Abstract

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Keywords: Simple points, topology preserving deformations, thinning.

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Characterizations of Simple Pixels in Binary Images

Gisela Klette*

Abstract

There are hundreds of publications on different aspects of image deformations that preserve topological properties of the original binary image. The notion of a “simple pixel” is of fundamental importance for these transformations. Simple pixels in 2D binary images have been characterized in a number of ways. This paper reviews some of these characterizations and it points out that some of them are actually equivalent.

Keywords: Simple points, topology preserving deformations, thinning

1. Introduction

The basic notion of “simple pixel” is used in topology preserving digital deformations in order to characterize a single element $p$ of a digital image $I$ which can change the value $I(p)$ without destroying the topology of the image. Simple pixels are defined in the literature using different basic concepts. One way to characterize simple pixels is based on adjacency or surroundingness relations between the connected components of $I(p) = 1$ and $I(p) = 0$ using the grid point model. The concept of “attachment set” using the grid cell model is a different option. The definition of crossing numbers can be used in both the grid point and the grid cell model. Further possibilities to describe simple pixels are the use of masks or Boolean expressions. A more general characterization of a simple pixel can be given by using topological properties such as homotopy equivalence.

The used notation in this paper follows [8] and [9]. A digital image $I$ is a function defined on a discrete set $\mathbb{C}$, which is called the carrier of the image. The elements of $\mathbb{C}$ are grid points or grid cells, and the elements $(p, I(p))$ of an image are pixels (2D case). The range of a binary image is $[0, 1]$. We only use binary images $I$ in this report. Let $\langle I \rangle$ be the set of all pixel locations with value 1, i.e. $\langle I \rangle = I^{-1}(1)$.

The image carrier is defined on an orthogonal grid in 2D. There are two options: using the grid cell model a 2D pixel location $p$ is a closed square (2-cell) in the Euclidean plane, where edges are of length 1 and parallel to the coordinate axes, and centers have integer coordinates. As a second option, using the grid point model a 2D pixel location is a grid point.

Two pixel locations $p$ and $q$ in the grid cell model are called $B$-adjacent if $p \neq q$ and they share at least one vertex (which is a $B$-cell). Note that this specifies 8-adjacency in 2D if the grid point model is used. Two pixel locations $p$ and $q$ in the grid cell model are called $I$-adjacent if $p \neq q$ and they share at least one edge (which is a $I$-cell). Note that this specifies 4-adjacency in 2D if the grid point model is used. Any of these adjacency relations $A_{\alpha}, \alpha \in \{0, 1, 4, 8\}$, is irreflexive and symmetric on an image carrier $\mathbb{C}$. The $\alpha$-neighborhood $N_{\alpha}(p)$ of a pixel location $p$ includes $p$ and its $\alpha$-adjacent pixel locations.

Coordinates of 2D grid points are denoted by $(i, j)$, with $1 \leq i \leq n$ and $1 \leq j \leq m$; $i, j$ are integers and $n, m$ are the numbers of rows and columns of $\mathbb{C}$.

Based on neighborhood relations we define connectedness as usual [6]: two points $p, q \in \mathbb{C}$ are $\alpha$-connected with respect to $M \subseteq \mathbb{C}$ and neighborhood relation $N_{\alpha}$ iff there is a sequence of points $p = p_0, p_1, p_2, ... , p_n = q$ such that $p_i$ is $\alpha$-adjacent to $p_{i+1}$ for $1 \leq i \leq n$, and all points on this sequence are either in $M$ or all in the complement of $M$. A subset $M \subseteq \mathbb{C}$ of an image carrier is called $\alpha$-connected iff $M$ is not empty and all points in $M$ are pairwise $\alpha$-connected with respect to set $M$. An $\alpha$-component of a subset $S$ of $\mathbb{C}$ is a maximal $\alpha$-connected subset of $S$. It is standard practice to use different types of connectedness for pixels $p \in \langle I \rangle$ and for pixels $p \in \langle \overline{I} \rangle$. For brevity, we call them 1's and 0's. We use $\alpha$-connectivity for the 1's and $\alpha'$-connectivity for the 0's where $(\alpha, \alpha') = (4, 8) \text{or} (8, 4)$. In case of the grid cell model, a component is the union of closed squares (2D case). The boundary of a 2-cell is the union of its four edges. An edge of a 2-cell of an 8-connected component includes its two vertices at the two ends.

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An edge of a 2-cell of an 4-connected component is an open set distinct from the vertices at the ends.

Let \( U, V, W \) be pairwise disjoint sets of pixels. We say that \( V \)-\( 4-(\alpha) \)-separates \( U \) from \( W \) if any \( 4-\alpha \)-path from a pixel of \( U \) to a pixel of \( W \) must intersect \( V \) (i.e. must contain a pixel of \( V \)). The border of a set \( M \) of \( 1 \)'s (0's) is the set of pixels of \( M \) that are \( \alpha - \alpha \)-adjacent to the complement of \( M \).

For practical purposes it is easy to use neighborhood operations (called local operations) on a digital image \( I \) which define a value at \( p \in \mathbb{C} \) in the transformed image \( I' \) based on pixel values in \( I \) at \( p \in \mathbb{C} \) and its immediate neighbors in \( N_{\alpha}(p) \). Note that the set \( N_{\alpha}(p) \) consists of \( p \) and its \( \alpha \)-adjacent pixel locations. Let \( A_{\alpha}(p) = N_{\alpha}(p) \setminus \{p\} \) be the \( \alpha \)-adjacency set of \( p \). The elements \( x_i = I(p) \) of the adjacency set are numbered in the following way:

\[
\begin{align*}
x_1 & = x_4 \\
x_2 & = x_3 \\
x_3 & = x_5 \\
x_4 & = x_6 \\
x_5 & = x_7 \\
x_6 & = x_8 \\
\end{align*}
\]

A component is \( 4 \)-adjacent to \( p \) iff it contains a \( 4 \)-adjacent pixel location of \( p \). Rutovitz defines in [10] the crossing number \( X_{\alpha}(p) \) of point \( p \) as follows:

**Definition 1** The number of transitions from a 0 to a 1 or vice versa when the points of \( A_{\alpha}(p) \) are traversed in counterclockwise order is called \( \alpha \)-crossing number \( X_{\alpha}(p) \):

\[
X_{\alpha}(p) = \sum_{i=1}^{8} |x_{i+1} - x_i| \quad \text{where } x_9 = x_1.
\]

Let \( X_{\alpha}(p) \) be the number of distinct \( 4 \)-components of \( 1 \)'s in \( A_{\alpha}(p) \). Then \( X_{\alpha}(p) = X_{\alpha}(p) / 2 \) if there is at least one 0 in \( A_{\alpha}(p) \). Hilditch [3] defined the crossing number as follows:

**Definition 2** The number of times of crossing over from a 0 to a 1 when the points in \( A_{\alpha}(p) \) are traversed in order, cutting the corner between \( 8 \)-adjacent \( \alpha \)-neighbors of 1's, is called \( \alpha \)-crossing number \( X_{\alpha}(p) \):

\[
X_{\alpha}(p) = \sum_{i=1}^{4} c_i
\]

where

\[
c_i = \begin{cases} 
1 & \text{if } x_{2i-1} = 0 \text{ and } (x_{2i} = 1 \text{ or } x_{2i+1} = 1) \\
0 & \text{otherwise} \end{cases}
\]

The \( \alpha \)-crossing number is equivalent to the number of distinct \( 8 \)-components of 1's in \( A_{\alpha}(p) \) in case there is at least one 0 in \( A_{\alpha}(p) \).

There are straightforward ways to compute crossing numbers \( X_{\alpha}(p) \) and \( X_{\alpha}(p) \). Definitions of simple pixels based on crossing numbers or connectivity numbers (see below) are frequently used for the design of thinning algorithms in order to find a decision whether \( I(p) = 1 \) can be changed to \( I'(p) = 0 \) in the transformed image \( I' \) which is topologically equivalent to \( I \).

In [4] the notion of a connectivity number is introduced.

**Definition 3** The number of distinct \( \frac{1}{4} \)-adjacent \( \frac{1}{4} \)-components of 1's (0's) is called connectivity number \( X_{\alpha}(p) / \overline{X}_{\alpha}(p) \) with:

\[
X_{\alpha}(p) = \sum_{i=1}^{4} a_i
\]

where

\[
a_i = x_{2i-1} - x_{2i-1} * x_{2i+1},
\]

and

\[
\overline{X}_{\alpha}(p) = \sum_{i=1}^{4} b_i
\]

where

\[
b_i = \overline{V}_{2i-1} - \overline{V}_{2i-1} * \overline{V}_{2i+1} \quad \overline{V}_{2i} = 1 - x_i \in A_{\alpha}(p).
\]

For the grid cell model, Kong [5] defined the \( I \)-attachment set of a pixel \( p \).

**Definition 4** The set of all points on the boundary of \( p \) that also lie on the boundary of at least one other \( 2 \)-cell \( q \) with \( I(p) = I(q), p \neq q \) is the \( I \)-attachment set of \( p \) in \( I \).

Examples for \( I \)-attachment sets, crossing numbers and connectivity numbers are shown in Figure 1. Note that the \( \alpha \)-crossing number is always equal to the number of distinct \( 4 \)-adjacent \( 4 \)-components of 0's in \( A_{\alpha}(p) \).

2. Characterizations of simple pixels

Simple pixels have in common that changing from 1 to 0 or vice versa preserves the topology of the image. There are different ways to define this important property. Some of the characterizations in the literature are abstract such as Kong’s definition [5] using homotopy equivalence in order to include pixels of 3- and higher dimensional images. However for the design of algorithms it is necessary to determine whether
<table>
<thead>
<tr>
<th>I-attachment sets</th>
<th>$X_n(p)$</th>
<th>$X_u(p)$</th>
<th>$X_v(p)$</th>
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<tr>
<td>1 0 1</td>
<td>$X_v(p)=8$</td>
<td>$X_u(p)=4$</td>
<td>$X_n(p)=0$</td>
</tr>
<tr>
<td>1 1 0</td>
<td>$X_v(p)=2$</td>
<td>$X_u(p)=1$</td>
<td>$X_n(p)=1$</td>
</tr>
<tr>
<td>1 0 1</td>
<td>$X_v(p)=4$</td>
<td>$X_u(p)=2$</td>
<td>$X_n(p)=2$</td>
</tr>
<tr>
<td>1 1 0</td>
<td>$X_v(p)=4$</td>
<td>$X_u(p)=2$</td>
<td>$X_n(p)=2$</td>
</tr>
</tbody>
</table>

**Figure 1. 2D-examples: I-attachment sets, crossing numbers, connectivity numbers, 2D.**

A given pixel in an image $I$ is simple in $I$. This paper reviews a few characterizations which are easy to compute or easy to visualize for 2D digital images. We analyze some properties and start with the following characterization of a simple pixel [8].

**Definition 5** A pixel $p$ of an image $I$ is called a-simple if it is $a$-adjacent to exactly one $a$-component of 1’s in $A_a(p)$ and it is $a'$-adjacent to exactly one $a'$-component of 0’s in $A_b(p)$.

Note that a simple pixel $p$ can be 4-adjacent to exactly one 4-component of 1’s in $A_4(p)$ and 8-adjacent to exactly 4-components of 1’s in $A_8(p)$.

In the example below, $p$ is a 4-simple 1 or an 8-simple 0.

```
1 1 1
0 p 0
1 0 1
```

Let $p \in a$-component in $A_a(p)$. We say $p$ is a-simple if the result of changing 1 to 0 or vice versa is an $a$-component in $A_a(p)$. In [8] it is explained that a change of the value of a simple pixel delivers a topologically equivalent image. The same publication defines simple deformation as follows: Two images differ by simple deformation if one can be obtained from the other one by repeatedly changing simple pixels from 1 to 0 or vice versa.

Thinning or shrinking procedures are one-way simple deformations (from 1 to 0). In the literature about thinning, shrinking, or skeletonization processes different characterizations of “simple pixel” are based on different assumptions (i.e., 8-connectivity is used for the 1’s and 4-connectivity for the 0’s). Simple pixels are restricted to simple 1’s in these publications.

In [10] the following characterization of a 4-simple 1 is used.

**Definition 6** A 1 of an image $I$ is 4-simple in $I$ iff $X_{4n}(p) = 2$.

Changing 1 to 0 of an image $I$ preserves 4-connectivity of $I$ if there is exactly one change from 0 to 1 and vice versa in $A_4(p)$. In $A_8(p)$ is exactly one component of 1’s and exactly one component of 0’s. Note that all 4-simple 1’s are 8-simple. In [3] an 8-simple 1 is defined as follows:

**Definition 7** A 1 of an image $I$ is 8-simple in $I$ iff $X_{8n}(p) = 1$.

This definition is equivalent to Hall’s characterization [2] where a 1 is 8-simple if there is exactly one distinct 8-component of 1’s in $A_8(p)$ and $p$ is a border 1. Note that 8-simple 1’s are not always 4-simple. Another well known characterization of simple pixels is equivalent, see [7].

**Definition 8** A 1 of an image $I$ is 8-simple in $I$ iff both of the following conditions hold: The union of the pixels in $I \setminus \{p\}$ that is 8-adjacent to $p$ is nonempty and connected, $p$ is 4-adjacent to a 0.

An easy to visualize characterization is using the concept of an I-attachment set of a pixel $p$ [5].

**Definition 9** A 1 at $p$ of an image $I$ is 8-simple in $I$ iff the I-attachment set of $p$, and the complement of that set in the boundary of $p$, are non-empty and connected.

This characterization can be simplified for 2D images in the following way:

**Definition 10** A 1 at $p$ of an image $I$ is 8-simple in $I$ iff the I-attachment set of $p$ is non-empty and connected, and it is not the entire boundary of $p$.

Kong has shown in [5] that the last two characterizations are equivalent for 2D images.

### 3. Equivalences and properties

In case the pixel is a border 1 (0) we can simplify definition 5: Let $p$ be a border 1 (0) of an image $I$. Then $p$ is a-simple if it is $a$-adjacent to exactly one $a$-component of 1’s (0’s) in $A_a(p)$. The characterization in Definition 5 is equivalent to the following:
Theorem 1 A 1 (0) of an image I is per Definition 5 \(j\)-simple iff \(X_B(p) = 1\) and a 1 (0) of an image I is per Definition 5 8-simple iff \(\overline{X_B}(p) = 1\).

Proof. Let us consider a 4-simple 1. First we assume \(X_B(p) = 0\) then \(a_i = 0, 1 \leq i \leq 4\). We have \(a_i = 0\) if \(x_{2i+1}^+ = 0\) or \(x_{2i} = x_{2i+1} = 1\). In case \(x_{2i-1}^+ = 0\) for \(1 \leq i \leq 4\) there is a contradiction to the property of having exactly one 4-adjacent 4-component of 1's. In case \(x_{2i-1}^+ = 0\) for \(1 \leq i \leq 4\) there is a contradiction to the property of having exactly one 8-component of 0's in \(A_8(p)\). In case one \(x_{2i-1}^+ = 0\) and \(x_{2i} = x_{2i+1} = 1\) it follows that \(x_{2i+3} = 0\) and \(x_{2i} = x_{2i+1} = 1\) in order to get \(a_i = 0, 1 \leq i \leq 4\). But \(x_{2i} = x_{2i+1} = 0\) and this is a contradiction to \(X_B(p) = 0\). In case one \(x_{2i} = x_{2i+1} = 0\) then \(x_{2i+3} = 1\) (contradiction to \(X_B(p) = 0\) or \(x_{2i} = x_{2i+1} = 0\) what is a contradiction to at least one 4-adjacent pixel location has value 1. Now we assume \(X_B(p) > 1\). Then there exist at least two \(a_i\)'s with \(a_i = a_j = 1, i \neq j, 1 \leq i, j \leq 4\). It follows that there are at least two 4-adjacent 4-components of 1's and two 8-adjacent 8-components of 0's which is a contradiction to \(p\) which is 4-simple. It follows \(X_B(p) = 1\).

Let \(X_B(p) = 1\) then one \(a_i = 1\), say \(a_1 = 1\) and \(a_j = 0, 2 \leq j \leq 4\). \(a_i = 1\) iff \(x_{2i} = 1\) and \(x_3 = 0\) or \(x_3 = 0\). There is at least one 4-adjacent 4-component of 1's and at least one 8-adjacent 8-component of 0's. For \(x_{2j+1} = 0, 2 \leq j \leq 4\) it follows that \(x_{1} = 1\) is an element of the only one 4-adjacent 4-component of 1's and there is exactly one 8-adjacent 8-component of 0's because the remaining 4-adjacent 0's are 8-connected. For \(x_{2j+1} = 0, 2 \leq j \leq 4\) it follows that \(x_{0} = 0\) is the only 8-adjacent 8-component of 0's. Now we consider \(x_{2j-1} = 0\) and \(x_{2j+1} = 1\), \(j = 2, 3\) then \(x_{2j+1} = 1\) is always 4-connected to \(x_{1} = 1\). In any case there is exactly one 4-adjacent 4-component of 1's and exactly one 8-adjacent 8-component of 0's. All other cases follow by symmetry.

Let \(p\) be an 8-simple 1. There is exactly one 4-adjacent 4-component of 0's and exactly one 8-adjacent 8-component of 1's. Based on the definition of \(X_B(p)\) the proof is analog. The proof is also analog for 4-8-simple 0's.

Note that for a 1 \((p)\) of an image I the crossing number \(X_B(p)\) is always equal to \(X_B(p)\). It follows the characterization of an 8-simple 1 in Theorem 1 is equivalent to characterization in Definition 7.

Let us consider the definitions for 8-simple 1's based on I-attachment sets.

Theorem 2 Let \(b \in A_i\) of an image I. Then \(b\) is per Definition 10 8-simple in \(I\) iff the neighborhood of \(p\) matches one of the following masks (simple point masks 1 to \(j\), from left to right, empty squares can be either 0 or 1)

\[
\begin{array}{cccc}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0
\end{array}
\]

or one of their 90° rotations.

Proof. These masks represent the only four ways in which the I-attachment set is non-empty and connected and not the entire boundary.

Earlier thinning algorithms have used the characterization of Definition 6 in order to preserve 4-connected subsets of the original image. These algorithms determine only pixels which have exactly one 4-component of 1's and exactly one 4-component of 0's and no other components in \(A_8(p)\). The 4-component of 1's can be 4-adjacent or not. For \((a, a') = (8, 4)\) and \((a, a') = (4, 8)\) a thinning algorithm using Definition 6 for connectivity preservation results in a 4-connected subset of the original set. Now let us only consider \((a, a') = (4, 8)\). In case two disjoint 4-components of 1's are 8-adjacent to each other then there exists a 1 which is 4-simple per Definition 5 and \(X_R(p) \neq 2\). It follows that the 1 is not 4-simple per Definition 6.

Theorem 4 For \((a, a') = (4, 8)\) a 1 of an image I is \(j\)-simple per Definition 5 and per Definition 6 iff all disjoint \(j\)-components of 1's of I are pairwise 8-separated by 0's to each other.

Proof. Let \(p\) be 4-adjacent to a 4-component of 1's and 8-adjacent to a 8-component of 0's in \(A_8(p)\) and \(X_R(p) = 2\). It follows we have exactly one 4-adjacent
4-component of 1’s and exactly one 4-component of 0’s and no other components in $A_8(p)$ for all these pixels $p$. Any 8-path from a 1 of a 4-component must intersect a 0. Let us consider an image where all 4-components of 1’s are pairwise 8-separated by 0’s. Then an 8-component of 0’s which is 8-adjacent to a pixel $p$ is a 4-component of 0’s in $A_8(p)$ and it follows $X_R(p) = 2$. □

4. Conclusions

Simple pixels are of fundamental interest for topology-preserving deformations of digital images. Thinning algorithms are only one-way topology-preserving deformations which change simple pixels from 1 to 0. Theorem 1 shows an easy way to determine whether a 1(0) in an image $I$ is 4-(8-)simple for two-way simple deformations. Thinning algorithms using the characterization of Definition 6 determine only a subset of all 4-simple 1’s. This characterization is useful for 8-separated 4-components.

References