frac{1}{\sqrt{(t_2-a_1)^2+b_1^2}} - \frac{(t_2-a_1)^2}{[(t_2-a_1)^2+b_1^2]^{3/2}} \\ & + \frac{1}{\sqrt{(t_2-a_2)^2+b_2^2}} - \frac{(t_2-a_2)^2}{[(t_2-a_2)^2+b_2^2]^{3/2}}\end{aligned}$$

This simplifies to

$$\frac{\partial^2 d_2}{\partial {t_2}^2} = \frac{b_1^2}{[(t_2-a_1)^2+b_1^2]^{3/2}} + \frac{b_2^2}{[(t_2-a_2)^2+b_2^2]^{3/2}} > 0 \tag{2}$$

All other cases follow analogously. □

Let l_i be a critical line, $p_i \in l_i$, where $i = 0, 1, 2, \ldots, n$. Let $d(t_0, t_1, \ldots, t_n) = \sum_{i=0}^{n-1} d_e(p_i, p_{i+1})$. Assume $n+1$ reals t_{i_0} $(i = 0, 1, \ldots, n)$ which define a minimum $d(t_{0_0}, t_{1_0}, \ldots, t_{n_0})$ of function $d(t_0, t_1, \ldots, t_n)$. By Lemma 1 we immediately obtain

Lemma 2. *For any two reals t_{i_1} and t_{i_2}, we have*

$$d(t_{0_0}, \ldots, t_{i_0}, \ldots, t_{n_0}) < d(t_{0_0}, \ldots, t_{i_1}, \ldots, t_{n_0}) < d(t_{0_0}, \ldots, t_{i_2}, \ldots, t_{n_0})$$

if $t_{i_0} < t_{i_1} < t_{i_2}$, and

$$d(t_{0_0}, \ldots, t_{i_1}, \ldots, t_{n_0}) > d(t_{0_0}, \ldots, t_{i_2}, \ldots, t_{n_0}) > d(t_{0_0}, \ldots, t_{i_0}, \ldots, t_{n_0})$$

if $t_{i_1} < t_{i_2} < t_{i_0}$.

Let e_1, e_2, and e_3 be three critical edges, and let l_1, l_2, and l_3 be their critical lines, respectively. Let p_1, p_2, and p_3 be three points such that p_i belongs to l_i, where $i = 1, 2, 3$. Let the coordinates of p_2 be $(x_2 + k_{x_2} t_2, y_2 + k_{y_2} t_2, z_2 + k_{z_2} t_2)$. Let $d_2 = d_e(p_1, p_2) + d_e(p_2, p_3)$.

Lemma 3. *The function $f(t_2) = \frac{\partial d_2}{\partial t_2}$ has a unique real root.*

Proof. Examine the proof of Lemma 1. Without loss of generality, we can assume that $a_1 \le a_2$. Then by Equation (1) we have $f(a_1) \le 0$ as well as $f(a_2) \ge 0$. The lemma follows with Equation (2). □

Let l_i be a critical line, $p_i \in l_i$, the coordinates of p_i be $(x_i + k_{x_i} t_i, y_2 + k_{y_i} t_i, z_i + k_{z_i} t_i)$, where $i = 1, 2, \ldots, n$. Let $d(t_0, t_1, \ldots, t_n) = \sum_{i=0}^{n-1} d_e(p_i, p_{i+1})$.

Theorem 2. *There is a unique* $(n+1)$*-tuple of reals* t_{i_0} $(i = 0, 1, \ldots, n)$ *defining the minimum* $d(t_{0_0}, t_{1_0}, \ldots, t_{n_0})$ *of* $d(t_0, t_1, \ldots, t_n)$*, with* $\frac{\partial d}{\partial t_i}(t_{0_0}, t_{1_0}, \ldots, t_{n_0}) = 0$*, for* $i = 0, 1, \ldots, n$*.*

Proof. From the proof of Lemma 1 we know that there are two reals a_{i_1} and a_{i_2} such that $a_{i_1} \le a_{i_2}$ and

$$\frac{\partial d}{\partial t_i} = \frac{t_i - a_{i_1}}{\sqrt{(t_i - a_{i_1})^2 + {b_{i_1}}^2}} + \frac{t_i - a_{i_2}}{\sqrt{(t_i - a_{i_2})^2 + {b_{i_2}}^2}}$$

for every $i \in \{0, 1, \ldots, \text{n}\}$. By Lemma 3, there is a unique real root $t_{i_0} \in [a_{i_1}, a_{i_2}]$ for $\frac{\partial d}{\partial t_i} = 0$, where $-\infty < t_i < \infty$. On the other hand, if there are m reals $t_i = t'_{i_0}$ $(i = 0, 1, \ldots, n)$ such that $d(t'_{0_0}, t'_{1_0}, \ldots, t'_{n_0})$ is a minimum of $d(t_0, t_1, \ldots, t_n)$ then $\frac{\partial d}{\partial t_i}(t'_{i_0}) = 0$. $\square$

Let e_1, e_2 and e_3 be three consecutive critical edges of a simple cube-curve g. Let $D(e_1, e_2, e_3)$ be the dimension of the linear space generated by e_1, e_2 and e_3. Let l_{13} be a line segment with its two end points at e_1 and e_3. Let $d_{e_i e_j}$ be Euclidean distance between e_i and e_j (i.e., the minimum distance between points p on e_i and q on e_j), where $i, j = 1, 2, 3$.

Lemma 4. *The line segment* l_{13} *is not completely contained in* ***g*** *if* $D(e_1, e_2, e_3) = 3$*,* $\min\{d_{e_1e_2}, d_{e_2e_3}\} \ge 1$ *and* $\max\{d_{e_1e_2}, d_{e_2e_3}\} \ge 2$*, or if* $D(e_1, e_2, e_3) \le 2$ *and* $\min\{d_{e_1e_2}, d_{e_2e_3}\} \ge 2$*.*

Proof. Case 1. Let $D(e_1, e_2, e_3) = 3$, $\min\{d_{e_1e_2}, d_{e_2e_3}\} \ge 1$ and $\max\{d_{e_1e_2}, d_{e_2e_3}\} \ge 2$. We only need to prove that the conclusion is true when $\min\{d_{e_1e_2}, d_{e_2e_3}\} = 1$ and $\max\{d_{e_1e_2}, d_{e_2e_3}\} = 2$. In this case, the parallel projection (denoted by $g'(e_1, e_2, e_3)$) of all of g's cubes contained between e_1 and e_3 is illustrated in Figure 2, where AB is the projective image of e_1, EF that of e_3, and C that of one of the end points of e_2. Note that line segment AF must intercept grid edge BC at a point G, and intercept grid edge CD at a point H. And note that line segment GH is not completely contained in $g'(e_1, e_2, e_3)$. Therefore, if l_{13} is a

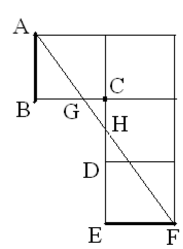

Fig. 2. Illustration of Case 1 in the proof of Lemma 4.

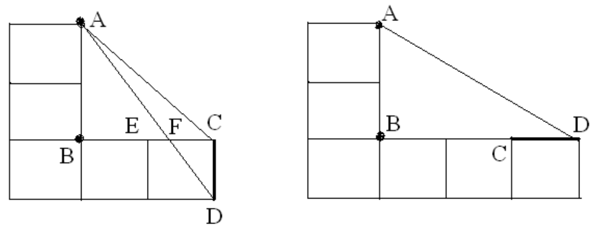

Fig. 3. Illustration of Case 2.1 in the proof of Lemma 4.

line segment with its two end points are on e_1 and e_3 respectively. Then l_{13} is not completely contained in **g**.

Case 2. Let $D(e_1, e_2, e_3) = 2$ and $\min\{d_{e_1e_2}, d_{e_2e_3}\} \geq 2$. Without loss of generality, we can assume that $e_1 \parallel e_2$.

Case 2.1. e_1 and e_2 are on the same grid line; we only need to prove that the conclusion is true when $d_{e_1e_2} = 2$ and $d_{e_2e_3} = 2$. In this case, the projective image (denoted by $g'(e_1, e_2, e_3)$) of all of g's cubes contained between e_1 and e_3 is illustrated in Figure 3.

Case 2.1.1. $g'(e_1, e_2, e_3)$) is as on the left in Figure 3, where A and B are the projective images of either one end point of e_1 or e_2, respectively, and CD that of e_3. Note that line segment AD must intercept grid edge EC at a point F. Also note that line segments AD and AC are not completely contained in $g'(e_1, e_2, e_3)$. Therefore, if l_{13} is a line segment where one end point is on e_1, and the other on e_3, then l_{13} is not completely contained in **g**. Similarly, we can show that the conclusion is also true for Case 2.1.2, with $g'(e_1, e_2, e_3)$) as illustrated on the right in Figure 3.

Case 2.2. Assume that e_1 and e_2 are on different grid lines. We only need to prove that the conclusion is true when $d_{e_1e_2} = \sqrt{5}$ and $d_{e_2e_3} = 2$. In this case, the projective image (denoted by $g'(e_1, e_2, e_3)$) of all of g's cubes contained between e_1 and e_3 is illustrated in Figure 4, where A (B) is the projective image of one end point of e_1 (e_2), and CD that of e_3. Note that line segment AD must

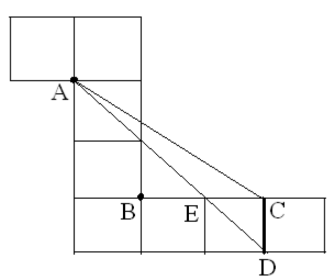

Fig. 4. Illustration of both subcases of Case 2.2 in the proof of Lemma 4.

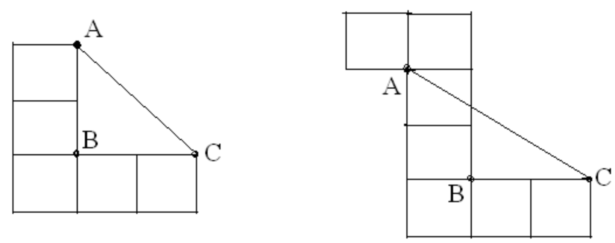

Fig. 5. Illustration of both subcases of Case 3 in the proof of Lemma 4.

intercept grid edge EC at a point E. Also note that line segments AD and AC are not completely contained in $g'(e_1, e_2, e_3)$. Therefore, if l_{13} is a line segment with one end point on e_1, and one on e_3, then l_{13} is not completely contained in **g**.

Case 3. Let $D(e_1, e_2, e_3) = 1$ and $\min\{d_{e_1e_2}, d_{e_2e_3}\} \geq 2$. Without loss of generality, we can assume that $e_1 \parallel e_2$.

Case 3.1. e_1 and e_2 are on the same grid line. We only need to prove that the conclusion is true when $d_{e_1e_2} = 2$ and $d_{e_2e_3} = 2$. In this case, the projective image (denoted by $g'(e_1, e_2, e_3)$) of all of g's cubes contained between e_1 and e_3 is illustrated on the left of Figure 5, where A, B, and C are projective images of one end point of e_1, e_2, and e_3, respectively. Note that line segment AC is not completely contained in $g'(e_1, e_2, e_3)$. Therefore, if l_{13} is a line segment with an end point on e_1 and another one on e_3, then l_{13} is not be completely contained in **g**.

Case 3.2. Now assume that e_1 and e_2 are on different grid lines. We only need to prove that the conclusion is true when $d_{e_1e_2} = \sqrt{5}$ and $d_{e_2e_3} = 2$. In this case, the projective image (denoted by $g'(e_1, e_2, e_3)$) of all of g's cubes contained between e_1 and e_3 is illustrated on the right in Figure 5, where A,B,and C are the projective image of one end point of e_1, e_2, and e_3, respectively. Note that line segment AC is not completely contained in $g'(e_1, e_2, e_3)$. Therefore, if l_{13} is a line segment with end points on e_1 and e_3, then l_{13} is not be completely contained in **g**. □

Let g be a simple cube-curve such that any three consecutive critical edges e_1, e_2 and e_3 do satisfy that either $D(e_1, e_2, e_3) = 3$, $\min\{d_{e_1e_2}, d_{e_2e_3}\} \geq 1$ and $\max\{d_{e_1e_2}, d_{e_2e_3}\} \geq 2$, or $D(e_1, e_2, e_3) \leq 2$ and $\min\{d_{e_1e_2}, d_{e_2e_3}\} \geq 2$. By Lemma 4, we immediately obtain

Lemma 5. *Every critical edge of g contains at least one vertex of g's MLP.*

Let g be a simple cube-curve, and assume that every critical edge of g contains at least one vertex of the MLP. Then we also have the following:

Lemma 6. *Every critical edge of g contains at most one vertex of g's MLP.*

Proof. Assume that there exists a critical edge e such that e contains at least two vertices v and w of the MLP P of g. Without loss of generality, we can assume that v and w are the first (in the order on P) two vertices which are on e. Let u be a vertex of P, which is on the previous critical edge of P. Then line segments uv and uw are completely contained in $\mathbf{g}$. By replacing $\{uv, uw\}$ by uw we obtain a polygon of length shorter than P, which is in contradiction to the fact that P is an MLP of g. □

Let g be a simple cube-curve such that any three consecutive critical edges e_1, e_2, and e_3 do satisfy that either $D(e_1, e_2, e_3) = 3$, $\min\{d_{e_1e_2}, d_{e_2e_3}\} \geq 1$ and $\max\{d_{e_1e_2}, d_{e_2e_3}\} \geq 2$, or $D(e_1, e_2, e_3) \leq 2$ and $\min\{d_{e_1e_2}, d_{e_2e_3}\} \geq 2$. By Lemma 5 and Lemma 6, we immediately obtain

Theorem 3. *The specified simple cube-curve g is first class.*

Let e_1, e_2, and e_3 be three consecutive critical edges of a simple cube-curve g. Let p_1, p_2, and p_3 be three points such that $p_i \in e_i$, for $i = 1, 2, 3$. Let the coordinates of p_i be $(x_i + k_{x_i} t_i, y_2 + k_{y_i} t_i, z_i + k_{z_i} t_i)$, where $k_{x_i}, k_{y_i}, k_{z_i}$ are either 0 or 1, and $0 \leq t_i \leq 1$, for $i = 1, 2, 3$. Let $d_2 = d_e(p_1, p_2) + d_e(p_2, p_3)$.

Theorem 4. $\frac{\partial d_2}{\partial t_2} = 0$ *implies that we have one of the following representations for t_3: we can have*

$$t_3 = \frac{-c_2 t_1 + (c_1 + c_2) t_2}{c_1} \tag{3}$$

if $c_1 > 0$; we can also have

$$t_3 = 1 - \sqrt{\frac{c_1^2 (t_2 - a_2)^2}{(t_2 - t_1)^2} - c_2^2} \quad \textit{or} \tag{4}$$

$$t_3 = \sqrt{\frac{c_1^2 (t_2 - a_2)^2}{(t_2 - t_1)^2} - c_2^2} \tag{5}$$

if a_2 is either 0 or 1, and c_1 and c_2 are positive; and we can also have

$$t_3 = 1 - \sqrt{\frac{(t_2 - a_2)^2 [(t_1 - a_1)^2 + c_1^2]}{(t_2 - b_1)^2} - c_2^2} \quad \textit{or} \tag{6}$$

$$t_3 = \sqrt{\frac{(t_2 - a_2)^2 [(t_1 - a_1)^2 + c_1^2]}{(t_2 - b_1)^2} - c_2^2} \tag{7}$$

if a_1, a_2, and b_1 are either 0 or 1, and c_1 and c_2 are positive reals.

Proof. We have that the coordinates of p_i are $(x_i + k_{x_i} t_i, y_2 + k_{y_i} t_i, z_i + k_{z_i} t_i)$, with $k_{x_i}, k_{y_i}, k_{z_i}$ equals 0 or 1, and $0 \leq t_i \leq 1$, for $i = 1, 2, 3$. Note that only one of values $k_{x_i}, k_{y_i}, k_{z_i}$ can be 1, and the other two must be 0. It follows that for every $i, j \in \{1, 2, 3\}$, $d_e(p_i, p_j) = \sqrt{(t_j - t_i)^2 + c^2}$ or $\sqrt{(t_i - a)^2 + (t_j - b)^2 + c^2}$, where a, b are 0 or 1, and $c > 0$. We have $c \neq 0$ because otherwise e_1 and e_2 would be on the same line, and that is impossible. Let $d_2 = d_e(p_1, p_2) + d_e(p_2, p_3)$. We have three possible cases:

Case 1. $d_2 == \sqrt{(t_2 - t_1)^2 + c_1^2} + \sqrt{(t_2 - t_3)^2 + c_2^2}$, with $c_i > 0$, for $i = 1, 2$. Then we have

$$\frac{\partial d_2}{\partial t_2} = \frac{t_2 - t_1}{\sqrt{(t_2 - t_1)^2 + c_1^2}} + \frac{t_2 - t_3}{\sqrt{(t_2 - t_3)^2 + c_2^2}}$$

and equation $\frac{\partial d_2}{\partial t_2} = 0$ implies the form of Equation (3).

Case 2. $d_2 == \sqrt{(t_2 - t_1)^2 + c_1^2} + \sqrt{(t_2 - a_2)^2 + (t_3 - b_2)^2 + c_2^2}$, with a_2, b_2 equals 0 or 1, and $c_i > 0$, for $i = 1, 2$. Then we have

$$\frac{\partial d_2}{\partial t_2} = \frac{t_2 - t_1}{\sqrt{(t_2 - t_1)^2 + c_1^2}} + \frac{t_2 - a_2}{\sqrt{(t_2 - a_2)^2 + (t_3 - b_2)^2 + c_2^2}}$$

and equation $\frac{\partial d_2}{\partial t_2} = 0$ implies the form of Equations (4) or (5).

Case 3. $d_2 = \sqrt{(t_2 - a_1)^2 + (t_1 - b_1)^2 + c_1^2} + \sqrt{(t_2 - a_2)^2 + (t_3 - b_2)^2 + c_2^2}$, with a_i, b_i equals 0 or 1, and $c_i > 0$, for $i = 1, 2$. Then we have

$$\frac{\partial d_2}{\partial t_2} = \frac{t_2 - a_1}{\sqrt{(t_2 - a_1)^2 + (t_1 - b_1)^2 + c_1^2}} + \frac{t_2 - a_2}{\sqrt{(t_2 - a_2)^2 + (t_3 - b_2)^2 + c_2^2}}.$$

and equation $\frac{\partial d_2}{\partial t_2} = 0$ implies the form of Equations (6) or (7). $\square$

The proof of Case 3 of Theorem 4 and Lemma 3 show the following:

Lemma 7. *Let g be a first class simple cube-curve. If e_1, e_2 and e_3 form a middle angle of g then the vertex of the MLP of g on e_2 can not be an endpoint (i.e., a grid point) on e_2.*

Lemma 8. *Let $f(x)$ be a continuous function defined on interval $[a, b]$, and assume $f(\xi) = 0$ for some $\xi \in (a, b)$. Then, for every $\varepsilon > 0$, there exist a' and b' such that for every $x \in [a', b']$ we have $|f(x)| < \varepsilon$.*

Proof. Since $f(x)$ is continuous at $\xi \in (a, b)$, so $\lim_{n \to \xi} f(x) = f(\xi) = 0$. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every $x \in (\xi - \delta, \xi + \delta)$ we have $|f(x)| < \varepsilon$. Let $a' = \xi - \frac{\delta}{2}$ and $b' = \xi + \frac{\delta}{2}$. Then for every $x \in [a', b']$ we have $|f(x)| < \varepsilon$. $\square$

Lemma 9. *Let $f(x)$ be a continuous function on an interval $[a, b]$, with $f(\xi) = 0$ at $\xi \in (a, b)$. Then for every $\varepsilon > 0$, there are two integers $n > 0$ and $k > 0$ such that for every $x \in [\frac{(k-1)(b-a)}{n}, \frac{k(b-a)}{n}]$, we have $|f(x)| < \varepsilon$.*

Proof. By Lemma 8, for every $\varepsilon > 0$, there exist a' and b' such that for every $x \in [a', b']$ we have $|f(x)| < \varepsilon$. Select an integer $n \geq \frac{2(b-a)}{b'-a'}$. Then $\frac{b-a}{n} \leq \frac{b'-a'}{2} \leq b' - a'$. So there is an integer j (where $j = 1, 2, \ldots, n-1$), such that $a' \leq \frac{j(b-a)}{n} \leq b'$. If $\frac{j(b-a)}{n} \leq \frac{b'-a'}{2}$, then $a' \leq \frac{j(b-a)}{n} \leq \frac{(j+1)(b-a)}{n} \leq b'$. If $\frac{j(b-a)}{n} \geq \frac{b'-a'}{2}$, then $a' \leq \frac{(j-1)(b-a)}{n} \leq \frac{j(b-a)}{n} \leq b'$. $\square$

3 Algorithm

This section contains main ideas and steps of our algorithm for computing the MLP of a first class simple cube-curve which has at least one end angle.

3.1 Basic Ideas

Let p_i be a point on e_i, where $i = 0, 1, 2, \ldots, n$. Let the coordinates of p_i be $(x_i + k_{x_i} t_i, y_2 + k_{y_i} t_i, z_i + k_{z_i} t_i)$, where $i = 0,\ 1,\ \ldots,$ and n. Then the length of the polygon $p_0 p_1 \ldots p_n$ is $d = d(t_0, t_1, \ldots, t_n) = \sum_{i=0}^{n} d_e(p_i, p_{i+1})$. If the polygon $p_0 p_1 \ldots p_n$ is the MLP of g, then (by Theorem 2) we have $\frac{\partial d}{\partial t_i} = 0$, where $i = 0, 1, \ldots, n$.

Assume that e_i, e_{i+1}, and e_{i+2} form an end angle, and also e_j, e_{j+1}, and e_{j+2}, and no other three consecutive critical edges between e_{i+2} and e_j form an end angle, where $i \leq j$ and $i, j = 0, 1, 2, \ldots, n$. By Theorem 4 we have $t_{i+3} = f_{i+3}(t_{i+1}, t_{i+2}), t_{i+4} = f_{i+4}(t_{i+2}, t_{i+3}), t_{i+5} = f_{i+5}(t_{i+3}, t_{i+4}), \ldots, t_j$, and $t_{j+1} = f_{j+1}(t_{j-1}, t_j)$. This shows that $t_{i+3}, t_{i+4}, t_{i+5}, \ldots, t_j$, and t_{j+1} can be represented by t_{i+1}, and t_{i+2}. In particular, we obtain an equation $t_{j+1} = f(t_{i+1}, t_{i+2})$, or

$$g(t_{j+1}, t_{i+1}, t_{i+2}) = 0, \tag{8}$$

where t_{j+1}, and t_{i+1} are already known, or

$$g_1(t_{i+2}) = 0. \tag{9}$$

Since e_i, e_{i+1}, and e_{i+2} form an end angle it follows that $e_{i+1} \perp e_{i+2}$. By Theorem 4 we can express $\frac{\partial d_2}{\partial t_{i+2}}$ either in the form

$$\frac{t_{i+2} - t_{i+1} - a_1}{\sqrt{(t_{i+2} - t_{i+1} - a_1)^2 + b_1^2}} + \frac{t_{i+2} - a_2}{\sqrt{(t_{i+2} - a_2)^2 + (t_{i+3} - b_2)^2 + c_2^2}} \tag{10}$$

or in the form

$$\frac{t_{i+2} - b_1}{\sqrt{(t_{i+1} - a_1)^2 + (t_{i+2} - b_1)^2 + c_1^2}} + \frac{t_{i+2} - a_2}{\sqrt{(t_{i+2} - a_2)^2 + (t_{i+3} - b_2)^2 + c_2^2}} \tag{11}$$

If t_{i+2} satisfies Equation (10), then $\frac{\partial d_2}{\partial t_{i+2}}(a_1') < 0$, and $\frac{\partial d_2}{\partial t_{i+2}}(a_2') > 0$, where $a_1' = \min\{t_{i+1} + a_1, a_2\}$, and $a_2' = \max\{t_{i+1} + a_1, a_2\}$. It follows that Equation (10) has a unique real root between a_1' and a_2'. If t_{i+2} satisfies Equation (11), then Equation (11) has a unique real root between a_2 and b_1. In summary, there are two real numbers a and b such that Equation (11) has a unique root in between a and b. If $g_1(a)g_1(b) < 0$, then we can use the bisection method (see [2, page 49]) to find an approximate root of Equation (11). Otherwise, by Lemma 9 (see also Appendix A), we can also find an approximate root of Equation (11). Therefore we can find an approximate root for $\frac{\partial d}{\partial t_k} = 0$, where $k = i{+}2, i{+}3, \ldots,$ and j, and an exact root for $\frac{\partial d}{\partial t_k} = 0$, where $k = i + 1$ and $j + 1$. In this way we will find an approximate or exact root t_{k_0} for $\frac{\partial d}{\partial t_k} = 0$, where $k = 1{,}2,\ \ldots,$ and n. Let $t'_{k_0} = 0$ if $t_{k_0} < 0$ and $t'_{k_0} = 1$ if $t_{k_0} > 1$, where $k = 1, 2, \ldots, n$. Then (by Theorem 2) we obtain an approximation of the MLP (its length is $d(t'_{1_0}, t'_{2_0}, \ldots, t'_{i_0}, \ldots, t'_{n_0})$) of the given first class simple cube-curve.

3.2 Main Steps of the Algorithm

The input is a first class simple cube-curve g with at least one end angle. The output is an approximation of the MLP and a calculated length value.

Step 1. Represent g by the coordinates of the endpoints of its critical edges e_i, where $i = 0, 1, 2, \ldots, n$. Let p_i be a point on e_i, where $i = 0, 1, 2, \ldots, n$. Then the coordinates of p_i should be $(x_i + k_{x_i} t_i, y_2 + k_{y_i} t_i, z_i + k_{z_i} t_i)$, where only one of the parameters k_{x_i}, k_{y_i} and k_{z_i} can be 1, and the other two are equal to 0, for $i = 0, 1, \ldots, n$.

Step 2. Find all end angles $\angle(e_j, e_{j+1}, e_{j+2}), \angle(e_k, e_{k+1}, e_{k+2}), \ldots$ of g. For every $i \in \{0, 1, 2, \ldots, n\}$, let $d_{i+1} = d_e(p_i, p_{i+1}) + d_e(p_{i+1}, p_{i+2})$. By Lemma 3, we can find a unique root $t_{(i+1)_0}$ of equation $\frac{\partial d_{i+1}}{\partial t_{i+1}} = 0$ if e_i, e_{i+1} and e_{i+2} form an end angle.

Step 3. For every pair of two consecutive end angles $\angle(e_i, e_{i+1}, e_{i+2})$ and $\angle(e_j, e_{j+1}, e_{j+2})$ of g, apply the ideas as described in Section 3.1 to find the root of equation $\frac{\partial d_k}{\partial t_k} = 0$, where $k = i+1, i+2, \ldots,$ and $j+1$.

Step 4. Repeat Step 3 until we find an approximate or exact root t_{k_0} for $\frac{\partial d}{\partial t_k} = 0$, where $d = d(t_0, t_1, \ldots, t_n) = \sum_{i=1}^{n-1} d_i$, for $k = 0, 1, 2, \ldots, n$. Let $t'_{k_0} = 0$ if $t_{k_0} < 0$, and $t'_{k_0} = 1$ if $t_{k_0} > 1$, for $k = 0, 1, 2, \ldots, n$.

Step 5. The output is a polygonal curve $p_0(t'_{1_0})p_1(t'_{2_0}) \ldots p_n(t'_{n_0})$ of total length $d(t'_{1_0}, t'_{2_0}, \ldots, t'_{i_0}, \ldots, t'_{n_0})$, and this curve approximates the MLP of g.

We give an estimate of the time complexity of our algorithm in dependency of the number of end angles m and the accuracy (tolerance ε) of approximation.

Let the accuracy of approximation be $\frac{1}{2^k}$. By [2, page 49], the bisection method needs to know the initial end points a and b of the search interval $[a, b]$. In the best case, if we can set $a = 0$ and $b = 1$ to solve all the forms of Equation (9) by the bisection method, then the algorithm completes each run in $O(mk^2)$ time. In the worst case, if we have to find out the values of a and b for every of the forms of Equation (9) by the bisection method, then by Lemma 9, and let us assume that we need 2^{k_0} steps to find out the values of a and b, the algorithm completes each run in $O(mk^2 2^{k_0})$ time.

4 Experiments

We provide one example where we compare results obtained with our algorithm with those of the rubber-band algorithm as described in [1].

4.1 The Example

We approximate the MLP of the first-class simple cube-curve of Figure 1.

Step 1. See Table 1 which lists the coordinates of the critical edges $e_0, e_1, \ldots, e_{21}$ of g. Let p_i be a point on the critical line of e_i, where $i = 0, 1, \ldots, 21$.

Critical edge	x_{i1}	y_{i1}	z_{i1}	x_{i2}	y_{i2}	z_{i2}
e_0	1	4	7	2	4	7
e_1	2	4	5	2	5	5
e_2	4	5	4	4	5	5
e_3	4	7	4	5	7	4
e_4	5	7	2	5	8	2
e_5	7	8	1	7	8	2
e_6	7	10	2	8	10	2
e_7	8	10	4	8	11	4
e_8	10	10	4	10	10	5
e_9	10	8	5	11	8	5
e_{10}	11	7	7	11	8	7
e_{11}	12	7	7	12	7	8
e_{12}	12	5	7	12	5	8
e_{13}	10	4	8	10	5	8
e_{14}	9	4	10	10	4	10
e_{15}	9	2	10	9	2	11
e_{16}	7	1	10	7	2	10
e_{17}	6	2	8	7	2	8
e_{18}	6	4	7	6	4	8
e_{19}	4	4	7	4	4	8
e_{20}	3	2	7	3	2	8
e_{21}	2	2	7	2	2	8

Table 1. Coordinates of endpoints of critical edges in Figure 1.

Step 2. We calculate the coordinates of p_i, where $i = 0, 1, \ldots 21$, as follows: $(1+t_0, 4, 7)$, $(2, 4+t_1, 5)$, $(4, 5, 4+t_2)$, $(4+t_3, 7, 4)$, $(5, 7+t_4, 2)$, $(7, 8, 1+t_5)$ ... $(2, 2, 7+t_{21})$.

Step 3. Now let $d = d(t_0, t_1, \ldots, t_{21}) = \sum_{i=0}^{21} d_e(p_i, p_{i+1(\text{mod } 22)})$. Then we obtain

$$\frac{\partial d}{\partial t_0} = \frac{t_0 - 1}{\sqrt{(t_0-1)^2 + t_{21}^2 + 4}} + \frac{t_0 - 1}{\sqrt{(t_0-1)^2 + t_1^2 + 4}} \tag{12}$$

$$\frac{\partial d}{\partial t_1} = \frac{t_1}{\sqrt{(t_0-1)^2 + t_1^2 + 4}} + \frac{t_1 - 1}{\sqrt{(t_1-1)^2 + (t_2-1)^2 + 4}} \tag{13}$$

$$\frac{\partial d}{\partial t_2} = \frac{t_2 - 1}{\sqrt{(t_1-1)^2 + (t_2-1)^2 + 4}} + \frac{t_2}{\sqrt{t_2^2 + t_3^2 + 4}} \tag{14}$$

$$\frac{\partial d}{\partial t_3} = \frac{t_3}{\sqrt{t_2^2 + t_3^2 + 4}} + \frac{t_3 - 1}{\sqrt{(t_3-1)^2 + t_4^2 + 4}} \tag{15}$$

$$\frac{\partial d}{\partial t_4} = \frac{t_4}{\sqrt{(t_3-1)^2 + t_4^2 + 4}} + \frac{t_4 - 1}{\sqrt{(t_4-1)^2 + (t_5-1)^2 + 4}} \tag{16}$$

and

$$\frac{\partial d}{\partial t_5} = \frac{t_5 - 1}{\sqrt{(t_4-1)^2 + (t_5-1)^2 + 4}} + \frac{t_5 - 1}{\sqrt{(t_5-1)^2 + t_6^2 + 4}} \tag{17}$$

By Equations (12) and (17) we obtain $t_0 = t_5 = 1$.

Similarly, we have $t_7 = t_{15} = 0$, and $t_{16} = 1$. Therefore we find all end angles as follows: $\angle(e_{21}, e_0, e_1)$, $\angle(e_4, e_5, e_6)$, $\angle(e_6, e_7, e_8)$, $\angle(e_{14}, e_{15}, e_{16})$, and $\angle(e_{15}, e_{16}, e_{17})$.

By Theorem 4 and Equations (13), (14), (15) it follows that

$$t_2 = 1 - \sqrt{\frac{(t_1 - 1)^2[(t_0 - 1)^2 + 4]}{t_1^2} - 4} \tag{18}$$

$$t_3 = \sqrt{\frac{t_2^2[(t_1 - 1)^2 + 4]}{(t_2 - 1)^2} - 4} \tag{19}$$

and

$$t_4 = \sqrt{\frac{(t_3 - 1)^2[t_2^2 + 4]}{t_3^2} - 4} \tag{20}$$

By Equation (16) we have

$$t_4^2[(t_5 - 1)^2 + 4] = (t_4 - 1)^2[(t_3 - 1)^2 + 4]$$

Let

$$g_1(t_1) = t_4^2[(t_5 - 1)^2 + 4] - (t_4 - 1)^2[(t_3 - 1)^2 + 4] \tag{21}$$

By Equation (18) we have $t_1 \in (0, 0.5)$, $g_1(0.4924) = 3.72978 > 0$, and also $g_1(0.4999) = -51.2303 < 0$. By Theorem 2 and the Bisection Method we obtain the following unique roots of Equations (21), (18), (19), and (20):

$t_1 = 0.492416$, $t_2 = 0.499769$, $t_3 = 0.499769$, and $t_4 = 0.507584$,

with error $g_1(t_1) = 4.59444 \times 10^{-9}$. These roots correspond to the two consecutive end angles $\angle(e_{21}, e_0, e_1)$ and $\angle(e_4, e_5, e_6)$ of g.

Step 4. Similarly, we find the unique roots of equation $\frac{\partial d}{\partial t_i} = 0$, where $i = 6, 7, \ldots, 21$. At first we have $t_6 = 0.5$, which corresponds to the two consecutive end angles $\angle(e_4, e_5, e_6)$ and $\angle(e_6, e_7, e_8)$; then we also obtain

$t_8 = 0.492582, t_9 = 0.494543, t_{10} = 0.331074, t_{11} = 0.205970, t_{12} = 0.597034, t_{13} = 0.502831, t_{14} = 0.492339$, which correspond to the two consecutive end angles $\angle(e_6, e_7, e_8)$ and $\angle(e_{14}, e_{15}, e_{16})$; followed by $t_{15} = 0, t_{16} = 1$, which correspond to the two consecutive end angles $\angle(e_{14}, e_{15}, e_{16})$ and $\angle(e_{15}, e_{16}, e_{17})$; and finally $t_{17} = 0.501527, t_{18} = 0.77824, t_{19} = 0.56314, t_{20} = 0.32265$, and $t_{21} = 0.2151$, which correspond to the two consecutive end angles $\angle(e_{15}, e_{16}, e_{17})$ and $\angle(e_{21}, e_0, e_1)$.

Step 5. In summary, we obtain the values shown in the first two columns of Table 2. The calculated approximation of the MLP of g is $p_0(t'_{1_0})p_1(t'_{2_0}) \ldots p_n(t'_{n_0})$, and its length is $d(t'_{1_0}, t'_{2_0}, \ldots, t'_{i_0}, \ldots, t'_{n_0}) = 43.767726$, where $t'_{i_0} = t_{i_0}$ for i limited to the set $\{0, 1, 2, \ldots, 21\}$.

Critical points	t_{i_0} (our algorithm)	t_{i_0} (Rubber-Band Algorithm)
p_0	1	1
p_1	0.492416	0.4924
p_2	0.499769	0.4998
p_3	0.499769	0.4998
p_4	0.507584	0.5076
p_5	1	1
p_6	0.5	0.5
p_7	0	0
p_8	0.492582	0.4926
p_9	0.494543	0.4945
p_{10}	0.331074	0.3311
p_{11}	0.205970	0.2060
p_{12}	0.597034	0.5970
p_{13}	0.502831	0.5028
p_{14}	0.492339	0.4923
p_{15}	0	0
p_{16}	1	1
p_{17}	0.501527	0.5015
p_{18}	0.77824	0.7789
p_{19}	0.56314	0.5641
p_{20}	0.32265	0.3235
p_{21}	0.2151	0.2157

Table 2. Comparison of results of both algorithms.

4.2 Comparison with Rubber-Band Algorithm

The Rubber-Band Algorithm [1] calculated the roots of Equations (12) through (17) as shown in the third column of Table 2. Note that there is only a finite number of iterations until the algorithm terminates. No threshold needs to be specified for the chosen input curve.

From Table 2 we can see that both algorithms converge to the same values.

5 Conclusions

We designed an algorithm for the approximative calculation of an MLP for a special class of simple cube-curves (first-class simple cube-curves with at least one end angle). Mathematically, the problem is equivalent to solving equations with one variable each. Applying methods of numerical analysis, we can compute their roots with sufficient accuracy. We illustrated by one non-trivial example that the Rubber-Band Algorithm also converges to the correct solution (as calculated by our algorithm).

Acknowledgements The authors thank Dr Thomas Bülow for providing the source code of and correct results of the Rubber-Band Algorithm. The IWCIA

reviewers' comments have been very helpful for revising an earlier version of this paper.

A Algorithm 1 (n-section method)

The following C++ code is used to find a root of $f(x) = 0$, where $f(x)$ is a continuous function on $[a, b]$, with $f(a)f(b) > 0$. The input are endpoints a and b, a tolerance TOL, and the maximum number N of iterations. The output is an approximate root p, or a fail-message.

```
int main( void )
{
     long int i=0;
     for(i =0;i <N;i++){
       if(function(i*(b-a)/N) < TOL){
         cout << "approximate root p = " << i*(b-a)/N << endl;
         return 0;
       }
     }
     //fail message
     cout <<"N is too small! N = "<< N << endl;
     cout <<"Try a bigger number!" << endl;

     return 0;
}
```

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