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External versus Internal Parameterizations for Lengths of Curves with Nonuniform Samplings

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External versus Internal Parameterizations for Lengths of Curves with Nonuniform Samplings

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Abstract. This paper* studies differences in estimating length (and also trajectory) of an unknown parametric curve $\gamma : [0,1] \to \mathbb{R}^n$ from an ordered collection of data points $q_i = \gamma(t_i)$, with either the t_i's known or unknown. For the t_i's uniform (known or unknown) piecewise Lagrange interpolation provides efficient length estimates, but in other cases it may fail. In this paper, we apply this classical algorithm when the t_i's are sampled according to first *α-order* and then when sampling is *ε-uniform*. The latter was introduced in [20] for the case where the t_i's are unknown. In the present paper we establish new results for the case when the t_i's are known for both types of samplings. For curves sampled ε-uniformly, comparison is also made between the cases, where the tabular parameters t_i's are known and unknown. Numerical experiments are carried out to investigate sharpness of our theoretical results. The work may be of interest in computer vision and graphics, approximation and complexity theory, digital and computational geometry, and digital image analysis.

1 Introduction

For $k \geq 1$, consider the problem of estimating the length $d(\gamma)$ of a C^k regular parametric curve $\gamma : [0,1] \to \mathbb{R}^n$ from *ordered* $(m+1)$-tuples

$$\mathcal{Q}_m = (q_0, q_1, \ldots, q_m)$$

of points $q_i = \gamma(t_i)$ on the curve γ. In this paper the *tabular parameters* t_i's are assumed to be either known or at least distributed in some specific manner.

The problem is easiest when the t_i's are chosen uniformly, namely $t_i = \frac{i}{m}$ (see [15] or [26]). In such a case it seems natural to approximate γ by a curve $\tilde{\gamma}_r$ that is piecewise polynomial of degree $r \geq 1$. The following result can be proved (see [20]):

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Theorem 1. *Let γ be C^{r+2}, with the t_i's be sampled uniformly. Then a piecewise-r-degree Lagrange polynomial $\tilde{\gamma}_r$ determined by $\mathcal{Q}_m$ satisfies*

$$d(\tilde{\gamma}_r) - d(\gamma) = \begin{cases} O(\frac{1}{m^{r+1}}) & \text{if } r \geq 1 \text{ is odd} , \\ O(\frac{1}{m^{r+2}}) & \text{if } r \geq 1 \text{ is even} , \end{cases} \quad (1)$$

and

$$\|\gamma - \tilde{\gamma}_r\|_\infty = O(\frac{1}{m^{r+1}}) \, . \quad (2)$$

As usual, $O(a_m)$, means a quantity whose absolute value is bounded above by some constant multiple of a_m as $m \to \infty$. Both asymptotic estimates appearing in (1) and (2) are *sharp* (see [20]), namely there exist C^{r+2} regular curves γ which, when sampled uniformly, yield lower bounds of convergence rates such as specified in the upper bounds (1) or (2).

Consider samplings of the following type.

Definition 1. *We say that sampling $\{t_i\}_{i=0}^m$ is of α-order, for some $0 < \alpha \leq 1$, if $t_i < t_{i+1}$ and the following holds*

$$t_{i+1} - t_i = O(\frac{1}{m^\alpha}) \, . \quad (3)$$

The second part of the present paper is mainly concerned with the case, where $\alpha = 1$.

We may ask whether Theorem 1 extends, either to an arbitrary sampling (3), or to some subclasses of (3) for both the t_i's known or unknown. More specifically, we examine the existence of some $\beta_1, \beta_2 > 0$ yielding

$$d(\tilde{\gamma}_r) - d(\gamma) = O(\frac{1}{m^{\beta_1}}) \quad \text{and} \quad \|\gamma - \tilde{\gamma}_r\|_\infty = O(\frac{1}{m^{\beta_2}}) \, . \quad (4)$$

Subsequently, the comparison and the analysis of underlying difference between *internal* and *external* parameterizations (the t_i's known versus unknown) will follow. Those two issues are treated in this paper in detail and some new results for internal parameterization are established.

Evidently, the knowledge of explicit distribution of the tabular points t_i's, provides extra information to the problem in question (including the order of the points in $\mathcal{Q}$). Thus, as expected and proved later in this paper a nonuniform case (3) together with internal parameterization yields a better result than its external parameterization counterpart. The latter is in contrast with the uniform case where the corresponding convergence rates coincide - see Theorem 1. Note that if the t_i's are unknown, the *order* of points in $\mathcal{Q}_m$ is also assumed to be given.

This work is relevant to some computer vision problems: tracking an object or its center of mass from satellite or video images, finding the boundary of planar objects (for example in medical image analysis or automated production line) or handling any data (such as a sequence of video images) parametrized by one parameter in decompressing, interpolation, or noise rectification processes.

There is another context of possible applications outside the scope of approximation theory. Recent research in digital and computational geometry and digital image analysis concerns analogous work for estimating lengths of digitized curves. Depending on the digitization model [11], γ is mapped onto a digital curve and approximated by a polygonal curve $\hat{\gamma}_m$ whose length is an estimator for $d(\gamma)$. Approximating polygons $\hat{\gamma}_m$ based on local configurations of digital curves do not ensure multigrid length convergence, but global approximation techniques yield *linearly* convergent estimates, namely $d(\gamma)-d(\hat{\gamma}_m)=O(\frac{1}{m})$ [1], [13], [14] or [25]. Recently, experimentally based results reported in [4], [5], [6], and [12] confirm a similar rate of convergence for $\gamma \subset \mathbb{R}^3$. In the special case of discrete straight line segment in $\mathbb{R}^2$ a stronger result is proved, for example, [8], where $O(\frac{1}{m^{1.5}})$ errors for asymptotic length estimates are established.

Our paper focuses on *curve interpolation* and asymptotical analysis is based on the number of interpolation points. On the other hand digital models assume *curve approximation* and the corresponding asymptotics is based on the size of image resolution. So strict comparisons cannot be made yet. However, as a special case we provide upper bounds for optimal rates of convergence when piecewise polynomials are applied to the digitized curves. Related work can also be found in [2], [3], [9], [10], [22], and [24]. There is also some interesting work on complexity [7], [23], and [27].

The layout of the present paper is as follows. The first part is mainly expository with some extension of standard result for 1-order case to α-order one (see Theorems 1 and 2). The second part discusses essential differences in estimating length and trajectory of γ between both cases with the interpolation times t_i's either known or unknown. In particular the above difference for ε-uniform sampling (constituting a special case of 1-order sampling) is empahsized in Theorem 3 and Theorem 4. Finally, as Theorem 4 also indicates, if the t_i's are known, the results in Theorem 2 covering also ε-uniform case (as a special 1-order one) can be strengthened.

2 Preliminaries

Let $\|\cdot\|$ be the Euclidean norm in $\mathbb{R}^n$, where $n \geq 1$, with $<\cdot,\cdot>$ the corresponding inner product. The *length* $d(\gamma)$ of a C^k parametric curve ($k \geq 1$) $\gamma : [0,1] \to \mathbb{R}^n$ is defined as

$$d(\gamma) = \int_0^1 \|\dot{\gamma}(t)\| dt \ ,$$

where $\dot{\gamma}(t) \in \mathbb{R}^n$ is the derivative of γ at $t \in [0,1]$. The curve γ is said to be *regular* when $\dot{\gamma}(t) \neq \mathbf{0}$, for all $t \in [0,1]$. A *reparameterization* of γ is a parametric curve of the form $\gamma \circ \psi : [0,1] \to \mathbb{R}^n$, where $\psi : [0,1] \to [0,1]$ is a C^k diffeomorphism. The reparameterization $\gamma \circ \psi$ has the same image and length as γ. For simplicity we assume here that ψ is C^∞. Let γ be regular: then so is any reparameterization $\gamma \circ \psi$. Recall that a regular curve γ is said to be *parameterized proportionally to arc-length* when $\|\dot{\gamma}(t)\|$ is constant for $t \in [0,1]$.

We want to estimate $d(\gamma)$ from *ordered* $(m+1)$-tuples

$$\mathcal{Q}_m = (q_0, q_1, q_2, \ldots, q_m) \in (\mathbb{R}^n)^{m+1},$$

where $q_i = \gamma(t_i)$, whose parameter values $t_i \in [0,1]$ are either known or unknown and sampled in some reasonably regular way.

We are going to discuss different ways of forming ordered samples

$$0 = t_0 < t_1 < t_2 < \ldots < t_m = 1$$

of variable size $m+1$ from the interval** $[0,1]$. The simplest procedure is *uniform sampling*, where $t_i = \frac{i}{m}$ (where $0 \leq i \leq m$). Uniform sampling is not invariant with respect to *reparameterizations*, namely order-preserving C^∞ diffeomorphisms $\phi : [0,1] \to [0,1]$. A small perturbation of uniform sampling is no longer uniform, but may approach uniformity in some asymptotic sense, at least after some suitable reparameterization. We define now a special subclass of (3) (see also [20]), namely a special type of 1-order sampling:

Definition 2. *For $0 \leq \varepsilon \leq 1$, the t_i's are said to be ε-uniformly sampled when there is an order-preserving C^∞ reparameterization $\phi : [0,1] \to [0,1]$ such that*

$$t_i = \phi(\frac{i}{m}) + O(\frac{1}{m^{1+\varepsilon}}) \, .$$

Note that ε-uniform sampling arises from two types of perturbations of uniform sampling: first via a diffeomorphism $\phi : [0,1] \to [0,1]$ combined subsequently with added extra distortion term $O(\frac{1}{m^{1+\varepsilon}})$. In particular, for ϕ the identity, and $\varepsilon = 0$ ($\varepsilon = 1$) the perturbation is *linear* (*quadratic*), which constitutes asymptotically a big (small) distortion of a uniform partition of $[0,1]$. The extension of Definition 2 to $\varepsilon > 1$ could also be considered. This case represents, however, a very small perturbation of uniform sampling (up to a ϕ-shift) which seems to be of less interest in applications. As mentioned the perturbation of uniform sampling via ϕ has no effect on both $d(\gamma)$ and geometrical representation of γ. The only potential nuisance stems from the second perturbation term $O(\frac{1}{m^{1+\varepsilon}})$.

Finally, note that ε-uniform sampling is invariant with respect to C^∞ order preserving reparameterizations $\psi : [0,1] \to [0,1]$. So suppose in all the following, without loss of generality, that γ is parameterized proportionally to arc-length.

We shall need later the following lemma (see [16]; Lemma 2.1):

** In the present context there is no real gain in generality from considering other intervals $[0,T]$.

Lemma 1. *Let $f : [a, b] \to \mathbb{R}^n$ be C^l, where $l \geq 1$ and assume that $f(t_0) = \mathbf{0}$, for some $t_0 \in (a, b)$. Then there exists a C^{l-1} function $g : [a, b] \to \mathbb{R}^n$ such that $f(t) = (t - t_0)g(t)$.*

Proof. For each i-th component of $f = (f_1, f_2, \ldots, f_n)$ consider $F_i : [0, 1] \to \mathbb{R}$ $F_i(u) = f_i(tu + (1 - u)t_0)$. By the Fundamental Theorem of Calculus

$$f_i(t) = F_i(1) - F_i(0) = (t - t_0) \int_0^1 f_i'(tu + (1 - u)t_0)\, du \ .$$

Take $g = (g_1, g_2, \ldots, g_n)$, where

$$g_i(t) = \int_0^1 f_i'(tu + (1 - u)t_0)\, du \ .$$

This proves Lemma 1. □

The proof of Lemma 1 shows also that $g = O(\frac{df}{dt})$, namely the uniform norm of g is bounded by a constant multiple of the uniform norm of $\frac{df}{dt}$. Here f may depend on some other parameter $m \to \infty$. If f has multiple zeros $t_0 < t_1 < \ldots < t_k$ then $k + 1$ applications of Lemma 1 give

$$f(t) = (t - t_0)(t - t_1)(t - t_2) \ldots (t - t_k)h(t) \ , \tag{5}$$

where h is $C^{l-(k+1)}$ and $h = O(\frac{d^{k+1}f}{dt^{k+1}})$.

3 Internal and External Clocks for α-order Samplings

We begin with some results for estimating $d(\gamma)$ and γ when piecewise-r-degree Lagrange interpolants are used with *internal parameterization* applied to arbitrary sampling of α-order (for proof see Appendix 1). When $\alpha = 1$ formula (6) is well-known.

Theorem 2. *Let γ be C^{r+2} and let the t_i's be given explicitly and sampled according to α-order. Then a piecewise-r-degree Lagrange polynomial $\tilde{\gamma}_r$, determined by $\mathcal{Q}_m$ yields*

$$d(\tilde{\gamma}_r) - d(\gamma) = O(\frac{1}{m^{\alpha(r+2)-1}}) \quad \textit{and} \quad \|\gamma - \tilde{\gamma}_r\|_\infty = O(\frac{1}{m^{\alpha(r+1)}}) \ . \tag{6}$$

Remark 1: Note that, if $\alpha \leq \frac{1}{r+2}$ then formula (6) does not guarantee convergence for $d(\gamma)$ estimation. On the other hand, the most interesting case when $\alpha = 1$ renders convergence for arbitrary $r > 0$ integer.

For the general case when the t_i's are unknown and sampling is of α-order, Lagrange interpolation for length estimation can behave badly. For example, consider the most interesting case when $\alpha = 1$. From now we shall call the derivation of $\tilde{\gamma}_2$ as a QS-Algorithm (Quadratic Sampler). The next example shows that for the t_i's *unknown* with $\alpha = 1$ in (3) and $r = 2$, the formula (4) may not hold even if γ is well approximated.

Example 1. Consider the following two families of the t_i's distributions:

$$t_i = \begin{cases} \frac{i}{m} & \text{if } i \text{ even },\\ \frac{i}{m} + \frac{1}{2m} & \text{if } i \text{ odd } \& \ i = 4k+1 ,\\ \frac{i}{m} - \frac{1}{2m} & \text{if } i \text{ odd } \& \ i = 4k+3 , \end{cases} \tag{7}$$

$$t_i = \frac{i}{m} + \frac{(-1)^{i+1}}{3m} , \tag{8}$$

with $t_0 = 0$ and $t_m = 1$. In order to generate synthetically sampling points $\mathcal{Q}_m$ assume temporarily that the t_i's distributions from (7) and (8) are known and that the analytic formulae for regular curves *semicircle* and *cubic curve* $\gamma_s, \gamma_c : [0,1] \to \mathbb{R}^2$

$$\gamma_s(t) = (\cos(\pi(1-t)), \sin(\pi(1-t))) \quad \text{and} \quad \gamma_c(t) = (\pi t, (\frac{\pi t + 1}{\pi + 1})^3) \tag{9}$$

are given. Consequently, upon deriving initial data $\mathcal{Q}_m$, the QS-Algorithm is used merely with $\mathcal{Q}_m$. As it turns out with uniform estimate $\hat{t}_i = i/m$ of the t_i's, QS-Algorithm yields a good trajectory estimation in either cases (see Figure 1). Note also that for synthetic generation of curve samplings proportional to arc-length parameterization is not needed. Only the existence of the latter (assured by the regularity of γ) is used to prove both Theorems 2 and 4.

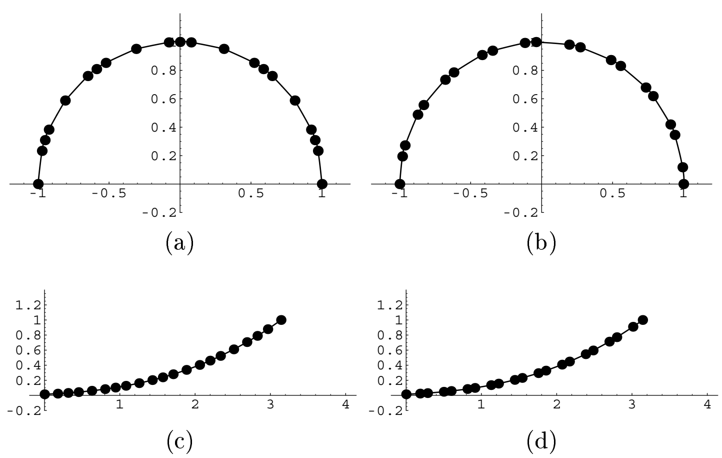

Fig. 1. (a) $\tilde{\gamma}_2$ for a semicircle γ_s and (7). (b) $\tilde{\gamma}_2$ for a semicircle γ_s and (8). (c) $\tilde{\gamma}_2$ for a cubic curve γ_c and (7). (d) $\tilde{\gamma}_2$ for a cubic curve γ_c and (8)

On the other hand the length estimation by QS-Algorithm (used with $\hat{t}_i = i/m$) for $d(\gamma_s) = \pi$ and $d(\gamma_c) = 3.3452$ yields a dual result (see Table 1), where $\rho^m_{d(\gamma)} = |d(\gamma) - d(\tilde{\gamma}_2)|$ and $\beta^{200}_{d(\gamma)}$ defines an estimate of β_1 (see (4)) found by linear regression applied to the pairs of points $(\log(m), -\log(\rho^m_{d(\gamma)}))$, with m running from 6 to 200. □

Table 1. $d(\gamma)$ estimation by QS-Algorithm with the t_i's unknown

curves:	semicircle γ_s		cubic curve γ_c	
samplings:	(7)	(8)	(7)	(8)
$\beta^{200}_{d(\gamma)}$:	1.44	n/a[a]	1.99	n/a[a]
$\rho^{200}_{d(\gamma)}$:	3.45x10^{-4}	0.1288	6.36x10^{-8}	0.1364

[a] not applicable: $\lim_{m\to\infty} d(\tilde{\gamma}_r)$ exists but is not equal to $d(\gamma)$.

In contrast, if the t_i's for both samplings (7) and (8) are *known*, then QS-Algorithm yields a better result for (4) (see Table 2). In the next section we discuss a similar problem of estimating $d(\gamma)$ with either internal or external parameterizations used and applied to the special subclass of 1-order samplings, namely the so-called ε-uniform ones.

Table 2. $d(\gamma)$ estimation by QS-Algorithm with the t_i's known

curves:	semicircle γ_s		cubic curve γ_c	
samplings:	(7)	(8)	(7)	(8)
$\beta^{200}_{d(\gamma)}$:	3.99	4.02	3.99	2.99
$\rho^{200}_{d(\gamma)}$:	4.52x10^{-9}	2.26x10^{-11}	5.54x10^{-9}	1.39x10^{-8}

In the last example, among all, the sharpness (6) for length estimation was confirmed when $\alpha = 1$, $r = 2$ with internal clock available. The validity of (6) can in fact be similarly verified for all r integer and $\alpha = 1$. The next example tests the case for some $0 < \alpha < 1$, $r = 2, 3$ and the t_i's *known*.

Example 2. Consider the following α-order samplings $t_i = (i/m)^\alpha$, for $0 < \alpha < 1$. For γ_c and γ_s defined in Example 1, the QS-Algorithm yields:
Similarly, for $r = 3$ (here $\tilde{\gamma}_3$ forms a piecewise cubic spline) and for γ_s and for a quartic curve $\gamma_{q_4}(t) = (\pi t, (\frac{\pi t+1}{\pi+1})^4)$ (where $t \in [0, 1]$) for which $d(\gamma_{q_4}) = 3.3909$, the results are shown in Table 4. Note that γ_c was replaced here by γ_{q_4} as otherwise piecewise cubic spline $\tilde{\gamma}_3$ coincides with γ_c thus yielding error equal zero.

The convergence rates in Table 3 (or in Table 4) are faster than the corresponding β_1 from Theorem 2 for $r = 2$ (or $r = 3$), namely: $\beta_1^{\alpha=1/2} = 1$

Table 3. $d(\gamma)$ estimation: $r = 2$ and the t_i's are known α-order samplings

curves:	semicircle γ_s		cubic curve γ_c	
samplings:	$\alpha = 1/2$	$\alpha = 1/3$	$\alpha = 1/2$	$\alpha = 1/3$
$\beta^{200}_{d(\gamma)}$:	2.46	1.61	2.09	1.43
$\rho^{200}_{d(\gamma)}$:	7.32x10^{-7}	0.71x10^{-4}	1.10x10^{-7}	6.03x10^{-6}

Table 4. $d(\gamma)$ estimation: $r = 3$ and the t_i's are known α-order samplings

curves:	semicircle γ_s		quartic curve γ_{q_4}	
samplings:	$\alpha = 1/2$	$\alpha = 1/3$	$\alpha = 1/2$	$\alpha = 1/3$
$\beta^{200}_{d(\gamma)}$:	2.46	1.60	2.64	1.81
$\rho^{200}_{d(\gamma)}$:	4.65x10^{-6}	2.41x10^{-3}	1.74x10^{-8}	1.18x10^{-6}

(or $\beta_1^{\alpha=1/2} = 1.5$) and $\beta_1^{\alpha=1/3} = 1/3$ (or $\beta_1^{\alpha=1/3} = 2/3$), respectively. As it stands now it remains an open problem whether for $0 < \alpha < 1$ and arbitrary r Theorem 2 indeed provides sharp estimates. □

In the next section we will establish sharp estimates for the special subclass of 1-order sampling, namely for ε-uniform with internal (when $r > 0$) and external parameterizations (when $r = 2$) used.

4 Internal & External Clocks for ε-Uniform Samplings

In this section we shall discuss the performance of QS-Algorithm ($r = 2$) for ε-uniformly sampled C^{r+2} curves. Note that both examples of 1-order samplings (7) and (8) are also 0-uniform samplings. As shown in Example 1 Lagrange interpolants for length estimation can behave badly for 0-uniform sampling and external parameterizations (where $\hat{t}_i = i/m$ is used to approximate t_i). The more elaborate algorithms of [17], [18] or [19] are needed for this case to correctly in parallel estimate both γ and the t_i's distribution. However, for $\varepsilon > 0$ and QS-Algorithm the following can be proved (see [20]):

Theorem 3. *Let the t_i's be unknown and sampled ε-uniformly, where $\varepsilon > 0$, and suppose that γ is C^4. Then QS-Algorithm used with $\hat{t}_i = i/m$ yields*

$$d(\tilde{\gamma}_2) = d(\gamma) + O(\frac{1}{m^{4\min\{1,\varepsilon\}}})\ , \quad \|\gamma - \tilde{\gamma}_2\|_\infty = O(\frac{1}{m^{1+2\min\{1,\varepsilon\}}})\ . \quad (10)$$

The estimates from Theorem 3 are sharp (see [20] and [21]). Note that for $\varepsilon = 0$ the proof of Theorem 3 fails and in fact as shown in Example 1 dual outcomes are possible.

Whereas Theorems 1, 2 permit length estimates of arbitrary accuracy (for r arbitrary large or $r > \frac{1}{\alpha} - 2$, respectively) Theorem 3 refers only to piecewise-quadratic estimates, and accuracy is limited accordingly. The proof of Theorem 3 shows that if $r > 2$ and the t_i's are unknown, then any convergence result for $\tilde{\gamma}_r$ and $\hat{t}_i = i/m$ requires ε to be large. The latter would force the sampling to be almost uniform which does not constitute the most interesting case. Note also that if $r = 1$ a piecewise linear interpolation provides the same quadratic convergence rates (see proof of Theorem 3) independently whether the t_i's are known or unknown. Equal convergence rates result from the existence of exactly one (and the same for the t_i's known and unknown) linear interpolant passing through two points in $\mathbb{R}^n$.

Note that if the t_i's are known for ε-uniform sampling (for which $\alpha = 1$) by sharpness of Theorem 1 and 2 the following hold $r + 1 \leq \beta_1 \leq r + 2$ (if r is even) and $r + 1 \leq \beta_1 \leq r + 1$ i.e. $\beta_1 = r + 1$ (if r is odd). It turns out that for ε-uniform samplings (a subclass of 1-order sampling (3)) a tighter result than claimed by Theorem 2 can be proved at least for r even (for a proof which constitutes a new result see Appendix 2).

Theorem 4. *If sampling is ε-uniform, $\varepsilon \geq 0$ and $\gamma \in C^{r+2}$ then with the t_i's known explicitly piecewise-r-degree Lagrange interpolation yields*

$$d(\tilde{\gamma}_r) - d(\gamma) = \begin{cases} O(\frac{1}{m^{r+1}}) & \text{if } r \geq 1 \text{ is odd}, \\ O(\frac{1}{m^{r+1+\min\{1,\varepsilon\}}}) & \text{if } r \geq 1 \text{ is even}, \end{cases} \tag{11}$$

and

$$\|\tilde{\gamma}_r - \gamma\|_\infty = O(\frac{1}{m^{r+1}}) . \tag{12}$$

Remark 2: Note that Theorem 4 can be applied to the extended definition of ε-uniform samplings namely: $-1 < \varepsilon < 0$, for which in fact $t_i = O(\frac{1}{m^\alpha})$ satisfying (3) with $0 < \alpha < 1$ and $\alpha = 1 + \varepsilon$. Then formula (32) is replaceable by $O(\frac{1}{m^{\alpha(r+2)}})$ and as $\alpha(r + 2) \leq r + 2$ we would have (33) of order $O(\frac{1}{m^{\alpha(r+2)}})$. This consequently yields the same length estimates as Theorem 2 with $0 < \alpha < 1$. There is still, however need for Theorem 1 as not all order preserving samplings (3) are of the form $t_i = O(\frac{1}{m^\alpha})$

Next we test the sharpness of the theoretical results in Theorem 4 with some numerical experiments which assume the t_i's to be *known*.

Example 3. Experiments as in the previous section were performed with Mathematica on a 700 MHZ Pentium III with 384 MB RAM. We show first the sharpness of (11) for $r = 2$ and γ_c sampled according to ε-uniform sampling:

$$t_i = \frac{i}{m} + \frac{(-1)^{i+1}}{3m^{1+\varepsilon}} , \tag{13}$$

with $d(\gamma_c) = 3.3452$. We use a similar notation in Table 5 as in Example 1. Note that computed rates $\beta^{200}_{d(\gamma)}$ nearly coincide with those asserted by

Table 5. $d(\gamma)$ estimation: $r = 2$ and the t_i's known from (13)

	computed β_1 for γ_c							
ε	2	1	1/2	1/3	1/10	5/100	1/100	0
$\beta^{200}_{d(\gamma)}$	4.00	4.01	3.48	3.32	3.09	3.04	3.00	3.00

the Theorem 4, namely: for $\varepsilon = 2, 1, 1/2, 1/3, 1/10, 5/100, 1/100$, and 0 we have $\beta_1^{\varepsilon=2} = 4$, $\beta_1^{\varepsilon=1} = 4$, $\beta_1^{\varepsilon=1/2} = 3.5$, $\beta_1^{\varepsilon=1/3} = 10/3$, $\beta_1^{\varepsilon=1/10} = 3.1$, $\beta_1^{\varepsilon=5/100} = 3.05$, $\beta_1^{\varepsilon=1/100} = 3.01$, and $\beta_1^0 = 3$, respectively. Similar sharp results can be obtained for $r = 4$ and (13) with $\varepsilon = 0, 0.5, 2$ yielding $\beta_1 = 4.91, 5.31, 5.88$, respectively. Here the cubic curve (9) is replaced by a quintic curve $\gamma_{q_5}(t) = (\pi t, (\frac{\pi t+1}{\pi+1})^5)$, with $t \in [0, 1]$ and $d(\gamma_{q_5}) = 3.4319$. Otherwise a piecewise quartic spline $\tilde{\gamma}_4$ coincides with γ_c thus yielding error equal zero. The computed estimates are slightly less than (11) with $r = 4$ (they should be at least 5, 5.5, and 6, respectively) as only a small number of interpolation points was considered before reaching machine precision during integration. Of course, the asymptotical nature of Theorem 4 requires m to be sufficiently large. Finally, for $r = 3$ and γ_s we have for $\varepsilon = 1, 0.5, 0$ the following values $\beta_1 = 3.99, 4.02$, and 3.92, respectively. The latter coincides with $\alpha = 4$ claimed by Theorem 4 which strongly confirms the sharpness of the last theorem also for r odd. □

5 Conclusions

We examined here a class of α-order and ε-uniform samplings for piecewise Lagrange interpolation to give length (and trajectory) estimates converging to $d(\gamma)$, including investigation of convergence rates for both internal (with the t_i's known) and external (with $\hat{t}_i = i/m$ taken as estimates of t_i) parameterizations. Our results are confirmed to be sharp or nearly sharp for both classes of samplings.

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7 Appendix 1

In this Appendix we shall prove Theorem 2. Part of the proof from this section shall be used also in Appendix 2 to justify Theorem 4.

Proof. Suppose that γ is C^k, where $k = r + 2$ with $r \geq 1$, and (without loss of generality) that m is a multiple of r. Then $\mathcal{Q}_m$ gives $\frac{m}{r}$ $(r+1)$-tuples of the form

$$(q_0, q_1, \ldots, q_r), (q_r, q_{r+1}, \ldots, q_{2r}), \ldots, (q_{m-r}, q_{m-r+1}, \ldots, q_m) \ .$$

The j-th $(r+1)$-tuple is interpolated by the r-degree Lagrange polynomial $P_r^j : [t_{(j-1)r}, t_{jr}] \to \mathbb{R}^n$, here $1 \leq j \leq \frac{m}{r}$:

$$P_r^j(t_{(j-1)r}) = q_{(j-1)r}, \ldots, P_r^j(t_{jr}) = q_{jr} \ .$$

Clearly, each P_r^j is defined in terms of a global parameterization $t \in [t_{(j-1)r}, t_{jr}]$. A simple inspection shows that

$$f = P_r^j - \gamma : [t_{(j-1)r}, t_{jr}] \to \mathbb{R}^n$$

is C^{r+2} and that it satisfies

$$f(t_{(j-1)r}) = f(t_{(j-1)r+1}) = \ldots = f(t_{jr}) = 0 \ .$$

Note also that P_r^j depends implicitly on m and thus f (and later h) should be understood as a sequence of f_m, while m varies. By Lemma 1 and (5) we have

$$f(t) = (t - t_{(j-1)r})(t - t_{(j-1)r+1}) \ldots (t - t_{jr}) h(t) \ , \tag{14}$$

where $h : [t_{(j-1)r}, t_{jr}] \to \mathbb{R}^n$ is C^1. Still by proof of Lemma 1

$$h(t) = O(\frac{d^{r+1}f}{dt^{r+1}}) = O(\frac{d^{r+1}\gamma}{dt^{r+1}}) = O(1) \ , \tag{15}$$

because $deg(P_r^j) \leq r$ and $\frac{d^{r+1}\gamma}{dt^{r+1}}$ is $O(1)$. Thus by (3), (14), and (15) we have

$$f(t) = O(\frac{1}{m^{\alpha(r+1)}}) \ ,$$

for $t \in [t_{(r-1)j}, t_{rj}]$. This completes the proof of the second formula in (6).

Furthermore, differentiating function h (defined as a $(r+1)$-multiple integral of $f^{(r+1)}$ over the compact cube $[0,1]^{r+1}$; see proof of Lemma 1) yields

$$\dot{h}(t) = O(\frac{d^{r+2}f}{dt^{r+2}}) = O(\frac{d^{r+2}\gamma}{dt^{r+2}}) = O(1) \ , \tag{16}$$

as $deg(P_r^j) \leq r$. Thus by (3), (14), and (16) $\dot{f} = O(\frac{1}{m^{\alpha r}})$ and hence for $t \in [t_{(j-1)r}, t_{jr}]$

$$\dot{\gamma}(t) - \dot{P}_r^j(t) = \dot{f}(t) = O(\frac{1}{m^{\alpha r}}) \ . \tag{17}$$

Let $V_{\dot{\gamma}}^{\perp}(t)$ be the orthogonal complement of the line spanned by $\dot{\gamma}(t)$. Since $\|\dot{\gamma}(t)\| = d(\gamma)$ (as γ can be parameterized proportionally to arc-length)

$$\dot{P}_r^j(t) = \frac{< \dot{P}_r^j(t), \dot{\gamma}(t) >}{d(\gamma)^2}\dot{\gamma}(t) + v(t) \ , \tag{18}$$

where $v(t)$ is the orthogonal projection of $\dot{P}_r^j(t)$ onto $V_{\dot{\gamma}}^{\perp}(t)$. As $\dot{P}_r^j(t) = \dot{f}(t) + \dot{\gamma}(t)$ and $\|\dot{\gamma}(t)\| = d(\gamma)$, by (18) we have

$$v(t) = \dot{f}(t) - \frac{< \dot{f}(t), \dot{\gamma}(t) >}{d(\gamma)^2}\dot{\gamma}(t) \ .$$

The latter combined with (17) yields $v = O(\frac{1}{m^{\alpha r}})$. Hence as by (17) and (18)

$$\dot{P}_r^j = (1 + \frac{< \dot{f}(t), \dot{\gamma}(t) >}{d(\gamma)^2})\dot{\gamma}(t) + v(t)$$

and as $< \dot{\gamma}(t), v(t) >= 0$, the Binomial Theorem yields

$$\begin{aligned} \|\dot{P}_r^j(t)\| &= \|\dot{\gamma}(t)\|\sqrt{1 + 2\frac{< \dot{f}(t), \dot{\gamma}(t) >}{d(\gamma)^2} + O(\frac{1}{m^{2\alpha r}})} \\ &= \|\dot{\gamma}(t)\|(1 + \frac{< \dot{f}(t), \dot{\gamma}(t) >}{d(\gamma)^2}) + O(\frac{1}{m^{2\alpha r}}) \ . \end{aligned} \tag{19}$$

Note that by (17) $|2\frac{<\dot{f}(t),\dot{\gamma}(t)>}{d(\gamma)^2}+O(\frac{1}{m^{2\alpha r}})|<1$ holds asymptotically. Integration by parts with (19) renders

$$\int_{t_{(j-1)r}}^{t_{jr}} (\|\dot{P}_r^j(t)\| - \|\dot{\gamma}(t)\|)\ dt = \int_{t_{(j-1)r}}^{t_{jr}} \frac{<\dot{f}(t),\dot{\gamma}(t)>}{d(\gamma)}dt + O(\frac{1}{m^{\alpha(2r+1)}})$$

$$= -\int_{t_{(j-1)r}}^{t_{jr}} \frac{<f(t),\ddot{\gamma}(t)>}{d(\gamma)}\ dt + O(\frac{1}{m^{\alpha(2r+1)}})\ . \tag{20}$$

Since γ is compact and at least C^3 by (15), (16), and $h=O(1)$ we have

$$<h(t),\ddot{\gamma}(t)>=O(1)\ ,\quad <h(t),\gamma^{(3)}(t)>=O(1)\ \text{ and }\ <\dot{h}(t),\ddot{\gamma}(t)>=O(1)\ .$$

Hence, by (14) and Taylor's Theorem applied to $r(t)=<h(t),\ddot{\gamma}(t)>$ at $t=t_{(j-1)r}$, we get

$$<f(t),\ddot{\gamma}(t)>=(t-t_{(j-1)r})\dots(t-t_{jr})(a+O(\frac{1}{m^\alpha}))\ , \tag{21}$$

where a is constant in t and $O(1)$. Note that it is important that a is of order $O(1)$ as it varies with m changed. Thus by (20) and (21) we arrive at

$$\int_{t_{(j-1)r}}^{t_{jr}} (\|\dot{P}_r^j(t)\| - \|\dot{\gamma}(t)\|)\, dt = O(\frac{1}{m^{\alpha(r+2)}})\ .$$

As already defined take $\tilde{\gamma}_r$ to be a track-sum of the P_r^j, i.e.

$$d(\tilde{\gamma}_r) = \Sigma_{j=0}^{\frac{m}{r}-1} d(P_r^j) = d(\gamma) + O(\frac{1}{m^{\alpha(r+2)-1}})\ .$$

This proves the Theorem 1. $\Box$

8 Appendix 2

In this Appendix we justify Theorem 4.

Proof. The second formula (12) results directly from Theorem 2 by setting $\alpha=1$ (as each ε-uniform sampling with $\varepsilon\geq 0$ is also a 1-order sampling). Furthermore, upon repeating the argument from Theorem 2 up to (21) we obtain

$$<f(t),\ddot{\gamma}(t)>=(t-t_{(j-1)r})\dots(t-t_{jr})(a+O(\frac{1}{m}))\ , \tag{22}$$

where a is constant in t and $O(1)$. Upon substitution $(t_{(j-1)r},t_{(j-1)r+1},\dots,t_{jr}) = (t_0,t_1,\dots,t_r)$ let $\chi_i:\mathbb{R}^{r+1}\to\mathbb{R}$ be defined as

$$\chi_i(\boldsymbol{h}) = \int_{t_0}^{t_r} (t-t_0)\dots(t-t_r)\, dt\ , \tag{23}$$

where $i = (j-1)r$, $t_k = \phi(\frac{i+k}{m}) + h_k$ (for $0 \le k \le r$) with $\boldsymbol{h} = (h_0, h_1, \ldots, h_r) \in \mathbb{R}^{r+1}$ satisfying $h_k = O(\frac{1}{m^{1+\varepsilon}})$, for each $0 \le k \le r$. By Taylor's Theorem and ε-uniformity there exists $\delta > 0$ such that for each $\boldsymbol{h} \in \bar{B}(0, \delta) \subset \mathbb{R}^{r+1}$

$$\chi_i(\boldsymbol{h}) = \chi_i(\boldsymbol{0}) + D_h \chi_i(\xi(\boldsymbol{h}))(\boldsymbol{h}) \; , \tag{24}$$

with $\xi(\boldsymbol{h}) = (\xi_0(\boldsymbol{h}), \xi_1(\boldsymbol{h}), \ldots, \xi_r(\boldsymbol{h})) \in \bar{B}(0, \delta)$ positioned on the line between $\boldsymbol{0} \in \mathbb{R}^{r+1}$ and $\boldsymbol{h} = O(\frac{1}{m^{1+\varepsilon}})$ (and thus here $\delta = O(\frac{1}{m^{1+\varepsilon}})$). Furthermore, the integral (23) at $\boldsymbol{h} = \boldsymbol{0}$ upon integration by substitution reads

$$\chi_i(\boldsymbol{0}) = \int_{\frac{i}{m}}^{\frac{i+r}{m}} (\phi(s) - \phi(\frac{i}{m})) \ldots (\phi(s) - \phi(\frac{i+r}{m})) \dot{\phi}(s) \, ds \; . \tag{25}$$

Again, Taylor's Theorem applied to each factor of the integrand of (25) combined with compactness of $[0,1]$ and ϕ being a diffeomorphism yields

$$\chi_i(\boldsymbol{0}) = b \int_{\frac{i}{m}}^{\frac{i+r}{m}} (s - \frac{i}{m} + \tilde{h}_0) \ldots (s - \frac{i+r}{m} + \tilde{h}_r)(\dot{\phi}(0) + O(\frac{1}{m})) \, ds \; ,$$

where $b = \prod_{k=0}^{r} \dot{\phi}(\frac{i+k}{m})$ is constant in s and $O(1)$ and $\tilde{h}_k = O(\frac{1}{m^2})$ (for $0 \le k \le r$). Furthermore,

$$\chi_i(\boldsymbol{0}) = c \int_{\frac{i}{m}}^{\frac{i+r}{m}} (s - \frac{i}{m}) \ldots (s - \frac{i+r}{m}) \, ds + O(\frac{1}{m^{r+3}}) \; , \tag{26}$$

where $c = b\dot{\phi}(0)$ is constant in s and $O(1)$. Again, as previously, it is vital that both b and c are of order $O(1)$, since they vary with m. A simple verification shows that the integral in (26) either vanishes for r even or otherwise is of order $O(\frac{1}{m^{r+2}})$. Hence

$$\chi_i(\boldsymbol{0}) = \begin{cases} O(\frac{1}{m^{r+2}}) & \text{if } r \ge 1 \text{ is odd} \; , \\ O(\frac{1}{m^{r+3}}) & \text{if } r \ge 1 \text{ is even} \; . \end{cases} \tag{27}$$

In order to determine the asymptotics of the second term in (24) let

$$\widetilde{f}_i(t, h_0, \ldots, h_r) = (t - \phi(\frac{i}{m}) - h_0) \ldots (t - \phi(\frac{i+r}{m}) - h_r) \; . \tag{28}$$

As $[\phi(\frac{i}{m}) + h_0, \phi((\frac{i+r)}{m}) + h_r]$ is compact and $\widetilde{f}_i(t, \boldsymbol{h})$ is C^1 we have

$$\frac{\partial \chi_i}{\partial h_k}(\boldsymbol{h}) = \int_{\phi(\frac{i}{m})+h_0}^{\phi(\frac{i+r}{m})+h_r} \frac{\partial \widetilde{f}_i}{\partial h_k}(t, \boldsymbol{h}) \, dt \; , \quad \text{for} \quad 1 \le k \le r-1 \; . \tag{29}$$

Similarly,

$$\frac{\partial \chi_i}{\partial h_0}(\boldsymbol{h}) = \int_{\phi(\frac{i}{m})+h_0}^{\phi(\frac{i+r}{m})+h_r} \frac{\partial \widetilde{f}_i}{\partial h_0}(t, \boldsymbol{h}) \, dt - \widetilde{f}_i(\phi(\frac{i}{m}) + h_0, \boldsymbol{h}) \; . \tag{30}$$

Note that by (28) the second term in (30) vanishes. Thus formulae (29) extend to $k=0$ and similarly to $k=r$. Hence by the Mean Value Theorem the second term in (24) satisfies

$$
\begin{aligned}
D_h\chi_i(\xi(\boldsymbol{h}))(\boldsymbol{h}) &= \sum_{k=0}^{r} h_k \int_{\phi(\frac{i}{m})+\xi_0(\boldsymbol{h})}^{\phi(\frac{i+r}{m})+\xi_r(\boldsymbol{h})} \frac{\partial \widetilde{f}_i}{\partial h_k}(t,\xi(\boldsymbol{h}))\,dt \\
&= \sum_{k=0}^{r} O(h_k) O(\phi(\frac{i+r}{m}) - \phi(\frac{i}{m}) + \xi_r(\boldsymbol{h}) - \xi_0(\boldsymbol{h})) O(\frac{\partial \widetilde{f}_i}{\partial h_k}(t,\xi(\boldsymbol{h}))) \;,
\end{aligned}
\tag{31}
$$

with $t \in \mathcal{I}_\xi = [\phi(\frac{i}{m})+\xi_0(\boldsymbol{h}), \phi(\frac{i+r}{m})+\xi_r(\boldsymbol{h})]$ and, where as in (24) $\boldsymbol{h} \in \bar{B}(0,\delta)$ and $\xi(\boldsymbol{h}) \in \bar{B}(0,\delta)$ is positioned on the line between $\mathbf{0}, \boldsymbol{h} \in \mathbb{R}^{r+1}$. By Taylor's Theorem $\phi(\frac{i+r}{m}) - \phi(\frac{i}{m}) = O(\frac{1}{m})$ and

$$
|\xi_r(\boldsymbol{h}) - \xi_0(\boldsymbol{h})| \leq 2\|h\| = O(\frac{1}{m^{1+\varepsilon}}) \;.
$$

Similarly, for each $0 \leq l \leq r$ we have $t - \phi(\frac{i+l}{m}) - \xi_l(\boldsymbol{h}) = O(\frac{1}{m})$ and thus as $t \in \mathcal{I}_\xi$ by (28) we have $\frac{\partial \widetilde{f}_i}{\partial h_k}(t,\xi(\boldsymbol{h})) = O(\frac{1}{m^r})$. Hence the asymptotics in (31) coincides with

$$
D_h\chi_i(\xi(\boldsymbol{h}))(\boldsymbol{h}) = \sum_{k=0}^{r} O(\frac{1}{m^{1+\varepsilon}}) O(\frac{1}{m}) O(\frac{1}{m^r}) = O(\frac{1}{m^{r+2+\varepsilon}}) \;. \tag{32}
$$

Coupling (27) and (32) with (24) renders

$$
\chi_i(\boldsymbol{h}) = \begin{cases} O(\frac{1}{m^{r+2}}) & \text{if } r \geq 1 \text{ is odd} \;, \\ O(\frac{1}{m^{r+2+\min\{1,\varepsilon\}}}) & \text{if } r \geq 2 \text{ is even} \;. \end{cases} \tag{33}
$$

Thus putting (33) into (23) and combining the latter with (20) and (22) yields

$$
\begin{aligned}
\int_{t_{(j-1)r}}^{t_{jr}} (\|\dot{P}_r^j(t)\| - \|\dot{\gamma}(t)\|)\;dt &= \int_{t_{(j-1)r}}^{t_{jr}} \frac{< \dot{f}(t), \dot{\gamma}(t) >}{d(\gamma)} dt + O(\frac{1}{m^{2r+1}}) \\
&= -\int_{t_{(j-1)r}}^{t_{jr}} \frac{< f(t), \ddot{\gamma}(t) >}{d(\gamma)}\,dt + O(\frac{1}{m^{2r+1}}) \\
&= \begin{cases} O(\frac{1}{m^{r+2}}) & \text{if } r \geq 1 \text{ is odd} \;, \\ O(\frac{1}{m^{r+2+\min\{1,\varepsilon\}}}) & \text{if } r \geq 2 \text{ is even} \;. \end{cases}
\end{aligned}
$$

Hence as $d(\widetilde{\gamma}_r) = \Sigma_{j=0}^{\frac{m}{r}-1} d(P_j^r)$, we finally obtain

$$
d(\gamma) - d(\widetilde{\gamma}_r) = \begin{cases} O(\frac{1}{m^{r+1}}) & \text{if } r \geq 1 \text{ is odd} \;, \\ O(\frac{1}{m^{r+1+\min\{1,\varepsilon\}}}) & \text{if } r \geq 2 \text{ is even} \;. \end{cases}
$$

This completes the proof of Theorem 4. $\square$