Digital flatness and related combinatorial problems*

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Abstract

In the present notes we define and study the notion of *digital flatness*. We extend to two dimensions various definitions and results about digital straightness, in particular, we resolve a conjecture of M. Nivat for the case of digital planes and define and characterize 2D Sturmian rays.

Keywords: *Digital line, digital plane, string/array periodicity/aperiodicity, Sturmian word*

Contents:

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1 Introduction

In this work we define and study the notion of *digital flatness*. We extend to two dimensions various notions and results related to digital straightness.

The manuscript organization is described in the preceeding contents. In order to make the paper self-contained, we start with comprehensive preliminaries (Section 2). There we first recall some basic notions and facts related to digital straightness, which we extend to two dimensions in the subsequent sections. As a background for our further considerations, we also outline related basic definitions and results from digital topology, digital geometry, lattice theory, combinatorics on strings and 2D arrays, theory of computation, and others. Readers familiar with the above subjects could skip the corresponding subsections. As further preparation, periodicity and symmetry in infinite 2D arrays are considered in Section 3.

Then, we define digital 2D rays and study their basic properties. In particular, we address a conjecture by M. Nivat for the case of digital planes and consider 2D Sturmian rays (Sections 4 and 5). Examples and related issues are presented and discussed in Sections 6 and 7. Some algorithmic aspects are concerned in Section 8. Further tasks are commented in Section 9.

2 Preliminaries

In this section we summarize as a background for our further considerations some knowledge related to *digital topology and geometry*, *digital straightness*, *integer lattices*, *combinatorics on words and 2D arrays*, and *computational models*. Most of these preliminaries will be used for references in the further sections.

2.1 Basic notions of digital topology and digital geometry

Discrete coordinate plane consists of unit squares (pixels), centered on the integer points of the two-dimensional Cartesian coordinate system in the plane. *Discrete coordinate space* consists of unit cubes (voxels), centered on the integer points of the three-dimensional Cartesian coordinate system in the space. The pixels'/voxels' coordinates are the coordinates of their centers. Sometimes they are called *discrete points*. The edges of a pixel/voxel are parallel to the coordinate axes. A set of discrete points is usually referred to as a *discrete object*.

A j-dimensional facet of a pixel/voxel will be called *j-facet*, for some j, $0 \leq j \leq n-1$ ($n = 2$ or 3). Thus the 0-facets of a voxel v are its vertices, while the 1-facets are its edges.

Two pixels/voxels are called *j-adjacent* if they share a j-facet. A *k-path* in a discrete object A is a sequence of pixels/voxels from A such that every two consecutive pixels/voxels are k-adjacent. Two pixels/voxels are *k-connected* if there is a k-path between them. A discrete object A is *k-connected* if there is a k-path connecting any two pixels/voxels of A. A discrete object is said to be *connected* if it is at least 0-connected. Otherwise it is *disconnected*.[1]

Let D be a subset of a discrete object A. If $A - D$ is not k-connected then the set D is said to be *k-separating* in A. Let a set of pixels/voxels A be k-separating in a discrete object B but not j-separating in B. Then A is said to have *j-tunnels* for any $j < k$. Discrete object without any k-tunnels is called *k-tunnel-free*[2].

A *k-component* is a maximal (non-extendable) k-connected set. Let D be a k-separating discrete object in Z^3 such that $Z^3 - D$ has exactly two k-components. A *k-simple point* in D is a discrete point p such that $D - p$ is k-separating. A k-separating discrete object in Z^3 is *k-minimal* (or *k-irreducible*) if it does not contain any k-simple point.

2.2 Digital rays and lines

Let $\gamma_{\alpha,\beta}$ be a ray or a straight line, i.e.,

$$\gamma_{\alpha,\beta} = \{(x, \alpha x + \beta) : 0 \leq x < +\infty\} \tag{1}$$

or

$$\gamma_{\alpha,\beta} = \{(x, \alpha x + \beta) : -\infty < x < +\infty\}. \tag{2}$$

The digitization of these objects over the grid points has been extensively studied in the recent decades. Below we briefly recall some basic definitions and related facts, following Rosenfeld and Klette [36]. Some denotations and formulations used in Sections 2.1, 2.2, and 2.3 are directly taken from [36].

2.2.1 Equivalent definitions and basic properties

Consider first more in detail the digitization of (1) in the set $\mathbf{N}^2$ of the grid points with nonnegative integer coordinates. W.l.o.g., assume that $0 \leq \alpha \leq 1$. Let $\rho_0, \rho_1, \rho_2, \ldots$ be the intersection points of $\gamma_{\alpha,\beta}$ with the vertical grid lines for $n \geq 0$. Let $(n, I_n) \in \mathbf{Z}^2$ be the grid point nearest to ρ_n. If the second coordinate of ρ_n is half-integer, we choose (n, I_n) to be the upper one. Formally, we define

$$I_{\alpha,\beta} = \{(n, I_n) : I_n = \lfloor \alpha n + \beta + 0.5 \rfloor, n \geq 0\}.$$

[1] Classically, 0-adjacent/connected (resp. 1-adjacent/connected) pixels are called 8-adjacent/connected (resp. 4-adjacent/connected). In dimension 3, 0-adjacent/connected (resp. 1 or 2-adjacent/connected) voxels are called 26-adjacent/connected (resp. 18 or 6-adjacent/connected).

[2] Classically, in dimension two, a 0-tunnel (resp. 1-tunnel) is called 8-tunnel (resp. 4-tunnel). In dimension three, a 0-tunnel (resp. 1- or 2-tunnel) is called 26-tunnel (resp. 18- or 6-tunnel).

α and β are the *slope* and the *intercept* of $I_{\alpha,\beta}$, respectively.

$I_{\alpha,\beta}$ will be considered as a *discrete ray.* We will also call it *discretization* of $\gamma_{\alpha,\beta}$. Sometimes, discretization of a ray γ will be denoted also by $discr(\gamma)$.

Now we define a *digital ray* with *slope* α and *intercept* β as follows:

$$i_{\alpha,\beta} = i_{\alpha,\beta}(0) i_{\alpha,\beta}(1) i_{\alpha,\beta}(2) \ldots,$$

where $i_{\alpha,\beta}(n)$ are called *chain codes* and defined for $n \geq 0$ by

$$i_{\alpha,\beta}(n) = I_{n+1} - I_n = \begin{cases} 0, & \text{if } I_{n+1} = I_n \\ 1, & \text{if } I_{n+1} = I_n + 1 \end{cases}.$$

Code 0 can be interpreted as a horizontal grid increment while 1 as a diagonal increment in the grid $\mathbf{N}^2$.

We will also call $i_{\alpha,\beta}$ *digitization* of $\gamma_{\alpha,\beta}$. Sometimes, digitization of a ray γ will be denoted also by $digit(\gamma)$. Discretization and digitization of a straight line are defined analogously.

An *8-arc* is a finite or infinite sequence of grid points such that any point is an 8-neighbor of its predecessor in the sequence. An 8-arc is *irreducible* iff its set of grid points does not remain 8-connected after removing a point which is not an end point. We have the following theorem.

Theorem 1 *(Rosenfeld 1974 [37]) A digital ray is an irreducible 8-arc.*

Remark 1 *The digital ray $i_{\alpha,\beta}$ is* generated *by the ray $\gamma_{\alpha,\beta}$. Clearly, if $\beta - \beta'$ is integer, then $i_{\alpha,\beta} = i_{\alpha,\beta'}$. Thus, w.l.o.g., we may assume that the intercepts are limited to $0 \leq \beta \leq 1$. Clearly, $i_{0,\beta} = 000\ldots$ and $i_{1,\beta} = 111\ldots$.*

We have the following theorem.

Theorem 2 *(Bruckstein 1991 [20]) For irrational α, $I_{\alpha,\beta}$ uniquely determines both α and β. For rational α, $I_{\alpha,\beta}$ uniquely determines α, and β is determined up to an interval.*

Remark 2 *Let $\gamma_{\alpha,\beta} = \{(x, \alpha x + \beta) : 0 \leq x < +\infty\}$ be a ray. Let α be a rational number $\frac{p}{q}$ which is an irreducible fraction (i.e., $gcd(p,q) = 1$). According to our assumption, $0 \leq \frac{p}{q} \leq 1$, i.e., $p \leq q$. The line equation $y = \alpha x + \beta$, $\alpha = \frac{p}{q}$, can be written as $-px + qy = qb$. Consider the ray determined by the line $g^{\lfloor\rfloor}_{\alpha,\beta} : -px + qy = \lfloor qb \rfloor$. Its digitization is clearly $I_{\alpha,\beta}$. In addition, the lines with the same digitization $I_{\alpha,\beta}$ are $g : -px + qy = \beta'$ with $\lfloor qb \rfloor - \frac{1}{2} \leq \beta' \leq \lfloor qb \rfloor + \frac{1}{2}$, if $\max(|p|, |q|)$ is an odd number, and $\lfloor qb \rfloor \leq \beta' < \lfloor qb \rfloor + 1$, if $\max(|p|, |q|)$ is an even number.*

A digital ray is *rational* if it has a rational slope, and *irrational* if its slope is an irrational number. We have the following theorem.

Theorem 3 *(Brons 1974 [19]) Rational digital rays are periodic and irrational digital rays are aperiodic. Moreover, if the slope of a digital ray is an irreducible rational fraction p/q, then the period length is equal to q.*

Remark 3 *At this point it is useful to mention that the rationality/irrationality of a line depends on the integer (or rational) points it contains. A line l contains exactly one or no one rational point if and only if it is irrational. If l contains more than one rational point, then it contains infinitely many rational points which are dense on the line. In this case l is parallel to or coincides with a line l' which contains infinitely many equidistant integer points. This is the case if and only if the line is rational, and according to Theorem 3, it is periodic. The period of a digital line corresponds to the segment between two consecutive integer points.*

There are several approaches to studying digital lines and rays. In the sequel we will use the following one.

Definition 1 2D arithmetic line *is a set of pixels* $L(a_1, a_2, \mu, \omega) = \{(x, y) \in \mathbf{Z}^2 | -\lfloor\frac{\omega}{2}\rfloor \leq a_1x+a_2y+\mu < \lfloor\frac{\omega}{2}\rfloor\}$*, where* $\omega \in \mathbf{N}$*.* ω *is called* arithmetic thickness *of the line and* μ *is called* internal translation constant*.*

An arithmetic line $L(a_1, a_2, \mu, \omega)$ *is* 0-connected *(classically,* 8-connected *or* naive*) if* $\omega = \max(|a_1|, |a_2|)$*, and* 1-connected *(classically,* 4-connected *or* standard*), if* $\omega = |a_1| + |a_2|$*.*

The naive line is the thinnest 2-tunnel-free arithmetic line. We have the following theorem.

Theorem 4 *(Reveilles 1991 [34]) An arithmetic line* $P(a_1, a_2, \mu, \max(|a_1|, |a_2|))$ *coincides with a discrete straight line with the same slope and intercept, and vice versa: any discrete straight line can be represented in the form* $P(a_1, a_2, \mu, \max(|a_1|, |a_2|))$ *for some parameters* a_1, a_2, μ*.*

Thus the points of a discrete straight line which correspond to a digital straight line, lie between or on two parallel lines having a distance less than 1, measured in the y-axis direction.

2.2.2 Other definitions. Sturmian words

For some basic notions of theory of words, such as *factor* of a word, *period* of a word, *periodic* and *eventually periodic* word, etc., we refer to [36].

Let w be a finite or infinite word on the alphabet $A = \{0, 1\}$. *Complexity function* of w is the function $P(w, n)$ defined as the number of different factors of w of length n. In particular, $P(w, 0) = 1$ (since the empty word can be considered as a factor), while $P(w, 1)$ is the number of the different letters appearing in w. It is well-known (see, e.g., [32, 23]) that a word w is eventually periodic if $P(w, n) \leq n$ for some $n \geq 0$. Thus, the complexity function of any aperiodic word w satisfies $P(w, n) \geq n + 1$.

Definition 2 *A word* w *is called* Sturmian *if* $P(w, n) = n + 1$*, i.e., if it is aperiodic and has minimal complexity.*

Definition 3 Height $h(w)$ *of a word* w *is the number of letters in* w *equal to 1. Given two words* v *and* w *of the same length,* $\delta(v, w) = |h(v) - h(w)|$ *is their* balance*. A set of words* X *is* balanced *if* $|v| = |w|$ *implies* $\delta(v, w) \leq 1$ *for all pairs of words* $v, w \in X$*. An infinite word* w *is* balanced *if its set of factors is balanced.*

For a finite or infinite word w, by $Sub(w)$ we denote the set of all *finite factors* of w.

2.2.3 Properties of periodic (rational) and aperiodic (irrational) rays/lines

As already mentioned, rational straight lines/rays are periodic. Detailed study of various points related to line/ray periodicity can be found in [23, 24, 32, 30]. We list some basic results.

Theorem 5 *(Lunnon and Pleasants 1991 [30]) (a) All digital lines with a rational slope* $\alpha \neq 0, \infty$ *are equivalent up to a translation.*

(b) All digital lines with an irrational slope α *contain the same set of factors* $Sub(\alpha)$*.*

Theorem 6 *(Coven and Hedlund 1973 [23]) Let* w *be a Sturmian word. Then every subword appearing in* w *appears infinitely many times in* w*.*

Theorem 7 *(Coven and Hedlund 1973 [23]) For an infinite word w over two letter alphabet, the following conditions are equivalent: (i) w is eventually periodic;*

(ii) $P(w, n) < n + 1$ for some $n \geq 1$;

(iii) $P(w, n)$ is bounded (by the period length of w).

Theorem 8 *(Morse and Hedlund 1940 [32]) For an infinite word w, the following conditions are equivalent: (i) w is Sturmian;*

(ii) w is balanced and (eventually) aperiodic;

(iii) w is an irrational digital line.

Remark 4 *Note that, by definition, the complexity $P(w, n)$ of a Sturmian word w is unbounded.*

2.2.4 Symmetry

It has been shown in [30] that if the slope of a digital line is *rational*, then each period is a symmetric word. Classification of the possible types of symmetry is provided, as well. Thus, in particular, if w is a word in a rational digital line, then the reverse of w is a word in that line, too.

About an irrational ray w, it has been shown in [23] that if certain block (finite word) appears in w, then its reverse appears in w, too.

2.2.5 Other notions and facts related to repetitions

In this section we follow [22]. A word is said to contain a *repetition of order* k, with a rational $k > 1$, if it contains a factor of the form

$$z \in pref(r^\omega), \ \frac{|z|}{|r|} = k,$$

where $pref(.)$ is a denotation for a prefix of a word. Let $k > 1$ be a *real* number. We say that a word u is:

k-free, if it does not contain as a factor a repetition of order at least k;
k^+-free, if, for any $k' > k$, it is k'-free;
k^--free, if it is k-free, but not k'-free for any $k' < k$.

Now consider the following two morphisms:

- *Thue-Mors morphism*

$$T : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 10 \end{cases},$$

which defines a word $w_T = abbabaabbaababbabaababbaabbabaab\ldots$;

- *Fibonacci morphism*

$$F : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 0 \end{cases},$$

which defines a word $w_F = abaababaabaababaababaabaababa\ldots$.

We have the following results.

Theorem 9 *(Thue 1912 [39, 40]) The Thue-Morse word w_T is 2^+-free, i.e., does not contain overlapping factors.*

Theorem 10 *(Mignosi and Pirillo 1992 [31]) The Fibonacci word w_F is $(2 + \phi)$-free, where ϕ is the golden ratio.*

2.3 Integer lattices

Consider the hyperplane

$$H : a_1x_1 + a_2x_2 + \ldots + a_nx_n = b,$$

where the vector $a = (a_1, \ldots, a_n)$ and the number b are rational. W.l.o.g., we can assume that they are integer. It is well-known that if $gcd(a_1, a_2, \ldots, a_n)$ divides b, then the hyperplane H contains infinitely many integer points which form an $(n-1)$-dimensional lattice $\Lambda(a, b)$ in $\mathbf{R}^n$, i.e., a lattice with a basis consisting of $n - 1$ linearly independent vectors. For $n = 2$, $\Lambda(a, b)$ is a 1-dimensional lattice, while for $n = 3$, $\Lambda(a, b)$ is 2-dimensional.

If $gcd(a_1, a_2, \ldots, a_n)$ is not a divisor of b, H contains rational points that are not integer. In this case, H is parallel to a plane which contains an $(n - 1)$-dimensional integer lattice.

2.4 2D arrays. Basic notions

Below we introduce some definitions and facts from combinatorics of 2D words. Some of them will be used in the sequel. Others elucidate certain periodicity issues in 2D, which may facilitate the better understanding of the further considerations. We use in part [14].

2.4.1 2D periodicity of finite 2D arrays

A *two-dimensional* array (or *2D-array*, for short) is any $m\times n$ rectangular array $X[0..m-1, 0..n-1]$ with $m > 1$ *rows* and $n > 1$ *columns*. Any rectangular subarray of X is a *block*. A block $W = X[0..k, 0..l]$, $0 \le k < m$ and $0 \le l < n$ is a *prefix* of X, while a block $W = X[k..m - 1, l..n - 1]$, $0 \le k < m$ and $0 \le l < n$ is a *suffix* of X. A *point* of X is a pair of integers (i, j) for a row i and a column j. An *element* of X at the point (i, j) is denoted by $X(i, j)$. The elements of X are symbols of some alphabet Σ.

Two-dimensional periodicity of arrays has been introduced and studied in [1, 2, 3, 27].

Definition 4 *A* symmetry vector *of X is a vector that maps, without a mismatch, one copy of X to another copy of its, positioned at a certain point of X.*

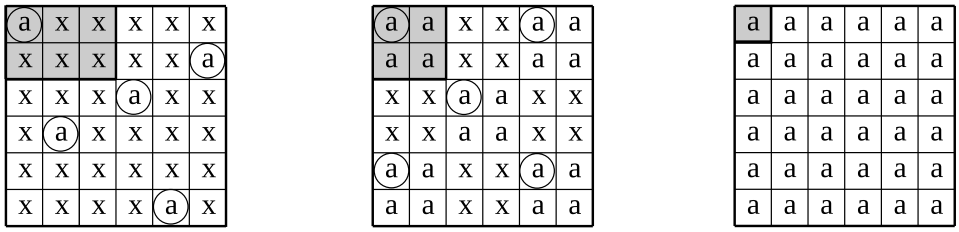
We will also use interchangebly an equivalent definition of symmetry vector in terms of blocks (called periods) which seems to us somewhat more demonstrative.

Definition 5 *A block W of X is a* period *of X if W is X itself or one of the following conditions is met:*

(1) $W = X[0..k - 1, 0..l - 1]$, $0 < k \le m$ and $0 < l < n$, and it is possible to rigidly superimpose two copies of X in such a way that $X(0, 0)$ is brought onto $X(k, l)$ (if $k < m$) or onto $X(0, l)$ (if $k = m$) without generating any mismatches;

(2) $W = X[m - k..m - 1, 0..l - 1]$, $0 < k < m$ and $0 < l \le n$, and it is possible to rigidly superimpose two copies of X in such a way that $X(m-1, 0)$ is brought onto $X(m-k-1, l)$ (if $l < n$) or onto $X(m - k - 1, 0)$ (if $l = n$) without generating any mismatches.

Thus a symmetry vector (k, l) in Definition 4 corresponds to a period block $W = X[0..k - 1, 0..l - 1]$ in Definition 5. An illustration to the above definitions is given in Figure 1. Period $W = X$ is said to be the *trivial period*. Following [1, 27], a period fulfilling Condition 1 is called *quad-I period*. In the particular case where $k = m$ but $l < n$, i.e., it is possible to superimpose without mismatches two copies of A in such a way that $A(0, 0)$ is brought onto $A(0, l)$. Then W is said to be a *horizontal period* of X; A period fulfilling Condition 2 is called *quad-II period*. If, in particular, $l = n$, but $k < m$, so



(a) (b) (c)

Figure 1: a) Lattice-periodic, primitive, non-tiled array. A period is darkened. b) Lattice-periodic, primitive, non-tiled array. A period is darkened. c) Every prefix is a period, and there is a unique tile a which is also the only primitive period.

that $A(m-1,0)$ can be brought without mismatches onto $A(m-k-1,0)$, then W is a *vertical period* of X.

In a string, a special role is played by the period of minimum size. With 2D arrays, there can be more than one such an elementary period, as the following definition suggests.

Definition 6 *A quad-I period $W = X[0..k-1, 0..l-1]$ is* minimal *if it does not contain any other period $W' = X[0..k'-1, 0..l'-1]$ with $k' \leq k$ and $l' \leq l$.*

Minimal quad-II periods are similarly defined. Note that W, being a minimal period for X, does not imply that any minimal period of W must coincide with W.

Definition 7 *X is* periodic *if one of the following conditions is met. Condition 1: the period W is a horizontal (respectively, vertical) period for X and its horizontal (resp. vertical) length does not exceed 1/2 of the corresponding dimension of X. Condition 2: Period W is neither horizontal nor vertical, but each of its dimensions does not exceed 1/2 of the corresponding dimension of X.*

A period such as W in Definition 7 is called a *short period.* An array X is *non-periodic* if it has no short periods. A pair (W_1, W_2) consisting of a short minimal quad-I period W_1 and a short minimal quad-II period W_2 of X (when it exists) is called a *basis* for X. The periods W_1 and W_2 are called quad-I and quad-II basis periods of X, respectively. It is not difficult to prove (or see [27]) that the existence of two short minimal quad-I periods for array X implies the existence of a short (minimal) quad-II period, hence also of a basis for X. Two periods $W_1 = X[0..k_1-1, 0..l_1-1]$ and $W_2 = X[0..k_2-1, 0..l_2-1]$ are *independent* if the points $(0,0)$, (k_1, l_1) and (k_2, l_2) are not collinear (i.e., $k_1/l_1 \neq k_2/l_2$). Independence of periods of other types is defined analogously.

The following definitions parallel those given in [1, 27]. Let X be a periodic array. X is *lattice-periodic* if it has at least one short quad-I period and at least one short quad-II period, or, equivalently, if it has a basis. X is *line-periodic* if it has only one independent short period. X is *radiant-periodic* if it is not lattice-periodic but it has at least two independent short quad-I periods or at least two short quad-II periods. In all other cases X is non-periodic.

2.4.2 Tiles, tiled arrays, and repetitions

We now introduce some additional notions.

a	b	a	b	a	a	b	a	b	a	a
c	d	c	d	c	c	d	c	d	c	c
a	b	a	b	a	a	b	a	b	a	a
a	b	a	b	a	a	b	a	b	a	a
c	d	c	d	c	c	d	c	d	c	c
a	b	a	b	a	a	b	a	b	a	a
a	b	a	b	a	a	b	a	b	a	a
c	d	c	d	c	c	d	c	d	c	c

Figure 2: The tile of a tiled array may be tiled itself.

Definition 8 *An array X is* primitive *if setting $X = \begin{array}{|ccc|} \hline W & \dots & W \\ \dots & \dots & \dots \\ W & \dots & W \\ \hline \end{array}$, where X has k rows and l columns, implies $k = 1$, $l = 1$.*

As Figure 1 (a, b) displays, an array can be periodic and primitive at the same time.

Definition 9 *A primitive block W of X is a* tile *of X if $X = \begin{array}{|ccc|c|} \hline W & \dots & W & W' \\ \dots & \dots & \dots & \dots \\ W & \dots & W & W' \\ \hline W'' & \dots & W'' & W''' \\ \hline \end{array}$, where W', W'', W''' are possibly empty prefixes of W of appropriate dimensions.*

Definition 10 *An array $X = \begin{array}{|ccc|c|} \hline W & \dots & W & W' \\ \dots & \dots & \dots & \dots \\ W & \dots & W & W' \\ \hline W'' & \dots & W'' & W''' \\ \hline \end{array}$ is* tiled *if it features at least two vertical or horizontal blocks W. If there are at least two W-blocks along both dimensions then X is called a* two-dimensional (2D) repetition*. An array is called* repetition-free *if it does not contain any repetitions.*

A tile may have periodicity type and properties of its own, in particular it can be a tiled array (see Figure 2). An array may have several tiles. Consider, for example, the array A made of two Fibonacci words *abaababa* one on top of the other. Then both *abaab* and *abaabab* are tiles of A. However, it is impossible for an array to be tiled in two different fashions.

Proposition 1 *[14] An array X can be tiled by only one tile W, and the size of W is $p \times q$, where p is the least common multiple (lcm) of the lengths of the shortest periods of all the rows of X, and q is the lcm of the lengths of the shortest periods of all the columns of X.*

Clearly, a tiled array is periodic. We have also that an array X is horizontally (vertically) periodic if and only if it is tiled horizontally (vertically), as the horizontal (vertical) length of the tile is equal to the length of the shortest horizontal (vertical) period of X.

In the general case, the lattice-periodicity of X (and of course the radiant or line-periodicity, as well) does not imply that X is tiled (Figure 1a,b). However, we have the following plain propositions.

Proposition 2 *[14] An array X is a 2D repetition if and only if X is lattice-periodic with one short horizontal and one short vertical period or, equivalently, if it has one short horizontal (vertical) period of height l (height k) and one short non-horizontal $k \times l$-quad-I period or one non-vertical $k \times l$-quad-II period.*

The simultaneous existence in X of a short horizontal and a short vertical period can be interpreted as the existence of an *orthogonal basis* for X. The following lemma shows that if an array X has a basis and if the dimensions of the array are sufficiently large compared to those of the basis periods, then X is a 2D repetition.

Proposition 3 *[14] Let X be a lattice-periodic $m \times n$ array with basis (W_1, W_2). If the array dimensions m and n are sufficiently large compared to the corresponding dimensions of W_1 and W_2 then X is a 2D repetition.*

2.5 Computational issues, models of computation

When computational issues are concerned, it matters what kind of data an algorithm takes as an input. If rational lines/planes are considered, then the classical computational model with bit-operations can be used. However, when processing lines and planes with irrational slopes/coefficients, one needs to perform the operations in a *real model of computation.* This kind of model has been widely used in various disciplines, such as algebraic complexity, scientific computing, computational geometry, and (although not explicitly) numerical analysis. In our study we conform mainly to the version presented in [13], known as the BSS-model (named after the authors Blum, Shub and Smale). In this model, the assumption is that all the real numbers in the input have unit size, and the basic algebraic operations $+, -, *, /$ and the relation $\leq$ are executable at unit cost. Thus, the algebraic complexity of a computation on a problem instance is the number of operations and branchings performed to solve the instance. For more details on the BSS-model and complexity theory over arbitrary rings, we refer to [13, 12].

3 Further preparation

3.1 Periodicity, repetitions and tilings of infinite arrays

As distinct from the one-dimensional case, a two-dimensional structure may feature many different shapes. In what follows, we will be mainly concerned with the case of a plane quadrant, which is the 2D counterpart of a ray. For definitenes, we will consider the first quadrant *quad I*. We will also consider the full plane, which is the 2D counterpart of a line.

Consider R^2_+ and the integer lattice Z^2_+. Often we will work with the corresponding set of pixels with centers at the integer points and sides parallel to the coordinate axes. The pixels' sides form a *grid*. An *array A on Z^2_+ over an alphabet* Σ is a mapping from Z^2_+ to Σ, i.e.,

$$A = \boxed{\begin{array}{cccc} \cdots & \cdots & \cdots & \cdots \\ a_{2,0} & a_{2,1} & a_{2,2} & \cdots \\ a_{1,0} & a_{1,1} & a_{1,2} & \cdots \\ a_{0,0} & a_{0,1} & a_{0,2} & \cdots \end{array}},$$

where $a_{ij} \in \Sigma$. *Array on* Z^2 is defined analogously.

Two elements of an array are called *adjacent* if the corresponding points (pixels) are adjacent. The *unit cell* on vectors u and v is the set of points w such that $w = \alpha u + \beta v$, for $0 \leq \alpha, \beta < 1$. The *lattice cell* at w on u and v is the set of points $w + p$ for p in the unit cell (i.e., the unit cell shifted by w). Two points p and q are *lattice congruent modulo u and v* if $p - q = iu + jv$ for integers i, j. A point that is lattice congruent modulo u and v to $(0, 0)$ is called *lattice point on u and v*.

A subset $s \subseteq Z^2_+$ is called a *shape*. Given an array A on Z^2_+, by $A[s]$ we denote the restriction of A to s. $A[s]$ is *connected* if s is connected. We will call $A[s]$ *factor of A on shape s*. In what follows, we will consider factors which are discretizations of lines, line segments, rectangles, parallelograms, etc. A rectangular factor of size $m \times n$ will be called *$m \times n$-factor*.

Below we slightly modify some definitions from Section 2.4 in order to adapt them to infinite arrays. We follow in part [25], where periodility in infinite 2D arrays is considered.

Definition 11 *Let A be an array on Z^2_+ (or on Z^2). Let $S = A[s]$ be a factor of A on shape s. (possibly, $s = Z^2_+$ and $S = A$). A vector v is a* symmetry vector *for S if $A(i, j) = A(v + (i, j))$ for any point $(i, j) \in s$ such that $v + (i, j)$ is still in s. (If $s = Z^2$, then clearly $v + (i, j) \in s$ for any point (i, j).)*

v is periodicity vector *(or a* period*) for S if for any integer k, the vector kv is a symmetry vector for S.*

Remark 5 *Since the considered arrays have infinite size, clearly all periods kv for a period vector v and a fixed integer k may be regarded as short periods.*

We list a simple proposition from [25].

Proposition 4 *A vector v is a symmetry vector for S if and only if $-v$ is a symmetry vector for S. Moreover, if v is a period for S, then for any $k \in Z$, kv is a period of S.*

Definition 12 *An array A on Z^n_+ is* lattice periodic *if there are two linearly independent vectors u and v such that $w = iu + jv$ is a period for A for any pair of integers i, j for which $w \in Z^n_+$. A is* line periodic *if all periods of A are parallel vectors.*

Note that this definition does not destinguish between quad I and quad II periods, and, in general, it is not equivalent to the definitions from [1, 27] outlined in Section 2.4.

Remark 6 *If A is line-periodic, then the set of its periods is generated by a single element v, whose length is a divisor of the lengths of all other periods.*

Let A be lattice-periodic and u and v two of its periods. Then there is a finite set $T \subset Z^2_+$ such that every point from Z^2_+ is congruent modulo u and v to a unique point from T.

We also have the following property.

Proposition 5 *Let A be a lattice periodic array on Z^2_+ or Z^2. Let u and v be two symmetry vectors both pointing to the first quadrant (i.e., with positive coordinates). Then A has symmetry vectors pointing to the forth quadrant (i.e., with a first coordinate positive number and a second coordinate negative number).*

The above statement holds also for finite lattice periodic arrays of sufficiently large size. Its meaning is roughly that in an enough large array, the existence of two quad I periods implies the existence of a quad II period.

Definition 13 *Let A be lattice periodic array on Z^n_+. The set of its symmetry vectors is a subset of (is extendable to) a sublattice of Z^2. Let Λ be the minimal one by inclusion. Then any basis of Λ will be considered as a* basis of A.

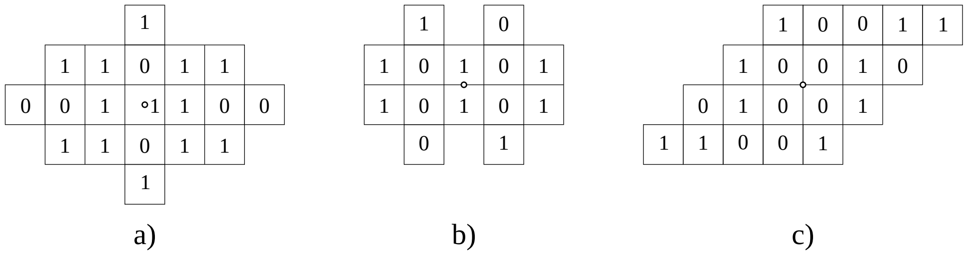


Figure 3: Illustration to 2D palindromes. a) The center of symmetry is an integer point.) The center of symmetry has one integer and one half-integer component. c) The center of symmetry has two half-integer components.

The definitions of tiles, tiled arrays, and 2D repetition can be transferred directly to infinite arrays. Propositions 1, 2 and 3 can be easily modified to hold for an array on Z^2 or Z_+^2. Thus we have:

Proposition 6 *Let X be a tiled array on Z_+^2. Then X can be tiled by only one tile W. If W is finite, then it is a $p \times q$ block, where p is the least common multiple (lcm) of the lengths of the periods of all rows of X, and q is the lcm of the lengths of the periods of all columns of X.*

If W is a strip that is infinite in one dimension (e.g., along the x-axis) and has size p in the other dimension (along the y-axis), then p is the lcm of the lengths of the periods of all rows of X.

Proposition 7 *An array X on Z_+^2 is a 2D repetition if and only if X is lattice-periodic with one horizontal and one vertical period.*

Proposition 8 *Any lattice-periodic array on Z_+^2 is a 2D repetition.*

3.2 Symmetric arrays and 2D palindromes

Let s be a shape in Z_+^2 and A be an infinite array on s.

s is *symmetric* if there is a point $c \in s$ (not necessarily integer), such that for every vector v with $c + v \in B$, it holds $c - v \in B$. A point c with this property will be called a *center of symmetry* for the shape s.

Now let $B \subseteq A$ be a factor of A on shape s. The factor B is a *2D palindrome* if its shape s is symmetric with respect to some center of symmetry c, and for every vector v with $c + v \in B$, the symbol at the point $c + v$ is the same as the one at $c - v$.

Note that it may happen that the center of a symmetric set of points is between two or four integer points (see Figure 3). In such a case the center of symmetry will be a point (x, y) such that one or both of its coordinates are half-integer (Figure 3b,c).

3.3 Discrete parallelogram tilings

Consider the plane with the integer grid defined in a previous section. Consider a parallelogram Q with integer vertices p_1, p_2, p_3, and p_4, such as in Figure 4. It is well-known that such a parallelogram has an integer area, equal to the determinant of the vectors $v_1 = p_2 - p_1$ and $v_2 = p_4 - p_1$. We will define a "discrete parallelogram tile" associated with Q. By the simple geometry of the considered objects, it

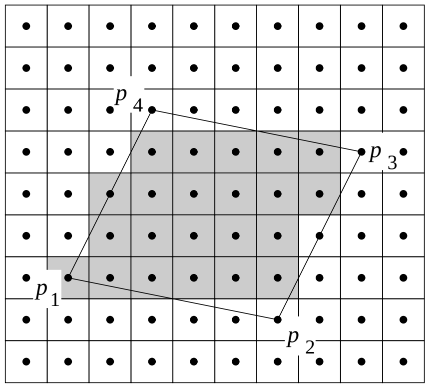


Figure 4: Discrete parallelogram tile.

is clear that two oposite sides of Q intersect the same number of pixels, and the corresponding pixels are intersected in the same way.

We construct a *discrete parallelogram tile* as follows. First, we include in Q' one of the pixels corresponding to one of the four vertices (e.g., to p_1). The other three pixels are not included. Next, include in Q' all pixels whose centers are inside Q. Finally, observe that if there is a pixel whose center lies on a side of Q, then the center of the corresponding pixel intersected by the oposite side of Q, lies on that side. In such a case, we include in Q' exactly one of the two pixels.

By simple geometrical arguments it is clear that the obtained pattern Q' has area equal to the one of Q. It is also clear that by such a tile one can tile the whole plane.

Note that we consider the discrete tile Q' as a useful technical construction to be used further in this paper, rather than as a discretization of Q that may be applied, e.g., in computer graphics. More reasonable discretizations of polygons have been suggested in [7, 15, 16].

4 Digital planes

4.1 Basic definition

We first observe that a connected linear set of points

$$\gamma_{\alpha,\beta} = \{(x, \alpha x + \beta) : x \in J \subseteq R\} \tag{3}$$

is a line segment, ray or a straight line depending on whether J is a line segment, ray or a straight line. Consider now the Euclidean plane

$$\mathcal{P}(\alpha_1, \alpha_2, \alpha_3, \beta) = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \beta\}. \tag{4}$$

W.l.o.g., assume that $\mathcal{P}$ makes with the coordinate plane Ox_1x_2 an angle

$$0 \leq \theta \leq \arctan \sqrt{2}. \tag{5}$$

(See Figure 5.) Then the coefficient α_1 of x_3 in (4) will be nonzero. Dividing both sides of (4) by α_3, we obtain the following equivalent formulation:

$$\mathcal{P}(a_1, a_2, b) = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 = a_1 x_1 + a_2 x_2 + b\}, \tag{6}$$

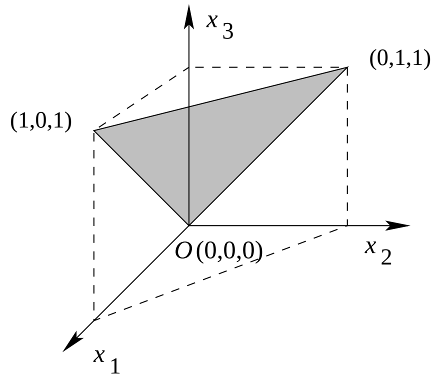


Figure 5: A plane forming an angle $\arctan\sqrt{2}$ with the plane Ox_1x_2.

where $a_1 = -\frac{\alpha_1}{\alpha_3}$, $a_2 = -\frac{\alpha_2}{\alpha_3}$, $b = \frac{\beta}{\alpha_3}$.

We will consider digitizations of the plane $\mathcal{P}$ or its portions in the set of grid points $\mathbf{Z}^3 = \{(i,j,k) : i,j,k \in \mathbf{Z}\}$. In terms of representation (6), we will digitize the third coordinate x_3 over the integer grid points $\mathbf{Z}^2 = \{(i,j) : i,j \in \mathbf{Z}\}$ in the coordinate plane Ox_1x_2.

In 2D, with a reference to (3), $\gamma_{\alpha,\beta}$ is a connected set of points (segment, ray or line) as long as the set J is an interval (finite or infinite). In 3D, the situation may be more complicated. To illustrate, let us write (6) in a more general form:

$$\mathcal{P}^D(a_1,a_2,b) = \{(x_1,x_2,x_3) \in \mathbf{R}^3 : x_3 = a_1x_1 + a_2x_2 + b, (x_1,x_2) \in D \subseteq \mathbf{R}^2\}. \tag{7}$$

We call $\mathcal{P}^D(a_1,a_2,b)$ the *restriction* of $\mathcal{P}(a_1,a_2,b)$ to the *domain* D.

We have that $\mathcal{P}^D(a_1,a_2,b)$ is connected as long as D is connected. Also, $\mathcal{P}^D(a_1,a_2,b)$ is bounded (resp. unbounded) if and only if D is bounded/unbounded. Note, however, that a 2D domain D admits many different shapes, whether D is bounded or not. (The possible unbounded shapes are even infinitely many.) As far as in our study periodicity properties of digitized planes are concerned, it is reasonable to restrict ourselves to a few cases.

When we consider a *finite domain* D, we will usually assume that it is the rectangle

$$D = \{(x_1,x_2) : m_1 \le x_1 \le n_1, m_2 \le x_2 \le n_2,\ m_1,n_1,m_2,n_2 \in \mathbf{Z}\}.$$

Then clearly the corresponding portion $\mathcal{P}^D(a_1,a_2,b)$ of $\mathcal{P}(a_1,a_2,b)$ will be a space rectangle. For an *infinite domain* D, one can consider the following three basic cases:

1. D is a quadrant;
2. D is a half-plane;
3. D is the whole plane.

Remark 7 *The first case corresponds to the case of a ray while the third one to the case of a line in the plane. Therefore, in Case 1, we will call* $\mathcal{P}^D(a_1,a_2,b)$ a 2D ray. *Note that the second case of a half-plane does not have a 1D counterpart.*

We will deal mostly with digitizations of 2D rays. The other cases can be handled in a similar (although not fully identical) way.

Below we explain how one can digitize $\mathcal{P}^D(a_1, a_2, b)$ when D is a 2D ray, i.e., the first quadrant of the plane. Formally, we have

$$\mathcal{P}^D(a_1, a_2, b) = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 = a_1x_1 + a_2x_2 + b, (x_1, x_2) \in D = \{(x_1, x_2) : 0 \leq x_1, x_2 \leq \infty\}\}. \tag{8}$$

We digitize $\mathcal{P}^D(a_1, a_2, b)$ in $\mathbf{Z}^3_+ = \{(i, j, k) : i, j, k \in \mathbf{Z}_+\}$, where $\mathbf{Z}_+$ is the set of nonnegative integers. We digitize the third coordinate x_3 over the nonnegative integer grid points $\mathbf{Z}^2_+ = \{(i, j) : i, j \in \mathbf{Z}_+\}$ in the first quadrant $QuadI$. Let

$$\boxed{\begin{array}{llll} \cdots & \cdots & \cdots & \cdots \\ \rho_{2,0} & \rho_{2,1} & \rho_{2,2} & \cdots \\ \rho_{1,0} & \rho_{1,1} & \rho_{1,2} & \cdots \\ \rho_{0,0} & \rho_{0,1} & \rho_{0,2} & \cdots \end{array}}$$

be the intersection points of the vertical *grid lines* with $\mathcal{P}^D(a_1, a_2, b)$. Let $(i, j, I_{i,j}) \in \mathbf{Z}^3$ be the grid point *nearest* to $\rho_{i,j}$. If there are two nearest points, we choose the upper one. Formally, we have that the *discretization* of $\mathcal{P}^D(a_1, a_2, b)$ over $\mathbf{Z}^3_+$ is

$$I_{a_1,a_2,b} = \{(i, j, I_{i,j}) : i, j \geq 0, I_{i,j} = \lfloor a_1x_1 + a_2x_2 + b + \frac{1}{2} \rfloor\}.$$

It has a *slope vector* (a_1, a_2) and *intercept* b. Sometimes, the discretization of a 2D ray R will alternatively be denoted $discr(R)$.

Remark 8 *The plane $\mathcal{P}$ intersects the coordinate planes Ox_1x_3 and Ox_2x_3 in straight lines with equations $x_3 = a_1x_1 + b, x_2 = 0$ and $x_3 = a_2x_2 + b, x_1 = 0$, respectively. Considered in the plane Ox_1x_3, the first line has slope a_1, while the second has slope a_2. The slope vector of the plane has the slopes of these two lines as coordinates.*

Now we define a *digital 2D ray* $r_{a_1,a_2,b}$ with a *slope vector* (a_1, a_2) and *intercept* b, as follows[3]:

$$r_{a_1,a_2,b} = \boxed{\begin{array}{llll} \cdots & \cdots & \cdots & \cdots \\ r_{a_1,a_2,b}(2,0) & r_{a_1,a_2,b}(2,1) & r_{a_1,a_2,b}(2,2) & \cdots \\ r_{a_1,a_2,b}(1,0) & r_{a_1,a_2,b}(1,1) & r_{a_1,a_2,b}(1,2) & \cdots \\ r_{a_1,a_2,b}(0,0) & r_{a_1,a_2,b}(0,1) & r_{a_1,a_2,b}(0,2) & \cdots \end{array}},$$

where $r_{a_1,a_2,b}(i, j)$ are called *cell codes* and defined for $i, j \geq 0$, as follows:

We set $r_{a_1,a_2,b}(0, 0) = I_{0,0}$.
Defining the 0-th digitized row:

$$r_{a_1,a_2,b}(0, j + 1) = I_{0,j+1} - I_{0,j} = \begin{cases} 0, & \text{if } I_{0,j+1} = I_{0,j} \\ 1, & \text{if } I_{0,j+1} = I_{0,j} + 1 \end{cases},$$

Defining the 0-th digitized column:

$$r_{a_1,a_2,b}(i + 1, 0) = I_{i+1,0} - I_{i,0} = \begin{cases} 0, & \text{if } I_{i+1,0} = I_{i,0} \\ 1, & \text{if } I_{i+1,0} = I_{i,0} + 1 \end{cases},$$

[3] An alternative way to define digital 2D rays is briefly sketched in Section 8.2.

Defining the i-th digitized row:

$$r_{a_1,a_2,b}(i,j+1) = I_{i,j+1} - I_{i,j} = \begin{cases} 0, & \text{if } I_{i,j+1} = I_{i,j} \\ 1, & \text{if } I_{i,j+1} = I_{i,j} + 1 \end{cases},$$

Alternatively, we can digitize the array columnwisely.
Defining the i-th digitized column:

$$r_{a_1,a_2,b}(i+1,j) = I_{i+1,j} - I_{i,j} = \begin{cases} 0, & \text{if } I_{i+1,j} = I_{i,j} \\ 1, & \text{if } I_{i+1,j} = I_{i,j} + 1 \end{cases}.$$

Note that the 0-th row and the 0-th column are the same both in the rowwise and the columnwise digitizations. Code 0 can be interpreted as a horizontal rowwise/columnwise grid increment while 1 as a vertical rowwise/columnwise increment in the grid $\mathbf{N}^3$.

Remark 9 *Because of assumption (5), horizontal/vertical move from one integer point to another in the domain D can increase the z-coordinate by at most 1. Once the 0-th row or column is generated, one can build the rest of the array either rowwisely or columnwisely.*

$r_{a_1,a_2,b}$ is called *digitization* of the 2D ray $\mathcal{P}^D$. Sometimes, the digitization of a 2D ray R will alternatively be denoted by $digit(R)$.

Note that the two digitizations defined by rows and by columns *may not be identical*, although they both correspond to the same 2D ray discretization $I_{a_1,a_2,b}$. This is illustrated by the following example. (Further details are given in Example 2.)

Example 1 *Consider the 2D ray $R_{\frac{1}{2},\frac{1}{3}}$ determined by $x_3 = \frac{1}{2}x_1 + \frac{1}{3}x_2$, $x_1, x_2 \geq 0$. The x_3-values at the integer points (x_1, x_2) for the first 10 rows and columns are presented in the following table.*

$R_{\frac{1}{2},\frac{1}{3}}$:

.	.	.	.	.	.	.	.	.	.	.	.	.
10	$3\frac{1}{3}$	$3\frac{5}{6}$	$4\frac{1}{3}$	$4\frac{5}{6}$	$5\frac{1}{3}$	$5\frac{5}{6}$	$6\frac{1}{3}$	$6\frac{5}{6}$	$7\frac{1}{3}$	$7\frac{5}{6}$	$8\frac{1}{3}$	.
9	$\bullet 3$	$3\frac{1}{2}$	$\bullet 4$	$4\frac{1}{2}$	$\bullet 5$	$5\frac{1}{2}$	$\bullet 6$	$6\frac{1}{2}$	$\bullet 7$	$7\frac{1}{2}$	$\bullet 8$	.
8	$2\frac{2}{3}$	$3\frac{1}{6}$	$3\frac{2}{3}$	$4\frac{1}{6}$	$4\frac{2}{3}$	$5\frac{1}{6}$	$5\frac{2}{3}$	$6\frac{1}{6}$	$6\frac{2}{3}$	$7\frac{1}{6}$	$7\frac{2}{3}$	.
7	$2\frac{1}{3}$	$2\frac{5}{6}$	$3\frac{1}{3}$	$3\frac{5}{6}$	$4\frac{1}{3}$	$4\frac{5}{6}$	$5\frac{1}{3}$	$5\frac{5}{6}$	$6\frac{1}{3}$	$6\frac{5}{6}$	$7\frac{1}{3}$	.
6	$\bullet 2$	$2\frac{1}{2}$	$\bullet 3$	$3\frac{1}{2}$	$\bullet 4$	$4\frac{1}{2}$	$\bullet 5$	$5\frac{1}{2}$	$\bullet 6$	$6\frac{1}{2}$	$\bullet 7$	.
5	$1\frac{2}{3}$	$2\frac{1}{6}$	$2\frac{2}{3}$	$3\frac{1}{6}$	$3\frac{2}{3}$	$4\frac{1}{6}$	$4\frac{2}{3}$	$5\frac{1}{6}$	$5\frac{2}{3}$	$6\frac{1}{6}$	$6\frac{2}{3}$	.
4	$1\frac{1}{3}$	$1\frac{5}{6}$	$2\frac{1}{3}$	$2\frac{5}{6}$	$3\frac{1}{3}$	$3\frac{5}{6}$	$4\frac{1}{3}$	$4\frac{5}{6}$	$5\frac{1}{3}$	$5\frac{5}{6}$	$6\frac{1}{3}$	.
3	$\bullet 1$	$1\frac{1}{2}$	$\bullet 2$	$2\frac{1}{2}$	$\bullet 3$	$3\frac{1}{2}$	$\bullet 4$	$4\frac{1}{2}$	$\bullet 5$	$5\frac{1}{2}$	$\bullet 6$	.
2	$\frac{2}{3}$	$1\frac{1}{6}$	$1\frac{2}{3}$	$2\frac{1}{6}$	$2\frac{2}{3}$	$3\frac{1}{6}$	$3\frac{2}{3}$	$4\frac{1}{6}$	$4\frac{2}{3}$	$5\frac{1}{6}$	$5\frac{2}{3}$	.
1	$\frac{1}{3}$	$\frac{5}{6}$	$1\frac{1}{3}$	$1\frac{5}{6}$	$2\frac{1}{3}$	$2\frac{5}{6}$	$3\frac{1}{3}$	$3\frac{5}{6}$	$4\frac{1}{3}$	$4\frac{5}{6}$	$5\frac{1}{3}$	.
0	$\bullet 0$	$\frac{1}{2}$	$\bullet 1$	$1\frac{1}{2}$	$\bullet 2$	$2\frac{1}{2}$	$\bullet 3$	$3\frac{1}{2}$	$\bullet 4$	$4\frac{1}{2}$	$\bullet 5$	.
	0	1	2	3	4	5	6	7	8	9	10	.

The points with integer x_3 values are marked by bullet sign. It is visible that they form a lattice with a basis $b^1 = (2,0,1)$ and $b^2 = (0,3,1)$. The next table illustrates the corresponding discretization $discr(R_{\frac{1}{2},\frac{1}{3}})$ composed by the values $I_{i,j}$. Any two rectangular blocks are equivalent modulo an integer *number.*

$discr(R_{\frac{1}{2},\frac{1}{3}})$:

.	.	.	.	.	.	.	.	.	.	.	.	.
10	3	4	4	5	5	6	6	7	7	8	8	.
9	•3	4	•4	5	•5	6	•6	7	•7	8	•8	.
8	3	3	4	4	5	5	6	6	7	7	8	.
7	2	3	3	4	4	5	5	6	6	7	7	.
6	•2	3	•3	4	•4	5	•5	6	•6	7	•7	.
5	2	2	3	3	4	4	5	5	6	6	7	.
4	1	2	2	3	3	4	4	5	5	6	6	.
3	•1	2	•2	3	•3	4	•4	5	•5	6	•6	.
2	1	1	2	2	3	3	4	4	5	5	6	.
1	0	1	1	2	2	3	3	4	4	5	5	.
0	•0	1	•1	2	•2	3	•3	4	•4	5	•5	.
	0	1	2	3	4	5	6	7	8	9	10	.

The next table presents the columnwise digitization $digit_{col}(R_{\frac{1}{2},\frac{1}{3}})$ *of* $R_{\frac{1}{2},\frac{1}{3}}$*. From it, one can recover the discretization* $discr(R_{\frac{1}{2},\frac{1}{3}})$ *of* $R_{\frac{1}{2},\frac{1}{3}}$*. This can be done as follows. Let the* x_3 *value of a point* (i,j) *be sought. First move along the* 0*th row counting the number of 1s until reaching the* j*th column. Then move up along the* j*th column counting the number of 1s until reaching the element* (i,j)*. The overall number of 1s counted is the value of* x_3 *at the point* (i,j)*. Note, however, that the result may be different if we first move vertically rather than horizontally. For instance, in the latter case the value at the point* $(7,4)$ *is 5 (the correct value), while in the former it is 1 (incorrect).*

$digit_{col}(R_{\frac{1}{2},\frac{1}{3}})$:

.	.	.	.	.	.	.	.	.	.	.	.	.
10	0	0	0	0	0	0	0	0	0	0	0	.
9	•0	1	•0	1	•0	1	•0	1	•0	1	•0	.
8	1	0	1	0	1	0	1	0	1	0	1	.
7	0	0	0	0	0	0	0	0	0	0	0	.
6	•0	1	•0	1	•0	1	•0	1	•0	1	•0	.
5	1	0	1	0	1	0	1	0	1	0	1	.
4	0	0	0	0	0	0	0	0	0	0	0	.
3	•0	1	•0	1	•0	1	•0	1	•0	1	•0	.
2	1	0	1	0	1	0	1	0	1	0	1	.
1	0	0	0	0	0	0	0	0	0	0	0	.
0	•0	1	•0	1	•0	1	•0	1	•0	1	•0	.
	0	1	2	3	4	5	6	7	8	9	10	.

The next table presents the rowwise digitization $digit_{row}(R_{\frac{1}{2},\frac{1}{3}})$ *of* $R_{\frac{1}{2},\frac{1}{3}}$*. From it, one can recover the discretization* $discr(R_{\frac{1}{2},\frac{1}{3}})$*, moving first upward along the 0th column until reaching the* i*th row, and then to the right on* i*th row until reaching the element* (i,j)*. Although the two digitization are different as 0/1 arrays, they have identical periodicity structure and represent the same discrete plane. Note that the digital 2D rays* $digit_{col}(R_{\frac{1}{2},\frac{1}{3}})$ *and* $digit_{row}(R_{\frac{1}{2},\frac{1}{3}})$ *are both tiled by a* 2×3 *tile, as the horizontal size of the tile (*2*) and the vertical one (*3*) are precisely the denominators of the coefficients of* x_1 *and* x_2*. Note also that this tiling is identical to the lattice determined by the integer points of* $R_{\frac{1}{2},\frac{1}{3}}$*. It is so because the denominators* 2 *and* 3 *are relatively prime. We will see in Example 2 that* *if this is not the case, the tiling and the lattice may be essentially different.*

$digit_{row}(R_{\frac{1}{2},\frac{1}{3}})$:

.	.	.	.	.	.	.	.	.	.	.	.	.
10	0	1	0	1	0	1	0	1	0	1	0	.
9	•0	1	•0	1	•0	1	•0	1	•0	1	•0	.
8	1	0	1	0	1	0	1	0	1	0	1	.
7	0	1	0	1	0	1	0	1	0	1	0	.
6	•0	1	•0	1	•0	1	•0	1	•0	1	•0	.
5	1	0	1	0	1	0	1	0	1	0	1	.
4	0	1	0	1	0	1	0	1	0	1	0	.
3	•0	1	•0	1	•0	1	•0	1	•0	1	•0	.
2	1	0	1	0	1	0	1	0	1	0	1	.
1	0	1	0	1	0	1	0	1	0	1	0	.
0	•0	1	•0	1	•0	1	•0	1	•0	1	•0	.
	0	1	2	3	4	5	6	7	8	9	10	.

□

Example 2 *Consider the 2D ray $R_{\frac{1}{4},\frac{1}{6}}$ determined by $x_3 = \frac{1}{4}x_1 + \frac{1}{6}x_2,\ x_1, x_2 \geq 0$. The table presents the x_3-values at the integer points (x_1, x_2) for the first 10 rows and columns.*

$R_{\frac{1}{4},\frac{1}{6}}$:

.	.	.	.	.	.	.	.	.	.	.	.	.
10	1.66	1.91	2.16	2.41	2.66	2.91	3.16	3.41	3.66	3.91	4.16	.
9	1.5	1.75	•2	2.25	2.5	2.75	•3	3.25	3.5	3.75	•4	.
8	1.33	1.58	1.83	2.08	2.33	2.66	2.91	3.16	3.41	3.66	3.91	.
7	1.16	1.41	1.66	1.91	2.16	2.41	2.66	2.91	3.16	3.41	3.66	.
6	•1	1.25	1.5	1.75	•2	2.25	2.5	2.75	•3	3.25	3.5	.
5	0.83	1.08	1.33	1.58	1.83	2.08	2.33	2.58	2.83	3.08	3.33	.
4	0.66	0.91	1.16	1.41	1.66	1.91	2.16	2.41	2.66	2.91	3.16	.
3	0.5	0.75	•1	1.25	1.5	1.75	•2	2.25	2.5	2.75	•3	.
2	0.33	0.58	0.83	1.08	1.33	1.66	1.91	2.16	2.41	2.66	2.91	.
1	0.16	0.41	0.66	0.91	1.16	1.41	1.66	1.91	2.16	2.41	2.66	.
0	•0	0.25	0.5	0.75	•1	1.25	1.5	1.75	•2	2.25	2.5	.
	0	1	2	3	4	5	6	7	8	9	10	.

Similar to Example 1, the integer points are marked by bullet sign. They form a 2D lattice with a basis $b^1 = (4, 0, 1)$ and $b^2 = (2, 3, 1)$. The next table illustrates the corresponding discretization composed by *the values $I_{i,j}$. Any two rectangular blocks are equivalent modulo an integer number.*

$discr(R_{\frac{1}{4},\frac{1}{6}})$:

.	.	.	.	.	.	.	.	.	.	.	.	.
10	2	2	2	2	3	3	3	3	4	4	4	.
9	2	2	$\bullet$2	2	3	3	$\bullet$3	3	4	4	$\bullet$4	.
8	1	2	2	2	2	3	3	3	3	4	4	.
7	1	1	2	2	2	2	3	3	3	3	4	.
6	$\bullet$1	1	2	2	$\bullet$2	2	3	3	$\bullet$3	3	4	.
5	1	1	1	2	2	2	2	3	3	3	3	.
4	1	1	1	1	2	2	2	2	3	3	3	.
3	1	1	$\bullet$1	1	2	2	$\bullet$2	2	3	3	$\bullet$3	.
2	0	1	1	1	1	2	2	2	2	3	3	.
1	0	0	1	1	1	1	2	2	2	2	3	.
0	$\bullet$0	0	1	1	$\bullet$1	1	2	2	$\bullet$2	2	3	.
	0	1	2	3	4	5	6	7	8	9	10	.

The next table presents the columnwise digitization of $R_{\frac{1}{4},\frac{1}{6}}$. From it, one can reconstruct the discretization of $R_{\frac{1}{4},\frac{1}{6}}$.

$digit_{col}(R_{\frac{1}{4},\frac{1}{6}})$:

.	.	.	.	.	.	.	.	.	.	.	.	.
10	0	0	0	0	0	0	0	0	0	0	0	.
9	1	0	$\bullet$0	0	1	0	$\bullet$0	0	1	0	$\bullet$0	.
8	0	1	0	0	0	1	0	0	0	1	0	.
7	0	0	0	0	0	0	0	0	0	0	0	.
6	$\bullet$0	0	1	0	$\bullet$0	0	1	0	$\bullet$0	0	1	.
5	0	0	0	1	0	0	0	1	0	0	0	.
4	0	0	0	0	0	0	0	0	0	0	0	.
3	1	0	$\bullet$0	0	1	0	$\bullet$0	0	1	0	$\bullet$0	.
2	0	1	0	0	0	1	0	0	0	1	0	.
1	0	0	0	0	0	0	0	0	0	0	0	.
0	$\bullet$0	0	1	0	$\bullet$0	0	1	0	$\bullet$0	0	1	.
	0	1	2	3	4	5	6	7	8	9	10	.

Note that the area of the rectangular tiles is equal to the product of the denominators of the coefficients of x_1 and x_2 (i.e., $4 \times 6 = 24$). The corresponding lattice has basis vectors $(4,0)$ and $(0,6)$, which are the projections on Ox_1x_2 of the vectors $(4,0,1)$ and $(0,6,1)$. Note, however, that this basis is not "minimal." An actual basis is, e.g., the one consisting of the vectors $\bar{b}^1 = (0,4)$ and $\bar{b}^2 = (2,3)$, which are the projections on Ox_1x_2 of the vectors $b^1 = (4,0,1)$ and $b^2 = (2,3,1)$. The area of the corresponding parallelogram tile is the least common multiple of the denominators of the coefficients of x_1 and x_2 (i.e., $lcm(4,6) = 12$). See Figure 6. Note that the rectangular tiling of Example 1 provided a basis for the considered lattice, since the denominators 2 and 3 of the two coefficients were relatively prime. □

4.2 Other definitions and properties

Arithmetic plane is a set of voxels $P(\alpha_1, \alpha_2, \alpha_3, \mu, \omega) = \{(x_1, x_2, x_3) \in \mathbf{Z}^3 | -\lfloor \frac{\omega}{2} \rfloor \leq \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \mu < \lfloor \frac{\omega}{2} \rfloor\}$, where $\omega \in \mathbf{N}$. ω is the *arithmetic thickness* of the plane and μ is its *internal translation constant.* An arithmetic plane $P(\alpha_1, \alpha_2, \alpha_3, \mu, \omega)$ is *naive* if $\omega = \max(|\alpha_1|, |\alpha_2|, |\alpha_3|)$, and

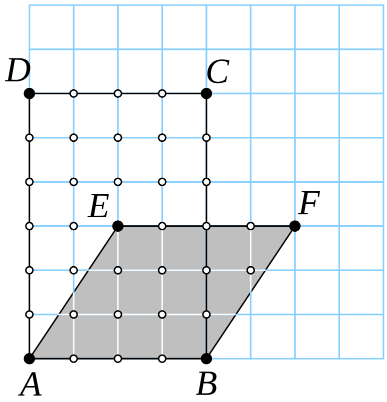


Figure 6: Rectangular tile with area 24 and a parallelogram tile with area 12. The latter corresponds to a lattice basis.

standard if $\omega = |\alpha_1| + |\alpha_2| + |\alpha_3|$. The naive plane which is centered about the continuous plane $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \mu = 0$ is $P(\alpha_1, \alpha_2, \alpha_3, \mu + \lfloor \frac{\omega}{2} \rfloor, \omega)$, $\omega = \max(|\alpha_1|, |\alpha_2|, |\alpha_3|)$. Such an arithmetic plane is called *regular*. In what follows, we consider arithmetic planes of this type. We have the following theorem.

Theorem 11 *(Reveilles 1991 [34]) arithmetic plane $P(\alpha_1, \alpha_2, \alpha_3, \beta, \max(|\alpha_1|, |\alpha_2|, |\alpha_3|))$ coincides with a set of grid points assigned to a discrete plane, and vice versa: for any discrete plane P there are numbers $\alpha_1, \alpha_2, \alpha_3$ and β such that the set of points of P is $P(\alpha_1, \alpha_2, \alpha_3, \beta, \max(|\alpha_1|, |\alpha_2|, |\alpha_3|))$.*

Corollary 1 *The points corresponding to a digital plane lie between or on two parallel planes having a distance less than one, measured in the x_3-axis direction.*

We also have the following fact which follows from a more general result about discrete hyperplanes.

Theorem 12 *(Andres and Acharya 1997 [4]) Let $P = P(\alpha_1, \alpha_2, \alpha_3, \beta, \omega)$ be an arithmetic plane. Assume without loss of generality that $|\alpha_1| \le |\alpha_2| \le |\alpha_3|$. Then,*

- *if $\omega < |\alpha_3|$, the plane has 2-tunnels;*
- *if $|\alpha_2| + |\alpha_3| \le \omega \le |\alpha_1| + |\alpha_2| + |\alpha_3|$, then the plane has 0-tunnels and is 1-separating;*
- *if $|\alpha_3| \le \omega \le |\alpha_2| + |\alpha_3|$, then the plane has 1-tunnels and is 2-separating;*
- *if $\omega \ge |\alpha_1| + |\alpha_2| + |\alpha_3|$, then the plane has no tunnels.*

We have the following theorem.

Theorem 13 *(analog of Theorem 1) A discretization of a plane $\mathcal{P}$ is 2-minimal in $\mathbf{Z}^3_+$.*

Proof By Theorem 11, the discretization of $\mathcal{P}$ is a naive plane for some parameters $\alpha_1, \alpha_2, \alpha_3, \beta$ and thickness $\omega = \max(|\alpha_1|, |\alpha_2|, |\alpha_3|)$. Then from Theorem 12 (3rd case) it follows that the discretization $I_{a_1,a_2,b}$ of $\mathcal{P}$ is 1-separating in $\mathbf{Z}^3_+$. By construction, to any integer point $(x_1, x_2) \in Ox_1x_2$ corresponds exactly one integer point $(x_1, x_2, x_3) \in \mathbf{Z}^3_+$ which belongs to $I_{a_1,a_2,b}$. Let us remove an integer point $Q = (x_1, x_2, x_3)$ from $I_{a_1,a_2,b}$. Then clearly $I_{a_1,a_2,b} - Q$ will contain a 2-tunnel, since a 2-connected path can penetrate through $I_{a_1,a_2,b} - Q$ and connect points which before removing of Q have been 2-separated by $I_{a_1,a_2,b}$. $\square$

Corollary 2 *A discretization of a 2D ray is 2-minimal in* $\mathbf{Z}_+^3$.

Remark 10 *The digital 2D ray* $r_{a_1,a_2,b}$ *is said to be* generated *by the 2D ray* $x_3 = a_1x_1 + a_2x_2 + b$.

If we have two 2D rays $r_{a_1,a_2,b}$ *and* $r_{a_1,a_2,b'}$ *and if* $b - b'$ *is integer, then clearly* $r_{a_1,a_2,b} = r_{a_1,a_2,b'}$. *Thus, without loss of generality we may assume that the intercepts are limited to* $0 \le b \le 1$.

We also have

$$r_{0,0,b} = \boxed{\begin{array}{cccc} \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \end{array}} \text{ and } r_{1,1,b} = \boxed{\begin{array}{cccc} \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots \\ 1 & 1 & 1 & \dots \\ 1 & 1 & 1 & \dots \end{array}}.$$

We have the following theorem.

Theorem 14 *(Barneva, Brimkov and Nehlig 1998 [7]) Let* $\mathcal{P} : a_1x_1 + a_2x_2 + a_3x_3 = b$ *be a continuous plane and* $P = P(a_1, a_2, a_3, b + \lfloor \frac{\omega}{2} \rfloor, \omega)$ *be the corresponding regular plane. Let without loss of generality* $0 \le a_1 \le a_2 \le a_3$ *and* $a_3 \ne 0$. *Let* $(x_1, x_2) \in \mathbf{Z}$. *Then* $(x_1, x_2, \bar{x}_3) \in \mathcal{P}$ *and* $(x_1, x_2, x_3) \in P$ *implies* $|\bar{x}_3 - x_3| \le \frac{1}{2}$.

In other words, the points of a plane discretization are within $\frac{1}{2}$ vertical distance from the continuous plane.

Let (a_1, a_2) be the slope-vector of a plane discretization $I_{a_1,a_2,b}$. $I_{a_1,a_2,b}$ (as well as the corresponding digital 2D ray $r_{a_1,a_2,b}$ and the Euclidean plane $x_3 = a_1x_1 + a_2x_2 + b$) is called *rational* if *both* a_1 and a_2 are rational numbers. Otherwise, it is called *irrational*.

We also have the following analog of the Bruckstein theorem (Theorem 2).

Theorem 15 *For irrational plane with a slope vector* (a_1, a_2), *the plane discretization* $I_{a_1,a_2,b}$ *uniquely determines both* (a_1, a_2) *and* b. *For rational* (a_1, a_2), $I_{a_1,a_2,b}$ *uniquely determines* (a_1, a_2), *and* b *is determined up to an interval.*

The proof is similar to the one of Theorem 2.

Remark 11 *Let* $\mathcal{P}^D(a_1, a_2, b) = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 = a_1x_1 + a_2x_2 + b, (x_1, x_2) \in D = \{(x_1, x_2) : 0 \le x_1, x_2 \le \infty\}\}$ *be a 2D ray. Let* (a_1, a_2) *be a pair of rational numbers* $\frac{p_1}{q_1}$ *and* $\frac{p_2}{q_2}$ *which are irreducible fractions (i.e., gcd*$(p_1, q_1) = 1$ *and gcd*$(p_2, q_2) = 1$*). According to our assumption,* $0 \le \frac{p_1}{q_1} \le 1$, *i.e.,* $p_1 < q_1$ *and* $0 \le \frac{p_2}{q_2} \le 1$, *i.e.,* $p_2 < q_2$. *Then the plane equation* $x_3 = a_1x_1 + a_2x_2 + b$ *can be written as*

$$-p_1q_2x_1 - p_2q_1x_2 + q_1q_2x_3 = q_1q_2b.$$

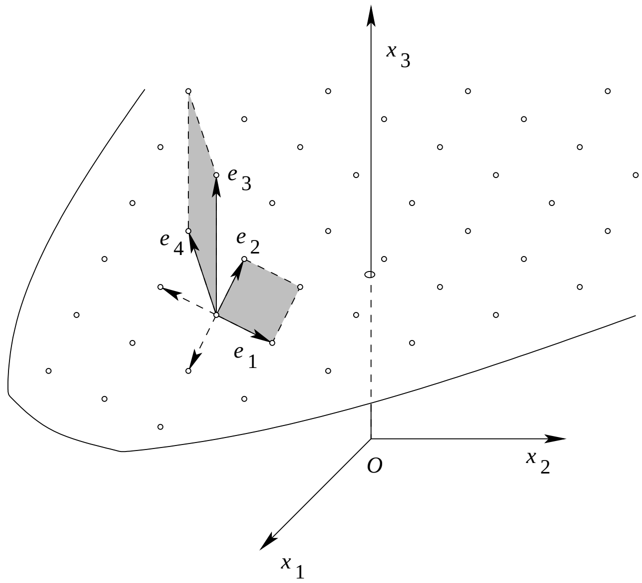
Consider the 2D ray determined by the plane

$$\mathcal{P}^{\lfloor\rfloor}_{a_1,a_2,b} : -p_1q_2x_1 - p_2q_1x_2 + q_1q_2x_3 = \lfloor q_1q_2b \rfloor.$$

Its discretization and digitization are the same as the one of $\mathcal{P}$. *Moreover, it is easy to prove that gcd*$(p_1q_2, p_2q_1, q_1q_2) = 1$, *hence* $P^{\lfloor\rfloor}_{a_1,a_2,b}$ *contains integer points. One can realize that all the planes with the same discretization/digitization are*

$$\mathcal{P}_{a_1,a_2,b} : -p_1q_2x_1 - p_2q_1x_2 + q_1q_2x_3 = b'$$

where $\lfloor q_1q_2b \rfloor - \frac{1}{2} \le b' < \lfloor q_1q_2b \rfloor + \frac{1}{2}$ *if* $\max(|\alpha_1|, |\alpha_2|, |\alpha_3|)$ *is an odd number, and* $\lfloor q_1q_2b \rfloor \le b' < \lfloor q_1q_2b \rfloor + 1$, *if* $\max(|\alpha_1|, |\alpha_2|, |\alpha_3|)$ *is an even number.*



y

Figure 7: Illustration to the proof of Theorem 16. The 2D integer lattice Λ in the plane P and some of its bases.

4.3 Periodicity properties of 2D digital rays

In this section we study the periodicity of 2D digital rays. We consider separately the cases of rational and irrational rays.

4.3.1 Rational digital 2D rays

Consider a rational 2D ray $R = \mathcal{P}^D(a_1, a_2, b) = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 = a_1x_1 + a_2x_2 + b, (x_1, x_2) \in D = \{(x_1, x_2) : 0 \leq x_1, x_2 \leq \infty\}\}$, its discretization $discr(R) = I_{a_1,a_2,b}$, and the corresponding digital 2D ray $digit(R) = r_{a_1,a_2,b}$. The coefficients a_1, a_2, b are rational numbers. Without loss of generality we may assume that they are integer and that R contains integer points. With a reference to Section 2.3, these integer points belong to a 2-dimensional integer lattice $\Lambda \in \mathbf{Z}^3$ in the plane $P = \mathcal{P}(a_1, a_2, b) = \{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_3 = a_1x_1 + a_2x_2 + b\}$.

Consider a basis B for Λ, i.e., a linearly independent system of integer vectors $B = \{e^1, e^2\}$, such that

$$\{x : a_1x_1 + a_1x_2 + a_1x_3 = b, x \in \mathbf{Z}^n\} = \{e^0 + \lambda_1e^1 + \lambda_2e^2, \lambda_1, \lambda_2 \in \mathbf{Z}\},$$

where e^0 is an arbitrary integer point in P. Note that Λ has different bases. For instance, in Figure 7, any one of the pairs of vectors $B_1 = \{e^1, e^2\}$ $B_2 = \{-e^1, e^2\}$, $B_3 = \{e^1, -e^2\}$, and $B_4 = \{-e^1, -e^2\}$ constitutes a basis. Geometrically, for a given basis $\{e^1, e^2\}$, the whole plane P is partitioned into parallelograms spanned on the basis vectors. (See Figure 7.) Any two parallelograms are equivalent up to translation. Every lattice point can be obtained from any other lattice point by consecutive passes along the vectors $e^1, e^2, -e^1$ or $-e^2$. The discretization $discr(P)$ and the digitization $digit(P)$ will be periodic as well. $discr(P)$ has period vectors e^1 and e^2, while $digit(P)$ has as period vectors the *projections* of e^1 and e^2 on the coordinate plane Ox_1x_2.

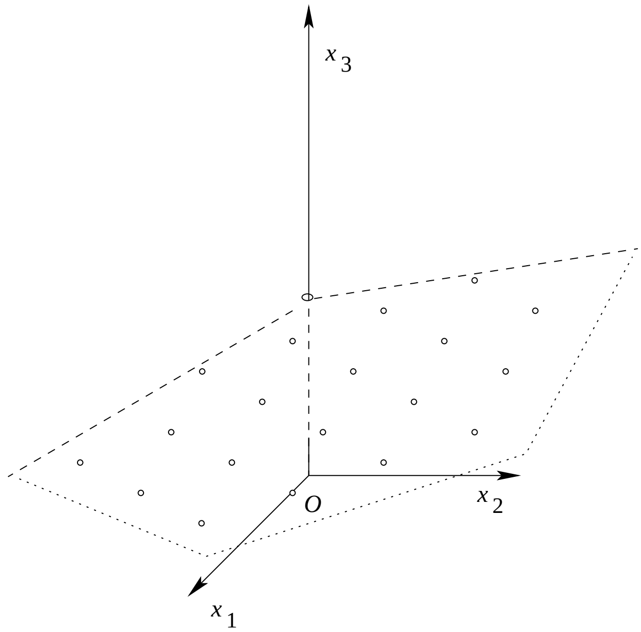


Figure 8: The points of the integer lattice Λ of the 2D ray R.

If we choose an integer point $e^0 \in \mathcal{P}^D(a_1, a_2, b)$, we can obtain identical periodicity picture of R, $discr(R)$ and $r_{a_1,a_2,b} = digit(R)$. (See Figure 8.) In view of the above discussion, one can consider $digit(P)$ and $digit(R)$ as tiled by a tile with a shape of parallelogram formed by the vectors of a given basis.

It follows from Proposition 3 that $r_{a_1,a_2,b}$ and $r^D_{a_1,a_2,b}$ are also tiled by a rectangular tile of suitable dimension. (See Figure 9 and Examples 1 and 2.) We have seen that the lattice of the integer points of a plane or a 2D ray can be generated by different bases, which feature different parallelogram partitions (Figure 7). Nevertheless, it is a well-known fact from lattice theory that the lattice cells have the same area for all possible bases. It equals the value $\max(|\alpha_1|, |\alpha_2|, |\alpha_3|)$, where $\alpha_1, \alpha_2, \alpha_3$ are the coefficients in the plane representation (4) with $gcd(\alpha_1, \alpha_2, \alpha_3) = 1$.

The above discussion leads us to the following 2D version of Brons theorem (Theorem 3) for the case of rational 2D rays.

Theorem 16 *Rational digital 2D rays are lattice-periodic. For a given basis of the lattice, the corresponding lattice cells are parallelograms. For all possible bases, the lattice cells have the same area which equals the absolute value of the maximal coefficient in the plane representation* $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \beta$ *with* $gcd(\alpha_1, \alpha_2, \alpha_3) = 1$.

Remark 12 *We have seen that in the case of digital plane or 2D ray, always lattice-periodicity holds. Note, however, that in case of a finite array, one may have radiant-periodicity (see Figure 10a). Note also that even if a digitized domain is infinite, depending on its particular shape, it may have periodicity structure different than lattice-periodicity (see Figure 10b).*

Remark 13 *It is also clear that:*

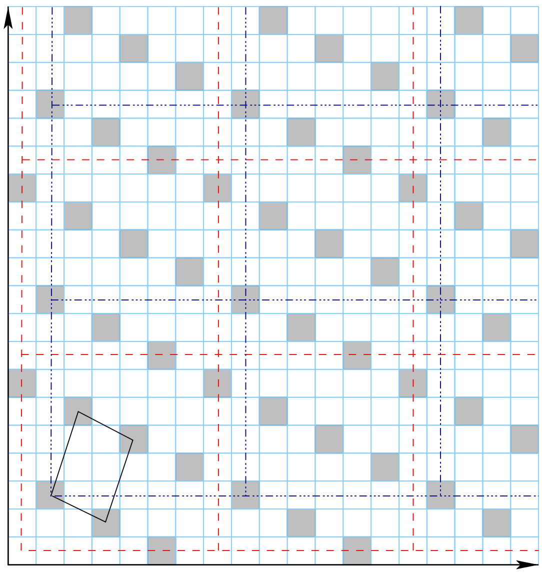


Figure 9: If a lattice-periodic array is enough large, then it can be tiled by a rectangular tile.

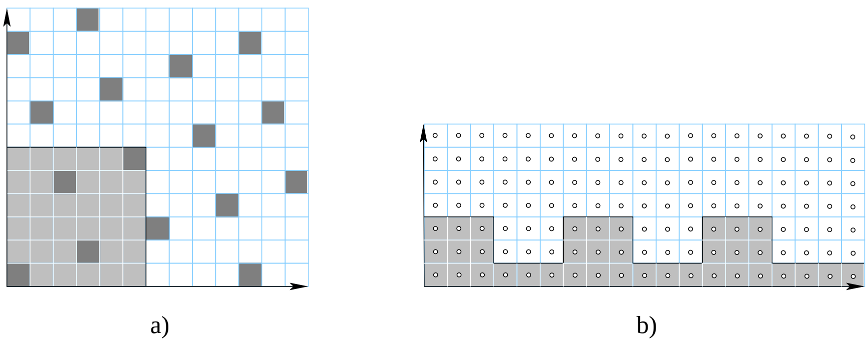


Figure 10: a) A small portion of a lattice-periodic array may be radiant-periodic. b) An infinite subdomain of a lattice-periodic array may not be lattice-periodic.

1. *All digital planes with the same rational slope vector are equivalent up to translation with an appropriate translation vector.*

2. *For any two digital 2D rays $r'_{a_1,a_2,b'}$ and $r''_{a_1,a_2,b''}$ with the same rational slope vector (a_1, a_2), there exist vectors v' and v'', such that $v' + r'_{a_1,a_2,b'} = r''_{a_1,a_2,b'}$ and $v'' + r''_{a_1,a_2,b'} = r'_{a_1,a_2,b'}$.*

Symmetry of rational digital 2D rays and planes

The study of the digital plane structure reveals certain symmetry properties of digital 2D rays and planes.

Fact 1 *1. Let $digit(P)$ be the digitization of a plane P with a rational slope vector. Then every point of $digit(P)$ which corresponds to an integer point of P is a center of symmetry for $digit(P)$.*

2. *Let B be a symmetric set of points of a plane digitization or a ray digitization. (In case of a plane, B may be finite or infinite, while in case of a 2D ray, B is always finite.) Then B is a 2D palindrome.*

3. *Any 2D parallelogram tile of a plane digitization or a 2D ray digitization is a 2D palindrome whose center of symmetry is the center of the period.*

4.3.2 Irrational digital 2D rays

We start with the note that, unlike an irrational line digitization which is always aperiodic, a plane digitization may be line-periodic. As a background for our further considerations we start by listing some useful facts.

Fact 2 *(A) Let P be an irrational plane. The possible periodicity structure as well as the possible symmetry of $digit(P)$ depends on the integer (or rational) points which P contains. There are several possibilities:*

1. *P contains* no integer or rational *points.*

2. *P contains a* single integer or rational *point.*

3. *P contains* infinitely many *equidistant* integer *points which belong to a straight line on P, or P is parallel to a line which contains infinitely many equidistant integer points. (In the latter case, P contains* infinitely many rational *points, which belong to parallel straight lines on P and form dense sets on these lines.)*

(B) A 2D ray R may feature the following possibilities:

1. *R contains* no integer or rational *points.*

2. *R contains a* single integer or rational *point.*

3. *R contains* infinitely many *equidistant* integer *points which belong to a ray on R, or R is parallel to a ray which contains infinitely many equidistant integer points. (In the latter case, R contains* infinitely many rational *points, which belong to parallel rays on R and form dense sets on these rays. See Figure 11a.)*

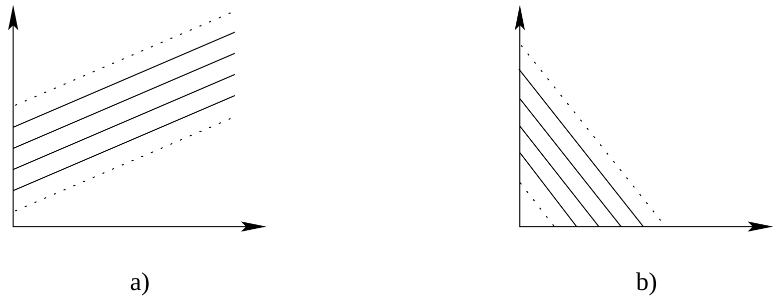


Figure 11: a) Sample orientation of the rational rays, Fact 2 (B-3). b) Sample orientation of the rational segments, Fact 2 (B-4).

4. R contains finitely many *equidistant* integer *points which belong to a straight line segment on R, or R is parallel to a segment which contains finitely many equidistant integer points. (In the latter case, R contains* infinitely many rational *points which belong to these straight line segments and form dense sets on them. See Figure 11b.)*

The following are other plain facts.

Fact 3 *A line/plane contains* exactly one *integer/rational point if and only if it is irrational.*

Fact 4 *If a plane/2D ray Q contains* exactly one *integer/rational point, then there is no line parallel to Q and containing infinitely many equidistant integer points.*

Fact 5 *A plane/2D ray Q contains* infinitely many *equidistant integer points which belong to a straight line on Q, and no other integer points, if and only if Q is irrational.*

Fact 6 *A plane/2D ray Q contains* infinitely many rational *points which belong to (and are dense on) a straight line/straight line segment on Q, and no other rational points, if and only if Q is irrational.*

The presence of integer and/or rational points on a plane depends on its coefficients (the slope vector). We list some related facts.

Fact 7 *A plane with equation* $x_3 = a_1x_1+a_2x_2+b$ *may contain a line with infinitely many equidistant integer points on it, even if all coefficients* a_1, a_2 *and* b *are irrational numbers. Consider, for instance, the plane with equation*

$$\sqrt{2}x_1 + \sqrt{3}x_2 + (\sqrt{2}+\sqrt{3})x_3 = \sqrt{2}+2\sqrt{3},$$

or, equivalently,

$$x_3 = -\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}}x_1 - \frac{\sqrt{3}}{\sqrt{2}+\sqrt{3}}x_2 + \frac{\sqrt{2}+2\sqrt{3}}{\sqrt{2}+\sqrt{3}}.$$

This plane contains the integer points $(0,1,1)$ *and* $(1,2,0)$, *hence it contains infinitely many equidistant integer points.*

Fact 8 *Let a plane P have equation* $x_3 = a_1x_1 + a_2x_2 + b$, *such that one of the coefficients (say,* a_2*) is irrational, while the other is integer (or rational).*

1. *If b is integer or rational, then the plane contains a line with infinitely many equidistant integer points on it. More precisely, this is the line $x_3 = a_1x_1+b$, $x_2 = 0$. (If $a_1 = p_1/q_1$, $gcd(p_1, q_1) = 1$, and $b = p/q$, $gcd(p, q) = 1$, then the line equation is equivalent to $qq_1x_3 = p_1x_1 + p$. Clearly, $gcd(qq_1, p_1) = 1$ and the equation has infinitely many integer solutions).*

2. *If b is an irrational number such that it is not a rational multiple of a_1, then the plane contains no integer or rational points. Nevertheless, P is parallel to a line containing infinitely many equidistant integer points.*

Fact 9 *Let a plane have equation $x_3 = a_1x_1 + a_2x_2 + b$, such that both coefficients a_1 and a_2 are rational (i.e., the plane has a rational slope vector), while b is irrational. Then the plane contains no integer or rational points. Nevertheless, P is parallel to two lines with linearly independent directions, each of them containing infinitely many equidistant integer points. In view of Theorem 16 and the related discussion, the digitization of P is lattice-periodic.*

Fact 10 *Let a plane P have equation $x_3 = a_1x_1 + a_2x_2 + b$, such that both coefficients a_1 and a_2 are irrational, while b is rational. Then the plane contains a single integer or rational point.*

In the following remark we describe some constructions and fix some denotations to be used in the sequel.

Remark 14 *(Constructions and denotations) Let $R : x_3 = a_1x_1 + a_2x_2 + b, x_1, x_2 \geq 0$ be a 2D ray, $discr(R)$ its discretization and $digit(R)$ its digitization.*

The row indexes $i = 0, 1, 2, \ldots$ run on the integer points of x_2 axis, while the column indexes $j = 0, 1, 2, \ldots$ run on the integer points of x_1 axis.

We denote by $discr(R)^i$ the ith row of $discr(R)$ and by $discr(R)^j$ the jth column of $discr(R)$.

Also, let us denote by $digit_{row}(R)^i$ a generic ith row of $digit_{row}(R)$ and by $digit_{row}(R)^j$ a generic jth column of $digit_{row}(R)$. Analogously, denote by $digit_{col}(R)^i$ a generic ith row of $digit_{col}(R)$ and by $digit_{col}(R)^j$ a generic jth column of $digit_{col}(R)$. These are sequences of 0s and 1s.

Consider $discr(R)^i$. Consider the two vertical planes P_1 and P_2 which bound $discr(R)^i$. They intersect R in two parallel lines g_1 and g_2. Consider the ray g on R which is between g_1 and g_2 and is in equal distance from both of them. Let g' be the orthogonal projection of g on Ox_1x_3. Let $discr(g')$ denote the discretization of g' on the integer points of Ox_1x_3, and $digit(g')$ the corresponding digitization on the integer points of Ox_1x_3.

In an analogous way we define a discretization $discr(h')$ and digitization $digit(h')$ of a ray h corresponding to a column $discr(R)^j$. Such a correspondence holds for any row or column of $discr(R)$.

After this preparation, we are able to prove the following theorem.

Theorem 17 *Irrational digital 2D rays are either aperiodic or line-periodic.*

Proof Consider a plane P with an irrational slope vector (a_1, a_2) and the corresponding 2D ray R. We will consider three different cases.

1. *Both a_1 and a_2 are irrational and the plane $P : x_3 = a_1x_1 + a_2x_2 + b$ does not contain more than one integer or rational point and is not parallel to a line containing infinitely many equidistant integer points.*

   Consider $discr(R)^i$, g' and $digit(g')$ defined in Remark 14. Let v be an arbitrary voxel of $discr(R)$. This means that the vertical distance from the center $c(v)$ of v to R, is $\leq \frac{1}{2}$. Clearly,

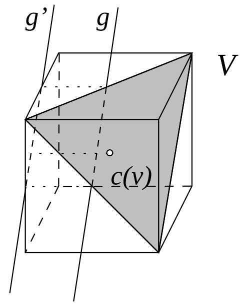


Figure 12: Illustration to the proof of Theorem 16. g' is the projection of g on the facet of v which is parallel to the plane Ox_1x_3.

equality is possible only if R passes through the center of the square which is an upper 2D-facet of v. (See Figure 12.) According to our assumption, this may happen with either *no one* voxels of $discr(R)$ (Case (a)) or with *exactly one* voxel of $discr(R)$ (Case (b)). In the former case, the vertical distance from $c(v)$ to R is strictly less than $\frac{1}{2}$. In the latter case, according to the definition of $discr(R)$, $c(v)$ is above R. Consider now the projections v' and $c(v)'$ of v and $c(v)$, respectively, on Ox_1x_3 (v' being a unit square and $c(v)'$ an integer point in Ox_1x_3). We investigate their position with respect to the line g'. In view of the definition of g', we have in Case (a) that the vertical distance between g' and $c(v)'$ is strictly less than $\frac{1}{2}$. In Case (b), we have that g' is above $c(v)'$ and that the vertical distance between them is equal to $\frac{1}{2}$. Hence, the projections of all voxels of the row $discr(R)^i$ form a discrete line, which is exactly the discrete line $discr(g')$. (According to the Bruckstein theorem (Theorem 2), a continuous straight line is uniquely determined by a given discretization, if the line has an irrational slope, which is the case.)

Since the slope vector (a_1, a_2) of R has *two* irrational components and the plane P contains *at most* one integer or rational point, the same applies to the lines g and g'.

According to Brons theorem (Theorem 3), the digitization $digit(g')$ of g' is aperiodic.

Consider now the digitization of the ith row of $discr(R)$. By definition, it is the ith row $digit_{row}(R)^i$ of $digit_{row}(R)$. We observe that $digit_{row}(R)^i$ is identical to $digit(g')$, with a possible difference in the first element (since building $digit_{row}(R)^i$ may start either from 0 or from 1, depending on the corresponding element in the 0th column). Thus we obtain that the ith row $digit_{row}(R)^i$ of $digit_{row}(R)$ is aperiodic.

In a similar way one can show that for any j, the jth column $digit_{col}(R)^j$ of $digit_{col}(R)$ is aperiodic.

The same applies to any column of $digit_{row}(R)$. If we assume that there is a column $digit_{row}(R)^j$ of $digit_{row}(R)$ that is periodic, it will follow that the corresponding jth column $digit_{col}(R)^j$ of $digit_{col}(R)$ is periodic with the same periodicity as $digit_{row}(R)^j$ – a contradiction. (In fact, this would mean that R is parallel to a line which contains equidistant integer points, and whose orthogonal projection on Ox_1x_2 is parallel to the Ox_2 axis. Then $digit_{row}(R)$ would be "vertically periodic," i.e., all its columns would be periodic, including the 0th one. However, $digit_{row}(R)$ and $digit_{col}(R)$ have the same 0th column, and we have seen that the one of $digit_{col}(R)$ is aperiodic, which is a contradiction.)

2. *The coefficient a_1 is irrational, while a_2 is rational.*

   Here the rowwise situation is identical to the previous case, i.e., all digital rows $digit_{row}(R)^i$ are aperiodic. Instead, all the continuous lines h associated with the columns of $digit_{col}(R)$ have integer slope a_2, and according to Brons theorem (Theorem 3), they are periodic, with the same period length. Furthermore, from Lunnon-Pleasant theorem (Theorem 5), we have that all collumns' digitizations are equivalent up to translation. Then, by arguments similar to those used in the case when both a_1 and a_2 were irrational, it follows that the columns of $digit_{row}(R)$ are periodic, as well.

3. *Both a_1 and a_2 are irrational and the plane P contains more than one integer point or is parallel to a line containing infinitely many equidistant integer points.*

   Here the rowwise and columnwise structures are identical to the first case, i.e., all digital rows $digit_{row}(R)^i$ and columns $digit_{row}(R)^j$ of $digit_{row}(R)$ are aperiodic.

   We have that R contains or is parallel to a ray or a segment of a line L with integer coefficients containing infinitely many equidistant integer points (or finitely many, in the case of segment). We have also that L is not parallel to a coordinate axis. Thus, its discretization does not constitute a row or a column of $discr(R)$. If R does not contain ray/segment with integer points but is parallel to a line containing such a ray/segment, then R is parallel to infinitely many lines of this type which may be arbitrarily close to R.

   W.l.o.g., consider, for the sake of definiteness, the case when R contains integer points. We discretize L as follows. Let $L_{x_1x_2}$ be the orthogonal projection of L on the plane Ox_1x_2. Let $discr(L_{x_1x_2})$ be the discretization of L (in other words, the *naive* line associated to it). Because of condition (5), for each integer point of $L_{x_1x_2}$ there exists exactly one integer point in $discr(R)$. Denote the set of all these points by $discr(L)$. Note that, as shown, e.g., in [7], $discr(L)$ may be disconnected (see also Section 6). Clearly, the set of voxels $discr(L)$ is periodic, as the period corresponds to the segment between two integer points on L.

   It is easy to realize that the discretizations of all straight lines on R which are parallel to L, are equal up to translation. So that, they all are periodic with the same period length as the one of L. Then the same applies to the corresponding digitizations of L.

We are now ready to conclude the proof of the theorem. We reason about $digit_{row}(R)$; the considerations about $digit_{col}(R)$ are analogous.

The above discussion reveals that:

(a) In Cases 2 and 3, the digital 2D ray $digit_{row}(R)$ is line-periodic, with a horizontal or vertical period in Case 2 and a non-horizontal non-vertical period in Case 3.

(b) In Case 1, $digit_{row}(R)$ does not have a period. To prove this, assume the opposite, i.e., that $digit_{row}(R)$ has a periodicity vector w. We suppose that w is not parallel to neither of the coordinate axes (the opposite case is more trivial).

   Let l be the ray in Ox_1x_2 determined by w. Since it is determined by an integer periodicity vector, it has integer coefficients. Let $discr(l)$ be the discretization of l in the integer points of Ox_1x_2. Assign 0s or 1s to the pixels of $discr(l)$ in such a way that they are the corresponding digits in $digit_{row}(R)$. That is, $discr(l)$ is a digital ray in Ox_1x_2, which is periodic with a period determined by w. Consider the set of voxels $L \subseteq digit_{row}(R)$ which correspond to $discr(l)$. L appears to be discretization of some ray $\bar{l}$ on P, such that l is the orthogonal projection of $\bar{l}$ on Ox_1x_2. (As mentioned, $discr(l)$ may be a disconnected set.) Because of this correspondence between $discr(l)$ and L, one can conclude that L is periodic with a period vector $\bar{w}$, such that w is the projection of $\bar{w}$ on Ox_1x_2. Since the points of L belong to $digit_{row}(R)$, this is possible only if R is parallel to a line with an integer slope, containing integer points. This contradicts the assumption.

(c) In all the three cases $digit_{row}(R)$ is not lattice-periodic, i.e., it does not have two linearly independent period vectors. To prove this, assume the opposite, i.e., that $digit_{row}(R)$ is lattice-periodic with linearly independent periodicity vectors w_1 and w_1. Since $digit_{row}(R)$ is infinite, it satisfies the conditions of Proposition 6. Hence, $digit_{row}(R)$ is a 2D repetition. Then its rows and columns will be periodic as well. By arguments similar to those used in the case of rational slope vector, one can conclude that for every row (column), the corresponding central line g (h) must be rational, which is a contradiction.

This completes the proof of the theorem. $\square$

5 Sturmian planes and 2D rays

5.1 Definitions

Let $r = r_{a_1,a_2,b}$ be a digital 2D ray. Let $k, l \geq 0$ be integers. We call a *(k, l)-suffix of* r the sub-array of r determined by its rows and columns with indexes greater than or equal to k and l, respectively. *k, l-prefix* of r is determined by the rows and columns with indexes not greater than k and l, respectively. Digital 2D ray r is called *ultimately periodic* if there are integers $k, l \geq 0$ such that the (k, l)-suffix of r has a period vector. r is *uniformly recurrent* if for every integer $n > 0$ there is an integer $N > 0$ such that every square factor of size $N \times N$ contains every square factor of size $n \times n$.

Given a 2D array A (finite or infinite), we define *complexity function* $P_A(m, n)$ of A as the number of different $(m \times n)$-factors of A. In particular, we have $P_A(0, 0) = 1$ (the empty word is the unique factor in this case), while $P_A(1, 1)$ is the size of the alphabet. Thus for a binary alphabet $\{0, 1\}$ we have $P_A(1, 1) = 2$. In what follows we will consider arrays on the alphabet $\{0, 1\}$.

The above definitions naturally extend analogous notions about usual one-dimensional words. Recall that the *complexity function* $P_w(n)$ of such a word w is defined as the number of different n-factors of w. A word with $P_w(n) \leq n$ for some n, is (ultimately) periodic. Sturmian words are the words that have lowest complexity among the non-ultimately periodic words, i.e., of complexity $P_w(n) = n + 1$ for any $n \geq 0$. It is also well-known that any Sturmian word is a digitization of an irrational straight line and is ultimately recurrent. See [36] for further details. In higher dimensions the situation is more complicated. For instance, it is still unknown whether a notion of minimal complexity can be reasonably defined (see [9] and the discussion therein). To a certain extend the same applies to the notion of 2D Sturmian word. Initially it has been expected that 2D words of minimal complexity are digitizations of irrational planes with no rational direction. Such words were believed to have complexity $mn + 1$. However, it has been recently shown that a 2D word of complexity $mn + 1$ cannot be uniformly recurrent and does not appear to be a digitization of any plane [21]. Therefore, it makes sense to call 2D Sturmian words the ones that appear to be digitizations of irrational planes which do not have a rational direction. Such kind of words obtained within a number of diverse digitization schemes have been widely investigated by S. Ito, M. Ohtsuki, L. Vuillon, V. Berthé, R. Tijdeman among others, and various interesting and sophisticated results have been obtained. See, e.g. [41, 6, 9, 10, 21, 11, 8, 29]. Here we study digital 2D rays in the framework of our simple digitization scheme described in the previous section.

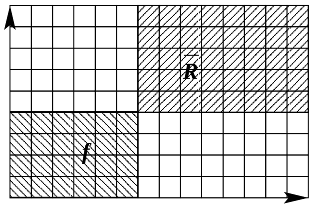


Figure 13: Illustration to the proof of Theorem 10.

5.2 Basic results

First we list a number of properties, which, as a matter of fact, are based on the Kronecker theorem.

Proposition 9 *All digital planes/rays with an irrational slope vector* (a, b) *contain the same set of rectangular factors.*

Proof The same proof works both for digital planes and rays. Let r' be an irrational digital 2D ray determined by a 2D ray R' with an irrational slope vector (a_1, a_2). Let w' be a $(p \times q)$-factor of r'. r' corresponds to a discretization I' of R'. Let x be an integer point of I' which is in a shortest vertical distance d' from R'. If there are more than one such kind of points (this may happen only if r' is line-periodic), we choose an arbitrary one. Let the point of r' corresponding to x have indexes i and j with respect to the ray r'. Now consider a 2D ray R'' which has the same slope vector (a_1, a_2) but another intercept. Let I'' and r'' be its discretization and digitization, respectively. Since the slope vector (a_1, a_2) is irrational, by arguments implied by the Kronecker theorem, it follows that one can find an integer point $y \in I''$, such that its vertical distance d'' from R'' is arbitrary close to h'. Consider in r'' the digital image $\bar{y}$ of y (which is a discrete point labeled 0 or 1) and take a $(p \times q)$-factor w'' of r'' in such a way that $\bar{y}$ has indexes i and j with respect to r''. Now it is easy to realize that if $|d' - d''|$ is small enough then $w' = w''$, i.e., the word w' appears also in r''. □

Proposition 10 *Let* r *be an irrational digital 2D ray. Then every rectangular block appearing in* r*, appears in it infinitely many times.*

Proof Let r be an irrational digital 2D ray determined by a 2D ray R with a slope vector (a_1, a_2). Let w be a $(p \times q)$-factor of r and I the discretization of R. Assume that w appears finitely many times in r. Then it is contained in a (finite) prefix f of r. Consider the rest of r, i.e., the digital domain $r - f$. It is a digitization of an infinite portion of R and contains a digital 2D ray $\bar{R}$ (see Figure 13) with the same slope vector (a_1, a_2) as R. By arguments similar to those used in the proof of Proposition 9, we get that the digitization of $\bar{R}$ will contain a factor equivalent to w – a contradiction. □

Proposition 11 *Any factor of an irrational digital plane is a factor of a rational digital plane.*

Proof Let r be an irrational digital 2D ray determined by a 2D ray R with a slope vector (a_1, a_2). Let I be the corresponding discretization of R and w a $(p \times q)$-factor of r. Let R_w be the rectangular portion of r corresponding to the digital rectangle w and I_w the corresponding portion of I. Let $x \in I_w$ be an integer point which is the closest one to R and does not belong to R. If there are more than one point with this property, we choose an arbitrary one. Consider a plane r' passing through x and parallel to R. Consider the corresponding portion R'_w defined similarly to R_w. Clearly, one can choose a rational plane $\bar{R}$ which is "sandwiched" between p and R' in the region restricted by the factor w. Then, the digitization of that region of $\bar{R}$ will be identical to w. □

We now investigate some other properties.

Proposition 12 *If the digital 2D ray r is irrational then $P_r(m,n)$ is unbounded.*

Proof Since r is irrational, by construction either its columns or rows (or both) will be irrational digital lines. Since the complexity function of an irrational digital line is unbounded, the same follows for any digital 2D ray containing such lines as columns or rows. $\square$

An important array characteristic is its balance. Let $h(U)$ denote the number of 1's in a binary array U. Given two binary arrays U and V of the same size $m \times n$, $\delta(U,V) = |h(U) - h(V)|$ is their *balance*. A set X of arrays is said to be *α-balanced* for a certain *constant* $\alpha > 0$, if $\delta(U,V) \leq \alpha$ for all pairs of $(m \times n)$-arrays $U, V \in X$, where m and n are arbitrary positive integers. An infinite array A is said to be *α-balanced* if its set of factors is α-balanced. Array balances are familiar from studies in number theory, ergodic theory, and theoretical computer science. For recent study on balance properties of multidimensional words on two or three letter alphabets see, e.g., [8]. Within the digitization scheme adopted here, we are able to state the following.

Proposition 13 *Let Q be a rowwise digitization of a 2D ray. Then $\delta(U,V) \leq m$ for any pair of $(m \times n)$-factors of Q, $m,n \geq 0$.*

Proof Suppose first that Q is irrational. Then, according to Proposition 9, all rows contain the same set of factors. In particular, the set of n-factors is the same for all rows. Let X and Y be two $(m \times n)$-factors of r. Since r is irrational, the rows of r are irrational digital lines with the same slope. Then they are all 1D Sturmian words. Therefore, the number of 1s in the jth row of X differs from the number of 1s in the jth row of Y by at most 1 (see [36]). Hence, the number of 1s in X differs from the number of 1s in Y by at most m.

Since a rational digital ray is 1-balanced as well (see [36]), the above argument applies also to the case when Q is rational. $\square$

Note that the bound of Proposition 13 is reachable, hence within the considered digitization scheme the 2D digital rays are, overall, non-balanced. To see this, consider the 2D ray $R_{0,\frac{1}{3}}$ defined by $x_3 = 2x_2,\ x_1, x_2 \geq 0$. Its digitizations $digit_{row}(R_{0,\frac{1}{3}})$ and $digit_{col}(R_{0,\frac{1}{3}})$ are identical and consist of rows composed either only by 0s or only by 1s, which change alternatively. Then, for an arbitrary integer $k \geq 1$, there will be infinitely many pairs of $(1 \times k)$-factors whose balance will be equal to k.

Before presenting the other results of this section, we provide a brief discussion on the structure of 2D ray discretization. First we recall that an (m,n)-*window* at a point $(p,q) \in Z^2$ is a set of points $(i,j) \in Z^2$ with $p \leq i < p+m$ and $q \leq j < q+n$. An (m,n)-*cube* at a point $(i,j) \in Z^2$ of a discrete plane P is the set $\{(x,y,z) \in P : i \leq x \leq i+m-1 \textit{ and } j \leq y \leq j+n-1\}$. Two (m,n)-cubes at two different points (i,j) and (i',j') of a discrete plane are geometrically equivalent if each of them can be obtained from the other by an appropriate translation. By $C_R(m,n)$ we will denote the number of different (m,n)-cubes over the points of $discr(R)$. $C_R(m,n)$ is an important parameter characterizing a discrete plane structure (see, e.g., [35]) and is closely related to the complexity function of a plane digitization. In what follows we will use the following lemma.

Lemma 1 *(a) If R is a 2D ray then $C_R(m,n) \leq mn$.*
(b) If R is rational, then $C_R(m,n) \leq lcm(q_1,q_2)$, where q_1 and q_2 are the denominators of the coefficients of x_1 and x_2 in plane representation (3).

Proof (a) Consider an arbitrary (m,n)-window B of Z_+^2. Let R_B be the corresponding quadrilateral portion of R and $discr(R_B) \subset discr(R)$ the corresponding discretization of R_B. We denote by r_B

the digitization of R_B. Recall that by Theorem 11, the integer points of $discr(R)$ are between two Euclidean planes R_1 and R_2 parallel to each other and to the plane containing R, so that R is in the same vertical distance $\frac{1}{2}$ both from R_1 and R_2. As already mentioned in Section 3, if $b' - b''$ is integer, then the discretized 2D rays $I_{a_1,a_2,b'}$ and $I_{a_1,a_2,b''}$ are equivalent. Hence, it is enough to study the structure of $discr(R)$ for planes between R_1 and R_2.

Now imagine that we continuously move R upward with a translation vector parallel to the Ox_3 axis, until it touches an integer point. At this moment, the discretization $discr(R_B)$ of the translated quadrilateral R_B changes (as its digitization r_B does, as well). The same happens if we move R downward until it touches an integer point. All Euclidean planes between these two positions form an equivalence class of planes whose restrictions to B have the same discretization. When moving R upward or downward, clearly at most mn changes are possible and at most mn different equivalence classes may arise: one for each point from $discr(R_B)$. Let these classes be $C_B(k)$, $1 \le k \le mn$. They can also be numbered by the pairs (i,j), $1 \le i \le m$, $1 \le j \le n$. Thus, there are at most mn different discrete rectangles corresponding to planes between R_1 and R_2 and restricted to the block B.

It remains to show that no other (m,n)-cubes may appear in $discr(R)$. Consider another (m,n)-window B' of Z^2_+. Let x' be the integer point which is above $R_{B'}$ and closest to $R_{B'}$. Let x' be over the ith row and jth column of B'. Then the corresponding integer point below R with the same coordinates (i,j) will be the point farthest to $R_{B'}$ among all points of $discr(R_{B'})$. Similarly, let y' be the integer point which is below the plane R and closest to R. Let y' be over the i'_0th row and j'_0th column of B'. We have $(i_0, j_0) \neq (i'_0, j'_0)$. Now consider the equivalence class $C_{B'}(k_0)$ to which $discr(R_{B'})$ belongs for some k_0, $1 \le k_0 \le mn$. This class corresponds to the pair of indexes (i'_0, j'_0), if we consider the equivalence classes as determined by upward moving, or to the pair of indexes (i_0, j_0) if the classes are determined by downward moving. Suppose the latter is the case. Consider the class $C_B(k_0)$ corresponding to the integer point from $discr(R_{B'})$ with the same indexes (i_0, j_0). Then $C_B(k_0)$ and $C_{B'}(k_0)$ are equivalent in a sense that they give rise to the same discretizations, i.e., for any plane $Q \in C_B(k_0)$ and any plane $Q' \in C_{B'}(k_0)$ we have $digit(Q'_{B'}) = digit(Q_B)$. To see this, consider the class $C_B(k_0)$ determined by the indexes (i_0, j_0), i.e., the integer point x over (i_0, j_0) is the closest one to R_B among all points of $discr(R_B)$. If the distance from x to R is larger than the one from x' to R, we can move R upward until both distances become equal. If R' is the translated plane, then clearly R'_B and R_B will have the same discretization. If the distance from x to R is smaller than the one from x' to R, we can move R downward until both distances become equal, thus obtaining the same result.

(b) Let R be rational. We have seen (Theorem 16) that in this case $discr(R)$ can be partitioned into parallelogram patches with sides corresponding to a basis of the lattice Λ. Every parallelogram involves $lcm(q_1, q_2)$ elements, where q_1 and q_2 are the denominators of the coefficients of x_1 and x_2 . Let P_1 and P_2 be two arbitrary parallelograms of the partition. Consider the translation T that brings P_1 onto P_2. The pair of points that are corresponding to each other under T, are considered equivalent. In the same way we define equivalence of points for every two parallelograms. Clearly, every point of any parallelogram is equivalent to exactly one point from any other. Thus the points of $discr(R)$ are partitioned into $lcm(q_1, q_2)$ equivalence classes. Then for any positive integers m and n the number of different (m,n)-cubes at points of $discr(R)$ is bounded by $lcm(q_1, q_2)$. □

Lemma 2 *If a digital 2D ray r is irrational and aperiodic, then $P_r(m,n) \ge mn$.*

Proof Keeping in mind the proof of Lemma 1, it is easy to see that in the considered case $C_R(m,n) = mn$. It is also clear that $C_R(m,n) \le P_r(m,n)$, from where the lemma follows. □

We conclude the present study with results related to a well-known conjecture by M. Nivat about periodicity of infinite binary 2D words. He conjectured that if for some integers $m, n \ge 0$ an infinite

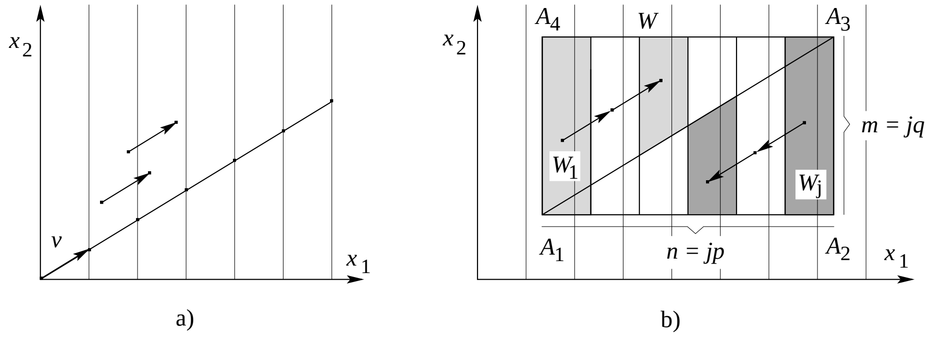


Figure 14: Illustration to the proof of Theorem 19.

bi-dimensional 0/1 array A has complexity $P_A(m,n) \leq mn$, then A has at least one period vector [33]. Note that the converse is not true, in general: an array may be periodic but its complexity may be higher than mn (see [9]). Only partial results for small values of m and n have been proved regarding this conjecture. In [25] a weaker statement is proved under the condition $P_A(m,n) \leq \frac{1}{100}mn$. Here we consider the Nivat's conjecture for the special case of arrays that are digital 2D rays.

Theorem 18 *A digital 2D ray r has a period vector if and only if for some integers $m, n \geq 0$, $P_r(m,n) < mn$.*

Proof From Lemmas 1 and 2 and their proofs it is clear that the inequality $P_r(m,n) < mn$ holds for some $m, n \geq 0$ if and only if $discr(R_B)$ contains two points that are in equal vertical distances from R. In view of theorems 16 and 17 and the related discussion, this may happen iff $discr(R)$ and, in turn $digit(R)$, have a period vector. As we have seen, this may happen when R is either rational or irrational digital ray. □

Remark 15 *If for some $m, n \geq 0$ an equality $P_r(m,n) = mn$ holds, it seems to imply the condition $P_r(m, n+1) < m(n+1)$, under which Theorem 18 applies. To prove this remains as a further task.*

The next theorem provides an asymptotic result in terms of $C_R(m,n)$. We will say that a vector v is a symmetry vector of $discr(R)$ if for any voxel $w \in discr(R)$, $v + w \in discr(R)$. v is a period vector of $discr(R)$ if for any integer k, kv is a symmetry vector of $discr(R)$.

Theorem 19 *Let R be a 2D ray. Then $discr(R)$ has a period vector if and only if*

$$\lim_{m,n \to \infty} \frac{C_R(m,n)}{mn} = 0. \tag{4}$$

Proof Let (4) holds. Then there exist positive integers m_0, n_0 such that for any pair of integers m, n with $m \geq m_0$ and $n \geq n_0$ we have $\frac{C_R(m,n)}{mn} < 1$, i.e., $C_R(m,n) < mn$. Then by Theorem 18, $digit(R)$ has a period vector, as $discr(R)$ does.

Now let $v = (p,q,r)$, $p \geq q$, be a period vector of $discr(R)$, where p, q and r are fixed integers. Let $v' = (p,q)$ be its projection on the coordinate plane Ox_1x_2. Because of condition (2), there is a one-to-one correspondence between the voxels of $discr(R)$ and the points of Z^2_+. So to obtain certain quantitative estimations, one can work with projections of (m,n)-cubes over Ox_1x_2 rather than with the (m,n)-cubes themselves.

Consider the set of nonnegative integer points of the form $u^{(i)} = i \cdot v = (ip, iq)$ for $i = 0, \pm 1, \pm 2, \ldots$. They are projections on Ox_1x_2 of points of $discr(R)$, generated by the period v. The points $u^{(i)}$ belong to a line determined by v' and induce a partition of Z_+^2 into a set S of vertical strips delimited by the vertical rays $x_1 = ip$, $x_2 \geq 0$, for $i = 0, \pm 1, \pm 2, \ldots$ (Figure 14a). Since v is a symmetry vector of $discr(R)$, any two strips from S correspond to regions of $digit(R)$ that are equivalent up to translation by the vector v.

Now consider an (m, n)-window $W = A_1A_2A_3A_4$ of Z_+^2 with $m = jp$ and $n = jq$ (see Figure 14b). It corresponds to an (m, n)-cube C of $discr(R)$. Partition W into j rectangles W_t $(t = 1, 2, \ldots, j)$ of width p and height jq and consider their pre-images C_t $(t = 1, 2, \ldots, j)$ from $discr(R)$ under the orthogonal projection onto Ox_1x_2. Now we notice with the help of Figure 14b that the set of voxels from C_1 corresponding to W_1 completely determines (through translation by the vector v) all the other C_t's portions that correspond to W_t's portions over the diagonal A_1A_3. Similarly, the set of voxels from C_j corresponding to W_j completely determines (through translation by vector $(-v)$) all the other C_t's portions that correspond to W_t's portions below the diagonal A_1A_3. Thus the sets of voxels from C_1 and C_j are sufficient to completely recover the whole (m, n)-cube C.

Because of the one-to-one correspondence between voxels from $discr(R)$ and elements of Z_+^2, the number of voxels in a set C_t equals the number of integer points in a strip W_t, so C_1 and C_j contain overall $2(p.jq)$ voxels. From this last fact and recalling the proof of Lemma 1, one can conclude that vertical perturbations of the digital 2D ray R through the window W can induce no more than $2(p.jq)$ different (m, n)-cubes. Then for the ratio of $C_R(m, n)$ and mn we have the upper bound

$$\frac{C_R(m, n)}{mn} \leq \frac{2pjq}{j^2pq} = \frac{2}{j} = \frac{2p}{n},$$

which approaches 0 as n approaches infinity. □

5.3 Symmetry

It is shown in [30] that if the slope of a digital line is *rational*, then each period is a symmetric word. Thus, if w is a word in a digital line, then the reverse of w is also a word in that digital line.

As mentioned in Remark 2, a straight line l with irrational slope may pass either through exactly one or through no one rational point. In the cases when l passes through an integer or semi-integer point, we can state the following proposition.

Proposition 14 *Let a straight line l with irrational slope passes through a point $x = (x_1, x_2)$, such that x_1 is either integer or half-integer and x_2 is either integer or half-integer. Then the Sturmian sequence corresponding to l is symmetric with respect to the point P.*

Proof The proof follows from the fact that in all possible cases, the corresponding integer points on both sides of x in the discretization of l are in equal distances from l. □

About rays we can state the following proposition.

Proposition 15 *Let a ray R with irrational slope pass through a point $x = (x_1, x_2)$, such that x_1 is either integer or half-integer and x_2 is either integer or half-integer. Let r be the Sturmian word corresponding to R. Let r' be the portion of the ray from its beginning to x, and r'' its portion symmetric to r' with respect to x. Then the part w' of w corresponding to r' is symmetric to the part w'' corresponding to r''.*

The proof is similar to the one of Proposition 14. We clarify the symmetry issue through the following two remarks.

Remark 16 *If the line l (resp. the ray R) does not pass through a point with integer/half-integer coordinates, some sort of symmetry still exists.*

Let, for instance, l be a line with irrational slope. Let $x = (x_1, x_2)$ be a point with integer or half-integer coordinates. Consider, for definiteness, the case when both x_1 and x_2 are half-integers. Consider the Sturmian sequence L related to l. If l passes through x, then by Proposition 14, L will be symmetric w.r.t. x. Now let us slightly translate l with a translation vector t orthogonal to l. Then, if $||t||$ is small enough, L will be symmetric in a certain neighborhood of x. It is also clear that the smaller $||t||$, the longer the symmetric factor around the point x.

Given a line l, one can choose integer or half-integer points arbitrarily close to l. Therefore, L contains infinitely many symmetric factors, and one can find arbitrarily large symmetric factors.

Remark 17 *The observations of Remark 16 can be extended to the case of digital 2D planes and rays. Specifically, let a plane P with an irrational slope vector pass through a point x with integer/half-integer coordinates. Then the corresponding digital plane is symmetric w.r.t. x. Further, let P' be a plane parallel to P and in a distance d from P. Then, if d is small enough, the corresponding digitization $digit_{row}(P)$ (or $digit_{col}(P)$) will be a 2D palindrome in a certain neighborhood of x. It is clear also that the smaller d, the larger the 2D palindrome centered at x.*

Given a plane P, one can choose integer or half-integer points arbitrarily close to it. Therefore, $digit_{row}(P)$ and $digit_{col}(P)$ contains infinitely many and arbitrary large 2D palindromes.

Recent studies on symmetries of digital planes can be found in [11].

6 Examples: Fibonacci 2D rays

In this section we illustrate with examples some of the theoretical results of the previous sections. First we define Fibonacci 2D rays, as follows.

Example 3 Consider a sequence which is a digitization of the ray $y = \phi x, x \geq 0$, where ϕ is one of the golden ratio numbers: $\phi = \frac{\sqrt{5}-1}{2} = \frac{1}{\tau}$, $\tau = \frac{1+\sqrt{5}}{2} = 0.618033988\ldots$.

We have $I_0 = \lfloor 0.\tau + 1/2 \rfloor = 0$, $I_1 = \lfloor 1.\tau + 1/2 \rfloor = 1$, $I_2 = \lfloor 2.\tau + 1/2 \rfloor = 1$, $I_3 = \lfloor 3.\tau + 1/2 \rfloor = 2$, $I_4 = \lfloor 4.\tau + 1/2 \rfloor = 2$, $I_5 = \lfloor 5.\tau + 1/2 \rfloor = 3$, $I_6 = \lfloor 6.\tau + 1/2 \rfloor = 4$, $I_7 = \lfloor 7.\tau + 1/2 \rfloor = 4$, $I_8 = \lfloor 8.\tau + 1/2 \rfloor = 5$, ..., and then $i(0) = I(0) = 0$, $i(1) = I_1 - I_0 = 1$, $i(2) = I_2 - I_1 = 0$, $i(3) = I_3 - I_2 = 1$, $i(4) = I_4 - I_3 = 0$, etc. (See the last row of the figure below containing the first elements of the sequence $0, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, 0, 1, 1, 0, 1, 1, 0, 1, \ldots$.)

Now we consider a digitization of a 2D ray $R_{Fib} : x_3 = \phi x_1 + \phi x_2$, $x_1, x_2 \geq 0$. Its projections on the coordinate planes Ox_1x_3 and Ox_2x_3 are the rays with equations $x_3 = \phi x_1, x_1 \geq 0, x_2 = 0$ and $x_3 = \phi x_2, x_2 \geq 0, x_1 = 0$, respectively. According to our digitization process, we obtain the following digitization of the lower-left corner of R_{Fib}:

$digit(R_{Fib})$:

	.	.																				
	.	.	.																			
	.	.	.	.																		
19	1	0	.	.	.																	
18	0	1	0	.	.	.																
17	1	0	1	0	.	.	.															
16	1	1	0	1	0	.	.	.														
15	0	1	1	0	1	0	.	.	.													
14	1	0	1	1	0	1	0	.	.	.												
13	1	1	0	1	1	0	1	0	.	.	.											
12	0	1	1	0	1	1	0	1	0	.	.	.										
11	1	0	1	1	0	1	1	0	1	0	.	.	.									
10	0	1	0	1	1	0	1	1	0	1	0	.	.	.								
9	1	0	1	0	1	1	0	1	1	0	1	0	.	.	.							
8	1	1	0	1	0	1	1	0	1	1	0	1	0	.	.	.						
7	0	1	1	0	1	0	1	1	0	1	1	0	1	0	.	.	.					
6	1	0	1	1	0	1	0	1	1	0	1	1	0	1	0	.	.	.				
5	1	1	0	1	1	0	1	$\underline{0}$	1	1	0	1	1	0	1	0	.	.	.			
4	0	1	1	0	1	1	0	1	0	1	1	0	1	1	0	1	0	.	.	.		
3	1	0	1	1	0	1	1	0	1	0	1	1	0	1	1	0	1	0	.	.	.	
2	0	1	0	1	1	0	1	1	0	1	0	1	1	0	1	1	0	1	0	.	.	.
1	1	0	1	0	1	1	0	1	1	0	1	0	1	1	0	1	1	0	1	0	.	.
0	0	1	0	1	0	1	1	0	1	1	0	1	0	1	1	0	1	1	0	1	.	.
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	.	.

Note that, since the array is symmetric with respect to the line $x_1 = x_2$, we have $digit_{col}(R_{Fib}) = digit_{row}(R_{Fib})$. From the digital Fibonacci ray one can find the corresponding x_3-coordinate at any point, thus recover $discr(R_{Fib})$. For example, the integer point corresponding to the underlined point $(7, 5)$ has value 7.

Fibonacci array has several interesting properties, some of which are considered next.

6.1 Repetitions in Fibonacci 2D rays

Let X be a rectangular factor of array A (finite or infinite). By $|X|_h$ and $|X|_v$ we denote the horizontal and vertical size of X, respectively. By X_h^w we denote a half-strip of infinitely many adjacent horizontal replicas of X from some starting point to the right. (See Figure 15.) X_h^w is called *infinite horizontal power* of X. *Infinite vertical power* of X is defined similarly.

Let A denote a finite or infinite array corresponding to the first quadrant. Every set of consecutive columns of A starting from the first one, forms a *horizontal prefix* of A. *Vertical prefix* is defined similarly.

We say that a 2D array A (finite or infinite) contains a *horizontal repetition of order k* where $k > 1$ is a rational number, if it contains a rectangular factor z, such that for some rectangular array y, z is a horizontal prefix of y^w, and $\frac{|z|_h}{|r|_h} = k$. Vertical repetition of order k is defined analogously.

Let $k > 1$ be a *real* number. We say that a 2D array is:

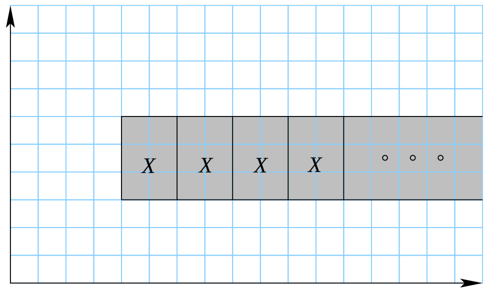


Figure 15: Infinite horizontal power of a rectangular block X.

k-free, if it does not contain as a factor a repetition (horizontal or vertical) of order at least k;
k^+-free, if, for any $k' > k$, it is k'-free;
k^--free, if it is k-free, but not k'-free for any $k' < k$.

We have the following 2D version of Mignosi-Perillo theorem (Theorem 10).

Theorem 20 *The Fibonacci digital 2D ray $digit(R_{Fib})$ is $(2 + \phi)$-free, where ϕ is the golden ratio.*

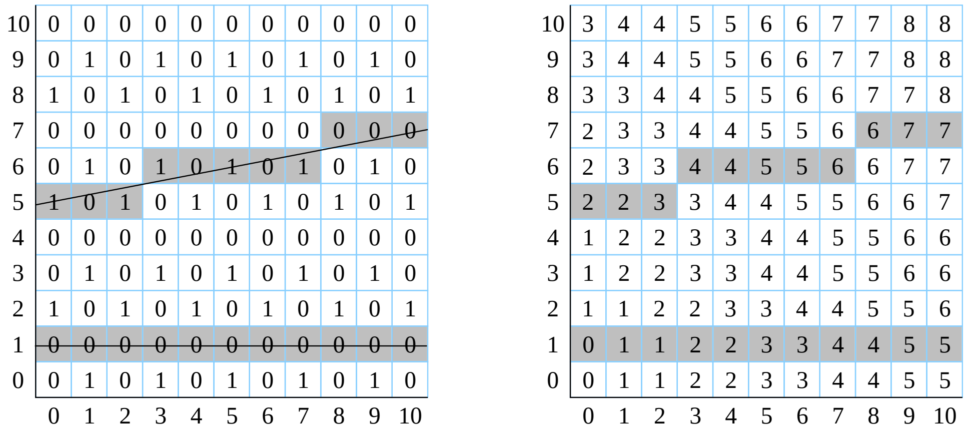
The proof follows from the corresponding result about Fibonacci words and by the construction of Fibonacci 2D ray.

We remark that in [5] we have constructed Fibonacci arrays in somewhat different fashion, and have demonstrated some interesting extremal properties of theirs. Those Fibonacci arrays, however, do not correspond to digital planes. Nevertheless, the digital Fibonacci arrays presented above are expected to possess similar properties.

Thue-Morse arrays can be defined analogously to the Fibonacci rays, and a similar result about 2^+-freedom can be obtained.

7 Digital flatness and 3D digital straightness

Let R be a 2D ray, $discr(R)$ its discretization, and $digit(R)$ its digitization. Let l be a straight line on R and l' its projection on Ox_1x_2. Let $discr(l')$ be the discretization of l' in Ox_1x_2. (See Figure 16a.) Now consider the set of voxels $discr(l) \subset discr(R)$ whose projections on Ox_1x_2 are the pixels of $discr(l')$. Consider the elements of $digit(R)$ corresponding to $discr(l')$. We denote this set by $digit(l)$ and call it *digitization of the straight line l relative to $digit(R)$*. $discr(l)$ is the discretization of l. Note that a digitization of a straight line l relative to a certain digital 2D ray R retains some information about the real line only if it is considered in relation with $digit(R)$. Then the x_3-coordinates of the voxels of l can be recovered, considering $digit(l)$, pixel by pixel, and computing the corresponding values, as illustrated in Example 1 from Section 4.1. For instance, the horizontal line in the figure has digitization consisting entirely of 0s. Nevertheless, it corresponds to a discrete line with x_3-coordinates presented in Figure 16b. Depending on the type of R (rational, irrational without period vector, irrational with period vector) and the particular position of l on R, the digitization $digit(l)$ considered as a sequence of 0s and 1s, may be periodic or aperiodic. Thus, if R is rational, $digit(l)$ is always periodic. If R is irrational and $digit(R)$ does not have a period vector, then $digit(l)$ is aperiodic. If R is irrational and $digit(R)$ has a period vector, then $digit(l)$ may be periodic, which is the case only if the direction of l' is the same as the one of the period vector.



a) b)

Figure 16: a) Discrete lines in $digit_{col}(R_{\frac{1}{2},\frac{1}{3}})$. b) The corresponding x_3-values of the lines' pixels in $discr(R_{\frac{1}{2},\frac{1}{3}})$.

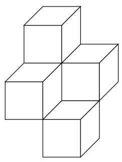


Figure 17: Possible configuration of voxels in a discrete plane satisfying Condition (5).

$discr(l)$ could be regarded as a sort of 3D discrete line, called *pseudoline* in [7]. It is well-known that $discr(l)$ can be disconnected, since $discr(r)$ may contain configurations as the one depicted in Figure 17. Such kind of configuration was called a *jump* in [16]. Locations of jumps are denoted by star signs in the figure exposing the x_3-coordinate values in the discretization of the Fibonacci 2D ray.

	.																						
	.	.																					
	.	.	.																				
19	12	.	.	.																			
18	11	12	.	.	.																		
17	11	11	12	.	.	.																	
16	10	11	11	12	.	.	.																
15	$\star$9	10	11	11	12	.	.	.															
14	9	$\star$9	10	11	11	12	.	.	.														
13	8	9	$\star$9	10	11	11	12	.	.	.													
12	$\star$7	8	9	$\star$9	10	11	11	12	.	.	.												
11	7	$\star$7	8	9	$\star$9	10	11	11	12	.	.	.											
10	6	7	$\star$7	8	9	$\star$9	10	11	11	12	.	.	.										
9	6	6	7	$\star$7	8	9	$\star$9	10	11	11	12	.	.	.									
8	5	6	6	7	$\star$7	8	9	$\star$9	10	11	11	12	.	.	.								
7	$\star$4	5	6	6	7	$\star$7	8	9	$\star$9	10	11	11	12	.	.	.							
6	4	$\star$4	5	6	6	7	$\star$7	8	9	$\star$9	10	11	11	12	.	.	.						
5	3	4	$\star$4	5	6	6	7	$\star$7	8	9	$\star$9	10	11	11	12	.	.	.					
4	$\star$2	3	4	$\star$4	5	6	6	7	$\star$7	8	9	$\star$9	10	11	11	12	.	.	.				
3	2	$\star$2	3	4	$\star$4	5	6	6	7	$\star$7	8	9	$\star$9	10	11	11	12	.	.	.			
2	1	2	$\star$2	3	4	$\star$4	5	6	6	7	$\star$7	8	9	$\star$9	10	11	11	12	.	.	.		
1	1	1	2	$\star$2	3	4	$\star$4	5	6	6	7	$\star$7	8	9	$\star$9	10	11	11	12	.	.	.	
0	0	1	1	2	$\star$2	3	4	$\star$4	5	6	6	7	$\star$7	8	9	$\star$9	10	11	11	12	.	.	.
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	.	.	.

If an Euclidean plane (resp. a 2D ray) is rational, then the locations of jumps form in it a lattice in the plane. Otherwise, if the plane/2D ray is irrational, this is not the case, as illustrated by the Fibonacci digital 2D ray above. It is easy to realize that jumps appear only in configurations of the form

$$\boxed{\begin{array}{cc} m+1 & m+2 \\ \star\, m & m+1 \end{array}}\,.$$

Let R have equation $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \beta, x_1, x_2 \geq 0$. Assume that $|\alpha_1| \leq |\alpha_2| \leq |\alpha_3|$. We have the following theorem.

Theorem 21 *The plane discretization $discr(R)$ contains jumps if and only if $|\alpha_3| < |\alpha_1| + |\alpha_2|$.*

The above statement has been proved in [16] for the case of rational planes, but the proof can be easily modified to hold also for irrational planes.

An analytical discrete plane with thickness $\omega = \max(|\alpha_3|, |\alpha_1| + |\alpha_2|)$ is called *graceful.* This is the thinnest discrete analytical plane without jumps. It is "sandwiched" between the naive and the standard planes. If the maximum above is reached for $|\alpha_3|$, then the graceful plane is a naive plane. Otherwise, it contains additional voxels, appearing exactly in a way to "fill up" the jumps. For instance, the Fibonacci array above has jumps, since in the corresponding plane equation $x_3 = \phi x_1 + \phi x_2$ the maximal coefficient ($\alpha_3 = 1$) is smaller than the sum of the other two ($|\alpha_3| = 1 < |\alpha_1| + |\alpha_2| = 0.618033988\ldots + 0.618033988\ldots = 1.236067976\ldots$). The corresponding graceful plane will contain one more voxel on the top of every voxel marked by a star. We call such a pair of voxels *tandem.* In this case, the graceful plane is thicker than the naive, but thinner than the standard.

Given a line l on a plane R, consider the corresponding graceful plane G and the set of its voxels $G(l)$ obtained in the same way as the voxels of a pseudoline in a naive plane. The set $G(l)$ is called *graceful line*. The graceful lines are the thinnest possible connected discrete lines obtained under this discretization scheme. Their 0/1 digitization is identical to the one of pseudolines, with the only difference that some of the 0/1 points correspond to *two* voxels which form tandems.

8 Algorithmic aspects

Given a straight line/plane, an important question is to recognize the periodicity type of the corresponding digital line/plane and (if applicable) to determine its periodicity or center of symmetry.

8.1 The case of rational planes

With a reference to Remark 11, for a given plane P with equation $x_3 = a_1x_1 + a_2x_2 + b$ there is a family of planes with the same coefficients and different right-hand sides, which determine the same digital plane as P.

Given a plane $P : x_3 = a_1x_1 + a_2x_2 + b$, one can verify whether it contains integer points by testing if $gcd(a_1, a_2, a_3)$ divides b. If it does not, then we consider the corresponding plane $P^{\sqcup}_{a_1,a_2,b}$: $a_1x_1 + a_1x_2 + a_1x_3 = \bar{b}$ defined in Remark 11. Then we find a basis for the set of solutions of the plane equation, i.e., a linearly independent system of integer vectors e^0, e^1, e^2 such that

$$\{(x_1, x_2, x_3) : a_1x_1 + a_1x_2 + a_1x_3 = \bar{b}, x \in \mathbf{Z}^n\} = \{e^0 + \lambda_1 e^1 + \lambda_2 e^2, \lambda_1, \lambda_2 \in \mathbf{Z}\}.$$

This can be done efficiently in polynomial time using well-known algorithms (see [38]). The orthogonal projections of the two basis vectors appear to be period vectors for the corresponding digital plane.

8.2 The case of irrational planes

8.2.1 Undecidebility results

We first mention the following fundamental fact.

Theorem 22 *(Brimkov and Dantchev 1997 [17]) There is no algorithm over R that verifies if a linear Diophantine equation with more than one unknown has an integer solution.*

Therefore, the search must be performed in certain (sufficiently large) subset of the line/plane, e.g., in the cube $\mathbf{0} \leq x \leq d$ for some vector d with sufficiently large norm $||d||$.

8.2.2 Recognition of the periodicity type

The following theorem shows that within the real number model, an equation with two or three unknowns can be solved efficiently in any restricted domain.

Theorem 23 *A linear Diophantine equation $ax = b$ with two or three unknowns and with real coefficients can be solved within the real number model with $O(\log d)$ operations, where $\mathbf{0} \leq x \leq d$. The algebraic complexity of the problem is $\Theta(\log d)$, i.e, the complexity bound is tight (best possible within a constant factor).*

Moreover, $r + 1$ affine independent integer solutions to $ax = b$ can be found with $O(\log \lambda_r)$ operations, where λ_r is the minimal integer such that the set $\{x : ||x|| \leq \lambda_r\}$ contains $r + 1$ affine *independent integer solutions. The algebraic complexity of the problem is* $\Theta(\log \lambda_r)$*, i.e, the complexity bound is tight.*

The above theorem is a corollary of more general results about systems of linear equations and inequalities of arbitrary dimension [17, 18].

9 Further work

9.1 Studying other properties

In these notes we did not consider possible 2D generalizations of some other interesting concepts and results related to digital straightness (see [36], Section 2). This remains as a further task.

9.2 Other digitization schemes

Another possible direction of research is to look for other reasonable digitizations of 2D rays and planes. An alternative way to digitize a 2D ray R is the following. Let R satisfy Condition (5) from Section 3.1. Then to any voxel v from $discr(R)$ corresponds exactly one pixel p from the discrete coordinate plane Ox_1x_2. If R intersects a lower 2-facet of v (including the case when the intersection point is a vertex of v), we label p by 1, otherwise by 0. Thus we obtain a 0/1 digitization of R.

It would be interesting to study the relation between the above digitization and the row-wise/columnwise digitization detailed in the present work. We expect that the properties of the two are essentially equivalent, and analogous statements hold in the framework of both digitization schemes.

9.3 Extension to higher dimensions

The presented theory can be extended to higher dimensions. A starting point can be the well-known fact that in $\mathbf{R}^n$, a hyperplane with real coefficients either contains no integer points, or it contains a k-dimensional lattice of integer points, where $0 \leq k \leq n-1$. Digitizations of hyper-planes containing no one or a single integer point can be considered as generalizations of Sturmian sequences. nD Sturmian arrays, power function and other notions can be defined similarly to the $3D$ case. Digitizations of hyper-planes containing an $(n-1)$-dimensional integer lattice are rational hyperplanes. Various properties, analogous to those valid in $3D$, can be proved in a similar fashion.

9.4 Graph representation, Fibonacci graphs

Consider a $(k \times k)$-prefix r_k of a digital 2D ray r, for some $k \geq 0$. r_k can be regarded as an incidence matrix of a graph $G_k(r)$ of order k. In the general case, the matrix r_k is non-symmetric, therefore the graph $G_k(r)$ is directed. If the equation $x_3 = a_1x_1 + a_2x_2 + b$ is with $a_1 = a_2$, then r_k is symmetric and $G_k(r)$ is an undirected graph. Clearly, a graph $G_k(r)$ is a subgraph of a graph $G_l(r)$, for $l > k$.

Graph representation of digital 2D rays may be useful for certain purposes. For instance, some properties of the digital planes can be studied by employing the rich arsenal of the graph theory. In particular, there is an evidence that the graphs corresponding to rational planes possess certain symmetry, while those corresponding to irrational arrays are not symmetric but possess certain regularity. One can expect that graphs representing digital arrays with certain optimal properties (e.g., Fibonacci arrays), may possess optimal properties that may be useful, e.g., in communications or other applications.

9.5 Nivat's conjecture

We believe that some of the ideas and results of this work, combined with those of Epifanio, Koskas, and Mignosi [25], may lead to a complete proof of Nivat's conjecture for the case of *arbitrary* digital arrays (not necessarily digitizations of 2D rays).

9.6 Penrose tilings and quasicrystals

It has been observed that the irrational digital lines and rays possess certain properties that are germain to ones of the Penrose tilings of the plane. (See, e.g., [28, 6]. On the other hand, Penrose tilings have been found relevant to the structure of the quasicrystals. It would be worth to explore such kind of interesting relations when irrational digital planes/2D rays are involved.

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