Curves, Hypersurfaces, and Good Pairs of Adjacency Relations

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Abstract. In this paper we propose several equivalent definitions of digital curves and hypersurfaces in arbitrary dimension. The definitions involve properties such as one-dimensionality of curves and $(n-1)$-dimensionality of hypersurfaces that make them discrete analogs of corresponding notions in topology. Thus this work appears to be the first one on digital manifolds where the definitions involve the notion of dimension. In particular, a digital hypersurface in nD is an $(n-1)$-dimensional object, as it is in the case of continuous hypersurfaces. Relying on the obtained properties of digital hypersurfaces, we propose a uniform approach for studying good pairs defined by separations and obtain a classification of good pairs in arbitrary dimension.
Keywords: *digital geometry, digital topology, digital curve, digital hypersurface, good pair*

1 Introduction

A regular orthogonal grid subdivides $\mathbb{R}^n$ into n-dimensional hypercubes (e.g., unit squares for $n = 2$) defining a class $\mathbb{C}_n^{(n)}$. Let $\mathbb{C}_n^{(k)}$ be the class of all k-dimensional facets of n-dimensional hypercubes, for $0 \leq k < n$. The grid-cell space $\mathbb{C}_n$ is the union of all these classes $\mathbb{C}_n^{(k)}$, for $0 \leq k \leq n$.

In this paper we study digital curves, hypersurfaces, and good pairs of adjacency relations in grid-cell spaces $\mathbb{C}_n$ ($n \geq 2$), equipped with adjacencies A_α (e.g., $\alpha = 0, 1$ for $n = 2$, and $\alpha = 0, 1, 2$ for $n = 3$)[3]. A *good pair*[4] combines two adjacency relations on $\mathbb{C}_n$. The reason for introducing the first good pairs (α, β) in [8], with (α, β) equal to (1,0) or (0,1), were observations in [28]. (A_α is the adjacency relation for 1*s*, which are the pixels with value 1, and A_β is the adjacency relation for 0*s*.) The benefit of two alternative adjacencies was then

[3] In 2D, 0- and 1-adjacency correspond to 8- and 4-adjacency, respectively, while in 3D, 0-, 1-, and 2-adjacency correspond to 26-, 18- and 6-adjacency, respectively. The latter are traditionally used within the grid-point model on $\mathbb{Z}^n$.

[4] The name was created for the oral presentation of [15]. Note that the same term has been used already with different meaning in topology.

formally shown in [25]: (1,0) or (0,1) define region adjacency graphs for binary pictures which form a rooted tree. This simplifies topological studies of binary pictures.

Good pairs may induce a digital topology[5] on $\mathbb{C}_n^{(n)}$ (and not vice-versa in general). For example, using the good pair (1,0) (or (0,1)) is equivalent to regarding 1-components of 1*s* as open regions and 0-components of 0*s* as closed regions in $\mathbb{C}_n^{(n)}$ (or vice versa). [9] shows that there are two digital topologies on $\mathbb{C}_2$ (where one corresponds to (1,0) or (0,1)), five on $\mathbb{C}_3$, and [16] shows that there are 24 on $\mathbb{C}_4$ (all up to homeomorphisms). This paper provides a complete characterization of good pairs, showing that there are $2n-1$ good pairs on $\mathbb{C}_n$.

The study of good pairs is directed on the understanding of separability properties: which sets defined by one type of adjacency allow to separate sets defined by another type of adjacency. These separating sets can be defined in the form of digital curves in 2D, or as digital surfaces in 3D. In this way, studies of good pairs and of (separating) surfaces are directly related to one-another. Topology of incidence grids is one possible approach: frontiers of closed sets of n-cells define hypersurfaces, consisting of $(n-1)$-cells.

Digital surfaces have been studied under different points of view. The approximation of n-dimensional manifolds by graphs is studied in [29, 30], with a special focus on topological properties of such graphs defined by homotopy and on homology or cohomology groups. [13] defined digital surfaces in $\mathbb{Z}^3$ based on adjacencies of 3-cells. The approximation of boundaries of finite sets of grid points (in n dimensions) based on "continuous analogs" was proposed and studied in [20]. [12] discusses local topologic configurations (stars) for surfaces in incidence grids. Digital surfaces in the context of arithmetic geometry were studied in [4].

A Jordan surface theorem for the Khalimsky topology is proved in [18]. For discrete combinatorial surfaces, see [10]. For obtaining α-surfaces by digitization of surfaces in $\mathbb{R}^3$, see [6]. It is proved in [21] that there is no local characterization of 26-connected subsets S of $\mathbb{Z}^3$ such that its complement $\overline{S}$ consists of two 6-components and every voxel of S is adjacent to both of these components. [21] defines a class of 18-connected surfaces in $\mathbb{Z}^3$, proves a Jordan surface theorem for these surfaces, and studies their relationship to the surfaces defined in [22]. [3] introduces a class of *strong surfaces* and proves that both the 26-connected surfaces of [22] and the 18-connected surfaces of [21] are strong. For 6-surfaces, see [5].

Frontiers in cell complexes (and related topological concepts such as components and fundamental group) were studied in [1]. For characterizations of and algorithms for curves and surfaces in frontier grids, see [11, 19, 27, 31]. G.T. Herman and J.K. Udupa used frontiers in the grid cell model, and V. Kovalevsky generalized these studies using the model of topologic abstract complexes, that can also be modelled by incidence grids. [7] define curves in incidence grids.

In this paper we present alternative definitions of digital hypersurfaces, partially following ideas already published in the cited references above, and prove

[5] A digital topology on $\mathbb{C}_n^{(n)}$ is defined by a family of open subsets that satisfy a number of axioms (see, e.g., Section 6.2 in [14]).

their equivalence. In short, a digital α-hypersurface is composed by (closed) α-curves; two of such curves are either disjoint and non-adjacent, or disjoint but adjacent, or they have overlapping portions. The main contributions of this paper are as follows ($n \geq 2$):

- We define digital manifolds in arbitrary dimensions, as the definitions involve the notion of *dimension of a digital object* [23]. Thus a digital curve is a one-dimensional digital manifold, while a digital hypersurface in nD is an $(n-1)$-dimensional manifold, in conformity to topology (see, e.g., the topological definitions of curves by Urysohn and Menger, as discussed in [14]). To our knowledge of the available literature, this is the first work involving dimensionality in defining these notions in digital geometry.
- We show that there are two and only two basic types of α-hypersurfaces, one for $\alpha = n-1$ and one for $\alpha = n-2$.
  - For $\alpha = n-2$, the hypersurface S has $(n-2)$-gaps which appear on $(n-2)$-curves that build S and, possibly, between adjacent pairs of such $(n-2)$-curves.
  - For $\alpha = n-1$, the hypersurface S is $(n-2)$-gapfree,[6] but may still have $0, 1, \ldots, (n-4)$ or $(n-3)$-gaps, which may appear between adjacent pairs of $(n-1)$-curves rather than on the curves themselves. The last possibility is when an $(n-1)$-hypersurface is i-gapfree for any $0 \leq i \leq n-2$.
- We investigate combinatorial properties of digital hypersurfaces, showing that any digital hypersurface defines a matroid.
- Relying on the obtained properties of digital hypersurfaces, we define and study good pairs of adjacency relations in arbitrary dimension. We define nD good pairs through separation by digital hypersurfaces and show that there are exactly $2n-1$ good pairs of adjacency relations. We also provide a short review and comments on some other approaches for defining good pairs which have been communicated elsewhere.

Some of the proofs of results reported in this paper follow directly from the definitions, while others are technical and rather lengthy, and cannot be reported in this brief conference submission. Complete proofs will be included in a full-length journal version of this paper.

2 Preliminaries

We start with recalling basic definitions; notations follow [14]. In particular, the grid point space $\mathbb{Z}^n$ allows a refined representation by an incidence grid defined on the cellular space $\mathbb{C}_n$ (as defined above).

[6] This was also called "tunnel-free" in earlier publications (e.g., in [2, 24]). The Betti number β_1 defines the number of tunnels in topology. Informally speaking, the location of a tunnel cannot be uniquely identified in general; there is only a unique way to count the number of tunnels. Locations of gaps are identified by defining sets. However, our hypothesis is that tunnel-freeness (i.e., $\beta_1 = 0$) and gap-freeness (in the sense of [2, 24]) are equivalent concepts.

2.1 Some Definitions

Elements in $\mathbb{C}_n^k$ are k-cells, for $0 \leq k \leq n$. An m-dimensional facet of a k-cell is an m-cell, for $0 \leq m \leq k-1$. Two k-cells are called *m-adjacent* if they share an m-cell. Two k-cells are *properly m-adjacent* if they are m-adjacent but not $(m+1)$-adjacent.

A digital object S is a finite set of n-cells. An *m-path* in S is a sequence of n-cells from S such that every two consecutive n-cells are m-adjacent. The *length* of a path is the number of n-cells it contains. A *proper m-path* is an m-path in which at least two consecutive n-cells are not $(m+1)$-adjacent. Two n-cells of a digital object S are *m-connected* (in S) iff there is an m-path in S between them. A digital object S is *m-connected* iff there is an m-path connecting any two n-cells of S. S is *properly m-connected* iff it contains two n-cells such that all m-paths between them are proper. An *m-component* of S is a maximal (i.e., non-extendable) m-connected subset of S.

Let M be a subset of a digital object S. If $S \setminus M$ is not m-connected then the set M is said to be *m-separating* in S. (In particular, the empty set m-separates any set S which is not m-connected.) Let a digital object M be m-separating but not $(m-1)$-separating in a digital object S. Then M is said to have *k-gaps* for any $k < m$. A digital object without any m-gaps is called *m-gapfree*.

Although the above definition has been used in a number of papers by different authors, one can reasonably argue that it requires further refinement. Consider, for instance, the following example.

Let M_1 and M_2 be two digital objects that are subsets of a superset S, and assume that $M_1 \cap M_2 = \emptyset$ (we may think that M_1 and M_2 are "far away" from each other). In addition, assume that M_1 has a k-gap with respect to an adjacency relation A_α, while M_2 is a closed digital hypersurface that k-separates S. Then it turns out that the digital set $M_1 \cup M_2$, that consists of (at least) two connected components, has no k-gap with respect to A_α.

Despite such kind of phenomena, the above definition is adequate for the studies that follow. Further work by authors will be aimed at contributing to a more restrictive definition which will exclude "counterintuitive" examples as the one above. For this, one can take advantage of some of the results presented in the subsequent sections.

Let M be an m-separating digital object in S such that $S \setminus M$ has exactly two m-components. An *m-simple cell* in M (with respect to S) is an n-cell c such that $M \setminus \{c\}$ is still m-separating in S. An m-separating digital object in S is *m-minimal* (or *m-irreducible*) if it does not contain any m-simple cell (with respect to S).

For a set of n-cells S, by $\overline{S}$ we denote the complement of S to the whole digital space $\mathbb{C}_n^{(n)}$ of all n-cells.

$J^+(A)$ is the outer Jordan digitization (also called *supercover*) of a set $A \subseteq \mathbb{R}^n$, which consists of all n-cells intersected by A.

By $B_\alpha(c)$ we denote the *unit α-ball* with center c consisting of all α-neighbors of c. Furthermore, let $B^*_\alpha(c) = B_\alpha(c) \setminus \{c\}$.

For a given set $M = \{c_1, c_2, \ldots, c_m\} \subseteq \mathbb{C}_n^{(n)}$ of n-cells, we define its α-adjacency graph $G_M^\alpha(V, E)$ with a set of vertices $V = \{v_1, v_2, \ldots, v_m\}$ and a set of edges $E = \{(v_i, v_j) : c_i$ and c_j are $\alpha -$ adjacent$\}$. (In the above definition a graph vertex v_i corresponds to the element $c_i \in M$.)

2.2 Dimension

Mylopoulos and Pavlidis [23] proposed definition of dimension of a (finite or infinite) set of n-cells S with respect to an adjacency relation A_α (for its use see also [14]). Let $\overline{B}_\alpha(c)$ be the union of $B_\alpha(c)$ with all n-cells c' for which there exist $c_1, c_2 \in B_\alpha(c)$ such that a shortest α-path from c_1 to c_2 not passing through c passes through c'. For example, $\overline{B}_1(c) = \overline{B}_0(c) = B_0(c)$ for $n = 2$, and $\overline{B}_2(c) = \overline{B}_1(c) = B_1(c)$ and $\overline{B}_0(c) = B_0(c)$ for $n = 3$.

In what follows we will use the definition of dimension from [23]. Let S be a digital object in $\mathbb{C}_n^{(n)}$ and A_α an adjacency relation on $\mathbb{C}_n^{(n)}$. The *dimension* $\dim_\alpha(S)$ is defined as follows:

(1) $\dim_\alpha(S) = -1$ if $S = \emptyset$,

(2) $\dim_\alpha(S) = 0$ if S is a totally α-disconnected nonempty set (i.e., there is no pair of cells $c, c' \in S$ such that $c \neq c'$ and $\{c, c'\}$ is α-connected),

(3) $\dim_\alpha(S) = 1$ if $\text{card}(B^\star(c) \cap S) \leq 2$ for all $c \in S$, and there is at least one $c \in S$ with $\text{card}(B^\star(c) \cap S) > 0$,

(4) $\dim_\alpha(S) = \max_{c \in S} \dim_\alpha(\overline{B}(c) \cap S) + 1$ otherwise.

If in the last item of the definition the maximum is reached for an n-cell c, we will also say that S is $\dim_\alpha(S)$-dimensional at c.

An *elementary grid triangle* in $\mathbb{C}_2^{(2)}$ is a set $T = \{(i, j), (i+1, j), (i, j+1)\}$, or a 90, 180, or 270 degree rotation of such a T. A 0-connected set $M \subseteq \mathbb{C}_2^{(2)}$ is two-dimensional with respect to adjacency relation A_0 iff it contains an elementary grid triangle as a proper subset. Similarly, a 1-connected set $M \subseteq \mathbb{C}_2^{(2)}$ is two-dimensional with respect to adjacency relation A_1 iff it contains as a proper subset a 2×2 square of grid points. See [14]. These properties generalize to an arbitrary dimension n, as follows.

Lemma 1. *(a) An α-connected set $M \subseteq \mathbb{C}_n^{(n)}$, $0 \leq \alpha \leq n-2$, is two-dimensional iff it contains as a proper subset an* elementary grid triangle *consisting of three cells c_1, c_2, c_3, such that any two of them are α-adjacent.*

(b) An $(n-1)$-connected set $M \subseteq \mathbb{C}_n^{(n)}$ is two-dimensional iff it contains as a proper subset an elementary grid square *consisting of four cells c_1, c_2, c_3, c_4 with coordinates $c_1 = (i, i, \ldots, i, i)$, $c_2 = (i+1, i, \ldots, i, i)$, $c_3 = (i+1, i+1, \ldots, i, i)$, $c_4 = (i, i+1, \ldots, i, i)$, for some $i \in \mathbb{Z}$.*

3 Digital Curves and Hypersurfaces

In what follows we consider digital analogs of simple closed curves and of hypersurfaces that separate the superspace $\mathbb{C}_n^{(n)}$. We will consider analogs of either bounded closed Jordan hypersurfaces or unbounded hypersurfaces (such as hyperplanes) that separate $\mathbb{R}^n$. (The latter can also be considered as "closed" in the infinite point.) We will not specify whether we consider closed or unbounded hypersurfaces whenever the definitions and results apply to both cases and no confusions arise. We also omit the word "digital" where possible.

The considerations take place in the n-dimensional space $\mathbb{C}_n^{(n)}$ of n-cells. We allow adjacency relations A_α as defined above. We are interested to establish basic definitions for this space that:

- reflect properties which are analogous to the topological connectivity of curves or hypersurfaces in Euclidean topology,
- reflect the one- or $(n-1)$-dimensionality of a curve or hypersurface, respectively, and
- characterize hypersurfaces with respect to gaps.

A digital curve (hypersurface), considered in the context of an adjacency relation A_α, will be called an α-curve (α-hypersurface).

3.1 Digital Curves

A set $\tau \subset \mathbb{C}_n^{(n)}$ is an α-curve iff it is α-connected and one-dimensional with respect to A_α. (Note that Urysohn-Menger curves in $\mathbb{R}^n$ are defined to be one-dimensional continua.) Figure 1 presents examples and counterexamples for $\mathbb{C}_2^{(2)}$.

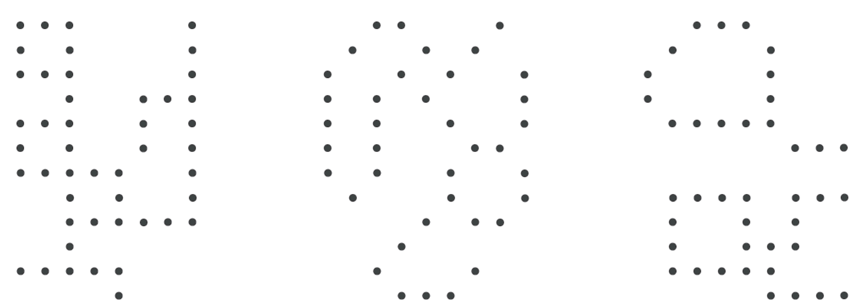

Fig. 1. Examples of a 1-curve (left), 0-curve (middle), and two 0-connected sets in the digital plane that are neither 0- nor 1-curves (right).

In the rest of this section we define and study digital analogs of simple closed curves (i.e., those that have branching index 2 at any point). The following lemma provides necessary and sufficient conditions for a set of n-cells to be connected and a loop with respect to adjacency relation A_α.

Lemma 2. *Let $\rho = \{c_1, c_2, \ldots, c_l\}$ be a set of n-cells. The following properties are equivalent:*

(A1) *c_i is α-adjacent to c_j iff $i = j \pm 1(modulo\ l)$.*

(A2) *ρ is α-connected and $\forall\ c \in \rho$, $\mathrm{card}(B^*_\alpha(c) \cap \rho) = 2$.*

(A3) *The α-adjacency graph $G^\alpha_\rho(V, E)$ is a simple loop.*

The following lemma provides conditions which are equivalent to the one-dimensionality of a set of n-cells.

Lemma 3. *Let $\rho = \{c_1, c_2, \ldots, c_l\}$ be a set of n-cells. The following properties are equivalent:*

(B1) *ρ is one-dimensional with respect to A_α.*

(B2) *If $0 \leq \alpha < n - 1$, then ρ does not contain as a proper subset an elementary grid triangle such that any two of its n-cells are α-adjacent; if $\alpha = n - 1$, then ρ does not contain as a proper subset an elementary grid square.*

(B3) *$\forall c \in \rho$, the set $B^*_\alpha(c) \cap \rho$ is totally disconnected.*

We list one more condition.

(B4) If $0 \leq \alpha < n - 1$, then $l \geq 4$; if $\alpha = n - 1$, then $l \geq 8$.

Lemma 4. *Let $\rho = \{c_1, c_2, \ldots, c_l\}$ be a set of n-cells. Then all property pairs* ((Ai), (Bj)), *for $1 \leq i \leq 3$ and $1 \leq j \leq 4$, are equivalent.*

Thus we are prepared to give the following general definition, summarizing twelve equivalent ways for defining a simple α-curve.

Definition 1. *A* simple α-curve $(0 \leq \alpha \leq n - 1)$ *of length l is a set $\rho = \{c_1, c_2, \ldots, c_l\} \subseteq \mathbb{C}^{(n)}_n$, satisfying properties* (Ai) *and* (Bj)*, for some pair of indexes i, j, with $1 \leq i \leq 3$ and $1 \leq j \leq 4$.*

Note that for any $n \leq 2$, four n-cells whose centers form a 1×1 square do not form a digital curve, since such a set of cells would be two-dimensional.

A simple α-curve will also be called a *one-dimensional α-manifold*. In $\mathbb{C}^{(2)}_2$ we have the following:

Proposition 1. *A finite set ρ of pixels is a simple α-curve in $\mathbb{C}^{(2)}_2$* $(\alpha = 0, 1)$ *iff it is α-minimal in $\mathbb{C}^{(2)}_2$.*

Note that this last result does not generalize to higher dimensions since a one-dimensional digital object cannot separate $\mathbb{C}^{(n)}_n$ if $n > 2$.

A simple α-curve ρ $(0 \leq \alpha < n - 1)$ is a *proper* α-curve (or a proper one-dimensional α-manifold), if it is not an $(\alpha + 1)$-curve.

Example 1. A proper 0-curve in $\mathbb{C}^{(2)}_2$ is a 0-curve which is not a 1-curve (see Figure 2, left) It follows that any closed 0-curve is a proper 0-curve.

A *proper* 0-curve in $\mathbb{C}^{(3)}_3$ is a 0-curve which is not a 1- or 2-curve, and a *proper* 1-curve is a 1-curve which is not a 2-curve.

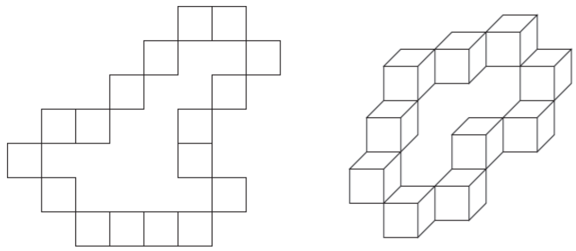

Fig. 2. A proper 0-curve in 2D (left) and an improper 0-curve in 3D (right).

Any 1-curve is a proper 1-curve. This follows from the facts that the curve is closed and one-dimensional with respect to 1-adjacency. If we assume the opposite, we will obtain that the curve is either an infinite sequence of voxels (e.g., of the form $(0,0,1),(0,0,2),(0,0,3),\ldots$) or that it is two-dimensional.

However, a closed 0-curve does not need to be proper (see Figure 2, right).

A *simple* α*-arc* σ is an α-connected proper subset of a simple α-curve. It contains exactly two n-cells c, c' such that $\mathrm{card}(B^*_\alpha(c) \cap \rho) = \mathrm{card} B^*_\alpha(c') \cap \rho) = 1$.

3.2 Digital Hypersurfaces

We consider digital analogs of hole-free hypersurfaces. Accordingly, we are interested in hypersurfaces without $(n-1)$-gaps, although the theory can be extended to cover this case, as well. However, in the framework of our approach, a hypersurface with $(n-1)$-gaps can be an $(n-2)$-dimensional set of n-cells, while we want a digital hypersurface to be $(n-1)$-dimensional, in conformity with the continuous case.

We give the following recursive definition.

Definition 2. (i) *M is a 1-dimensional* $(n-1)$-manifold *in $\mathbb{C}_n^{(n)}$ if it is an $(n-1)$-curve in $\mathbb{C}_n^{(n)}$.*

M is a k-dimensional $(2 \le k \le n-1)$ $(n-1)$-manifold *in $\mathbb{C}_n^{(n)}$ if*

(1) M is $(n-1)$-connected (or, equivalently, M consists of a single $(n-1)$-component), and

(2) for any $x \in M$ the set $B^*_0(x) \cap M$ is a $(k-1)$-dimensional $(n-1)$-manifold in $\mathbb{C}_n^{(n)}$.

(ii) *M is a k-dimensional* α-manifold $(0 \le \alpha \le n-2)$ *in $\mathbb{C}_n^{(n)}$ if*

(1) M is α-connected (or, equivalently, M consists of a single α-component), and

(2) for any $x \in M$ the set $B^*_0(x) \cap M$ is a $(k-1)$-dimensional α-manifold in $\mathbb{C}_n^{(n)}$ but is not a $(k-1)$-dimensional $(\alpha+1)$-manifold in $\mathbb{C}_n^{(n)}$.

(*Such an α-manifold will also be called* proper.)

In the particular case when S is an $(n-1)$-dimensional α-manifold in $\mathbb{C}_n^{(n)}$ for $\alpha = n-2$ or $n-1$, we say that S is a digital α*-hypersurface*. S is a proper α-hypersurface for $\alpha = n-2$ if it is not an $(n-1)$-hypersurface for $\alpha = n-1$.

It is also clear that any proper one-dimensional α-manifold is an α-curve.

We remark that if Condition (1) is missing, then S may have more than one connected component. In such a case Condition (2) implies that any connected component of S is an α-hypersurface.

Note that in the definition of an α-hypersurface we use the ball $B_0^*(x)$ rather than $B_\alpha^*(x)$, since the latter could cause certain incompatibilities. This can be easily seen in the 3D case: if we use B_2^* to define a 2-surface, $B_2^*(x) \cap S$ may be a 1-curve rather than a 2-curve. Similarly, if we use B_1^* to define a 1-surface, $B_1^*(x) \cap S$ may be a 0-curve rather than a 1-curve. This is avoided by using B_0^* in all cases.

The so defined digital hypersurfaces have the following properties, among others.

Proposition 2. *An* α*-hypersurface* S *is* $(n-1)$*-dimensional at any* n*-cell in* S *with respect to adjacency relation* A_α*.*

Proposition 3. *(a) An* $(n-2)$*-hypersurface* S *has* $(n-2)$*-gaps and is* $(n-1)$*-gapfree. Moreover, it is* $(n-1)$*-minimal.*

(b) An $(n-1)$*-hypersurface* S *is* $(n-2)$*-gapfree. Note that* S *may have or may not have* k*-gaps for* $0 \leq k \leq n-3$*. If* S *has* k*-gaps but no* $(k+1)$*-gaps for* $0 \leq k \leq n-3$*, then* S *is* $(k+1)$*-minimal. If* S *is* k*-gapfree for* $0 \leq k \leq n-3$*, then* S *is 0-minimal.*

Part (b) of this last proposition suggests to distinguish $n-1$ types of hypersurfaces: those that are 0-gapfree, those with 0-gaps but with no 1-gaps, etc., up to those with $(n-2)$-gaps but with no $(n-1)$-gaps.

We label them as $(n-1)_{(0)}$, $(n-1)_{(1)}$,. . .,$(n-1)_{(n-2)}$-hypersurfaces.

In fact, the concept of minimality itself can provide a complete characterization of a digital hypersurface, as follows.

Definition 3. *A set* $S \subset \mathbb{C}_n^{(n)}$ *is a* k^*-hypersurface*, for* $k = 0, 1, 2, \ldots, n-1$*, if* S *is* k*-minimal in* $\mathbb{C}_n$*.*

Theorem 1. *(a)* S *is an* $(n-1)_{(i)}$*-hypersurface* $(0 \leq i \leq n-2)$ *iff it is an* i^**-hypersurface.*

(b) S *is an* $(n-2)$*-hypersurface iff it is an* $(n-1)^*$*-hypersurface.*

We remark that a k^*-hypersurface cannot have "singularities," which may appear, e.g., in case of a 3D "pinched sphere" or a "strangled torus." In fact, surfaces of that kind would either be non-simple or three-dimensional or both, so they would not satisfy our definition of surface.

In summary, we have two types of hypersurfaces: $(n-1)$ and $(n-2)$ hypersurfaces, as the $(n-1)$ hypersurfaces can be classified (with respect to their gaps) as $(n-1)$-hypersurfaces of types $0, 1, 2, \ldots, n-2$, respectively.

Indeed, one can consider more general digital hypersurfaces which are not covered by the above definitions. If, for instance, we do not require in Definition 2 the manifold $B_0^*(x) \cap S$ to be proper, we may have a hypersurface where subsets can be of varying hypersurface type. We are interested (see next section) in combinatorial properties of the considered hypersurfaces. More general digital hypersurfaces would be just "mixtures" of patches of hypersurfaces of some of the considered types, and their combinatorial study would lose its focus.

Now let Γ be a closed surface in $\mathbb{R}^n$ and $J^+(\Gamma)$ its outer Jordan digitization.

Definition 4. *Let $\mathcal{D}_k(\Gamma)$ be the family of all subsets of $J^+(\Gamma)$ that are k-minimal, for some $0 \leq k \leq n-1$. We call a set of n-cells $D_k(\Gamma) \in \mathcal{D}_k(\Gamma)$ a k-*digitization* of Γ if the Hausdorff distance $H_d(\Gamma, V(D_k(\Gamma)))$ is minimal, over all the elements of $\mathcal{D}_k(\Gamma)$.*

Proposition 4. *Any k^*-hypersurface is a k-digitization of certain hypersurface Γ.*

Examples of k-digitization (and thus of k^*-hypersurfaces) were actually already provided by the following theorem from [4]:

Theorem 2. *A digital hyperplane P^k defined by $P^k = P^k(b, a_1, a_2, \ldots, a_n, \omega) = \{x \in \mathbb{Z}^n | -\frac{\omega}{2} \leq b + \sum_{i=1}^n a_i x_i < \frac{\omega}{2}\}$, where $\omega = \sum_{i=k+1}^n a_i$, $0 \leq k \leq n-1$, is a k-digitization of the hyperplane $\gamma : b + a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$.*

The above result is related to a theorem from [2] that characterizes the gaps of analytically defined digital hyperplanes. Specifically, let P^k be a digital hyperplane as in Theorem 2. If $\omega < a_n$, then P^k has $(n-1)$-gaps; for $0 < k < n$, if $\sum_{i=k+1}^n a_i \leq \omega < \sum_{i=k}^n a_i$, then P^k has $(k-1)$-gaps and is k-separating; and if $\omega \geq \sum_{i=1}^n a_i$, P^k is gapfree.

3.3 Hypersurface Digitization Matroid

In this section we briefly investigate the structure of digital hypersurfaces from a combinatorial point of view.

Let E be a finite set and $\mathcal{F}$ a family of subsets of E. Recall that $(E, \mathcal{F})$ is a *matroid*[7] if the following axioms are satisfied:

(1) $\emptyset \in \mathcal{F}$,

(2) if $F_2 \in \mathcal{F}$ and $F_1 \subseteq F_2$, then $F_1 \in \mathcal{F}$,

(3) if $F_1, F_2 \in \mathcal{F}$ and $\text{card}(F_1) < \text{card}(F_2)$, then there is an element $x \in F_2$ such that $F_1 \cup \{x\} \in \mathcal{F}$.

The last condition can be substituted by the following:

(3′) all maximal elements of $\mathcal{F}$ have the same cardinality.

[7] For getting acquainted with matroid theory the reader is referred to the monograph by Welsh [32].

Theorem 3. *For a given k, $0 \leq k \leq n-1$, all k-digitizations of a closed hypersurface Γ and their subsets form a matroid.*

We call it the *hypersurface digitization matroid*. This theorem demonstrates in particular the possibility to generate closed digital hypersurfaces by greedy-type algorithms.

4 Good pairs

As already mentioned, studies on digital surfaces naturally interfere with studies on good pairs of adjacency relations. An important motivation for studying good pairs is seen in the possibility that some results of digital topology may hold uniformly for several pairs of adjacency relations. Thus one could obtain a proof which is valid for all of them by proving a statement just for a single good pair of adjacencies.

4.1 Variations of the Notion "Good Pair"

Different approaches in the literature lead to diverse proposals of good pairs (note: they may be called differently, but address the same basic concept). It seems to be unrealistic to define good pairs in such a way that this will cover all previous studies. Therefore, instead of looking for a universal definition, it might be more reasonable and useful to propose and study a number of definitions related to the fundamental concepts of digital topology. The rest of this section reviews several possible approaches.

Good pairs in terms of strictly normal digital picture spaces have been considered in [17]. In that framework, it is shown that adjacencies (1,0) and (0,1) in 2D, and (2,0), (0,2), (2,1) and (1,2) in 3D define strictly normal digital picture spaces, while (1,1) and (0,0) in 2D and (2,2), (1,1), (0,0), (1,0) and (0,1) in 3D do not.

In [14] good pairs have been defined for 2D as follows: (β_1, β_2) is called a *good pair* in the 2D grid iff (for $(i,k) \in \{(1,2),(2,1)\}$) any simple β_i-curve β_k-separates its (at least one) β_k-holes from the background and any totally β_i-disconnected set cannot β_k-separate any β_k-hole from the background. It follows that (1,0) and (0,1) are good pairs, but (1,1) and (0,0) are not. [14] does not generalize this definition to higher dimensions, but suggests the use of (α,β)-separators for the case $n=3$. ($M \subseteq \mathbb{Z}^3$ is called an (α,β)-*separator* iff M is α-connected, M divides $\mathbb{Z}^3 \setminus M$ into (exactly) two β-components, and there exists a $p \in M$ such that $\mathbb{Z}^3 \setminus (M \setminus \{p\}) = (\mathbb{Z}^3 \setminus M) \cup \{p\}$ is β-connected.) (α,β)-and (β,α)-separators exist for (α,β) = (0,2),(2,0), (1,2), (2,1), and (1,1). However, there are some difficulties with the case $(\alpha,\beta) = (1,1)$, as an example from [14] illustrates. Further "strange" examples of separators in $\mathbb{Z}^3$ suggest to refine this notion.

Another approach is based on the following digital variant of the Jordan curve theorem due to A. Rosenfeld [26].

Theorem 4. *If C is the set of points of a simple closed 1-curve (0-curve) and* $\mathrm{card}(C) > 4$ ($\mathrm{card}(C) > 3$), *then $\overline{C}$ has exactly two 0-components (1-components).*

This theorem defines good pairs of adjacency relations in 2D, as follows. (α, β) *is a* 2D good pair *if for a simple closed α-curve C, $\overline{C}$ has exactly two β-components.* It follows that $(1, 0)$ and $(0, 1)$ are good pairs. It is also easy to see that $(1, 1)$ and $(0, 0)$ are not good pairs.

This above definition can be extended to 3D, as follows: (α, β) *is a* 3D good pair *if for a simple closed α-surface S, $\overline{S}$ has exactly two β-components.* We remark that in view of the definition of an α-surface from Section 3.2, a 0-digital surface would not be a true surface and should not be called "surface" since it would have 2-gaps. In fact, 3D digital surfaces need to be at least 1-connected. Thus (0,2) would not be a good pair in the framework of the Jordan surface theorem approach.

Another approach is based on separation through surfaces (see, e.g., [11, 14]).

Theorem 5. *A simple closed 1-curve (0-curve) γ 0-separates (1-separates) all pixels inside γ from all pixels outside γ. More precisely, we have that a simple closed 1-curve has exactly one 0-hole and a simple closed 0-curve has exactly one 1-hole. A simple closed 1-curve 0-separates its 0-hole from the background and a simple closed 0-curve 1-separates its 1-hole from the background.*

Based on this last theorem, we can give the following definition: (α, β) *is called a* 2D good pair *if any simple closed α-curve β-separates its β-holes from the background*.

Clearly, $(1, 0)$ and $(0, 1)$ are good pairs, while (0,0) is not a good pair. Note that here also (1,1) is a good pair, as distinct from the case of good pairs defined trough Jordan curve theorem.

Let us mention that in a definition from [14] both (α, β) and (β, α) are required to satisfy the conditions of a good pair. To avoid confusion, we suggest to treat this case as a special event: (α, β) *is called a* perfect pair *in 2D if any simple closed α-curve β-separates its β-holes from the background and any simple closed β-curve α-separates its α-holes from the background*.

In what follows we consider good pairs defined by this last approach that seems the most reasonable to the authors.

4.2 Good Pairs for the Space of n-Cells

Definition 5. (α, β) *is called a* good pair *of adjacency relations in* $\mathbb{C}_n^{(n)}$ *if any closed α-hypersurface β-separates its β-holes from the background.*

Here α is a (possibly composite) label of the hypersurface type in accordance with our hypersurface classification above, while β is an integer representing an adjacency. More precisely, $\alpha = (n-1)_{(i)}$, $0 \le i \le n-2$ or $\alpha = n-2$, and $0 \le \beta \le n-1$.

Theorem 6. *There are* $2n-1$ *good pairs in the* n*-dimensional digital space:* $((n-1)_{(i)}, i)$ *for* $0 \leq i \leq n-2$*,* $((n-1)_{(i)}, n-1)$ *for* $0 \leq i \leq n-2$*, and* $(n-2, n-1)$*, where the first component of such a pair labels the type of the hypersurface and the second is an adjacency relation.*

Alternatively, the good pairs are (i^*, i) *for* $0 \leq i \leq n-1$*, and* $(i^*, n-1)$ *for* $0 \leq i \leq n-2$*.*

Note that in a pair of the form $((n-1)_{(i)}, i)$, $0 \leq i \leq n-1$ the first component also specifies an adjacency relation corresponding to the type of the hyperplane.

We illustrate the last theorem for $n = 2$ and $n = 3$. For $n = 2$, the good pairs are $(1_{(0)}, 0)$, $(1_{(0)}, 1)$, and $(0^*, 1)$, which correspond to (1,0), (1,1), and (0,1), respectively. See Figure 3.

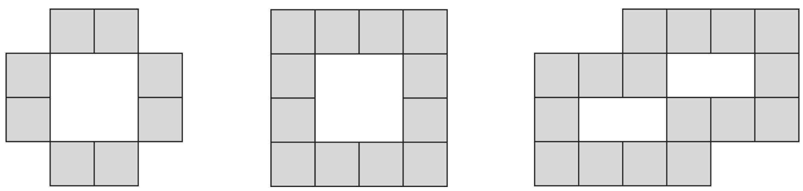

Fig. 3. Illustration to good pairs in 2D: $(0, 1)$ (left), $(1, 1)$ (middle), and $(1, 0)$ (right).

For $n = 3$ the good pairs are $(2_{(0)}, 0)$, $(2_{(1)}, 1)$, $(2_{(0)}, 2)$, $(2_{(1)}, 2)$, and $(1^*, 2)$, which correspond to (2,0), (2,1), (2,2), (2,2), and (1,2), respectively. Note that pair (2,2) is counted twice since there are two different structures corresponding to it.

5 Concluding Remarks

In this paper we proposed several equivalent definitions of digital curves and hypersurfaces in arbitrary dimension. The definitions involve properties (such as one-dimensionality of curves and $(n-1)$-dimensionality of hypersurfaces) that characterize them to be digital analogs of definitions for Euclidean spaces. Further research may pursue designing efficient algorithms for recognizing whether a given set of n-cells is a digital curve or hypersurface.

We also proposed a uniform approach to studying good pairs defined by separation and, in that framework, obtained a classification of good pairs in arbitrary dimension. A future task is seen in extending the obtained results under other reasonable definitions of good pairs.

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References

1. J.C. Alexander and A.I. Thaler. The boundary count of digital pictures. *J. ACM*, **18**:105–112, 1971.
2. E. Andres, R. Acharya, and C. Sibata. Discrete analytical hyperplanes. *Graphical Models Image Processing*, **59**:302–309, 1997.
3. G. Bertrand and R. Malgouyres. Some topological properties of surfaces in $\mathbb{Z}^3$. *J. Mathematical Imaging Vision,* **11**:207–221, 1999.
4. V.E. Brimkov, E. Andres, and R.P. Barneva. Object discretizations in higher dimensions. *Pattern Recognition Letters*, **23**:623–636, 2002.
5. L. Chen, D.H. Cooley, and J. Zhang. The equivalence between two definitions of digital surfaces. *Information Sciences,* **115**:201–220, 1999.
6. D. Cohen-Or, A. Kaufman, and T.Y. Kong. On the soundness of surface voxelizations. In: T.Y. Kong and A. Rosenfeld, editors. *Topological Algorithms for Digital Image Processing*, 181–204. Elsevier, Amsterdam, The Netherlands, 1996.
7. M. Couprie and G. Bertrand. Tessellations by connection. *Pattern Recognition Letters*, **23**:637–647, 2002.
8. R.O. Duda, P.E. Hart, and J.H. Munson. Graphical-data-processing research study and experimental investigation. TR ECOM-01901-26, Stanford Research Institute, Menlo Park, California, March 1967.
9. U. Eckhardt and L. Latecki. Topologies for the digital spaces $\mathbb{Z}^2$ and $\mathbb{Z}^3$. *Computer Vision Image Understanding*, **90**:295–312, 2003.
10. J. Françon. Discrete combinatorial surfaces. *Graphical Models Image Processing*, **57**:20–26, 1995.
11. G.T. Herman. Boundaries in digital spaces: Basic theory. In: T.Y. Kong and A. Rosenfeld, editors. *Topological Algorithms for Digital Image Processing*, 233–261. Elsevier, Amsterdam, The Netherlands, 1996.
12. Y. Kenmochi, A. Imiya, and A. Ichikawa. Discrete combinatorial geometry. *Pattern Recognition*, **30**:1719–1728, 1997.
13. C.E. Kim. Three-dimensional digital line segments. *IEEE Trans. Pattern Analysis Machine Intelligence*, **5**:231–234, 1983.
14. R. Klette and A. Rosenfeld. *Digital Geometry - Geometric Methods for Digital Picture Analysis.* Morgan Kaufmann, San Francisco, 2004.
15. T.Y. Kong. Digital topology. In: L.S. Davis, editor. *Foundations of Image Understanding*, 33–71. Kluwer, Boston, Massachusetts, 2001.
16. T.Y. Kong. Topological adjacency relations on $\mathbb{Z}^n$. *Theoretical Computer Science*, **283**:3–28, 2002.
17. T.Y. Kong, A.W. Roscoe, and A. Rosenfeld. Concepts of digital topology. *Topology and its Applications*, **46**:219–262, 1992.
18. R. Kopperman, P.R. Meyer, and R. Wilson. A Jordan surface theorem for three-dimensional digital spaces. *Discrete Computational Geometry*, **6**:155–161, 1991.
19. V. Kovalevsky. Multidimensional cell lists for investigating 3-manifolds. *Discrete Applied Mathematics*, **125**:25–44, 2003.
20. J.-O. Lachaud and A. Montanvert. Continuous analogs of digital boundaries: A topological approach to isosurfaces. *Graphical Models*, **62**:129–164, 2000.
21. R. Malgouyres. A definition of surfaces of $\mathbb{Z}^3$: A new 3D discrete Jordan theorem. *Theoretical Computer Science*, **186**:1–41, 1997.
22. D.G. Morgenthaler and A. Rosenfeld. Surfaces in three-dimensional digital images. *Information Control*, **51**:227–247, 1981.

23. J.P. Mylopoulos and T. Pavlidis. On the topological properties of quantized spaces. I. The notion of dimension. *J. ACM*, **18**:239–246, 1971.
24. J.-P. Reveillès. Géométrie discrète, calcul en nombres entiers et algorithmique. Thèse d'état, Université Louis Pasteur, Strasbourg, France, 1991.
25. A. Rosenfeld. Adjacency in digital pictures, *Information and Control* **26**, 24–33, 1974
26. A. Rosenfeld. Compact figures in digital pictures. *IEEE Trans. Systems, Man, Cybernetics,* **4**:221–223, 1974.
27. A. Rosenfeld, T.Y. Kong, and A.Y. Wu. Digital surfaces. *CVGIP: Graphical Models Image Processing*, **53**:305–312, 1991.
28. A. Rosenfeld and J.L. Pfaltz. Sequential operations in digital picture processing. *J. ACM*, **13**:471–494, 1966.
29. G. Tourlakis. Homological methods for the classification of discrete Euclidean structures. *SIAM J. Applied Mathematics*, **33**:51–54, 1977.
30. G. Tourlakis and J. Mylopoulos. Some results in computational topology. *J. ACM*, **20**:430–455, 1973.
31. J.K. Udupa. Connected, oriented, closed boundaries in digital spaces: Theory and algorithms. In: T.Y. Kong and A. Rosenfeld, editors. *Topological Algorithms for Digital Image Processing*, 205–231. Elsevier, Amsterdam, The Netherlands, 1996.
32. D.J.A. Welsh. *Matroid Theory.* Academic Press, London, 1976.