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General formula for distribution of random variables in DSSS-CDMA

Technical report

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Summary

The Direct sequence spectrum (DSSS) systems are using a sequence to present a bit. These sequences are orthogonal. Based on the orthogonality of these sequences, multiple users can use a single channel simultaneously. There are different methods to generate these orthogonal sequences. One of the well-known methods is using chaotic sequences. A chaotic sequence is generated by using a seed value and a mapping function. The seed acts as an initial value for the mapping function. Then, the mapping function makes a sequence with an infinite number of elements. In addition, by using different seeds, it can make orthogonal sequences.

The common sequences have a determined probability density function (pdf). So, in the performance analysis of DSSS systems, they are considered as a random variable (r.v) with known pdf. Also, in analyzing DSSS systems’ performance, it is common to find the distribution of the correlator output which represents the decision variable. The decision variable can be expressed as a combination of multiple random variables (r.v.s). In addition, it is very frequent in DSSS studies to find the distribution of r.v.s combination. Therefore, we have been motivated by these characteristics of the decision variable to develop a general formula for calculating the mean and variance of combinations of r.v.s. These two parameters are sufficient to characterize the density function of the correlator output in the case when this function is Gaussian. This case happens when the decision variable is a sum of a large number of random variables.
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Introduction

The wireless sensor networks (WSN) are widely used in different applications. However, WSNs have constraints in many accepts such as: limited energy, small computational memory and short transmission range. Thus, a lot of algorithms have been introduced in WSNs at different OSI layers to enhance its capabilities. In the design of WSNs’ physical layer, two main transmission techniques are commonly used: orthogonal frequency division multiplexing (OFDM) [1] and code division multiple access (CDMA) [2]. They both can provide the WSN’s special needs. One of the main types of CDMA system called direct sequence spread spectrum CDMA (DSSS-CDMA) which has been used IEEE 802.15.4 standard [3].

In analysis of the DSSS-CDMA system it is often required to deal with distribution of r.v.s combination. Here, we have demonstrated a procedure that it is required to calculate the distribution function of the combination of random variables.

The DSSS receiver correlator

On of the places which r.v.s combination is used is in the DSSS receiver correlator. The block schematic of DSSS-CDMA system for a single user is shown in Fig. 1. As mentioned before, a bit is presented by a sequence in DSSS system. In Fig. 1, $b^j$, which is the jth bit of the user one is spread by $c^1_j$, where $c^1_j$ is a sequence exclusive to user one. The element of a sequence is called chip. Then, the spread bit passes through the interleaver. The interleaver might change the chips order in a sequence. The changing of the chips order sometimes becomes necessary to mitigate the effect of channel fading [4]. Next, the signal is modulated and sent through the channel. In the channel, noise, fading and delay will change the transmitted signal. Also, in the case of jamming attack, the attacker’s signal is added to the received signal as well.
In the receiver side, the signal is de-modulated and de-interleaved. Then, it multiplied by a locally generated sequence, which is aligned with the received signal. This alignment is adjusted in the synchronization block. Next, the outcome is entered to the receiver correlator. The output of the receiver correlator ("w"), presenting the decision variable, is used for analyzing the DSSS-CDMA system performance. For instance, assuming that the decision variable is Gaussian, what is the case when conditions for the central limit theorem are fulfilled, the probability of error [5] is given by

\[ P_e(w) = \frac{1}{2} \text{erfc} \left( \frac{E[w]}{\sqrt{2 \cdot \text{var}[w]}} \right). \]  

(1)

Table I shows the correlator output (decision variables) in different channel characteristics.

<table>
<thead>
<tr>
<th>Channel scenarios</th>
<th>Absent of jammer</th>
<th>Under jamming attack</th>
</tr>
</thead>
<tbody>
<tr>
<td>WGN</td>
<td>( \sqrt{E_i \sum c_i^2} + \sqrt{E_i \sum n_i \cdot c_i} )</td>
<td>( \sqrt{E_i \sum c_i^2} + \sqrt{E_i \sum n_i \cdot c_i} + \sqrt{E_i \sum k \cdot c_i} )</td>
</tr>
<tr>
<td>WGN + fading</td>
<td>( \sqrt{E_i \sum a_i c_i^2} + \sqrt{E_i \sum n_i \cdot c_i} )</td>
<td>( \sqrt{E_i \sum a_i c_i^2} + \sqrt{E_i \sum n_i \cdot c_i} + \sqrt{E_i \sum a_i \cdot j \cdot c_i} )</td>
</tr>
<tr>
<td>WGN + fading + Interleaver</td>
<td>( \sqrt{E_i \sum a_i c_i^2} + \sqrt{E_i \sum n_i \cdot c_i} )</td>
<td>( \sqrt{E_i \sum a_i c_i^2} + \sqrt{E_i \sum n_i \cdot c_i} + \sqrt{E_i \sum a_i \cdot j \cdot c_i} )</td>
</tr>
</tbody>
</table>
Where the variables are:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_i$</td>
<td>chips</td>
</tr>
<tr>
<td>$E_c$</td>
<td>Energy of each chirps</td>
</tr>
<tr>
<td>$2\beta$</td>
<td>Spreading factor</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Fading factor</td>
</tr>
<tr>
<td>$n$</td>
<td>Gaussian noise</td>
</tr>
<tr>
<td>$j$</td>
<td>jammer</td>
</tr>
<tr>
<td>$E_n$</td>
<td>Energy of each chirps</td>
</tr>
<tr>
<td>$E_j$</td>
<td>Energy of each chirps</td>
</tr>
</tbody>
</table>

Since $\beta$ is relatively a big a number (ranging from 50 to 200). Therefore, according to central limit theorem (CLT) $W$ has a Gaussian distribution. A Gaussian distribution is described with its mean and variance value. Also, as can be seen from table I, the receiver correlator output is a combination of r.v.s. Therefore, in DSSS system analysis, it is very frequent to calculate the mean and variance of r.v.s combination. This motivates us to develop a general formula for the mean and variance value of the correlator output.

**The mean and variance of the combination of r.v.s**

In this section we develop a general procedure to calculate the distribution of combinations of r.v.s in the different cases.

**Multiplication of r.v.s**

The Multiplication of r.v.s can be express as:

$$W_i = \prod_{j=1}^{2} \alpha_i C_{ij} B_{ij} A_{ij} \tag{2}$$

$\alpha_i$ is a constant and $C_{ij}$, $B_{ij}$ and $A_{ij}$ are independent and identically distributed (i.i.d.).

We also define:

$$i = E \left[ C_{ij} \right] E \left[ B_{ij} \right] E \left[ A_{ij} \right] \tag{3}$$
\[
\begin{align*}
E[(C_i)^\gamma] E[(B_i)^\gamma] E[(A_i)^\gamma] \quad (4)
\end{align*}
\]

The mean value of \( W_i \) is equal to:

\[
W_i = \sum_{i=1}^{2\beta} (a_i E[C_i] E[B_i] E[A_i]) = 2\beta a_i E[C_i] E[B_i] E[A_i] = 2\beta a_i \bar{\lambda}_i. \quad (5)
\]

Also, the variance of \( W_i \) is given by:

\[
\begin{align*}
\text{var}(W)_i &= (a_i)^2 \left[ E\left[ \left( \sum_{i=1}^{2\beta} C_i B_i A_i \right)^\gamma \right] - E\left[ \sum_{i=1}^{2\beta} C_i B_i A_i \right]^\gamma \right] \\
&= (a_i)^2 \left[ 2\beta E\left[ (C_i)^\gamma \right] E\left[ (B_i)^\gamma \right] E\left[ (A_i)^\gamma \right] \right. \\
&
\quad \left. + 2\beta (2\beta - 1) E^2\left[ C_i \right] E\left[ B_i \right] E\left[ A_i \right] \right] \\
&= (a_i)^2 \left[ 2\beta \gamma_i + 2\beta (2\beta - 1)(\bar{\lambda}_i)^2 - 4\beta^2 (\bar{\lambda}_i)^2 \right] \\
&= (a_i)^2 2\beta \left( \gamma_i - (\bar{\lambda}_i)^2 \right).
\end{align*}
\]

Therefore,

\[
W_i = \left( 2\beta a_i \bar{\lambda}_i, (a_i)^2 2\beta \left( \gamma_i - (\bar{\lambda}_i)^2 \right) \right). \quad (7)
\]

\section*{The Summation of Multiplication of r.v.s}

For Summation of Multiplication of r.v.s, (2) can be express as

\[
W = \sum_{i=1}^{2\beta} a_i C_i B_i A_i + \sum_{i=1}^{2\beta} a_i C_i B_i A_i + \sum_{i=1}^{2\beta} a_i C_i B_i A_i = K_1 + K_2 + \ldots + K_N. \quad (8)
\]

We develop the mathematical expression for this scenario in two different conditions: 1) when terms \( K_1, K_2, \ldots, K_N \) are independent; 2) when terms \( K_1, K_2, \ldots, K_N \) are dependent.

\section*{The independent summation}

In the case of independency, we can express the correlator output as:
\[ W = \sum_{i=1}^{2^\beta} a_i C_i B_i A_i + \sum_{i=1}^{2^\beta} a_i C_i B_i A_i \ldots + \sum_{i=1}^{2^\beta} a_i C_i B_i A_i \]
\[ = \sum_{j=1}^{2^\beta} \sum_{i=1}^{2^\beta} a_j C_j B_j A_j. \]  \hfill (9)

since the variables are independence. Mean value of \( W \) can be express as:

\[ E[W] = 2^\beta a_1 \lambda_1 + 2^\beta a_2 \lambda_2 + \ldots + 2^\beta a_N \lambda_N \]
\[ = \sum_{i=1}^{N} (2^\beta a_i \lambda_i). \]  \hfill (10)

Also, the variance of \( W \) is:

\[ \text{var}(W) = (a_1)^2 2^\beta (\lambda_1 - (\lambda_1)^2) + (a_2)^2 2^\beta (\lambda_2 - (\lambda_2)^2) + \ldots + (a_N)^2 2^\beta (\lambda_N - (\lambda_N)^2) \]
\[ = \sum_{i=1}^{N} [(a_i)^2 2^\beta (\lambda_i - (\lambda_i)^2)]. \]  \hfill (11)

Therefore, the collator output in case of independency is given as

\[ W = \left[ 2 \sum_{i=1}^{N} (a_i), 2 \sum_{i=1}^{N} (a_i)^2 \left( \begin{array}{c} \lambda_i \\ (\lambda_i)^2 \end{array} \right) \right]. \]  \hfill (12)

### The dependent summation

In the case of statistical dependence between the terms in (9), it can be expressed as:

\[ W = \sum_{i=1}^{2^\beta} a_i C_i B_i A_i + \sum_{i=1}^{2^\beta} a_i C_i B_i A_i \ldots + \sum_{i=1}^{2^\beta} a_i C_i B_i A_i \]
\[ = \sum_{j=1}^{2^\beta} \sum_{i=1}^{2^\beta} a_j C_j B_j A_j. \]  \hfill (13)

We also define:

\[ h_l = E\left[ C_i B_i A_i C_i B_i A_i \right]. \]  \hfill (14)

Same as (10), the mean value of (13) is

\[ E[W] = 2 a_1 + 2 a_2 + \ldots + 2 a_N \]
\[ = 2 \sum_{i=1}^{N} (a_i). \]  \hfill (15)
The variance of (13) can be expressed as:

$$\text{var}(W) = \text{var}\left(\sum_{j=1}^{N} \sum_{i=1}^{2} a_j C_{ji} B_{ji} A_j A_i + \sum_{j=1}^{N} \sum_{i=1}^{2} a_j C_{ji} B_{ji} A_j a_i C_{m} B_{m} A_m\right)$$  \hspace{1cm} (16)

Where

$$\text{cov}\left(\sum_{j=1}^{N} \sum_{i=1}^{2} a_j C_{ji} B_{ji} A_j A_i, \sum_{j=1}^{N} \sum_{i=1}^{2} a_j C_{ji} B_{ji} A_j a_i C_{m} B_{m} A_m\right) = E\left[\sum_{j=1}^{N} \sum_{i=1}^{2} a_j C_{ji} B_{ji} A_j A_i \cdot E\left[\sum_{j=1}^{N} \sum_{i=1}^{2} a_j C_{ji} B_{ji} A_j a_i C_{m} B_{m} A_m\right]\right]$$  \hspace{1cm} (17)

$$= 4 \sum_{j=1}^{N} a_j a_j + 4 \sum_{i=1, j \neq m}^{N} a_i a_i$$

$$\text{var}(W) = \sum_{i=1}^{2} \left(\begin{array}{c} (a_i)^2 \end{array}\right) 2\beta \left(\begin{array}{c} (y_i - (\lambda_i)^2) \end{array}\right) + 4\beta^2 \sum_{i=1, j \neq m}^{N} a_i a_i \eta_{ij} - 4\beta^2 \sum_{i=1, j \neq m}^{N} a_i a_i \lambda_{ij} \lambda_{ij}$$  \hspace{1cm} (18)

The distribution then is defined by

$$W = \left(2 \sum_{i=1}^{2} (a_i), \sum_{i=1}^{2} (a_i)^2 \right) + 4 \sum_{i=1, j \neq m}^{N} a_i a_i \eta_{ij} + 4 \sum_{i=1, j \neq m}^{N} a_i a_i \lambda_{ij}$$  \hspace{1cm} (19)

**Special cases**

In this section, two special cases which also frequent in DSSS system are investigated.

**Special case I: partial summation**

As mentioned before, the correlator adds all the chips belong to a bit. Therefore, the summation would be form 1 to 2. However, sometimes, number of elements in a sequence is less than 2/$\beta$. For instance, the fast jammer switch between ON and OFF state with the duration less than a bit duration [6]. Therefore, it hits some chips and misses the other chips. The correlator output in this scenario would be:

$$W = \left(\sum_{i=1}^{2} a_i C_{i} B_{i} A_i + \sum_{i=3}^{2} a_i C_{i} B_{i} A_i + \cdots + \sum_{i=N}^{2} a_i C_{i} B_{i} A_i\right) + \sum_{i=1}^{2} a_i C_{i} B_{i} A_i$$  \hspace{1cm} (20)

The distribution of $W_{\text{special case I}}$ is:

$$W_{\text{special case I}} = \left(l(a_1), \left(l(a_1)^2 \{ - \left( a_1 \right) \} \right)\right)$$.  \hspace{1cm} (21)
According to the property of the Gaussian distribution, \( W \) the dependent summation and \( W \) special case distributions and be added to gather. Therefore, (20) can be express as:

\[
W = \left\{ 2 \sum_{i=1}^{N} (a_i) + l(a_{\omega,\omega}) \cdot \sum_{i=1}^{N} \left( (a_i)^2 \cdot \left( \frac{1}{(\omega - \omega)} \right) \right) \right. \\
+ 4 \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} a_i a_j \left. \left( \frac{1}{(\omega - \omega)} \right) + l(a_{\omega,\omega}) \cdot \left( \frac{1}{(\omega - \omega)} \right) \right\}
\]

\( (22) \)

**Special case II: Multiuser system**

The main objective of DSSS-CDMA system is multiuser compatibility. Therefore, in the DSSS-CDMA systems, sometime we expose to the double summation. For instance, the output of the correlator for the multiuser system in case of the Gaussian noise and jammer is given by:

\[
W = E \sum_{i=1}^{N} c_i a_{\omega,\omega} + E \sum_{i=1}^{N} c_i a_{\omega,\omega} + \sum_{i=1}^{N} c_i a_{\omega,\omega} + \sum_{i=1}^{N} c_i a_{\omega,\omega} + l(a_{\omega,\omega}) \left( \frac{1}{(\omega - \omega)} \right)
\]

\( (23) \)

Where, \( N \) is the number of users in the system. The general format for special case II can be express as:

\[
W = \sum_{\omega=1}^{N} \sum_{i=1}^{M} a_i c_i a_{\omega,\omega} + \sum_{i=1}^{M} a_i c_i a_{\omega,\omega} + \sum_{i=1}^{M} a_i c_i a_{\omega,\omega} + \sum_{i=1}^{M} a_i c_i a_{\omega,\omega} + l(a_{\omega,\omega}) \left( \frac{1}{(\omega - \omega)} \right)
\]

\( (24) \)

In the case the mean and variance of the first component of (24) is equal to:

\[
W_{i} = \left[ 2\beta M a_{\omega,\omega} M^2 (a_i) \right] 2b \left( \gamma_1 - (a_i) \right)
\]

\( (25) \)

Same as the first special case, \( W_i \) can be added to mean and variance of the rest of (24).

**Example of r.v.s combination’s distribution**

In this section we use an example to demonstrate our mathematical finding from previous section. We choose correlator output for chaotic multiuser DSSS-CDMA system sequences in the present of Gaussian noise and switching jammer which is given by:

\[
W_{syn} = E \sum_{i=1}^{N} \left( c_i^2 \right)^2 + E \sum_{\omega=1, \omega \neq \omega}^{N} c_i^2 c_j^2 + E \sum_{i=1}^{N} c_i^2 + E \sum_{\omega=1, \omega \neq \omega}^{N} c_i^2
\]

\( = F + G + H + I \)

\( (26) \)
where $\zeta$ and $\psi$ are representing the Gaussian noise and Gaussian Jammer. Node that, noise and the Gaussian jammer have mean value equal to zero and their energy is equal to $N_0/2$ and $N_j/2$ respectively. For more detail about Gaussian noise and Gaussian jammer, refer to [6]. Also, $\varphi$ is called hitting factor, which is a constant.

**Conventional method**

First we use the conventional approach (step by step calculation) for finding the mean and variance of (28). Therefore, the mean can be found as follows

The mean value of $w_{syn}$ is:

\[
E[w_{syn}] = \sqrt{E_\zeta \sum_{i=1}^{2} \left( c_i \right)^2} \\
= 2 \ E\left( c_i^2 \right) \sqrt{E_\zeta} = 2 \ E_\zeta. 
\]  
(27)

Also, the variable of (26) is given by:

\[
E[w_{syn}^2] = E\left( w_{syn}^2 \right) = \ E\left( (F + G + H + I)^2 \right) \\
= E\left[ F^2 \right] + E\left[ G^2 \right] + E\left[ H^2 \right] + E\left[ I^2 \right]. 
\]  
(28)

Thus,

\[
E[F^2] = \left[ E\left( \sum_{i=1}^{2} \left( c_i \right)^2 \right)^2 \right] \\
= 2 \ E_\zeta \left[ E\left( c_i^2 \right)^4 \right] + 2 \left( 1 E\left( c_i^2 \right)^4 \right) \left[ E\left( c_i^2 \right)^4 \right], 
\]  
(29)

\[
E[G^2] = E\left[ \left( \sum_{g=1, g \neq k}^{N} \sum_{j=0}^{2} c_j^g \right)^2 \right] \\
= 2 \ N \ E_\zeta \left[ E\left( c_i^g \right)^4 \right] + 2 \left( 1 E_\zeta \left[ E\left( c_i^g \right)^4 \right] \right) + N^2 \left( 1 E_\zeta \left[ E\left( c_i^g \right)^4 \right] \right).
\]
(30)
\[
E[H^2] = E\left[\left(\sum_{i=0}^{N/2} c_i^k\right)^2\right] = 2^{N/2} E\left[(c_i^k)^2\right],
\]
and
\[
E[I^2] = E\left[\left(\sum_{i=0}^{N/2} c_i^j\right)^2\right] = 2^{N/2} E\left[(c_i^j)^2\right].
\]

Furthermore,
\[
\varphi_{\infty}^2 = 4^{-2} E c^2 \left[(c_i^k)^2\right]
\]
So, (30) can be express as:
\[
\sigma_{\text{mean}}^2 = 2\beta E\left[(c_i^k)^2\right] + (2\beta - 1) E\left[(c_i^j)^2\right] E\left[(c_i^k)^2\right] + 2\beta NE\left[(c_i^j)^2\right] E\left[(c_i^k)^2\right] + \phi^{N/2} E\left[(c_i^j)^2\right] E\left[(c_i^k)^2\right] - 4\beta^2 E E^2 \left[(c_i^j)^2\right]
\]

In this example we are using chaotic sequences which uses logistic map. This chaotic sequence has mean value of zero and its power of sequence is 0.5. In order to have a fair comparison with classical binary systems which have power equal to one, the chaotic sequence is multiplied by \(\sqrt{2}\). Also, chaotic sequence \((c_i)\) have probability density function equal to:
\[
f_{c, (c_i)} = \frac{1}{\pi \sqrt{2 - c_i^2}}, \quad \text{for } -\sqrt{2} \leq c_i \leq \sqrt{2}.
\]
Thus,
\[
E\left[(c_i^j)^2\right] = \int_{-\sqrt{2}}^{\sqrt{2}} \frac{1}{\sqrt{2} - c_i^2} dc_i = 1
\]
and

\[ E\left[ \left( c_i^L \right)^4 \right] = \int_0^{\infty} (c_i^L)^4 \frac{1}{\sqrt{2}} \left( c_i^L \right)^4 d(c_i^L) = \frac{3}{2} \]  

(37)

By replacing (28) and (29) in (34), the variance can be express:

\[ z_{s, \alpha} = E_z + 2 \, N E_z + 2 \frac{N_0}{2} + 2 \frac{N_j}{2} \]  

(38)

Therefore the distribution of (26) can be given as:

\[ w_{s, \alpha} \left( 2 \sqrt{E_z}, E_z + 2 \, N E_z + N_0 + N_j \right) \]  

(39)

### Using general formula

Now, we use our general formula to find the mean and variance of (26). This case is a dependent summation of r.v.s with special case II.

1. \[ z = E\left[ c_i^L \right] = 1 \]  
   (40)

2. \[ z = z = z = 0 \]  
   (41)

3. \[ z = E\left[ (c_i^L)^2 \right] = \frac{3}{2} \]  
   (42)

4. \[ z = N \times E\left[ (c_i^L)^2 \right] E\left[ (c_i^L)^2 \right] = N \]  
   (43)

5. \[ z = E\left[ (c_i^L)^2 \right] E\left[ (c_i^L)^2 \right] = \frac{N_0}{2} \]  
   (44)

6. \[ z = E\left[ (c_i^L)^2 \right] E\left[ (c_i^L)^2 \right] = \frac{N_j}{2} \]  
   (45)
Therefore, (26) can express as

\[ w_{\text{ns}} = \left( 2 \sqrt{E_r} \cdot E_c + 2 \cdot NE_r + N_0 + N_j + 0 + 0 \right) = \left( 2 \sqrt{E_r} \cdot E_c + 2 \cdot NE_r + N_0 + N_j \right) \]  

(48)

Which matches exactly to the step-by-step calculation. The mathematical finding on this report can be directly used to calculate the distribution of the r.v combination in our future publications.

**Conclusion**

The summation of r.v.s is very common output of a DSSS-CDMA systems’ correlator. To analyze the performance of a DSSS-CDMA system it is necessary to find the distribution of the correlator outcome. This report introduced a general formula to express distribution of summation of r.v.s. Therefore, the mathematical finding can be used when the distribution a correlator in a DSSS-CDMA system is required.

**References**


