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Combinatorics on Adjacency Graphs and Incidence Pseudographs

Reinhard Klette [1]

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1 Introduction

Many results in combinatorial geometry can be shown by analyzing the given structures as graphs or pseudographs (graphs with loops), without making any use of a specific geometric embedding of the graphs, i.e. without properties defined in a metric space, such as angles or distances. Examples of such results in combinatorial geometry are the Pick formulas for calculating the area of a planar grid polygon [16] (by dividing it into simple grid polygons, and counting grid points in the interior and on the frontier of each polygon), the studies on local property measurements for regular grids (see [5, 15]), and the elementary result that the plane allows only three different regular grids: orthogonal, trigonal, and hexagonal.

The Descartes-Euler polyhedron theorem $\alpha_0 - \alpha_1 + \alpha_2 = 2$ is another example: originally it was proved only for convex polyhedra (i.e. in a metric space), but today it is generalized to connected finite planar graphs $G = [S, E]$, with $\alpha_0 = card(S)$, $\alpha_1 = card(E)$, and α_2 the number of faces (of any planar embedding) of G. The Descartes-Euler theorem represents the beginning of research on spatial subdivisions. In 1813 *A. Cauchy* generalized the Descartes-Euler polyhedron theorem by studying intercellular faces within a given simple polyhedron. Such a spatial subdivision was defined to be a *complex* by *J. B. Listing* in 1861. Cell complexes are the subject of combinatorial topology.

This paper presents some results in combinatorial topology which have been obtained in the context of digital image analysis. It starts with basic formulas for adjacency graphs (one-dimensional complexes), which model adjacencies between pixels, followed by results for the "more refined model" of cell complexes, here in the form of incidence pseudographs, which model 2D and 3D image grids.

2 Oriented Adjacency Graphs

A countable set S and an adjacency relation A (a symmetric and irreflexive relation) define an *adjacency structure* $[S, A]$, where $\{p, q\}$ is an adjacency pair iff $p, q \in S$ and pAq. Two nodes $p, q \in S$ are connected with respect to $M \subseteq S$ [17] iff there is a path $\gamma = (p_1, p_2, \ldots, p_n)$, $p_1 = p$ and $p_n = q$, in $[S, A]$ where the nodes of the path are either all in M or all in $\overline{M} = S \setminus M$. A subset of S is connected (with respect to M) iff every pair of its nodes is connected. Maximal disjoint connected subsets of S are the *components* (if contained in M) or the *complementary components* (if contained in $\overline{M}$) of M. Figure 1 illustrates "finite rectangular windows" of the infinite digital plane. The grid point model is used on the left (with alternating 6-adjacencies) and in the middle (with 4-adjacency). The grid cell model is used on the right, where the squares have been moved slightly apart from one another to illustrate edge adjacency by short connecting lines.

An adjacency structure is called an *adjacency graph* iff it satisfies the following axioms:

A1: The adjacency set $A(p) = \{q : q \in S \wedge pAq\}$ is finite for any $p \in S$.
A2: S is connected with respect to relation A.
A3: Any finite subset $M \subseteq S$ has at most one infinite complementary component.

The object nodes in the three parts of Fig. 1 define adjacency graphs, which are finite subgraphs of $[\mathbb{Z}^2, A_6]$ or $[\mathbb{Z}^2, A_4]$.

A *region* M is a connected, nonempty, finite subset of S. A node $p \in M$ is an *inner node* iff $A(p) \subseteq M$; otherwise, it is called a *border node*. The set of inner nodes of M is called the *inner set* of M and the set of border nodes of M is called the *border* of M.

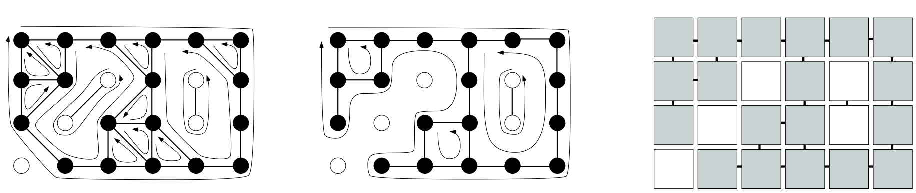

Fig. 1. A binary image with (left) 6- and (middle) 4-adjacency in the grid point model, and (right) edge-adjacency in the grid cell model.

A *local circular order* $\xi(p) = \langle q_1, \ldots, q_n \rangle$ at node $p \in S$ lists the nodes in $A(p)$ exactly once each. We use this order to trace the edges in adjacency graph $[S, A]$ as follows: if we arrive at p from $q_i \in A(p)$ we move next to q_k, where $k = i+1$ (modulo n). Any move from a node p to one of its neighbors q *initiates a path* defined by the local circular order. This path is a *cycle* if finite.

An *oriented adjacency graph* $[S, A, \xi]$ is defined by an adjacency graph $[S, A]$ and an *orientation* ξ on $[S, A]$, defined by local circular orders of the adjacency sets, which satisfies axiom

A4: Any directed edge initiates a cycle (and not an infinite path).

For example, assume that $\xi(p)$ is represented by the clockwise orientation of all outgoing edges. Figure 1 (left and middle) indicates orientations of cycles by arrows. Finite oriented adjacency graphs specify an alternative definition for *two-dimensional combinatorial maps*; see, e.g., [14] for such maps in the context of image analysis.

Let $\alpha_0 = card(S)$, $\alpha_1 = card(A)$, $\nu(p) = card(A(p))$ and $\lambda(\gamma)$ the length of cycle γ. Elementary graph theory says that

$$\sum_{p \in S} \nu(p) = \sum_{\gamma} \lambda(\gamma) = 2\alpha_1 \ . \tag{1}$$

Furthermore, for the *Euler characteristic* $\chi = \alpha_0 - \alpha_1 + \alpha_2$ we obtain

$$\chi \leq 2 \ , \tag{2}$$

for any finite oriented adjacency graph, see [23]. An oriented adjacency graph is called *planar* iff either it is finite and has $\chi = 2$, or it is infinite and any of its nonempty finite connected oriented subgraphs has $\chi = 2$. For example, the planar embeddings $[\mathbb{Z}^2, A_6, \xi_6]$ and $[\mathbb{Z}^2, A_4, \xi_4]$ (with clockwise local orders) of $[\mathbb{Z}^2, A_6]$ and $[\mathbb{Z}^2, A_4]$, respectively, define planar infinite oriented adjacency graphs.

An oriented adjacency graph is *regular* iff $\lambda(\gamma)$ and $\nu(p)$ are constants for all cycles γ and all nodes p of the graph. For infinite sets of nodes there are (up to isomorphy) only three regular planar infinite oriented adjacency graphs $G_{\nu,\lambda}$ [23]: $\mathbb{Z}^2$ with $\lambda(\gamma) = \nu(p) = 4$, with $\lambda(\gamma) = 3$ and $\nu(p) = 6$; and with $\lambda(\gamma) = 6$ and $\nu(p) = 3$. The graph $[\mathbb{Z}^2, A_8, \xi_8]$ is regular, but the Euler characteristic of a finite simply connected set has no lower bound; see Fig. 2.

A subset $M \subseteq S$ induces a *substructure* $[M, A_M, \xi_M]$ of an oriented adjacency graph $[S, A, \xi]$, where A_M contains only those adjacency pairs $\{p, q\}$ such that $\{p, q\} \in A$ and $p, q \in M$, and where for $p \in M$, $\xi_M(p)$ is the *reduced local circular order*, defined by deleting from $\xi(p)$ all points which are not in M. Such a substructure is again an oriented adjacency graph iff M is connected with respect to A_M.

The cycles of $[M, A_M, \xi_M]$ may differ from the cycles of $[S, A, \xi]$. Let (p, q) be a directed edge in $[M, A_M, \xi_M]$, let γ_1 be the cycle generated by (p, q) in $[M, A_M, \xi_M]$, and let γ_2 be the cycle generated by (p, q) in $[S, A, \xi]$. γ_1 is an

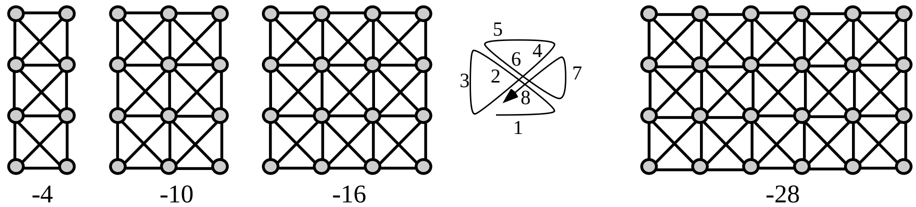

Fig. 2. Unlimited decrease of the Euler characteristic in case of 8-adjacency. The infinite oriented 8-adjacency graph is regular with $\nu(p) = \lambda(\gamma) = 8$.

atomic cycle iff $\gamma_1 = \gamma_2$, and a *border cycle* otherwise. For example, consider the "black" adjacency graphs in Fig. 1 as subgraphs of $[\mathbb{Z}^2, A_6, \xi_6]$ and $[\mathbb{Z}^2, A_4, \xi_4]$. There are 9 atomic cycles and 3 border cycles on the left, and 2 atomic cycles and 2 border cycles in the middle.

Let (r, q) be a directed edge in $[S, A]$, $M \subseteq S$, $q \in \delta M$ and $r \in (S \setminus M)$. We call (r, q) a *directed invalid edge* from $\overline{M} = S \setminus M$ to M. An undirected edge between $\overline{M}$ and M is *invalid* iff one of its two directions is invalid. In Fig. 1 we deleted all the invalid edges in $[\mathbb{Z}^2, A_6]$ and $[\mathbb{Z}^2, A_4]$, assuming that the object pixels shown in the figure specify the set M.

A directed invalid edge (r, q) *points to a cycle* in $[M, A_M, \xi_M]$ if (q, p) is the directed edge such that p is the first point of M that follows r in the (original) local circular order $\xi(q)$. Every directed invalid edge points to exactly one border cycle in $[M, A_M, \xi_M]$. This defines a partition of all (directed or undirected) invalid edges into equivalence classes; each class is the set of all (directed or undirected) invalid edges that are *assigned* to a given border cycle.

Theorem 1. [23] *Let $[S, A, \xi]$ be a (finite or infinite) planar oriented adjacency graph and M a nonempty finite connected proper subset of S. Then $[S, A, \xi]$ splits into at least two non-connected substructures when we delete all undirected invalid edges assigned to any border cycles of M.*

Let M be a finite connected subset of an infinite oriented adjacency graph $G = [S, A, \xi]$. M has exactly one infinite complementary component (the *background*). Any finite complementary component of M is called a *hole* of M in G. If $[\mathbb{G}, A, \xi]$ is also planar, M has exactly one border cycle, called its *outer border cycle*, which separates M (see Theorem 1) from its infinite complementary component. All other border cycles of M are *inner border cycles*. If complementary component A of M is separated from M by border cycle γ of M, we say that *A is assigned to γ*. Any complementary component of M assigned to one of its inner border cycles is a *proper hole* of M, and any finite complementary component assigned to the outer border cycle is an *improper hole* of M. For example, in Fig. 1, left, we have two proper holes, and in the middle we have two improper holes and one proper hole.

3 Infinite Regular Planar Oriented Adjacency Graphs

This section reviews a few results from [19] for a finite subgraph $[M, A_M, \xi_M]$ of an infinite regular planar oriented adjacency graph $G_{\nu,\lambda}$. First let us consider the case in which M only has one border cycle of length l. We call such a set M *simply connected*. Let k be the number of invalid undirected edges between $\overline{M}$ and M. We obtain in this case

$$\nu l - \lambda k + \nu\lambda = (2\nu + 2\lambda - \nu\lambda)\alpha_1 \ . \tag{3}$$

For all three versions of $G_{\nu,\lambda}$ we obtain $2\nu + 2\lambda - \nu\lambda = 0$, i.e., for any simply connected set M we have $k = \nu + \nu l/\lambda$. In other words, the relationship between the number of invalid edges and the length of the border cycle depends only on the parameters ν and λ.

Now assume that M has $r \geq 1$ border cycles. Let l_i be the length of border cycle i, and k_i the number of invalid edges connecting $\overline{M}$ with nodes on this border cycle, for $1 \leq i \leq r$. It follows that

$$\nu \sum_{i=1}^{r} l_i - \lambda \sum_{i=1}^{r} k_i - (r-2)\nu\lambda \ = \ \nu L - \lambda K - (r-2)\nu\lambda \ = \ 0 \ . \tag{4}$$

The outer border cycle of M coincides with the outer border cycle of its *cover* $C(M)$, which is the union of M with all of its proper holes. It follows that we always have $\nu l_r - \lambda k_r + \nu\lambda = 0$, where r is assumed to be the index of the outer border cycle of M. Subtracting this equation from Equation (4) it follows that

$$\nu \sum_{i=1}^{r-1} l_i - \lambda \sum_{i=1}^{r-1} k_i - (r-1)\nu\lambda \ = \ \nu(L - l_r) - \lambda(K - k_r) - (r-1)\nu\lambda \ = \ 0 \ . \tag{5}$$

All the $r-1$ inner border cycles of M can be regarded as independent events, and Equation (5) splits into $r-1$ equations $\nu l_i - \lambda k_i + \nu\lambda = 0$, for $1 \leq i \leq r-1$. Thus for a connected set M and any of its border cycles it follows that $k = \pm\nu + \nu l/\lambda$, where the outer border cycle implies the positive sign, and any inner border cycle the negative sign. This provides a simple algorithmic rule for deciding whether a traced border cycle is either inner or outer, by keeping track of k and l during border cycle tracing.

From Equation (4) it follows that $r = 2 + L/\lambda - K/\nu$. The total length L of all border cycles and the number K of all invalid edges allow us to calculate the number r of border cycles, which is a topological invariant of the given finite connected set M. Note that L and K can be accumulated by passing through all adjacency sets $A(p)$ of points in M, i.e. border cycle tracing is not necessary for calculating L and K.

Let f be the number of atomic cycles of set M. It follows that

$$\alpha_0 \ = \ \lambda f/\nu \ + \ l/2 \ + 1 \ , \tag{6}$$

for any simply connected set M. This result in [19] is a graph-theoretic generalization of Pick's formula for simple grid polygons. For an inner border cycle γ

let α_0 denote the number of nodes of $\mathbb{Z}^2 \setminus M$ surrounded by γ ('in the interior of γ'), and f be the number of atomic cycles of the oriented planar adjacency graph defined by these α_0 nodes and the nodes on the inner border cycle. Then we have

$$\alpha_0 \;=\; \lambda f/\nu \;-\; l/2 \;+1 \;. \tag{7}$$

Note that one inner border cycle may separate several proper holes from M, and this formula specifies a result for the union of these holes.

Let m be the total number of atomic cycles of all proper holes assigned to a given inner border cycle. The remaining $f-m$ atomic cycles, defined by nodes in the complementary set $\overline{M}$ and on the inner border cycle, are *boundary cycles*. The λm edges of all boundary cycles split into k invalid edges, the length l of the inner border cycle of M, and the sum of the lengths l_i of all the outer border cycles of all $n > 0$ proper holes assigned to the given inner border cycle of M:

$$\lambda m \;=\; 2k \;+\; l \;+\; \sum_{i=1} nl_i \;. \tag{8}$$

This implies that

$$n \;=\; 1 \;+\; k \;-\; m \;, \tag{9}$$

which is another example of how a topological invariant (the number n of proper holes) can be calculated by accumulating local counts of invalid edges and boundary cycles.

In the case of $G_{6,3}$ exactly one complementary component is assigned to any (inner or outer) border cycle of a finite connected set M. As a conclusion, a connected set in $G_{6,3}$ does not have any improper holes, and a region adjacency graph generated by any subset of $G_{6,3}$ is a tree. Any directed invalid edge in this planar oriented adjacency graph generates exactly one boundary cycle; for any inner border cycle of a connected subset of $G_{6,3}$.

4 Incidence Pseudographs

Let $\{S_0, S_1, ..., S_m\}$ be a family of pairwise disjoint, nonempty sets, where $c \in S_i$ is an i-dimensional subset of an n-dimensional space, for $0 \leq i \leq m$ and $n \geq m$. Let

$$S = \bigcup_{0 \leq i \leq m} S_i \;.$$

Two elements $c, c' \in S$ are called *set-theoretically incident* (notation: cIc') iff $c \subseteq c'$ or $c' \subseteq c$. The relation I is reflexive and symmetric. For example, any grid vertex (0-cell) in 3D space is incident with six grid edges (1-cells), a grid square (2-cell) is incident with four grid edges, a grid cube (3-cell) is incident with 12 grid edges, etc.

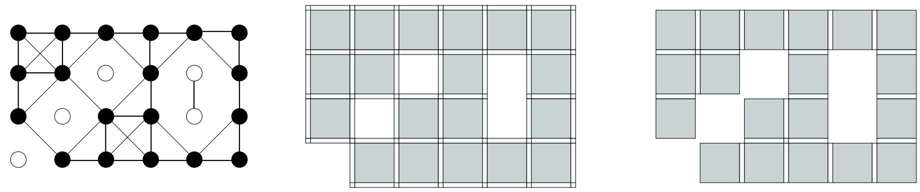

Fig. 3. A binary image with (left) $(8,4)$-adjacency in the grid point model, and (middle) black pixels closed or (right) white pixels closed in the grid cell topology.

Actually, these incidence relations can also be interpreted at a more abstract level; see Fig. 3: the geometric representations of 0-cells are small closed squares, those of 1-cells closed rectangles, those of 2-cells closed squares, etc. [1]

An *incidence structure* $[S, I, dim]$ is defined by a countable set S, an *incidence relation* I on S which is reflexive and symmetric, and a function dim defined on S and into a finite set $\{0, 1, ..., m\}$ of natural numbers. The definition of the function dim depends upon the context. For example, an i-cell in a regular orthogonal grid has dimension i: the geometric representation of a 0-cell may be a vertex or a small square, of a 1-cell an edge or a small rectangle etc. - the function dim just assigns integers to these cells.

Let $[S, I, dim]$ be a finite incidence pseudograph. If $c' \in I(c)$ with $c \neq c'$ and $dim(c) < dim(c')$, then we define $c < c'$. In general, for $c, c' \in S$ let $c \leq c'$ iff $c < c'$ or $c = c'$. This defines an (abstract) *cell complex* $[S, \leq, dim]$ [2]. Cell complexes are popular for describing pictorial information; see [1, 3, 4, 6–8, 12, 13].

Let $G = [S, I, dim]$ be an incidence structure with a finite or infinite set S. [2] Let $I(c) = \{c' : c' \in S \wedge c'Ic\}$ for $c \in S$. If $dim(c) = i, c$ is called an *i-node*. The maximum value of $dim(c)$, for any $c \in S$, defines the *index dimension* $n = ind(G)$ of G. Any node of G of dimension smaller than n is a *marginal node*, and any node of dimension $ind(G)$ is a *principal node* of G.

For example, let S be the set $\mathbb{C}_2$ of 0-, 1-, 2-cells in the orthogonal grid in the Euclidean plane, and the set $\mathbb{C}_3$ of these plus 3-cells in the orthogonal grid in 3D Euclidean space, and let I be an incidence relation. This defines incidence structures $[\mathbb{C}_2, I, dim]$ and $[\mathbb{C}_3, I, dim]$ of dimensions 2 and 3, respectively, of which pixels and voxels are the principal nodes.

Two nodes $c_1, c_2 \in S$ of an incidence structure $[S, I, dim]$ are called *i-adjacent* (notation: $c_1 A_i c_2$, or $\{c_1, c_2\} \in A_i$, or $c_1 \in A_i(c_2)$) iff $c_1 \neq c_2$ and there is an i-node $c \in S$, $c \neq c_1$ and $c \neq c_2$, such that $c_1 \in I(c)$ and $c \in I(c_2)$, for $0 \leq i \leq ind(S)$. As usually done in digital geometry [17], the transitive closure of i-adjacency defines i-connectedness and i-paths, and (for a subset

[1] This allows us to define topological equivalence of sets of cells based on unions of squares and rectangles. Note that, e.g., a closed square minus one of its vertices is homeomorphic to the real plane.

[2] [21] discussed incidence structures with finite base sets S only.

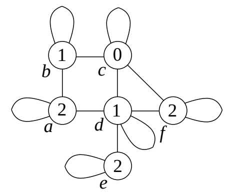

Fig. 4. Example of an incidence pseudograph. Nodes $a, \ldots, f$ have values $dim(a), \ldots, dim(f)$.

of S) i-components and complementary i-components. An incidence structure $G = [S, I, dim]$, with $n = ind(G)$, is an *incidence pseudograph* iff it satisfies the following axioms:

I1: $I(c)$ is finite for any $c \in S$.
I2: The set of all principal nodes in G is $(n-1)$-connected.
I3: Any finite subset $M \subseteq S$ of principal nodes has at most one infinite complementary $(n-1)$-component of principal nodes.
I4: If $c' \in I(c)$ and $c \neq c'$ then $dim(c) \neq dim(c')$.
I5: Any i-node $c \in S$ with $i < n = ind(G)$ is incident with at least one n-node in G.

If $\{c, c'\} \in I$, we also say that c and c' are (graph-theoretically) *incident*. The function dim partitions S into pairwise disjoint classes. See Fig. 4 for an example of a finite incidence pseudograph. The numbers in the circles specify the node dimensions.

$[\mathbb{C}_2, I, dim]$ and $[\mathbb{C}_3, I, dim]$ are examples of infinite incidence pseudographs. Consider $[\mathbb{C}_3, I, dim]$: two 1-cells may be 0-, 2-, or 3-adjacent, but not 1-adjacent. A 1-path in $[\mathbb{C}_3, I, dim]$ can contain 0-, 2- or 3-cells, but not 1-cells. Of course, an i-path or an i-component can be restricted to contain only cells of one chosen dimension.

Let $G = [S, I, dim]$ be an incidence pseudograph, and $M \subseteq S$. We define the *supplement* M^s of M (with respect to G) as the smallest subset of S which has the following two properties:

(i) all nodes in M are also in M^s
(ii) if $c' \in M^s$ for all $c' \in I(c)$ with $dim(c') > dim(c)$, then $c \in M^s$.

Note that condition (ii) leads to a recursive procedure for adding nodes: first $(ind(G)-1)$-nodes, then $(ind(G)-2)$-nodes, etc. In Fig. 5 we add three 0-nodes after having added six 1-nodes. A subset M of an incidence pseudograph is *complete* iff $M = M^s$.

The remainder of this article follows related definitions and results in [21], reformulates them in a cellular model, and completes them by discussing not only open but also closed subsets of an incidence pseudograph.

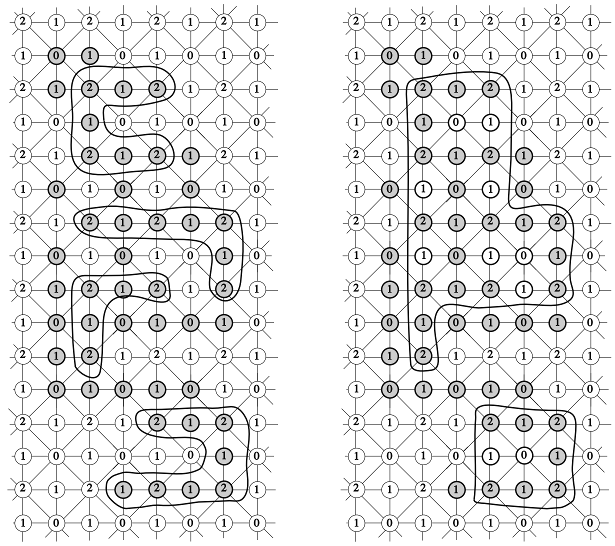

Fig. 5. Left: original nodes in a set M that has four 1-components of 2-nodes. Right: supplement M^s, having two 1-components of 2-nodes (loops omitted).

5 Connectedness, Components, and Regions

We consider complete subsets of incidence pseudographs $G = [S, I, dim]$, with $n = ind(G)$. Our main interest is in principal nodes of dimension n. The set of all n-nodes of a set $M \subseteq S$ defines the *core* of M. Let M be a complete subset of S. A *component* of M in G is the union of an $(n-1)$-component C of the core of M with the set of all nodes of M which are incident with at least one n-node in C. A complete subset M is *connected in* G iff it contains at least one n-node, and M coincides with the set of nodes of one component of M in G.

For example, the incidence pseudograph shown in Fig. 4 is connected (in itself); $C = \{a, e, f\}$ is its core, and the supplement of its node set is (of course) the node set again. The set M on the left in Fig. 5 is not connected; it is a proper subset of M^s. M^s consists of two components, which are the components of M in the incidence pseudograph. Note that one 0-node in M belongs to both components of M, i.e. components of M can share i-nodes, where $0 \leq i < n-1$.

It follows that any n- or $(n-1)$-node cannot be in more than one component of a complete set M. A complete set $M \subseteq S$ is not always a union of components; there may be i-nodes in M which are not incident with any n-node in M.

A *region* of an incidence pseudograph $G = [S, I, dim]$ is a nonempty, finite, complete, connected subset of S. A node c (of any dimension!) in a region $M \subseteq S$ is called an *inner node* iff $I(c) \subseteq M$; otherwise, it is called a *border node.* The set of all inner nodes of M is called the *inner set* of M and the set of border nodes of M is called the *border* of M.

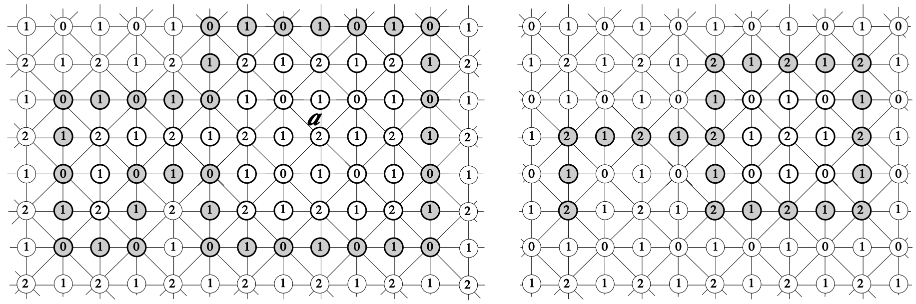

Fig. 6. Inner sets (bold, unfilled circles) and borders (bold, shaded circles) of a closed (left) and an open (right) region. The left region remains closed after deleting 2-node a, and this deletion creates an open hole.

The set S of an incidence pseudograph $G = [S, I, dim]$ has only inner nodes. The complete set M^s shown on the right in Fig. 5 can be split into two disjoint regions by deciding which region contains the 0-cell contained in both components. See Fig. 6 for two illustrations of inner sets and borders.

6 Open and Closed Regions

A subset $M \subseteq S$ of an incidence pseudograph $G = [S, I, dim]$ is called *closed in* G iff for any $c \in M$ and $c' \in I(c)$ with $dim(c') < dim(c)$ we have $c' \in M$. A subset $M \subseteq S$ of an incidence pseudograph $G = [S, I, dim]$ is *open in* G iff $\overline{M} = S \setminus M$ is closed. It follows that the sets S and $\emptyset$ are closed and open in G. Any closed or open set M is also complete. Figure 3 illustrates a given set of pixels (black dots) as close (middle) or open (right) region in $[\mathbb{C}^2, I, dim]$. Note that the closed representation corresponds to the good pair (8,4) (as used on the left), and the open representation corresponds to good pair (4,8). We have as a direct consequence of these definitions

Theorem 2. *A finite subset* $M \subseteq S$ *of an incidence pseudograph* $G = [S, I, dim]$*, with* $ind(G) = n$*, is a closed region in* G *iff its core is nonempty and* $(n-1)$*-connected, and for any* $c \in M$ *and* $c' \in I(c)$ *with* $dim(c') < dim(c)$ *we have* $c' \in M$*.*

An incidence pseudograph is called *monotone* (short for: in transitive correspondence with a monotonic chain of dimensions) iff $c'' \in I(c)$, if $c' \in I(c)$ and $c'' \in I(c')$, with $dim(c) \leq dim(c') \leq dim(c'')$. For example, the pseudograph in Fig. 4 is not monotone because, e.g., $c \in I(b)$ and $b \in I(a)$ but $c \notin I(a)$. The incidence pseudographs in Figs. 5 and 6 are monotone.

For an open region, any cell which is "enclosed" by higher-dimensional cells is also an element of the region:

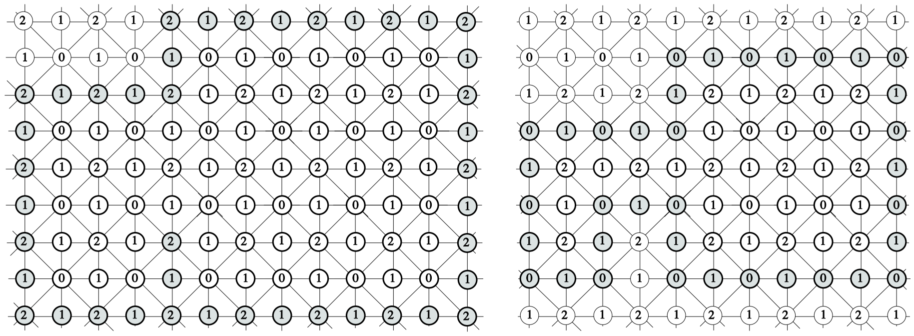

Fig. 7. Two regions (bold, unfilled circles) and their boundaries (bold, shaded circles). Left: closed region. Right: open region.

Theorem 3. *Let M be a finite subset of an incidence pseudograph $G = [S, I, dim]$, with $ind(G) = n$. M is an open region in G iff its core is nonempty and $(n-1)$-connected, and an i-cell $c \in \mathbb{C}_n$ with $i < n$ is in M iff all j-cells c' with cIc' and $i < j \leq n$ are in M. If G is monotone, M is an open region in G iff its core is nonempty and $(n-1)$-connected, and an i-cell $c \in \mathbb{C}_n$ is in M iff all n-cells in $I(c)$ are also in M.*

7 Holes and Boundaries

Let M be a closed region in an infinite incidence pseudograph. Then $\overline{M} = S \setminus M$ is the union of a finite number of pairwise disjoint open regions and of one infinite open subset of S. The (finite) open regions are *open holes*, and the infinite subset is the *open background* of M. Conversely, if M is an open region, we obtain *closed holes* and a *closed background*. See Fig. 6 for an example of an open hole (after removing node a).

Let $M \subseteq S$. A node $c \in S$ is *invalid* (with respect to M) iff $c \notin M$ but there is at least one n-node $c' \in M$ such that $c' \in I(c)$. Let $G = [S, I, dim]$ be an incidence pseudograph and let $M \subseteq S$. The set of all invalid nodes (with respect to M) defines the *boundary* of M. See Fig. 7 for two examples of boundaries. The numbers

$$b_{ij}^M(c) = \begin{cases} card\{c' \in I(c) : \; dim(c') = j \;\wedge\; c' \text{ is invalid}\} \\ \qquad\qquad\qquad\qquad \text{if } i = dim(c) \text{ and } c \in M \\ \qquad\qquad 0 \qquad\qquad \text{otherwise} \end{cases}$$

are called *boundary counts* for the cells in M, and

$$b_{ij}^M = \sum_{c \in S} b_{ij}^M(c) \qquad \text{for } 0 \leq i, j \leq ind(G)$$

are called the total boundary counts for a subset M of an incidence pseudograph $G = [S, I, dim]$.

node	a	b	c	d	e	f
i	2	1	0	1	2	2
a_{i2}	1	1	1	3	1	1
a_{i1}	2	1	2	1	1	1
a_{i0}	0	1	1	1	0	1

Table 1. Incidence counts for the pseudograph in Fig. 4.

8 The Matching Theorem

The following *Matching Theorem* is a basic combinatorial formula for finite incidence pseudographs of dimension $n \geq 0$. Let $G = [S, I, dim]$ be an incidence pseudograph, and let $c \in S$. We call

$$a_{ij}(c) = \begin{cases} card\{c' \in S : \ dim(c') = j \ \wedge \ \{c, c'\} \in I\ \} & \text{if} \quad i = dim(c) \\ 0 & \text{otherwise} \end{cases}$$

the *incidence count* of c in $[S, I, dim]$. Because of self-incidence and axiom I4 we have $a_{ii}(c) = 1$ for $i = dim(c)$. See Table 1 for examples of incidence counts.

Theorem 4. (Matching Theorem) $\sum_{c \in S} a_{ij}(c) = \sum_{c \in S} a_{ji}(c) \quad$ *for* $0 \leq i, j \leq n$.

Proof. See Fig. 8: if $i \neq j$, all the edges between i-nodes and j-nodes, and only those edges, are counted in the formula. All edges are undirected, and the numbers of endpoints are equal on the upper and lower row in the figure. If $i = j$, the sum is equal to the number of i-nodes in the pseudograph. □

The basic equation (1) of adjacency graphs follows from the Matching Theorem (one of the basic theorems for cell complexes) where $\nu(p) = a_{01}(p)$ for any node p of an adjacency graph $[S, A]$, and $E = \{\{p, q\} : (p, q) \in A\}$:

$$\sum_{p \in S} \nu(p) = \sum_{p \in S} a_{01}(p) = \sum_{e \in E} a_{10}(e) \ = \ 2\alpha_1 \ .$$

In a *regular incidence pseudograph* $[S, I, dim]$ we have $a_{ij}(c) = a_{ij}$ for all $c \in S$, where $dim(c) = i$ and $j \geq 0$. Complete finite graphs K_n are examples of

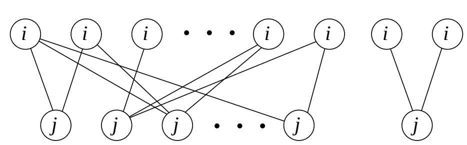

Fig. 8. All undirected edges connecting i-nodes of an incidence pseudograph with j-nodes.

finite regular incidence pseudographs of index dimension 1. The nodes of K_n are pseudograph nodes of dimension 0, the edges of K_n are pseudograph nodes of dimension 1, and every pseudograph node is self-incident.

For a finite subset M of an incidence pseudograph $[S, I, dim]$, with $ind(G) = n \geq 1$,

$$\alpha_i^M = card\{c : \ c \in M \ \wedge \ dim(c) = i\} \tag{10}$$

are called the *class cardinalities* of M, for $0 \leq i \leq n$. We usually omit the superscript M. From the Matching Theorem (4) we know that for finite regular incidence pseudographs, $\alpha_i a_{ij} = \alpha_j a_{ji}$ for $0 \leq i, j \leq n$. It follows that

$$\alpha_i a_{ik} - \alpha_k a_{ki} = 0 \ \text{ for } \ 0 \leq i \leq n \ , \tag{11}$$

for any (say: fixed) index k with $0 \leq k \leq n$. Hence the possible integer values of class cardinalities α_k and α_i define constraints (Diophantine equations) on the incidence counts a_{ki} and a_{ik}, and vice versa.

The infinite incidence pseudographs $[\mathbb{C}_2, I, dim]$ and $[\mathbb{C}_3, I, dim]$ are also regular. In the 3D case we have the incidence counts

$$\begin{array}{llll} a_{00} = 1, & a_{01} = 6, & a_{02} = 12, & a_{03} = 8, \\ a_{10} = 2, & a_{11} = 1, & a_{12} = 4, & a_{13} = 4, \\ a_{20} = 4, & a_{21} = 4, & a_{22} = 1, & a_{23} = 2, \\ a_{30} = 8, & a_{31} = 12, & a_{32} = 6, & \text{and } \ a_{33} = 1 \ . \end{array}$$

9 The Euler Characteristic

K. Voss generalized in [21] the definition of an Euler characteristic from oriented adjacency graphs (combinatorial maps) to incidence pseudographs. Let $G = [S, I, dim]$ be a finite incidence pseudograph, with $ind(G) = n \geq 1$. The *Euler characteristic* of G is defined by its class cardinalities:

$$\chi(G) = \sum_{i=0}^{n} (-1)^i \alpha_i \ .$$

For example, for the incidence pseudograph G shown in Fig. 4 we have $\chi(G) = 1 - 2 + 3 = 2$. Adding another edge, e.g. between nodes b and e, does not change the Euler characteristic. But deleting a node, e.g. 2-node e, results in an incidence pseudograph G' with $\chi(G') = 1 - 2 + 2 = 1$, and does change the Euler characteristic.

By the Matching Theorem, for regular finite incidence pseudographs G with index dimension n we have

$$\frac{\chi(G)}{\alpha_k} = \sum_{i=o}^{n} (-1)^i \frac{a_{ki}}{a_{ik}},$$

These $n+1$ equations, for $k = 0, 1, \ldots, n$, are all rational multiples of one another.

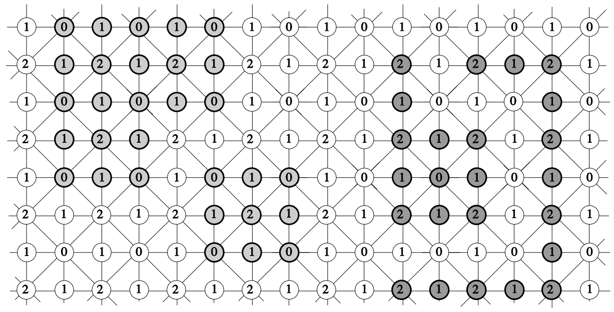

Fig. 9. Four components defining sub-pseudographs: two closed regions on the left and two open regions on the right. The Euler characteristic is 1 in all four cases.

Figure 9 shows examples of regions in the infinite regular incidence pseudograph $[\mathbb{Z}^2, I, dim]$, defining sub-pseudographs (loops omitted). For the upper left region we have $\chi = 3 - 10 + 8 = 1$, for the lower left region we have $\chi = 1 - 4 + 4 = 1$, for the region on their right we have $\chi = 5 - 5 + 1 = 1$, and for the remaining 1-path we have $\chi = 7 - 6 + 0 = 1$. For the set M^s (not a region!) shown on the right in Fig. 5 we have $\chi = 15 - 28 + 14 = 1$; note that removing marginal border nodes will change this value. In Fig. 6, for the closed region on the left we have $\chi = 12 - 33 + 22 = 1$, and for the open region on the right we have $\chi = 12 - 15 + 4 = 1$. Removing the "central" 2-node a from the closed region creates an 'open hole', and we obtain $\chi = 11 - 33 + 22 = 0$.

10 Incidence Grids

Consider the incidence pseudographs $[\mathbb{C}_2, I, dim]$ and $[\mathbb{C}_3, I, dim]$. These pseudographs $[S, I, dim]$ are monotone, and also have the following properties:

- If $c \in S$ and $dim(c) > 0$, there is a $c' \in S$ with $dim(c') < dim(c)$ and $\{c, c'\} \in I$.
- If $c, c' \in I$ with $\{c, c'\} \in I$ and $dim(c) - dim(c') > 1$, there is a $c'' \in S$ with $dim(c') < dim(c'') < dim(c)$ and $\{c, c''\} \in I$ and $\{c', c''\} \in I$.

These properties are the *closeness conditions* on incidence pseudographs. Generalizing $\mathbb{C}_2$ and $\mathbb{C}_3$, we will consider the n-dimensional case with $n \leq l$. This simplifies the formulation of combinatorial formulas.

In the set $\mathbb{Z}^n$ of grid points, let e_j be the straight segment which connects the origin o with grid point $(0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is in position j, for $1 \leq j \leq n$. The set $(0.5, 0.5) + \mathbb{Z}^n$ is the set of all *0-cells*. Let c be a k-cell in a k-dimensional subspace of $\mathbb{R}^n$, $0 \leq k < n$, and let e_j be in a $n - k$ dimensional subspace of $\mathbb{R}^n$. For any j with $1 \leq j \leq n$, the Minkowski sum $c \oplus e_j$ in $\mathbb{R}^n$ defines a $(k+1)$-cell. Let $\mathbb{C}_n^{(i)}$ be the set of all *i-cells*, $0 \leq i \leq n$, and let

$$\mathbb{C}_n = \bigcup_{i=1}^{n} \mathbb{C}_n^{(i)} .$$

Let $[\mathbb{C}_n, I, dim]$ be the regular incidence pseudograph of index dimension n, where $\mathbb{C}_n$ is the countably infinite set of all i-cells, for $0 \leq i \leq n$. This pseudograph is monotone and satisfies the closeness conditions. We have [8, 18]

$$a_{ij} = \begin{cases} 2^{j-i}\binom{n-i}{n-j} & \text{if } i < j \\ 1 & \text{if } i = j \\ 2^{i-j}\binom{i}{j} & \text{if } i > j \end{cases} \tag{12}$$

for the incidence counts in this general case. For example, an i-cell $c \in \mathbb{C}_n$ is incident with 2^{n-1} n-cells. Equation (12) allows us to show that

$$\frac{a_{ij}}{a_{ji}} = \frac{\binom{n}{j}}{\binom{n}{i}} \tag{13}$$

for $0 \leq i, j \leq n$; and these identities allow us to show that

$$\sum_{j=0}^{n} (-1)^j \frac{a_{ij}}{a_{ji}} = \frac{1}{\binom{n}{i}} \sum_{j=0}^{n} (-1)^j \binom{n}{j} = \frac{1}{\binom{n}{i}} \cdot 0 = 0 \tag{14}$$

for any i, $0 \leq i \leq n$, and for the incidence pseudograph $[\mathbb{C}_n, I, dim]$ with $n \geq 1$. Both equations will later prove to be very useful.

An incidence grid is an abstraction of the set-theoretical incidence relation in $\mathbb{C}_2$ or $\mathbb{C}_3$ which allows us to use the Euclidean topology in $\mathbb{R}^2$ or $\mathbb{R}^3$: $[S, I, dim]$ is an *incidence grid* iff (i) it is equal to $[\mathbb{C}_2, I, dim]$ or $[\mathbb{C}_3, I, dim]$, or (ii) it is a finite closed sub-pseudograph of one of these two infinite pseudographs, defined by a core which is either an $m \times n$ rectangular subset of the discrete plane $\mathbb{C}_2^{(2)}$ or an $l \times m \times n$ cuboidal subset of 3D discrete space $\mathbb{C}_3^{(3)}$. Nodes p in an incidence grid have assigned geometric locations in a Euclidean space; nodes p in an incidence pseudograph do not.

An incidence grid $\mathbb{G}$ is always connected in $[\mathbb{C}_2, I, dim]$ or $[\mathbb{C}_3, I, dim]$. A finite nonempty complete subset $M \subseteq S$ of an incidence grid $[S, I, dim]$ defines an incidence structure $[M, I', dim']$, where I' and dim' are the restrictions of I and dim to M.

11 The Region Matching Theorem

For a set $M \subseteq \mathbb{C}_n$ we have class cardinalities α_i^M, incidence counts a_{ij} (in the regular graph $[\mathbb{C}_n, I, dim]$), and boundary counts b_{ij}^M. We omit the superscript M.

Theorem 5. (The Region Matching Theorem) *Let M be an open or closed region in $[\mathbb{C}_n, I, dim]$. For $0 \leq i, j \leq n$ we have*

$$\alpha_i a_{ij} - b_{ji} = \alpha_j a_{ji} \quad \text{for } i < j \text{ if } M \text{ is closed, or for } i > j \text{ if } M \text{ is open}$$
$$\alpha_i a_{ij} = \alpha_j a_{ji} \quad \text{for } i = j$$
$$\alpha_i a_{ij} + b_{ji} = \alpha_j a_{ji} \quad \text{for } i > j \text{ if } M \text{ is closed, or for } i < j \text{ if } M \text{ is open.}$$

Proof. Let M be closed. The right sides of the equations for $i < j$ specify the number of j-cells in M times the number of i-cells which are incident with any of these j-cells, i.e. the number T of incidences between j-cells in M and i-cells in M. All these i-cells are also elements of M; see the first part of Theorem 2. The total number of incidences between i-cells in M and j-cells in M is, of course, identical to T. However, the i-cells in M are also incident with b_{ji} j-cells which are not in M, i.e. the count $\alpha_i a_{ij}$ on the left hand side of the equation must be reduced by b_{ji}.

The second equation is trivial, and is given only for completeness. Case $j < i$ follows for a closed region M by simply swapping i and j in the discussion of case $i < j$. In the case of an open region M we use the second part of Theorem 2. □

Note that the proof of this theorem makes no use of the connectedness of a region, but only of its being either closed or open.

Conclusion 1 *The formulas of the Region Matching Theorem also hold for any finite union of pairwise disjoint closed (or pairwise disjoint open) regions.*

For a closed region M (or a finite union of pairwise disjoint closed regions) the Region Matching Theorem implies

$$\alpha_i = \alpha_j \frac{a_{ji}}{a_{ij}} + \frac{b_{ji}}{a_{ij}} \quad \text{if } i < j$$
$$\alpha_i = \alpha_j \frac{a_{ji}}{a_{ij}} \quad \text{if } i = j$$
$$\alpha_i = \alpha_j \frac{a_{ji}}{a_{ij}} - \frac{b_{ji}}{a_{ij}} \quad \text{if } i > j\ .$$

Together with Equ. (14) it follows, for any j, $0 \leq j \leq n$, that the Euler characteristic

$$\chi\ (M) = \sum_{i=1}^{n} (-1)^i \alpha_i$$
$$= \alpha_j \left[\sum_{i=0}^{j-1} (-1)^i \frac{a_{ji}}{a_{ij}} + (-1)^j + \sum_{i=j+1}^{n} (-1)^i \frac{a_{ji}}{a_{ij}} \right] + \sum_{i=1}^{j-1} (-1)^i \frac{b_{ji}}{a_{ij}} - \sum_{i=j+1}^{n} (-1)^i \frac{b_{ji}}{a_{ij}}$$
$$= \sum_{i=1}^{j-1} (-1)^i \frac{b_{ji}}{a_{ij}} - \sum_{i=j+1}^{n} (-1)^i \frac{b_{ji}}{a_{ij}}$$

can be calculated by counting only invalid cells (cells in the boundary of the region). The incidence counts a_{ij} are constants in $\mathbb{C}_n$. Similarly, for an open region M (or a finite union of pairwise disjoint open regions) we obtain

$$\chi(M) = -\sum_{i=1}^{j-1} (-1)^i \frac{b_{ji}}{a_{ij}} + \sum_{i=j+1}^{n} (-1)^i \frac{b_{ji}}{a_{ij}} .$$

Note that the value of j can be any number in the set $\{0, 1, \ldots, n\}$, and the cases $j = 0$ and $j = n$ provide the simplest expressions. The following section will show that these expressions can be further simplified.

12 Euler Characteristics of Regions

We apply the Region Matching Theorem and Conclusion 1:

Lemma 1. *Let M be a finite union of pairwise disjoint closed regions in $\mathbb{C}_n$. For $0 \leq i, j \leq n$ we have*

$$\alpha_i = \alpha_n \frac{a_{ni}}{a_{in}} + \sum_{j=i}^{n-1} \frac{b_{j,j+1}}{a_{j+1,j}} \cdot \frac{a_{j+1,i}}{a_{i,j+1}} .$$

Proof. The proof is by downward induction, starting at $i = n$:

$$\alpha_n = \alpha_n \frac{a_{nn}}{a_{nn}} + 0 .$$

Assume that the equation is correct for $i \geq 1$. We show that it is also correct for $i - 1$. From the Region Matching Theorem for a closed region we have

$$\begin{aligned}
\alpha_{i-1} &= \alpha_i \frac{a_{i,i-1}}{a_{i-1,i}} + \frac{b_{i,i-1}}{a_{i-1,i}} \\
&= \alpha_n \frac{a_{ni} a_{i,i-1}}{a_{in} a_{i-1,i}} + \sum_{j=i}^{n-1} \frac{b_{j,j+1} a_{j+1,i} a_{i,i-1}}{a_{j+1,j} a_{i,j+1} a_{i-1,i}} + \frac{b_{i,i-1}}{a_{i-1,i}} \\
&= \alpha_n \frac{a_{n,i-1}}{a_{i-1,n}} + \sum_{j=i-1}^{n-1} \frac{b_{j,j+1} a_{j+1,i-1}}{a_{j+1,j} a_{i-1,j+1}} ,
\end{aligned}$$

where the simplification of products of b-values can be based on Equ. (13). □

Analogously, for open regions we obtain (the original formula in [21]):

Lemma 2. *Let M be a finite union of pairwise disjoint open regions in $\mathbb{C}_n$. For $0 \leq i, j \leq n$ we have*

$$\alpha_i = \alpha_0 \frac{a_{0i}}{a_{i0}} - \sum_{j=1}^{i} \frac{b_{j,j-1}}{a_{j-1,j}} \cdot \frac{a_{j-1,i}}{a_{i,j-1}} .$$

The following theorem was proved by *K. Voss* in 1993 for open regions.

Theorem 6. *Let M be a finite union of pairwise disjoint closed (or pairwise disjoint open) regions in $[\mathbb{C}_n, I, dim]$, with $ind(M) = n$. The Euler characteristic of M is*

$$\chi(M) = \frac{1}{2n}\sum_{i=1}^{n}(-1)^{i+1} b_{i,i-1} \text{ for open regions,}$$
and
$$\chi(M) = \frac{1}{2n}\sum_{i=0}^{n-1}(-1)^{i+1} b_{i,i+1} \text{ for closed regions.}$$

Proof. We prove this for open regions; closed regions can be treated analogously. Lemma 2 and the equation $\sum(-1)^i a_{0i}/a_{i0} = 0$ show that

$$\chi(M) = \sum_{i=0}^{n}(-1)^i\left(-\sum_{j=1}^{i}\frac{b_{j,j-1}a_{j-1,i}}{a_{j-1,j}a_{i,j-1}}\right).$$

The double sum can be rearranged: first take the sum for all j-values and then for all i-values. It follows that

$$\chi(M) = \sum_{j=1}^{n}\frac{b_{j,j-1}}{a_{j-1,j}}\sum_{i=j}^{n}(-1)^{i+1}\frac{a_{j-1,i}}{a_{i,j-1}}.$$

The formula for closed regions then follows from Equ. (13) and

$$-\sum_{i=m+1}^{n}(-1)^i\binom{n}{i} = \sum_{i=0}^{m}(-1)^i\binom{n}{i} = (-1)^m\binom{n-1}{m} \qquad (15)$$

for $0 \leq m < n$. □

The $b_{i-1,i}$'s and $b_{i+1,i}$'s in Theorem 6 can also be replaced by class cardinalities and (globally known) incidence counts, because

$$b_{i,i-1} = \alpha_{i-1}a_{i-1,i} - \alpha_i a_{i,i-1} \text{ for open regions, and}$$
$$b_{i,i+1} = \alpha_i a_{i,i+1} - \alpha_{i+1}a_{i+1,i} \text{ for closed regions.}$$

Let $n = 3$. For a closed region, b_{01} is the number of all invalid grid edges incident with grid vertices in the region, b_{12} is the number of all invalid grid squares incident with grid edges in the region, and b_{23} is the number of all invalid grid cubes incident with grid squares in the region. For open regions we use b_{10}, the number of all invalid grid vertices incident with grid edges in the region; b_{21}, the number of all invalid grid edges incident with grid squares in the region; and

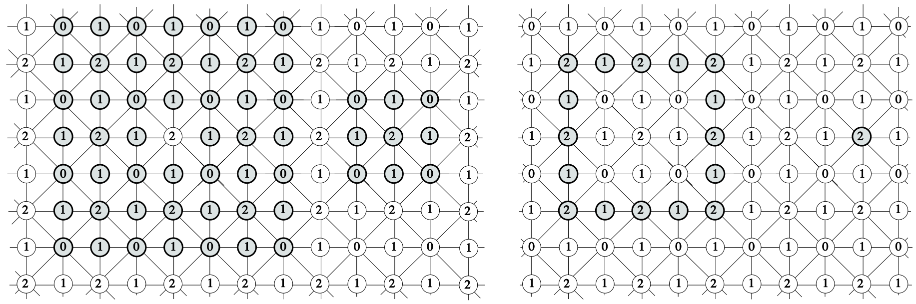

Fig. 10. Nodes in regions are represented by bold and shaded circles. Left: two closed regions with Euler characteristics 0 and 1. Right: two open regions, also with Euler characteristics 0 and 1.

b_{32}, the number of all invalid grid squares incident with grid cubes in the region. Note that the total boundary counts are over all cells in M, i.e. invalid cells can be counted repeatedly if they are incident with several cells in M.

Fig. 10 shows a 2D example. For the left closed region we have $b_{12} = 16$ and $b_{01} = 16$, and for the right closed region we have $b_{12} = 4$ and $b_{01} = 8$. For the left open region we have $b_{21} = 16$ and $b_{10} = 16$, and for the right open region (a single node) we have $b_{21} = 4$ and $b_{10} = 0$.

13 Concluding Remarks

Adjacency graphs have been studied in the context of image analysis since the 1970s, and the paper [17] stands at the beginning of this research. Oriented adjacency graphs defined by axioms A1-A4 have been studied in a sequence of papers, starting with [22] and ending with [9], completely documented in the book [20], briefly reviewed in the paper [10] and cited at some places in the book [21].

Incidence pseudographs are studied in graph theory, geometry, and combinatorial topology. The book [21] discussed incidence pseudographs for a model that assumes pixels or voxels to be grid points in incidence structures. Also, [21] only discussed the case of open sets, but without defining open or closed sets at all. The discussion of closed regions has been added in this article to complete the characterization of topological concepts in incidence pseudographs.

Acknowledgments This article presents material what will be part of a forthcoming book [11]. The author thanks *Azriel Rosenfeld* and *Klaus Voss* for comments which have been important in finalizing this article.

The author dedicates this paper to *Klaus Voss* on the occasion of his retirement in 2002. His outstanding creativity has been very inspiring for all who had or have the opportunity to work with him.

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