# Lovász theta-function of a class of graphs representing digital lines

Valentin E. Brimkov* Reneta P. Barneva* Reinhard Klette†

**Abstract**

We consider the problem of estimating the Shannon capacity of a circulant graph $C_{n,j}$ of degree four with $n$ vertices and chord length $j$, $2 \leq j \leq n$, by computing its Lovász theta function $\theta(C_{n,j})$. Our interest in this problem is driven by possible applications to error-free communication of data describing the structure of a digital line. The latter can be represented in terms of *spirographs* [12], which, as a matter of fact, are circulants of degree four. We present an algorithm for $\theta(C_{n,j})$ computation that takes $O(j)$ operations if $j$ is an odd number, and $O(n/j)$ operations if $j$ is even. On the considered class of graphs our algorithm strongly outperforms the known algorithms for theta function computation.

**Keywords:** *Shannon capacity, Lovász theta-function, Circulant graph, Digital line, Spirograph*

## 1 Introduction

In the present paper we consider the problem of estimating the Shannon capacity of a circulant graph of degree four by computing its Lovász theta function. In a famous paper of 1956 [26] Shannon first studied the amount that an information channel can communicate without error. He introduced the notion of zero-error capacity of a graph, known thereafter as the Shannon capacity. It was understood quite early that the exact determination of the Shannon capacity is a very difficult problem, even for small and simple graphs (see [15, 25]). In 1979 Lovász [20] introduced a function $\theta(G)$ with the aim of estimating the Shannon capacity. Despite a lot of work in the field, very little is known about classes of graphs for whose theta function either a formula or a very efficient algorithm is available. A sample for such a result is the Lovász's formula $\theta(C_n) = \frac{n \cos \frac{\pi}{n}}{1+\cos \frac{\pi}{n}}$ for an odd cycle $C_n$ with $n$ nodes [20]. Recently Brimkov et al. [11] generalized this last result by obtaining formulas for $\theta(G)$ for the special cases of circulant graphs of degree four with chord length two and three.

Our interest in Shannon capacity and Lovász theta function of circulant graphs is particularly driven by possible applications to error-free communication of data describing the structure of a digital line, the latter being the most fundamental primitive in computer graphics and image analysis. Computer representation of digital lines has been an active research topic for nearly half a century (see the recent survey [24] and the bibliography therein). In [12] Dorst and Duin have developped the theory of spirographs in order to establish links between digital straight lines and number theory. Spirograph is a diagram that models the distribution of the integer points constituting a digital line.

We observe that the spirographs appear to be circulant graphs of degree two or four. Various other applications of circulant graphs are known in counting and combinatorics [23], as well as in telecommunication networks, VLSI design, and distributed computing [9, 18, 19, 21]. Low-degree circulants provided a basis for some classical parallel and distributed systems [10, 27] as well as for

*State University of New York, Fredonia, NY 14063, USA, E-mail: {brimkov,barneva}@cs.fredonia.edu
†CITR Tamaki, University of Auckland, Building 731, Auckland, New Zealand, E-mail: r.klette@auckland.ac.nz

certain data alignment networks for complex memory systems [28]. Specifically, circulant graphs of degree four have been used in the design of local networks and interconnection subsystems [1, 9]. Recent work [7] presents a class of such graphs with minimal topological distances. These graphs (called Midimew networks) have been used as a basis for constructing an optimal interconnection network for parallel computers with a very high degree of fault-tolerance [7], as well as for designing networks for massively parallel computers [29] or optimal VLSI [16].

In the present note we use geometric approach to construct a very efficient algorithm for computing the theta function of *arbitrary* circulant graphs of degree four. For a circulant graph $C_{n,j}$ with $n$ vertices and chord length $j$, $2 \leq j \leq n$, the algorithm performs $O(j)$ operations when $j$ is odd and $O(n/j)$ operations when $j$ is even, and appears to be strongly superior to the known algorithm whose time complexity is of the order $O(n^4)$ [3].

The paper is organized as follows. In the next section we recall some graph-theoretic notions and results to be used in the sequel. In Section 3 we define spirographs and estimate their cardinality. In Section 4 we introduce some geometrical constructions used in designing our algorithm. The basic results are presented in Section 5. We conclud with some remarks in the final Section 6.

## 2 Some graph-theoretic notions and facts

Here we recall some well-known definitions from graph theory. (See [8] for details.) Let $G(V,E)$ be a simple graph. The *complement graph* of $G$ is the graph $\bar{G}(V,\bar{E})$, where $\bar{E}$ is the complement of $E$ to the set of edges of the complete graph on $V$. An *automorphism* of the graph $G$ is a permutation $p$ of its vertices such that two vertices $u,v \in V$ are adjacent iff $p(u)$ and $p(v)$ are adjacent. $G$ is *vertex symmetric* if its automorphism group is vertex transitive, i.e., for given $u,v \in V$ there is an automorphism $p$ such that $p(u) = v$. By $\omega(G)$ and $\chi(G)$ we denote the clique and the chromatic numbers of $G$, respectively. For any graph $G$ we have $\omega(G) \leq \chi(G)$. We also have the following classical result.

**Theorem 1** [8] *A connected simple graph $G$ with maximal degree $d$ is $d$-colorable, unless $d \neq 2$ and $G$ is a $(d+1)$-clique, or $h = 2$ and $G$ is an odd cycle.*

An *independent set* of $G$ is a set of vertices no two of which are adjacent. The cardinality of a maximal independent set is called the *independence number* of $G$ and denoted $\alpha(G)$. A graph $G'(V',E')$ is an *induced subgraph* of $G(V,E)$, if $E'$ contains all edges from $E$ that join vertices from $V' \subseteq V$. $G$ is *perfect* if $\omega(G_A) = \chi(G_A)$, $\forall A \subseteq V$, where $G_A$ is the induced subgraph of $G$ on $A$.

An $n \times n$ matrix $A = (a_{i,j})_{i,j=0}^{n-1}$ is called *circulant* if its entries satisfy $a_{i,j} = a_{0,j-i}$, where the subscripts belong to the set $\{0,1,\ldots,n-1\}$ and are calculated modulo $n$. In other words, any row of a circulant matrix can be obtained from the first one by a number of consecutive cyclic shifts, and thus the matrix is fully determined by its first row. A *circulant graph* is a graph with a circulant adjacency matrix. By $C_{n,j}$ we will denote a circulant graph of degree four, with vertex set $\{0,1,\ldots,n-1\}$ and edge set $\{(i, i+1 \bmod n), (i, i+j \bmod n), i = 0,1,\ldots,n-1\}$, where $1 < j \leq \frac{n-1}{2}$ is the *chord length.* See for illustration Fig. 2a presenting the circulant graph $C_{13,2}$. The Midimew networks mentioned in the Introduction are special circulant graphs of the form $C_{N,2k+1}$, where $N = k^2 + (k+1)^2$ and $k$ is the graph diameter.

Now consider a graph $G$ whose vertices are letters from a given alphabet and where adjacency indicates that two letters can be confused. In this setting, the maximal number of one-letter messages that can be communicated without danger of confusion equals the independence number $\alpha(G)$. Then the maximal number of $k$-letter messages that can be safely communicated is $\alpha(G^k)$, where $G^k$ is

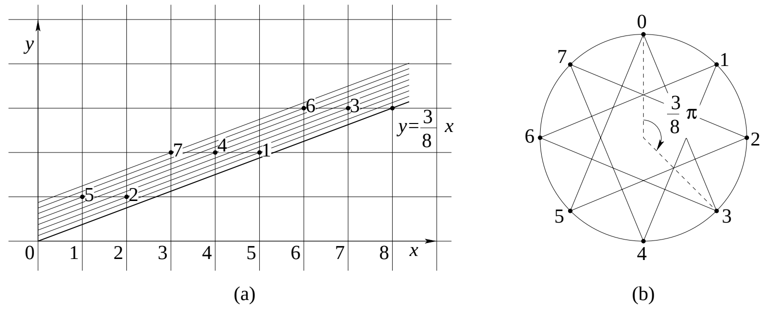


Figure 1: a) Digitization of a ray with a slope $\alpha = 3/8$ together with the level lines. b) The corresponding spirograph $S(3/8, 13)$.

the $k$-th power of $G$. It follows that $\alpha(G^k) \geq \alpha(G)^k$, as equality does not hold, in general [20]. The Shannon capacity of $G$ is then defined as the limit $\Theta(G) = \lim_{k\to\infty} \sqrt[k]{\alpha(G^k)}$. It satisfies $\Theta(G) \geq \alpha(G)$, where equality does not need to occur. Shannon proved that if $G$ is perfect, then $\Theta(G) = \alpha(G)$. As already mentioned, in order to estimate $\Theta(G)$, Lovász devised a function $\theta(G)$, known thereafter also as the Lovász number. Several equivalent definitions of the Lovász number are available [17]. We present here the one which requires only little technical machinery. Given a graph $G$, let $\mathbf{A}$ be the family of matrices $A$ such that $a_{ij} = 0$ if $v_i$ and $v_j$ are adjacent in $G$. Let $\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_n(A)$ be the eigenvalues of $A$. Then $\theta(A) = \max_{A\in\mathbf{A}}\{1 - \frac{\lambda_1(A)}{\lambda_n(A)}\}$.

Combining the fact that $\Theta(G) \leq \theta(G)$ with the easy lower bound $\Theta(C_5) \geq \sqrt{5}$, Lovász was able to determine the capacity of the pentagon $C_5$, which turnes out to be $\sqrt{5}$. We know very little about the Shannon capacity of other non-perfect graphs. For instance, $\Theta(C_7)$ is still unknown. $\theta(G)$, however, is computable in polynomial time with arbitrary precision, although being "sandwiched" between the clique number $\omega(G)$ and the chromatic number $\chi(G)$, whose computation is NP-hard for general graphs. More precisely, we have $\omega(G) \leq \theta(\bar{G}) \leq \chi(G)$. Because of this remarkable property and due to the relations to communication issues, the Lovász number is a subject of active study. For various results and applications see the surveys by Knuth [17] and Alizadeh [2] and the bibliography therein. See also [4, 5, 6, 13, 14] for a sample of the diversity of results and applications of $\Theta(G)$ and $\theta(G)$. Here we list a simple proposition for future reference.

**Proposition 1 (see [17])** *For every graph $G$ with $n$ vertices, $\theta(G) \cdot \theta(\bar{G}) \geq n$. If $G$ is vertex symmetric, then $\theta(G) \cdot \theta(\bar{G}) = n$.*

## 3 Digital lines and spirographs

Let $\gamma_{\alpha,\beta} = \{(x, \alpha x + \beta) : 0 \leq x < +\infty\}$ be a ray with a slope $\alpha$ and intercept $\beta$. *Digitization* of $\gamma_{\alpha,\beta}$ over the integer grid is defined as the set of integer points $I_{\alpha,\beta} = \{(n, I_n) : I_n = \lfloor \alpha n + \beta + 0.5 \rfloor, n \geq 0\}$. Because of the symmetry of the grid, one can assume that $0 \leq \alpha \leq 1$. Straight line digitization is defined analogously. The integer points of a digital ray $I_{\alpha,\beta}$ with rational $\alpha = p/q$ belong to *level lines* whose number equals $\alpha$'s denominator $q$, where $p/q$ is an irreducible fraction (see Figure 1a). It is well-known that a digital line/ray with rational coefficients is periodic with a period length $q$ [24]. Let, for simplicity, $I_{\alpha,0}$ be a digitization of a ray $y = \alpha x$ through the origin with a slope $\alpha = p/q$. The *spirograph* $S(\alpha, n)$ of $I_{\alpha,0}$ is a set of $n$ points on a circle with unit perimeter, marked $0, 1, \ldots, n-1$

and defined by the level lines of $I_{\alpha,0}$ of slope $\alpha$ intersecting the grid lines $x = 0, x = 1, \ldots, x = n-1$ at grid points. We mark on the circle a first point representing the intersection point for grid line $x = 0$. Then we proceed in clockwise orientation from the first vertex to a second vertex on the circle at radial distance $\alpha$ representing the intersection point for the grid line $x = 1$, and so on. If $\alpha$ is any rational number $p/q$, then the number of level lines, as well as their intersection points in $[0, 1)$, equals $q$. In turn, the number of different points on the circle generated through the above process is $q$, as well, i.e., we obtain a spirograph $S(\alpha, n)$ with $n = q$ vertices and chord length $p$. It is easy to see that the so constructed spirograph is isomorphic to a circulant graph. See Figure 1b and [12] for more details. Spirographs have been used to study the periodicity structure of digital lines as well as to handle some problems of practical importance (e.g., to determine the accuracy of estimating the slope and the intercept of a generating ray as a function of the length of a given digital line segment). Shannon capacity of a digital line is the capacity of the corresponding spirograph.

We notice that not every spirograph is a circulant of degree four: in fact, any spirograph $C_{q,1}$ that corresponds to a digital line with a slope $\alpha = 1/q$ is a circulant of degree two, i.e., a cycle $C_q$. If $q$ is even, then $C_q$ is perfect and its Shannon capacity and theta function equal $n/2$. For $n$ odd we have the Lovász formula $\theta(C_q) = \frac{q\cos\frac{\pi}{q}}{1+\cos\frac{\pi}{q}}$. Thus we can exclude cycles from further consideration.

We also observe that not every circulant graph $C_{n,j}$ with $j \geq 2$ corresponds to a digital line. However, it turns out that most of the graphs $C_{n,j}$ are spirographs. More precisely, we have the following fact.

**Proposition 2** *As $n$ tends to infinity, the number of spirographs of order $n$ is approximately* 0.6 *of the number of all circulant graphs $C_{n,j}$.*

**Proof** By the spirograph construction it follows that a circulant graph $C_{n,j}$ is a spirograph of some digital line if and only if $n$ and $j$ are relatively prime. Thus the number of disctinct spirographs of order $n$ is $\frac{\phi(n)}{2}$, where $\phi(n)$ is the Euler totient function. It is well-known that the latter tends to $\frac{6n}{\pi^2}$ as $n$ approaches infinity. Thus we obtain that the number of distinct spirographs of order $n$ is assimptotically equal to $\frac{1}{2} \cdot \frac{6n}{\pi^2} = \frac{3n}{\pi^2} = (0.30396\ldots) \cdot n \approx 0.3n$. The number of all distinct circulant graphs $C_{n,j}$ (including the cyclic graph $C_{n,1} = C_n$) is clearly $\lfloor \frac{n}{2} \rfloor$, from where the statement follows. □

The algorithm for theta function computation described in the following sections applies to circulant graphs of degree four. We notice that all circulants of order $\leq 5$ except the pentagon are perfect and their Shannon capacity is trivially determined. We also have $\Theta(C_{5,1}) = \theta(C_{5,1}) = \sqrt{5}$. Thus we can consider circulant graphs of order larger than 5. It is easy to see that for circulants $C_{n,j}$ with $n \geq 6$ it holds $\omega(C_{n,j}) \geq 2$ and $\chi(G) \leq 4$, hence $2 \leq \theta(\bar{C}_{n,j}) \leq 4$. Since the circulant graphs are vertex symmetric, by Proposition 1 we obtain the bounds $n/2 \geq \theta(C_{n,j}) \geq n/4$. In the subsequent sections we design an efficient algorithm for the exact computation of $\theta(C_{n,j})$.

# 4 LP formulation of $\theta(C_{n,j})$ and certain subsidiary geometrical constructions

Taking advantage of the particular properties of circulant matrices whose eigenvalues can be expressed in closed form, one can easily generalize the approach of [20]. Then the validity of the following minmax formulation of the $\theta$-function of circulant graphs of degree 4 can be derived.

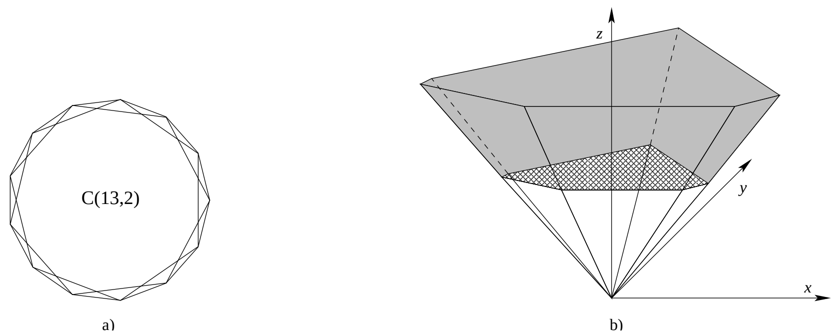


Figure 2: a) The graph $C_{13,2}$. b) The polyhedral cone related to $C_{13,2}$, cut at $z = 2$.

**Lemma 1 (see [11])** *Let $f_0(x, y) = n + 2x + 2y$ and, for some fixed value of $j$, $f_i(x, y) = 2x \cos \frac{2\pi i}{n} + 2y \cos \frac{2\pi ij}{n}$, $i = 1, 2, \ldots, n-1$. Then*

$$\theta(C_{n,j}) = \min_{x,y} \max_{i} \left\{ f_i(x, y), i = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor \right\}. \tag{1}$$

This is in turn equivalent to the following Linear Programming (LP) problem:

$$\theta(C_{n,j}) = \quad \min\{z \ : \ f_i(x, y) - z \leq 0, \ i = 0, 1, \ldots, \lfloor \tfrac{n}{2} \rfloor, z \geq 0\}. \tag{2}$$

We now observe that the equalities $f_1(x, y)-z = 0, \ldots, f_{n-1}(x, y)-z = 0$ define planes through the origin. Having in mind the specific coefficients of these planes in the different ortants, as well as the relations between the coefficients of two consecutive planes, one can see that the set $\max_i\{f_1(x, y), \ldots, f_n(x, y)\}$ is a polyhedral surface, namely a polyhedral cone $C$ with its apex at the origin. The cone belongs to the positive halfspace $z \geq 0$ and the $Oz$ axis is contained inside the cone. The faces of the cone are portions of consecutive planes with equations $z = f_i(x, y)$, $i = 1, 2, \ldots, n-1$, and thus the rays of $C$ are intersections of neighboring planes, obtained for pair of indices $i, i+1$. The other intersections are not of interest, since they all fall "below" the conic surface $\max_i\{f_i\}$ and thus are not part of it.

A more careful analysis can show that only one of the planes forming the cone has two positive coefficients, and only one of them has a positive coefficient for $y$ and a negative coefficient for $x$. In the other cases we may have arbitrary many planes. For example, in the ortant $x \leq 0, y \leq 0, z \geq 0$ (i.e., if the two coefficients are negative), we may have arbitrary many planes forming the cone.

We now consider the plane $f_0(x, y) = n + 2x + 2y$. Its intersection with the cone $C$ produces a new polyhedral surface. Roughly speaking, this is the upper part of the cone $C$, i.e., the part of the cone above the plane $f_0$ (see Fig. 1b). As it will turn out later, a part of the cone $C$ will be "cut out" and thus some of the planes (forming the faces of $C$) will be eliminated.

Clearly, the intersection points of the plane $f_0$ with $C$ are the possible candidates for solution of the problem. The theta function is the intersection point with minimal $z$. Consider the intersection of $C$ and $f_0$. This intersection is the boundary of some 2D convex polyhedron $P$ (possibly unbounded). As mentioned above, the solution is at some of the vertices of this intersection. Let this be the point $A = (x_0, y_0, z_0)$ (and thus $\vartheta = z_0$). Let us now assume that we have intersected $C$ by the plane $z = z_0$ (parallel to the $xy$-plane). The intersection is a (bounded) convex polygon $Q_{z_0}$. By construction, it follows that the polyhedron $P$ and the polygon $Q_{z_0}$ intersect at a single point, i.e., the point $A = (x_0, y_0, z_0)$. We will determine $A$ using the sides of $Q_{z_0}$, rather than the sides of $P$. Since the coefficients of $x$ and $y$ of the plane $z = n + 2x + 2y$ are equal (indeed they are both equal to 2), then it is not difficult to see that $A$ will be the vertex of $Q_{z_0}$, obtained as the intersection of the two sides of $Q_{z_0}$ which "sandwich" the straight line in $z = z_0$ passing through $A$, and with a slope of 45 degrees. These lines have equations $2x \cos\alpha + 2y \cos(\alpha J) = z_0$ and $2x \cos\beta + 2y \cos(\beta J) = z_0$, where $\alpha = \frac{2\pi i_1}{n}$ and $\beta = \frac{2\pi i_2}{n}$, for some indices $i_1$ and $i_2$. Once $i_1$ and $i_2$ are known, $z_0$ can be computed by solving the linear system

$$\begin{aligned} z &= 2x \cos\alpha + 2y \cos(2\alpha) \\ z &= 2x \cos\beta + 2y \cos(2\beta) \\ z &= n + 2x + 2y. \end{aligned}$$

Note that one can use any horizontal intersection of the cone, since all such intersections are homothetic to each other.

Through a more detailed analysis of the structure of the admissible region defined by the linear constraints, in the next section we propose an efficient computation of $\theta(C_{n,j})$. The general idea is to reduce significantly the set of constraints in the LP problem (2) and then apply existing efficient algorithms for LP in 3D (e.g., the Megiddo's algorithm which performs in time linear with respect to the number of constraints of the problem). We measure the complexity of a computation by counting the number of arithmetic operations in the set $S = \{+, -, *, /, \lfloor . \rfloor, \cos(\cdot)\}$ as a function of the number of constraints in the three-dimensional LP problem (2).

# 5 Computation of $\theta(G)$

Relying on 2, we will focus on the geometric unintuitive regularities of the polygon defined by the lines $l(k)$ of equation

$$x \cos(\alpha_k) + y \cos(j\alpha_k) = 1, \tag{3}$$

with $\alpha_k = \frac{2\pi}{n} k$. Let $a(k) = 1/\cos(\alpha_k)$ and $b(k) = 1/\cos(2\pi kj/n)$ be their $x$ and $y$ coordinates (axes cuts) respectively. We will refer to angle $\alpha_k$ as to the angle of line $l(k)$. We distinguish two cases: $j$ even and $j$ odd.

## 5.1 Odd chord lengths

As a first result let us prove the following lemma for arbitrary circulant graphs. It provides an immediate solution to the case of $C_{n,j}$ with $n$ even and $j$ odd.

**Lemma 2** *Let $C(n; j_1, j_2, \ldots, j_k)$ be a circulant graph of $n$ vertices and chord lengths $j_1, \ldots, j_k$ with $2 < j_1 < \ldots < j_k$. Assume that $n$ is even and all the chord lengths $j_i$ are odd. Then $C(n; j_1, \ldots, j_k)$ is perfect and $\theta(C(n; j_1, \ldots, j_k)) = n/2$.*

**Proof** Since every circulant graph is vertex symmetric we have

$$\theta(C(n; j_1, \ldots, j_k)) \cdot \theta(\bar{C}(n; j_1, \ldots, j_k)) = n.$$

Thus it is enough to show that $\theta(\bar{C}(n; j_1, \ldots, j_k)) = 2$. Bearing in mind the inequality $\omega(G) \leq \theta(\bar{G}) \leq \chi(G)$ which applies to any graph $G$, we obtain that it is enough to show that $\omega(C(n; j_1, \ldots, j_k)) = \chi(C(n; j_1, \ldots, j_k)) = 2$. In fact, the clique number is 2 since for $n \geq 6$ and $j_i \geq 3$ the minimal cycle in $C(n; j_1, \ldots, j_k)$ has length at least 4 (which bound is reached for $j = 3$). It is also not hard to see

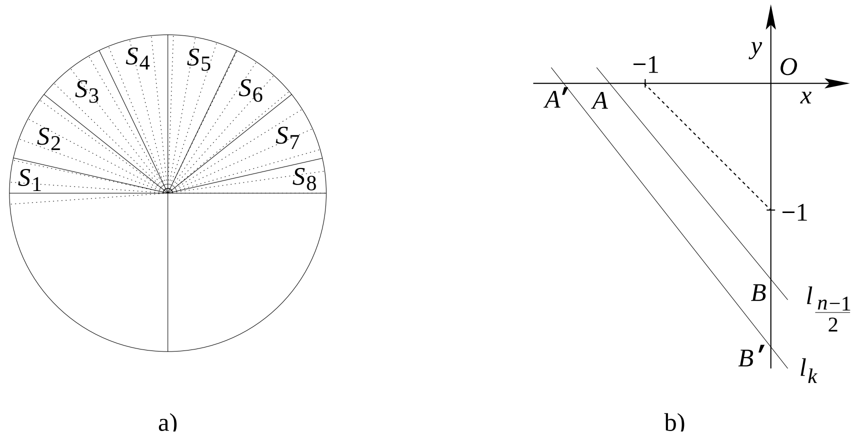


Figure 3: a) Pictorial description of the $S$-intervals for $j = 7$. b) Position of line $l(n-1/2)$ with respect to any $l_k$ with $\alpha_{l_k} \in S_1$.

that the vertices of $C(n; j_1, \ldots, j_k)$ can be alternatively colored with two colors only, if $n$ is even and the $j_i$'s are odd, which completes the proof. □

Now we obtain the following main result.

**Theorem 2** *Let $n, j$ be two odd numbers with $j \leq \frac{n-1}{2}$. Then $\theta(C_{n,j})$ can be computed with $O(j)$ operations by solving a 3D LP problem having $O(j)$ constraints.*

**Proof**

Here we prove that we can identify in constant time a set of at most $\lfloor \frac{j}{2} \rfloor + 1$ lines that define the polygon $Q_{z_0}$.

Let $\mathcal{S} = \{S_1, S_2, \ldots, S_{j+1}\}$ be a set of adjacent intervals covering $[0, \pi]$ defined as

$$S_1 = [\pi - \frac{\pi}{2j}, \pi], S_{j+1} = [0, \frac{\pi}{2j}] \text{ and } S_{k+1} = [\pi - (2k+1)\frac{\pi}{2j}, \pi - (2k-1)\frac{\pi}{2j}],$$

for $k = 1, 2, \ldots, j-1$. So, $S_1, S_{j+1}$ are intervals of width $\frac{\pi}{2j}$, whereas $S_2, S_3, \ldots, S_j$ are intervals of width $\frac{\pi}{j}$ (see Fig 3a). A quick analysis of the function $\cos(j\alpha)$ reveals that

**a)** it is periodic of period $2\pi/j$;

**b)** it nullifies on $S_k \cap S_{k+1}$, for $k = 1, 2, \ldots, j$, and

**c)** it is negative on the odd numbered intervals attaining $-1$ on their middle points.

Consider line $l(\lfloor \frac{n}{2} \rfloor)$ in the interval $S_1$. This verifies

$$\begin{cases} a(\lfloor \frac{n}{2} \rfloor) &= \frac{1}{\cos(\pi - \pi/n)} &= \max\{a(i) \mid a(i) < 0 \wedge i \geq 0\} < -1 \\ b(\lfloor \frac{n}{2} \rfloor) &= \frac{1}{\cos(j(\pi - \frac{\pi}{n}))} &= \frac{1}{\cos(\pi - j\frac{\pi}{n})} \end{cases}$$

This line defines a face of $Q_{z_0}$ because it is the one that intersects the $Ox$ axis in the closest point to $(-1, 0)$ (see Fig.3b). Furthermore, its inclination w.r.t the $Oy$ axis is less than 45 degrees and all other lines in $S_1$ have a lower $x$-cut and $y$-cut therefore falling outside $Q_{z_0}$.

Consider the even numbered intervals. Lines whose angle $\tau$ falls in these intervals all have positive $y$ coordinate because $\cos(j\tau) > 0$. Their $x$-coordinate can be either positive, and in that case they would not even cross the third quadrant or negative, in which case it must be

$$\frac{1}{\cos(\tau)} < \frac{1}{\cos(\alpha_{(n-1)/2})} = a(\lfloor\frac{n}{2}\rfloor).$$

So, we can conclude that those lines can not affect the solution.

Consider the odd numbered intervals $S_{2k-1}$, for $k = 1, 2, \ldots, (j+1)/2$ and let $\beta^{(k)} = \pi - (2k-2)\pi/j$. Angle $\beta^{(1)}$ is $\pi$ whereas, for $k > 1$, angles $\beta^{(k)}$ correspond to the centers of intervals $S_{2k-1}$ and all verify $\cos(j\beta^{(k)}) = -1$. We can observe that the only lines $l(i)$ that intersect $l(\lfloor\frac{n}{2}\rfloor)$ in the third quadrant are those for which

$$b(\lfloor\frac{n}{2}\rfloor) < b(i) < -1. \qquad (4)$$

Now, focus for a moment on the function $f(x) = 1/\cos(jx)$. It is periodic and assumes the same values within the odd numbered intervals

$$S_{2k-1} = [\beta^{(k)} - \pi/2j, \beta^{(k)} + \pi/2j].$$

Furthermore it is increasing over $[\beta^{(k)} - \pi/2j, \beta^{(k)}]$, decreasing over $[\beta^{(k)}, \beta^{(k)} + \pi/2j]$ and verifies:

$$f(\beta^{(k)}) = -1, \quad \lim_{x\to(\beta^{(k)}-\pi/2j)^+} = \lim_{x\to(\beta^{(k)}+\pi/2j)^-} = -\infty.$$

Observe that $b(i) = f(2i\pi/n)$, i.e., condition (4) can be rephrased as

$$f\left(\pi - \frac{\pi}{n}\right) < f\left(\frac{2\pi}{n} \cdot i\right) < -1 \ .$$

Since the behavior of $f(x)$ on the interval $[\pi - \pi/2j, \pi + \pi/2j]$ is the same as for all the other odd numbered intervals $[\beta^{(k)} - \pi/2j, \beta^{(k)} + \pi/2j]$ the condition

$$f\left(\pi - \frac{\pi}{n}\right) < f(x) < -1$$

will be verified only for

$$\mid x - \beta^{(k)} \mid < \frac{\pi}{n} \qquad (5)$$

where $\{\beta^{(k)} \mid \ k = 1, 2, \ldots, (j+1)/2\}$ is the set of the solutions to equation $f(x) = 1/\cos(jx) = -1$ on $[0, \pi]$.

Given that the angle does not vary with continuity, but assumes only a discrete set of values $\alpha_i = 2\pi i/n$, for $0 \leq i \leq (n-1)/2$, we can see that, if for some $u$, $\alpha_u$ satisfies condition (5), then $\alpha_{u+1} = \alpha_u + 2\pi/n$ can not.

Thus we can deduce that for each odd numbered interval $S_{2k-1}$ there can be at most one line verifying condition (4) and since we have $\lfloor\frac{j}{2}\rfloor$ odd numbered intervals to consider, there will be at most as many lines to select.

Now the obtained linear program can be solved in $O(j)$ time by the Megiddo linear programming algorithm which has linear complexity when the number of variables is fixed. □

**Corollary 1** *The theta-function of a Midimew network $C_{N,j}$ can be computed with $O(\sqrt{N})$ operations.*

**Proof** Follows from the fact that Midimew networks are circulant graphs of the form $C_{N,j}$ with $j = 2k+1$ and $N = k^2 + (k+1)^2$ which implies $j < \sqrt{2N}$. Then $\theta(C_{N,j})$ can be computed in $O(j) = O(\sqrt{N})$ time. □

### 5.2 Even chord lengths

We have the following theorem.

**Theorem 3** *Let $n$ be a positive integer and $j$ an even number with $j \leq \frac{n-1}{2}$. Then $\theta(C_{n,j})$ can be computed with $O(n/j)$ operations by solving a 3D LP problem having $O(n/j)$ constraints.*

**Proof** Consider once more the family of intervals $\mathcal{S} = \{S_1, S_2, \ldots, S_{j+1}\}$. Now the $j/2$ even numbered ones, $S_{2k}$, for $k = 1, 2, \ldots, j/2$, are those in which $\cos(j\alpha)$ is negative. Let $\beta^{(k)}$ denote the angle corresponding to the center of $S_{2k}$. Thus $\cos(j\beta^{(k)}) = -1$ for all $k$. First, notice that each interval contains no more than $\lceil \frac{n}{j} \rceil$ lines.

Let us focus on $S_1 \cup S_2$ and define an enumeration of consecutive (w.r.t. the corresponding increasing angle) lines $l_1, l_2, \ldots, l_s$, where $l_1$ is the line whose angle, $\alpha^{(1)}$ is the closest from below to the center of $S_2$ and $l_s$ is the line whose angle is the largest within $S_1$: $l(\lfloor n/2 \rfloor)$. It is not hard to see that those lines define a set $C_1$ of segments that, together with the $x$ and $y$ negative axes, bound a convex polygon $Q$.

We needed to consider also the lines in $S_1$, because those have $x$-coordinate that happen to be very close to point $(-1, 0)$ and, as proven above, they contribute to shaping polygon $Q$. Furthermore all the other lines in $S_2$ whose angle is smaller than $l_1$'s angle must have lower $x$-cut and $y$-cut and therefore do not intersect this polygon.

We can apply the same idea to the other even numbered intervals $S_{2k}$, $k = 2, 3, \ldots, j/2$, and define the corresponding finite sequences of lines $l_1^{(k)}, l_2^{(k)}, \ldots, l_{s_k}^{(k)}$, now ending with lines whose angle is within $S_{2k}$. (Note the asymmetry in the definition of the lines $\{l_i^{(k)}\}$, in that $l_{s_k}^{(k)}$ has the largest angle in $S_{2k}$, whereas the sequence $l_i$ is not limited to $S_2$ but goes on until the exhaustion of the interval $S_1$ adjacent to $S_2$.)

Now, as in the case for $j$ odd, it turns out that for all $k$ only $l_1^{(k)}$ might intersect $Q$. Furthermore this would occur only when the angle $\alpha^{(k)}$ of $l_1^{(k)}$ satisfies

$$| \alpha^{(k)} - \beta^{(k)} | < | \alpha^{(1)} - \beta^{(1)} | \leq \pi/n \ . \tag{6}$$

As a consequence, the search for the solution can be restricted to the vertices of the polygon formed by the two axes, the lines in $S_1 \cup S_2$ plus, possibly, the lines whose angle verify property 6. Thus the total number of lines to be considered is $O(n/j)$ Then the obtained LP problem can be solved in $O(n/j)$ time by the Megiddo linear programming algorithm. □

## 6 Concluding Remarks

We have presented efficient ways to compute the theta function of circulant graphs of degree four. In particular, the problem can be reduced to a 3-variable LP problem having at most $O(j)$ constraints when $j$ is odd, whereas for $j$ even the bound on the number of significant constraints was shown to be $O(n/j)$. Consequently, an application of the Megiddo algorithm allows to compute $\theta(C_{n,j})$ with $O(j)$ or $O(n/j)$ operations depending on the evenness of $j$. Megiddo algorithm solves any LP problem in linear time with respect to the number of constraints, provided that the number of variables is fixed. It is indeed known that its complexity would include an implicit factor of the order of $O(2^{s^2})$, where $s$ is the number of variables, which however is a small number for the considered dimension. Work in progress aims at providing efficient computation of the theta-function of circulant graphs of higher degree, e.g. of appropriately defined circulant graphs that represent digital planes.

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