Strategic Inventory Models for International Dairy Commodity Markets

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Abstract

Fonterra Co-operative Group Ltd is the largest exporter of dairy ingredient products in the world. Fonterra's production planning is subject to uncertain milk supply in a production year. We describe a model for Fonterra's supply chain, of which the important features are uncertain milk supply, price-demand curves and contracting. We derive the forecast milk supply for a production year assuming linear regional growth in milk supply over years, and then we develop an additive model and a multiplicative model for uncertain milk supply.

We present six optimization models assuming uncertain milk supply, of which two are multistage stochastic programming models. We describe an algorithm DOASA to solve the multistage stochastic programming models, give a mathematical proof of the almost-sure convergence of DOASA for the linear case, and give a discussion on convergence for quadratic objectives.

We assess the policies defined by the solutions of the optimization models in sampled milk supply scenarios in simulation. The policy generated from a multistage stochastic programming model has significant advantages over the other policies.

Another situation affected by uncertainty occurs when Fonterra sells products to international spot markets, and other agents' sales have an impact on the market prices. One of the main markets for Fonterra is the European dairy market where the regulations of the European Union Commission have an impact on Fonterra's earnings. In the second part of this thesis, we describe the European dairy market and present a game-theoretic inventory model for this market. The important

features include two leaders, Fonterra and Australia, a follower EU (representing the European Union), and price-demand curves. The model consists of an optimization problem for each leader and an optimization problem for each follower, and solves the leaders' optimization problems simultaneously for an equilibrium of the game. We describe an approach using EPECs and our algorithm to solve the model.

We perform computational experiments in a game with real data, which shows simple results. To illustrate the complicated game structure when the market situation is changed, we perform computational experiments in a game with fictitious data. First we present a simplified version of the game where Fonterra and Australia act together as a single leader, and describe five strategies for this leader. Then we present the game where Fonterra and Australia compete. We present five equilibria and illustrate them using the leaders' best responses. We also use sequential best response to illustrate how the leaders' strategies evolve given a leader's starting strategy. Finally we investigate the impact of inventory holding cost on equilibrium.

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Contents

Chapter 1

Introduction

The international market for dairy commodities is competitive, requiring dairy companies to optimize the profit from production and sales to ensure their survival. Fonterra Co-operative Group Ltd is a leading international dairy company based in New Zealand. Fonterra faces uncertain milk production in planning, mainly due to uncertainty in weather. It also faces uncertain behaviour of other agents in the dairy commodities market. Such uncertainties can lead to large losses, unless some account is taken of these in production planning.

For example, when Fonterra sells products by contracts, it makes the contracts several months in advance. After the contracts have been made, the actual milk production in the months before they are delivered may be lower than what has been forecast. If such a scenario was not considered when the contracts were made, then Fonterra may face inadequate inventory to deliver the contracts which incurs a large cost because it must either breach the contract or acquire the products from a third party (at considerable expense).

Another situation affected by uncertainty occurs when Fonterra sells products to international spot markets, and other agents' sales have an impact on the market prices. If this impact is not properly assessed in production planning, then Fonterra will face a lower market price than what it has expected which results in a lower revenue.

It is therefore of significant value for Fonterra to seek tools to mitigate the effects of these uncertainties in planning. A first step in doing this is to seek more accurate forecasts by collecting more historical data to better estimate the parameters in statistical forecasting models. Notwithstanding, there will always be some residual uncertainty in forecast milk supply, and it is important to account for it when making decisions.

This thesis has as its primary aim the development of computer models to enable Fonterra to make good decisions in the face of uncertainty. For production planning with uncertain milk supply, we develop a set of optimization models that produce policies that maximize expected earnings and compare the policies with existing policies using simulation. For the uncertain behaviour of the other agents in the market, we will not use probability distributions, but use game-theoretic inventory models to assess the inventory policies that emerge in a Nash equilibrium of the problem.

1.1 Fonterra

Fonterra Co-operative Group Ltd, an international dairy company based in New Zealand, is one of the leading companies in the international dairy industry. It was formed in 2001 as a merger of New Zealand Dairy Group, Kiwi Co-operative Dairies and the New Zealand Dairy Board, and it is now owned by 11,000 farmers. As stated in the company overview for the year ending on 31 May 2007[1], Fonterra collected 14 billion litres of milk, processed 1.2 billion kilograms of milksolids, distributed 2.9 million metric tonnes of products and achieved an annual revenue of 13.9 billion New Zealand dollars.

Fonterra produces and distributes top quality ingredient products under the NZMP brand and it is the largest exporter of ingredient products in the world. It produces four main categories of ingredient products – milk proteins, cheese

[1] Key facts and annual reports are available at Fonterra's home page http://www.fonterra.com.

ingredients, milk powders and cream products, which are used to produce up to 600 products.

New Zealand is Fonterra's main source of production of ingredient products. Fonterra collects milk from farmers, processes milk in manufacturing sites to produce products, sells products to international markets and stores products as inventory. Production, storage and transportation incur costs and sales generate revenue. Fonterra's production planning aims to maximize earnings.

1.2 Optimization models with uncertain milk supply

Fonterra's production planning is subject to uncertain milk supply in a production year. This prompts us to develop a set of optimization models to assess the effect of uncertain milk supply on Fonterra's earnings, assuming that Fonterra is a monopoly supplier in the market.

The underlying structure of our models represents Fonterra's supply chain. The important features are uncertain milk supply, inventory, contracting and a price-demand curve in the market. The solution of one of these models defines a policy in production planning which determines for each month, what to produce, what to sell, what contract to make, what to store with information of past contracts, past inventory and current milk supply.

We describe two possible models for uncertain milk supply. The first model is an additive model which has independent random variables. The second model is a multiplicative model which has Markov states and independent random variables.

With the models for uncertain milk supply, this is an optimization problem with uncertainty. We develop a multistage stochastic programming model to solve the problem, which can be used to generat a policy in production planning. Currently Fonterra implements a deterministic policy using forecasts. This policy solves a deterministic optimization problem with adaptive decision horizon and forecast, and implements the solution in each successive month until the end of the year. We would like to compare these two policies, along with some other possible policies in simulation experiment. Our hypothesis is that the policy from the multistage stochastic programming model is an improvement over the deterministic policy using forecasts.

To solve the multistage stochastic programming models, we develop a Dynamic Outer-Approximation Sampling Algorithm (DOASA). We give a mathematical proof of the almost-sure convergence of DOASA for linear objective functions and a discussion on that for quadratic objectives.

1.3 Game-theoretic inventory model for European dairy market

When Fonterra sells products to international spot markets, other agents' sales have an impact on the market prices. The uncertainty in market prices needs to be properly assessed in production planning, so that Fonterra's expected price matches the market price. One of the main markets for Fonterra is the European dairy market. Australia and the European Union are large suppliers in the market. The European Union Commission regulates the market which has an impact on Fonterra's earnings. Currently, under the regulations, the space for strategic selling is very limited, but the European Union Commission continually faces pressure to reform the regulations in favour of free trade, e.g., by choosing a lower intervention price, and thus we see the opportunity of strategic selling in the future. We develop a game-theoretic inventory model to assess Fonterra's strategy in the European dairy market.

In Part II of the thesis we describe the European dairy market and present a game-theoretic inventory model for the market. To simplify the analysis we need to make some assumptions. Particularly, in a multistage production year, Fonterra and Australia are leaders and EU is a follower in decision making, and they sell products to multiple markets in which market prices respond to sales. The model consists of an optimization problem for each leader and an optimization problem for each follower. We will describe an approach of solving the model using an EPEC and present our algorithm to solve the model that uses a grid search and sequential best response.

We will perform computational experiments in different game settings to assess the leaders' strategic behaviours. Our hypothesis is that strategic considerations are important to Fonterra in the European dairy market.

1.4 Literature review

We aim to develop multistage stochastic programming models for the dairy industry supply chain with uncertain milk supply and price-demand curves in markets, and we develop an algorithm to solve the models and give a proof of convergence of the algorithm. We also aim to develop a game-theoretic inventory model for the European dairy market. We give a literature review on these subjects as follows.

1.4.1 Stochastic programming in global supply chain

Our supply chain model is a stochastic global supply chain problem. (Here, global refers to the fact that production and sales are carried out by the same agent.) A recent literature review of supply chain modelling can be found in [35] and [78]. Stochastic supply chain problems are reviewed in [103] and global supply chain problems in [108]. (Hereafter we refer to supply chain as global supply chain.) The papers [77] and [49] give a literature review on supply chain design, and a rich literature studies supply chain design problems with uncertainty, for example [25]. Particularly, there is a lot of literature on capacity planning and location planning problems in supply chain design. For example, [1], [72] and [79] solve capacity planning with uncertain demand with two-stage stochastic programming models, and [102], [45] and [44] solve location planning problems with uncertain demand.

Stochastic supply chain problems also arise in networking and vehicle routing. The paper [28] gives a survey of these problems, and [37] and [13] are examples of solving instances of these problems.

A huge amount of literature on production planning in supply chains has been published; for example, [82] gives a literature review of these models. The papers [26] and [70] describe general stochastic programming models for production planning under uncertain demand. Some literature directly solves a particular practical stochastic supply chain problem. For example, [3] applies stochastic programming models in managing a chemical manufacturing supply chain, [104] performs evaluation on option contracts, [84] describes stochastic optimization in petroleum refinery complexes, [20] uses a two-stage stochastic linear programming model to investigate inventory deployment in the steel industry, [14] develops a scenario planning tool for government agencies to handle the logistics problem under flood emergency with uncertain weather factors, and [56] solves a problem of raw material procurement at pulp and paper mill with uncertain weather in Sweden.

In solving stochastic supply chain models, different solution approaches have been published. For example, [96] combines the sample average approximation scheme with an accelerated Benders decomposition algorithm, [67] uses a simulation-based approach, [90] describes a framework called reinforcement learning, and [60] uses queueing networks to solve a supply chain design problem. We also found some literature on multi-objective programming in supply chain with uncertainty, for example, [43] assesses profit and resulting demand satisfaction in the objective of supply chain management.

1.4.2 Stochastic programming in dairy industry

There is a lot of literature studying the effect of uncertainty in agriculture, fishery and horticulture. Most of this work uses heuristic approaches. Some papers use operations research models, for example, [111] presents operations research models in agriculture and forestry with uncertainty on the farm and regional-sector level.

Some literature uses stochastic simulation to assess the effect of uncertainty. Furthermore, there is literature applying stochastic programming in other industries that is applicable to the dairy industry. For example, [83] and [92] survey the literature on perishable/deteriorating products, and [69] presents a two-stage stochastic programming model for production planning on perishable products. However, to our knowledge, models with a direct focus on optimization in the dairy industry with uncertainty have not been developed.

1.4.3 Stochastic programming with price-demand curve

There is relatively little literature on stochastic programming with price-demand curves. Much of this literature has focused on electricity models, see [107] for a recent survey of electricity market modelling including stochastic optimization models with linear demand. In a different setting, [68] develops a model with stochastic yield and linear demand in agriculture following the same framework as [50]. The paper [62] describes similar work on agriculture. The paper [9] describes a model with a downward sloping demand curve in forest industry. The paper [18] studies stochastic models with a price-demand curve in managerial accounting which is surveyed in managerial accounting models in [19]. In a telecommunication setting, [34] surveys different optimization models and presents a two-stage stochastic programming model with a linear demand. More generally, [106] analyzes price and production postponement strategies in a two-stage model with uncertain demand curve in a monopolistic case, and [46] solves a pricing problem of end-of-season sales of fashion goods under stochastic demand function. Typically these applications apply large-scale nonlinear programming algorithms to deterministic-equivalent models.

There is relatively little literature on special algorithms for solving stochastic programming models that arise from models with price-demand curves. Exceptions are the papers [94] and [95] on solving piecewise linear-quadratic problems, and [91] a sequential quadratic programming approach. However there is a large amount of literature on algorithms for solving stochastic linear programming model with inelastic demand, and some of these algorithms can be used to solve problems with price-demand curves, for example, the methods described in [16] and [71].

One contribution of this thesis is to extend the portfolio of stochastic programming models (with demand curves) to a multistage setting. This approach is particularly useful when contracting decisions must be made periodically.

1.4.4 Multistage stochastic programming

Multistage stochastic programs are well known in the stochastic programming community, and are becoming more common in applications. The typical approach to solving these problems is to approximate the random variables using a finite set of outcomes forming a *scenario tree* and then solve a large-scale mathematical programming problem (see e.g. [8]). The scenario tree can be constructed to represent certain desired properties of the uncertain parameters (see e.g. [88]), or it can be (conditionally) sampled from some probability distribution (see e.g. [98]).

There are various approaches to solving multistage stochastic programs, such as Monte-Carlo methods described in [98] and [97], and multistage Benders decomposition sampling-based algorithms as described in [86].

One approach to solving multistage stochastic linear programs is based on the stochastic dual dynamic programming (SDDP) algorithm in [86]. This algorithm constructs feasible dynamic programming policies using an outer approximation of a (convex) future cost function that is computed using Benders cuts. The policies defined by these cuts can be evaluated using simulation, and their performance measured against a lower bound on their expected cost. This provides a convergence criterion that may be applied to terminate the algorithm when the estimated cost of the candidate policy is close enough to its lower bound (see [53]). The SDDP algorithm has led to a number of related methods (see [16],[23],[24],[52]) that are based on the same essential idea, but seek to improve the method by exploiting the structure of particular applications.

Since its publication in 1991, a number of papers have studied the convergence behaviour of SDDP and related algorithms. The PhD thesis [23] (and the paper [24]) states that "finite convergence of this algorithm follows from the finite convergence of the Nested Decomposition algorithm, since the scenarios from which the optimality cuts are generated are resampled at each iteration." This remark which, strictly speaking, should be a statement of convergence with probability 1, is not accompanied by a formal proof.

The first formal proof of the almost sure convergence of multistage sampling algorithms is published in [16] which is derived for the CUPPS algorithm. This proof is extended in [71] to cover other multi-stage sampling algorithms (SDDP [86], AND [24], ReSa [52]) that use outer approximation. However, the convergence proofs in [16] and [71] make use of an unstated assumption regarding the independence of sampled random variables and convergent subsequences of algorithm iterates. This assumption seriously weakens the analysis in these papers, and leaves open the question of convergence in general.

The DOASA algorithm includes SDDP, AND, ReSa and CUPPS as special cases. In chapter 5, we give a direct proof of the almost-sure convergence of DOASA for linear problems that does not require the assumption made in [16] and [71]. The proof follows the finiteness argument that is alluded to in the thesis [23], and makes this argument explicit, by basing it on two well-defined properties of the sampling procedure.

1.4.5 Game-theoretic inventory models

Game-theoretic models have been studied for many years and a rich literature has been published. The book [38] is a good recent publication on game theory. Game-theoretic inventory models arise from many situations, particularly in supply chain problems where there are different parties in production and sales. For example, [12] and [47] solve an inventory model in a two-stage supply chain, and [10] gives a broad view on inventory management in supply chains with competition. There is some literature on game-theoretic inventory models, where a different focus can be found.

For example, [115], [7] and [61] present stochastic game-theoretic inventory models, [12] and [101] present cooperative game-theoretic inventory models, [76], [4], [110] and [58] present game-theoretic inventory models with simultaneous-move players, and [100], [99] and [109] present game-theoretic inventory models with sequential-move players. On the other hand, studies on the properties of the equilibrium of the game and players' strategies, such as the existence and uniqueness of the equilibrium, and differences in open-loop and closed-loop strategies, is an active research field. Some examples of the literature in this area are [81], [36], [63] and [11]. Literature on algorithms in solving for equilibria in games are also available, for example, [33] describes sequential best response, [31] and [30] describe MCPs, [85] describes MPECs, and [54] describes EPECs.

1.4.6 Strategic models for European dairy market

There is very little literature on strategic issues arising in the European dairy market. Typically papers that study this use business models which do not explicitly quantify the effect of the strategic behaviour as a game-theoretic model does. For example, [21] discusses competitive strategies in the European dairy market, [27] describes the problem of deregulation of the European dairy market and provides a business model for strategy consideration, and [22] has a discussion on the strategy of a farm in exporting dairy products to Europe.

Most non-cooperative game-theoretic models have been developed from the perspective of competing firms within the same market. The models we develop in this thesis relate more to international trade strategy. We investigate the effect of different trade barriers on the strategic behaviour of large sellers of dairy products competing with the domestic seller.

1.5 Thesis outline

The thesis is organized as follows.

Part I, consisting of chapters 2 through 7, presents a suite of optimization models with uncertain milk supply. Chapter 2 describes a model for Fonterra's supply chain. Chapter 3 describes models for uncertain milk supply. Chapter 4 describes the optimization models. Chapter 5 describes the algorithm DOASA, and gives a mathematical proof of the almost-sure convergence of DOASA for linear problems and a discussion of its convergence for quadratic objective functions. Chapter 6 describes the implementation of the optimization models and DOASA, and presents some results of computational testing. Chapter 7 presents the results from some simulation experiments, and a comparison of two policies.

Part II (chapters 8 to 10) presents a game-theoretic model for the European dairy market. Chapter 8 describes the European dairy market. Chapter 9 presents the game-theoretic model and the algorithm we use to compute Nash equilibrium. In this model Fonterra and Australia are leaders, and the EU is a follower. Chapter 10 presents the results of computational experiments for different game settings.

Chapter 11 summarizes the thesis, describes how this research benefits Fonterra, and outlines the limitations and suggestions for future research.

Part I

Optimization models with uncertain milk supply

Chapter 2

A model for Fonterra's supply chain

In this chapter we describe a model for Fonterra's supply chain. The decision horizon is a twelve-month production year. Milk is collected from farmers and transported to factories for manufacturing products. Products are transported to stores. Products in stores are either retained as inventory, or sold to international spot markets or by contract. In production planning, a policy determines for each month, what to produce, what to sell, what contract to make and what to store, with information of past contracts, past inventory and current milk supply.

Fonterra collects milk in several regions in New Zealand and transports milk to manufacturing sites in the region for processing (see Figure 2.1). We assume in general that milk in each region is transported to a factory in that region. This means that the transportation cost is fixed for any milk supply outcome, and so it can be ignored in the optimization. This assumption is relaxed slightly by assuming that up to a pre-defined percentage of milk supply in a region can be transported to an adjacent region, which incurs some transportation cost to be accounted for in the objective functions. Given historical milk supply data for several years, we derive a model for uncertain milk supply for a production year, which will be presented in chapter 3.

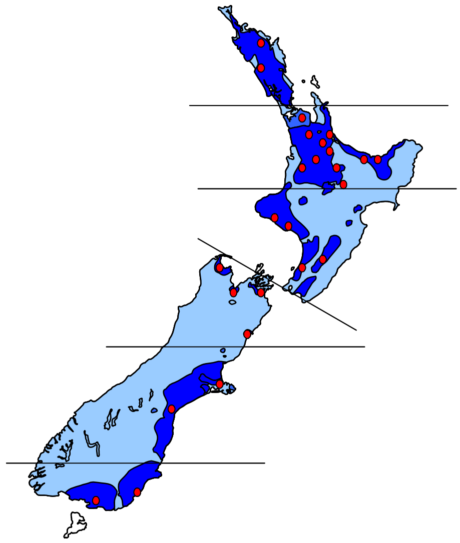

Figure 2.1: Regions of milk collection and production. Here blue indicates the density of milk production, the red dots are manufacturing sites and the lines divide New Zealand into six regions.

Fonterra has manufacturing sites around New Zealand, and each site operates different processes to produce different products. We aggregate these sites to form one factory in each region and each factory is able to operate all processes. We display a production schematic in Figure 2.2. Each process produces a mixture of products, and each factory has a daily process capacity for each process, which remains constant throughout the year. We assume that in each month, each factory operates a process at a constant daily rate under the daily process capacity. Products are categorized as perishable products or powder products. We assume products have infinite life times and the qualities of all products remain unchanged over time. In reality this assumption is not valid since perishable products deteriorate the longer that they are stored, but we observe in all the solutions that we have obtained (presented in chapter 7) that inventory is completely replenished in at most six months, which means that this deterioration can be ignored. The marginal production cost of a product is constant between processes and over months.

Products manufactured in factories are transported to storage places. Perishable products are stored in refrigerated stores, and powder products are stored in dry

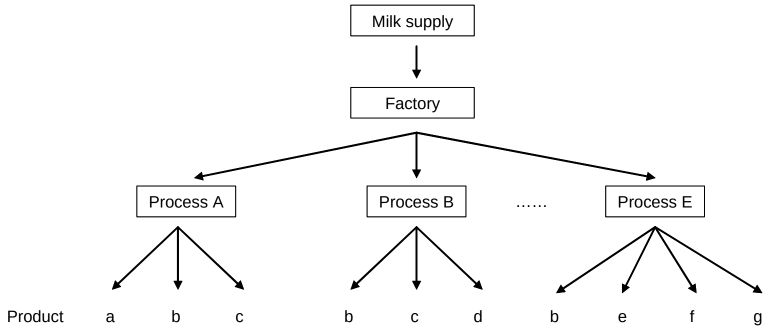

Figure 2.2: A production schematic.

stores. Fonterra has storage places around New Zealand, but we aggregate them into one refrigerated store and one dry store, and the storage capacity is set to be the total of the individual storage capacities. We assume that transportation cost to storage is low compared with the international freight cost and so we ignore it. The marginal costs for storage of different products are different, but remain constant throughout the year. Inventory is used to smooth variation in demands and thus it is an important feature in production planning.

Products can be sold to international *spot markets*. In a spot market, demand responds to market price and this is characterized by a price-demand curve. A generic price-demand curve is displayed in Figure 2.3, in which demand increases as market price decreases. These curves represent the residual demand in each region that Fonterra observes from the actual demand minus what is supplied by other producers. Note that we require these curves to be independent of Fonterra's sales policy, so we do not envisage any strategic selling by other producers. We assume that demand matches sales in the market, and assume a linear price-demand curve. The assumption of linearity is made not only for simplicity, but also for computational convenience, as this means that the revenue is a concave quadratic function of sales, which gives a convex programming problem with quadratic objective function to solve in each stage. Optimizing with a price-demand curve requires careful planning to balance production and market price to achieve the highest earnings.

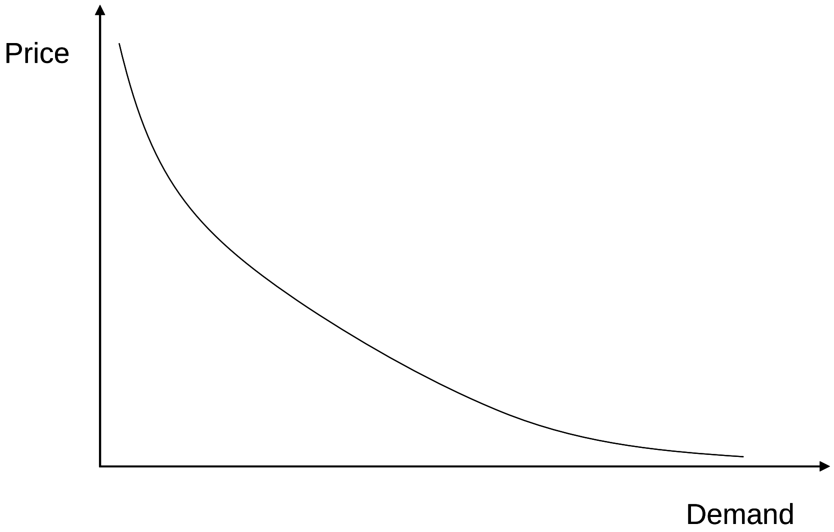

Figure 2.3: A generic price-demand curve.

There is a market capacity for sales of each product in each spot market and month. Since Fonterra has some long-term customers and their demands must be met, we assume there is a minimum demand in each market that is a lower bound on sales. We assume that products are transported from stores to spot markets via export ports. There is a marginal cost for transportation from a store to an export port (inland transportation), and there is a marginal cost for transportation from an export port to a spot market (overseas transportation). The marginal transportation costs are different between products and markets, but are constant over months.

Some products (called contract products) can be sold by *contract* at a predetermined price, while other products (non-contract products) are sold only at a spot price. Although there are some contracts with different durations, we assume that all contracts are arranged three months in advance. In each month there is one contract made for each product and the contract size is subject to an upper limit. Contracts are delivered at the exact amount, and variations are unacceptable. The contract price varies over months. There is a marginal transportation cost for delivery of contract products. Contracts require inventory, and thus contracting is an important factor affecting production planning.

At the beginning of a year, there is inventory and contracts that result from decisions made in the previous year. We call them *start-of-year inventory* and *start-of-year contracts*. On the other hand, inventory is required at the end of the year to enable Fonterra to meet minimum demands and contracts in the following year, particularly in the first several months when inventory is increasing only slowly. This gives an *inventory target* to be met or exceeded at the end of the year.

Given that milk supply is uncertain, production and inventory may not be able to meet minimum demand, contracts or the inventory target at the end of the year. To ensure feasibility in all outcomes we assume that products can be bought from other sources. Such *distress trading* is not encouraged in practice as it carries an implicit cost, which will vary with product and time of the year. We assume that this cost is a constant multiple of the highest spot price for the product in that period, and estimate the multiple to be the lowest value such that no distress trading occurs in a deterministic production plan with forecast milk supply. It is important to observe that the model of distress trading prices will be an important driver in the optimization, so in the absence of more exact information we choose the distress prices in our base model to be low to avoid over-estimating the value of well-hedged policies. We also test the sensitivity of the model to these prices by decreasing them below the base values and recomputing optimal policies.

Chapter 3

Models for uncertain milk supply

In this chapter we will describe our models for uncertain milk supply. We first derive the forecast milk supply for a production year, and then based on this we derive an additive model for uncertain milk supply and a multiplicative model for uncertain milk supply.

First of all we present our notation.

Sets and indices

i, i'	$=$	$1, 2, \ldots, I$	regions.
n	$=$	$1, 2, \ldots, I$	index for eigenvalues.
h	$=$	$1, 2, \ldots, H$	production years.
t	$=$	$1, 2, \ldots T$	months in a production year.
k	$=$	$1, 2, \ldots, K_t$	index of the values in Ω_t.
m	$=$	$1, 2, \ldots, M_t$	index of the values in Φ_t.
a_k	$\in$	Ω_t	set of independent random variable in month t.
b_m	$\in$	Φ_t	set of Markov states in month t.

Parameters and variables

$\bar{t}$	the first month with unpredictable milk supply.
ε_t	a vector of I random errors in month t.
ε_{it}	a random error in region i and month t.
ε_{th}	a vector of I random errors in month t and year h.

ε_0	the start-of-year random error.
ε_{i0}	the start-of-year random error in region i.
$\sigma_{it}, \hat{\sigma}_{it}$	standard deviation in region i in month t, and its estimate.
$D_t, \hat{D}_t$	a diagonal matrix of σ_{it}'s, and its estimate.
η_t	a vector of I multivariate normal random variables in month t.
η_{it}	ith component of η_t.
Υ_t	variance-covariance matrix of $D_t\varepsilon_t$.
erf	cumulative distribution function of a standard normal distribution.
$\Xi, \hat{\Xi}$	variance-covariance matrix of ε_t in month t, and its estimate.
$\Sigma, \hat{\Sigma}$	variance-covariance matrix of η_t in month t, and its estimate.
$\rho, \hat{\rho}$	serial correlation coefficient of ε_t, and its estimate.
Λ	diagonal matrix of eigenvalues of $\hat{\Xi}$ decreasing down the diagonal.
λ_n	the nth largest eigenvalues of Λ.
V	matrix of eigenvectors of $\hat{\Xi}$, ordered according to Λ.
V_{in}	the ith component in the eigenvector for λ_n.
ω_t	a vector of independent random variables in month t.
ω_{tn}	the nth random variable in ω_t.
$\Pr_{\Omega_t}$	probability of sampling ω_{tn} from Ω_t.
ϕ_t	a Markov state in month t.
$\Pr_{\Phi_t \mid \Phi_{t-1}}$	probability of sampling ϕ_t from Φ_t given ϕ_{t-1} in Φ_{t-1}.
$f_t, \hat{f}_t$	base of regional growth for month t, and its estimate.
f_{it}	base of regional growth for region i and month t.
$g_t, \hat{g}_t$	slope of regional growth for month t, and its estimate.
g_{it}	slope of regional growth for region i and month t.
N	the number of realizations of ω_{tn}.
s_t	milk supply in month t.
s_{it}	milk supply in region i and month t.
s'_{it}	dummy variable ensuring $s_{it} \geq 0$ for region i and month t.
s_{th}	milk supply in month t and year h.
s_{ith}	milk supply in region i, month t and year h.

3.1 Forecast milk supply

We observe that the regional growth over years in each region and month is close to linear, so we assume the forecast milk supply in year h, month t and region i is

$$s_{it} = f_{it} + hg_{it}, \quad i = 1, 2, \dots, I,$$
$$t = 1, 2, \dots, T,$$
$$h = 1, 2, \dots, H,$$

where f_{it} and g_{it} are constants. In vector form the forecast milk supply in month t in year h, $h = 1, 2, \dots, H$, is

$$s_t = f_t + hg_t, \quad t = 1, 2, \dots, T,$$

and the actual milk supply is

$$s_t = f_t + hg_t + D_t\varepsilon_t, \quad t = 1, 2, \dots, T,$$

where ε_t is a vector of I random errors, having variance-covariance matrix Ξ that captures the regional correlation in the random variation of milk supply.

f_t, g_t and D_t are estimated from historical data using maximum likelihood estimation (see [65]). Given Υ_t, the variance-covariance matrix of $D_t\varepsilon_t$, the ith diagonal component of D_t is the square root of the ith diagonal component in Υ_t. Given the estimates $\hat{f}_t$ of f_t and $\hat{g}_t$ of g_t, a maximum-likelihood estimate $\hat{\Upsilon}_t$ of Υ_t is

$$\hat{\Upsilon}_t(i, i') = \frac{1}{H} \sum_{h=1}^{H} (s_{ith} - \hat{f}_{it} - h\hat{g}_{it})\,(s_{i'th} - \hat{f}_{i't} - h\hat{g}_{i't}), \quad \forall i, i' = 1, 2, \dots, I, \tag{3.1}$$

where $\hat{\Upsilon}_t(i, i')$ is the component in the ith row and the i'th column of $\hat{\Upsilon}_t$. On the other hand, given $\hat{\Upsilon}_t$, we can obtain estimates $\hat{f}_t$ and $\hat{g}_t$ by solving

$$\min_{f_t, g_t} \sum_{h=1}^{H} (s_{th} - f_t - hg_t)^\intercal\, \hat{\Upsilon}_t^{-1}\, (s_{th} - f_t - hg_t). \tag{3.2}$$

This results in

$$\sum_{h=1}^{H} 2\hat{\Upsilon}_t^{-1}(s_{th} - \hat{f}_t - h\hat{g}_t) = 0$$
$$\sum_{h=1}^{H} 2h\hat{\Upsilon}_t^{-1}(s_{th} - \hat{f}_t - h\hat{g}_t) = 0,$$

which gives

$$\hat{\Upsilon}_t^{-1}\hat{f}_t \sum_{h=1}^{H} 1 + \hat{\Upsilon}_t^{-1}\hat{g}_t \sum_{h=1}^{H} h = \hat{\Upsilon}_t^{-1} \sum_{h=1}^{H} s_{th}$$
$$\hat{\Upsilon}_t^{-1}\hat{f}_t \sum_{h=1}^{H} h + \hat{\Upsilon}_t^{-1}\hat{g}_t \sum_{h=1}^{H} h^2 = \hat{\Upsilon}_t^{-1} \sum_{h=1}^{H} h\, s_{th}. \tag{3.3}$$

Observe that multiplying both sides of equation (3.3) by $\hat{\Upsilon}_t$ gives

$$\hat{f}_t \sum_{h=1}^{H} 1 + \hat{g}_t \sum_{h=1}^{H} h = \sum_{h=1}^{H} s_{th}$$
$$\hat{f}_t \sum_{h=1}^{H} h + \hat{g}_t \sum_{h=1}^{H} h^2 = \sum_{h=1}^{H} h\, s_{th},$$

which is independent of $\hat{\Upsilon}_t$, and can be solved for $\hat{f}_t$ and $\hat{g}_t$. Then we can calculate $\hat{\Upsilon}_t$ from observations of s_{ith}, and $\hat{f}_t$ and $\hat{g}_t$, using equation (3.1), and then $\hat{D}_t$ from $\hat{\Upsilon}_t$ and thus $\hat{\sigma}_{it}$, $i = 1, 2, \ldots, I$.

Note that the maximum likelihood estimation above is the same as performing a least-square estimation (see [65]) for $\hat{f}_t$ and $\hat{g}_t$ in each region, since $\hat{f}_t$ and $\hat{g}_t$ have one component for each region. If we assume that $\hat{f}_t$ and $\hat{g}_t$ are univariate over all regions, that is, the components in $\hat{f}_t$ and $\hat{g}_t$ have the same values, then we will have a different result from (3.2), which is dependent on $\hat{\Upsilon}_t$, and thus the estimation will differ from the least-square estimation.

We assume $\hat{\Xi}$ to be the variance-covariance matrix of $\hat{\sigma}_{it}^{-1}(s_{ith} - \hat{f}_{it} - h\hat{g}_{it})$ for all t, which is estimated from observations of s_{ith} using

$$\hat{\Xi}(i,i') = \frac{1}{TH} \sum_{t=1}^{T} \sum_{h=1}^{H} \hat{\sigma}_{it}^{-1}\hat{\sigma}_{i't}^{-1}(s_{ith} - \hat{f}_{it} - h\hat{g}_{it})(s_{i'th} - \hat{f}_{i't} - h\hat{g}_{i't}), \quad \forall i, i' = 1, 2, \ldots, I.$$

We give an example of processing the real data provided by Fonterra to illustrate this model. Figure 3.1 displays the actual milk supply and the forecast using the model in a particular region in January. For this region, $\hat{f}_{it}$ is 87 million and $\hat{g}_{it}$ is 18 million. In Figure 3.2, the estimate of $\hat{\Xi}$ from observations of s_{ith} (with six regions) shows strong correlation between adjacent regions.

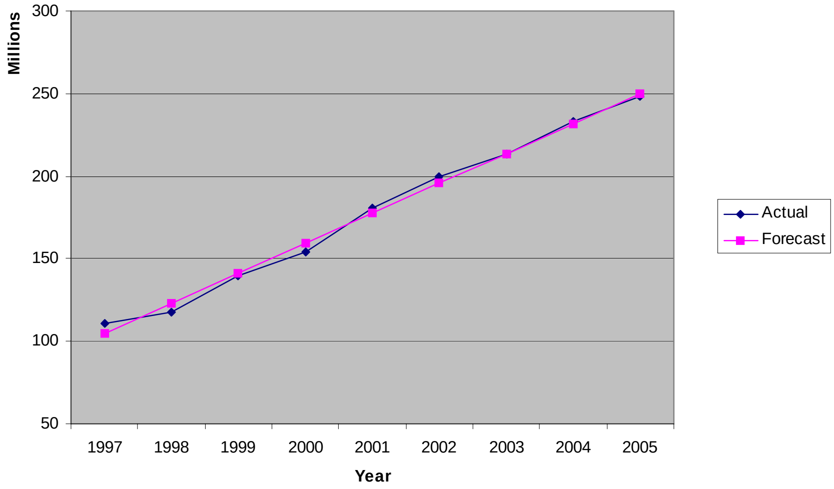

Figure 3.1: Actual milk supply and the forecast in one region in January.

1	0.52	0.14	-0.10	-0.03	0.18
0.52	1	0.46	0.28	0.31	0.32
0.14	0.46	1	0.68	0.54	0.63
-0.10	0.28	0.68	1	0.64	0.51
-0.03	0.31	0.54	0.64	1	0.70
0.18	0.32	0.63	0.51	0.70	1

Figure 3.2: The estimate of $\hat{\Xi}$ in an example.

3.2 Additive model

We derive an additive model for uncertain milk supply based on the forecast milk supply. The milk supply in month t in year h, $h = 1, 2, \ldots, H$, is

$$s_t = f_t + hg_t + D_t\varepsilon_t, \quad t = 1, 2, \ldots, T,$$

where the estimates $\hat{f}_t$, $\hat{g}_t$, $\hat{D}_t$ and $\hat{\Xi}$ are calculated as described above. We display ε_t in six regions in year 2005 in Figure 3.3. We observe strong serial correlations in ε_t's in each region, since ε_t's in each region follow the same path throughout the year. We also observe strong spacial correlations between ε_t's in each month, since ε_t's in each month move in the same direction.

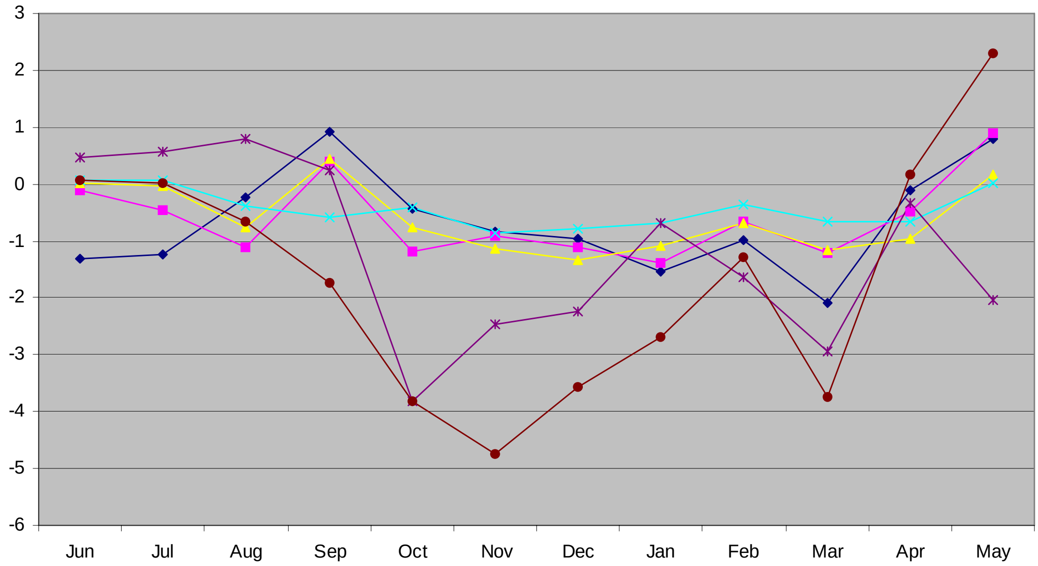

Figure 3.3: Residuals in six regions.

Accounting for these two types of correlations, we define an autoregressive model for ε_t. Given the *start-of-year random error* ε_0, let

$$\varepsilon_t = diag(\rho)\, \varepsilon_{t-1} + \eta_t, \quad t = 1, 2, \ldots, T, \tag{3.4}$$

in which ρ is a coefficient for linear serial correlation of ε_t, and η_t is a multivariate normal random variable with variance-covariance matrix Σ.

The estimation of parameters ρ and Σ is done using maximum likelihood. By equation (3.4), given that ε_{t-1} and η_t are independent of each other, we have

$$\begin{aligned}
E[\varepsilon_t^\intercal \varepsilon_t] &= E[(\rho\varepsilon_{t-1} + \eta_t)^\intercal (\rho\varepsilon_{t-1} + \eta_t)] \\
&= E[\rho^2 \varepsilon_{t-1}^\intercal \varepsilon_{t-1} + 2\rho\varepsilon_{t-1}^\intercal \eta_t + \eta_t^\intercal \eta_t] \\
&= \rho^2 E[\varepsilon_{t-1}^\intercal \varepsilon_{t-1}] + E[2\rho\varepsilon_{t-1}^\intercal \eta_t] + E[\eta_t^\intercal \eta_t] \\
&= \rho^2 E[\varepsilon_{t-1}^\intercal \varepsilon_{t-1}] + 2\rho E[\varepsilon_{t-1}^\intercal] E[\eta_t] + E[\eta_t^\intercal \eta_t] \\
&= \rho^2 E[\varepsilon_{t-1}^\intercal \varepsilon_{t-1}] + E[\eta_t^\intercal \eta_t]
\end{aligned}$$

which implies

$$\Xi = \rho^2 \Xi + \Sigma,$$

and thus given an estimate $\hat{\rho}$ of ρ, we estimate Σ by

$$\hat{\Sigma} = (1 - \hat{\rho}^2)\hat{\Xi}. \tag{3.5}$$

On the other hand, given $\hat{\Sigma}$, and observed errors ε_{th} and $\varepsilon_{t-1,h}$, $h = 1, 2, \ldots, H$, we can obtain a maximum likelihood estimate $\hat{\rho}$ by solving

$$\min_{\rho} \sum_{t=1}^{T} \sum_{h=1}^{H} (\varepsilon_{th} - \rho\varepsilon_{t-1,h})^\top \hat{\Sigma}^{-1} (\varepsilon_{th} - \rho\varepsilon_{t-1,h}). \tag{3.6}$$

We substitute equation (3.5) into (3.6), then $\hat{\rho}$ is given by solving

$$\min_{\rho} \sum_{t=1}^{T} \sum_{h=1}^{H} \frac{(\varepsilon_{th} - \rho\varepsilon_{t-1,h})^\top \hat{\Xi}^{-1} (\varepsilon_{th} - \rho\varepsilon_{t-1,h})}{(1-\rho^2)},$$

which results in

$$\min_{\rho} \frac{\rho^2 \sum_{t=1}^{T} \sum_{h=1}^{H} \varepsilon_{t-1,h}^\top \hat{\Xi}^{-1} \varepsilon_{t-1,h} - 2\rho \sum_{t=1}^{T} \sum_{h=1}^{H} \varepsilon_{th}^\top \hat{\Xi}^{-1} \varepsilon_{t-1,h} + \sum_{t=1}^{T} \sum_{h=1}^{H} \varepsilon_{th}^\top \hat{\Xi}^{-1} \varepsilon_{th}}{(1-\rho^2)}.$$

Let

$$a = \sum_{t=1}^{T} \sum_{h=1}^{H} \varepsilon_{t-1,h}^\top \hat{\Xi}^{-1} \varepsilon_{t-1,h}, \quad b = -2 \sum_{t=1}^{T} \sum_{h=1}^{H} \varepsilon_{th}^\top \hat{\Xi}^{-1} \varepsilon_{t-1,h}, \quad c = \sum_{t=1}^{T} \sum_{h=1}^{H} \varepsilon_{th}^\top \hat{\Xi}^{-1} \varepsilon_{th},$$

which are constants given $\hat{\Xi}^{-1}$, and observed errors ε_{th} and $\varepsilon_{t-1,h}$, $h = 1, 2, \ldots, H$. The maximum likelihood estimate $\hat{\rho}$ is given by

$$\begin{aligned}
\hat{\rho} &= \arg\min_{\rho \in [0,1]} \frac{a\rho^2 + b\rho + c}{(1-\rho^2)} \\
&= \frac{-a - c + \sqrt{(a+c)^2 - b^2}}{b}.
\end{aligned}$$

To construct a milk scenario, we sample a vector of I independent random variables ω_t with each random variable ω_{tn}, $n = 1, 2, \ldots, I$, being sampled independently from a standard normal distribution. We generate η_t using

$$\eta_t = (1 - \hat{\rho}^2)^{\frac{1}{2}} V \Lambda^{\frac{1}{2}} \omega_t, \quad t = 1, 2, \ldots, T.$$

Here Λ is a diagonal matrix of eigenvalues of $\hat{\Xi}$ with the nth component being larger than the $(n+1)$th component, and V is a matrix of eigenvectors of $\hat{\Xi}$. This gives a vector η_t with variance-covariance matrix

$$\begin{aligned}(1-\hat{\rho}^2)V\Lambda V^\top &= (1-\hat{\rho}^2)^\top\hat{\Xi} \\ &= \hat{\Sigma}.\end{aligned}$$

This is the *principal component* approach to obtain a vector of correlated random variables (η_t) from a vector of independent random variables (ω_t) (see [59]).

The model described above can be used to generate an infinite number of synthetic milk scenarios from which a finite number may be constructed or sampled to give a finite approximation to the stochastic programming problem. Ideally any solution to the stochastic programming problem would be tested with a set of *out-of-sample* milk scenarios from a new data set. Unfortunately such data were not available, and so we have chosen to test the policies (in chapter 7) using synthetic milk scenarios generated by the model above with all principal components included.

The finite scenario tree that we use to approximate the milk supply process is constructed as follows. We first approximate a standard univariate normal distribution with a discrete distribution by minimizing the *Wasserstein metric*. This is a well-known technique in stochastic programming (see [87]). Suppose we use a set of finite values for a random variable which are ordered from the smallest to the largest. We calculate the mid-point of every two successive values in the set. Then the sum of probabilities for the first j values is the cumulative probability of the jth mid-point, and the probability for the last value is one minus the total of probabilities of the previous values. We illustrate this using an example in Figure 3.4. The density curve for a standard normal distribution is displayed. The three blue dots are the values in a set, and the two black vertical lines sit on the mid-points between these. Under the Wasserstein metric, the probability of sampling the first value is defined to be the area (A) under the density function curve up to the first black line. Then the probability of sampling the second value is the area

(B) between the two black lines, and the probability of sampling the third value is the area (C).

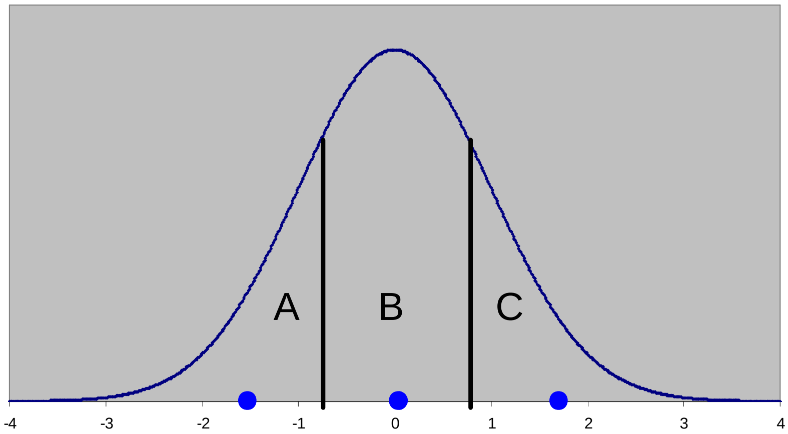

Figure 3.4: An example of constructing an approximation.

In general, suppose we use a set Ω_t of K_t values for ω_{tn}, $n = 1, 2, \ldots, I$, in the discrete distribution, then there are K_t^I outcomes of ω_t in each stage t and thus there are $\prod_{t=1}^{T} K_t^I$ milk scenarios. Hence even with a small value of K_t, the sizes of the scenario trees obtained are very large. To reduce the number of scenarios further we use only a subset of the principal components (see [59]). In other words, we restrict the component directions to the eigenvectors of $\hat{\Xi}$ corresponding to the $N < I$ largest eigenvalues in Λ, say λ_n for $n = 1, 2, \ldots, N$, and then we use ω_{tn}, $n = 1, 2, \ldots, N$, in generating η_t.

Since each ω_{tn} can take negative values in sampling, this may result in a negative milk supply which is physically impossible. To prevent this we use a dummy variable $s'_{it} \geq 0$ with a high penalty cost that makes $s_{it} = 0$ when the scenario milk supply is negative. In practice we observe that milk supply before month $\bar{t}$ which is the first month after the peak month is easy to predict, but milk supply in and after this month is not. Thus we assume s_{it} for $t < \bar{t}$ in year h is the forecast milk supply, and we assume there is random variation in milk supply in month $t \geq \bar{t}$. We define the set of random variables as

$$\Omega_t = \{0\}, \quad t < \bar{t},$$
$$\Omega_t = \{a_1, a_2, \dots, a_{K_t}\}, \quad t \geq \bar{t}.$$

Now we formally state the model for milk supply for year h as, given ε_{i0},

$$\begin{aligned}
s_{it} &= f_{it} + hg_{it} + \sigma_{it}\varepsilon_{it} + s'_{it}, && \forall i = 1, 2, \dots, I,\ t = 1, 2, \dots, T, \\
\varepsilon_{it} &= \rho\varepsilon_{i,t-1} + \eta_{it}, && \forall i = 1, 2, \dots, I,\ t = 1, 2, \dots, T, \\
\eta_{it} &= \textstyle\sum_{n=1}^{N}(1-\rho^2)^{\frac{1}{2}} V_{in}\lambda_n^{\frac{1}{2}}\omega_{tn}, && \forall i = 1, 2, \dots, I,\ t = 1, 2, \dots, T, \\
\omega_{tn} &\sim (\Omega_t, \Pr_{\Omega_t}), && \forall t = 1, 2, \dots, T,\ n = 1, 2, \dots, N,
\end{aligned}$$

with the probability distribution $(\Omega_t, \Pr_{\Omega_t})$ as

$$\begin{aligned}
&\Omega_t = \{0\}, \quad t < \bar{t}, \\
&\Pr_{\Omega_t}(0) = 1, \quad t < \bar{t}, \\
&\Omega_t = \{a_1, a_2, \dots, a_{K_t}\}, \quad t \geq \bar{t}, \\
&\sum_{j=1}^{k} \Pr_{\Omega_t}(a_j) = \operatorname{erf}\Big(\frac{a_k + a_{k+1}}{2}\Big), \quad k = 1, 2, \dots, K_t - 1, \quad t \geq \bar{t}, \\
&\Pr_{\Omega_t}(a_{K_t}) = 1 - \sum_{k=1}^{K_t - 1} \Pr_{\Omega_t}(a_k), \quad t \geq \bar{t}.
\end{aligned}$$

To illustrate this model, we give an example of processing the real milk data provided by Fonterra. A production year has twelve months starting from June. We display the real milk supply in a particular region over nine years in Figure 3.5. The figure shows that milk supply before and in the peak production month, October, is relatively deterministic, but after this month it is not. Similar behaviours are observed in the other regions. Thus November, which is the sixth month of the production year, is the first month with unpredictable milk supply.

We set $\varepsilon_{i0} = 0$ and choose $N = 2$, so ω_t has two components. For $t \geq 6$, we

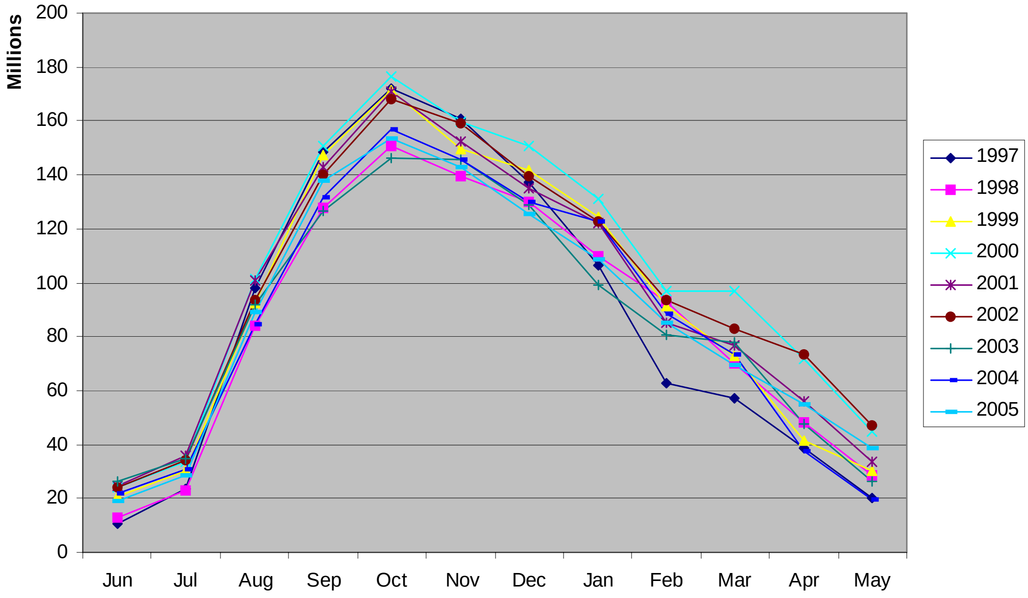

Figure 3.5: Real milk data in a particular region over nine years.

choose $K_t = 3$, giving a three-point probability distribution defined by

$$\Omega_t = \{-1.68, 0, 1.68\},$$

$$\Pr_{\Omega_t} = \begin{cases} 0.2, & \text{for } -1.68, \\ 0.6, & \text{for } 0, \\ 0.2, & \text{for } 1.68. \end{cases}$$

This results in 4.8 million milk scenarios. For the same region, we sample 100 milk scenarios from the approximation, and display the coverage of this sample along with the forecast milk supply in Figure 3.6. Note that we have assumed no random variation from June to October and thus milk supply in these months are the forecast milk supply. We observe large random variation in milk supply in each month.

We then superpose the coverage of the residuals of the sample onto the residuals of the nine years of real data from the forecast milk supply in Figure 3.7. Note that the residuals from June to October are zero. The bold curves are the upper bound and lower bound of the coverage. Most of the residuals of the real milk supply are

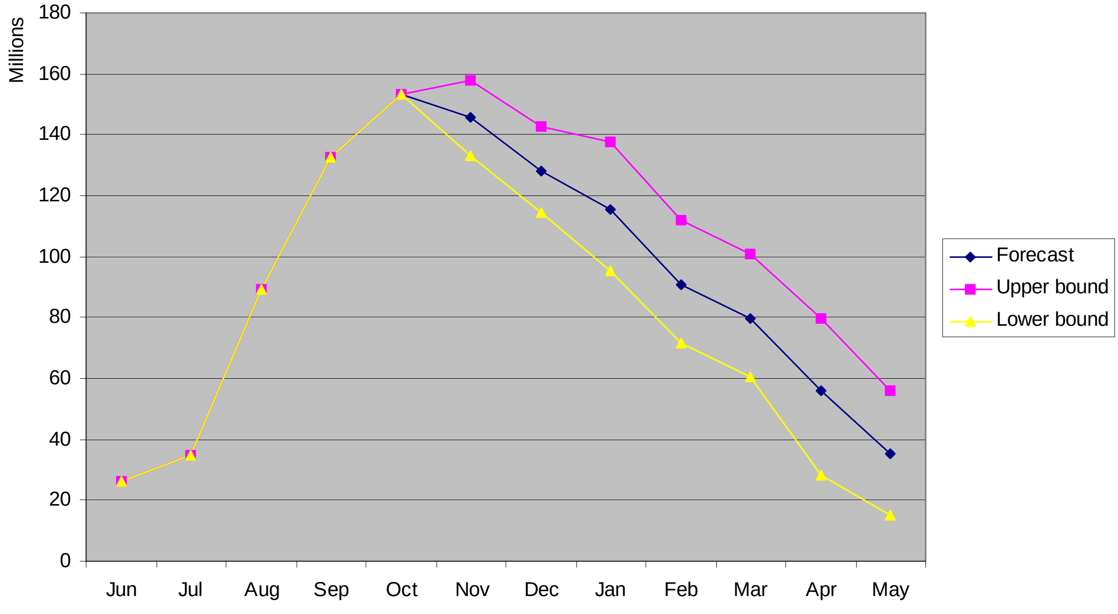

Figure 3.6: Coverage of a sample of milk scenarios from the additive model.

between the bounds.

3.3 Multiplicative model

In the previous section we presented a model for milk supply where the forecast milk supply $(f_t + hg_t)$ and the random error (ε_t) are in an additive form. Here we derive a different model for milk supply in which the forecast milk supply and the random error are in a multiplicative form. In this model milk supply varies from forecast milk supply by a random proportion ε_t. We assume this proportion is constant over regions and thus milk supply in each region has the same deviation in proportion from the forecast milk supply in each month. The milk supply in month t in year h, $h = 1, 2, \ldots, H$, is

$$s_t = (f_t + hg_t)\varepsilon_t, \quad t = 1, 2, \ldots, T.$$

Here the estimates $\hat{f}_t$ of f_t and $\hat{g}_t$ of g_t are already known, and ε_t is a scalar.

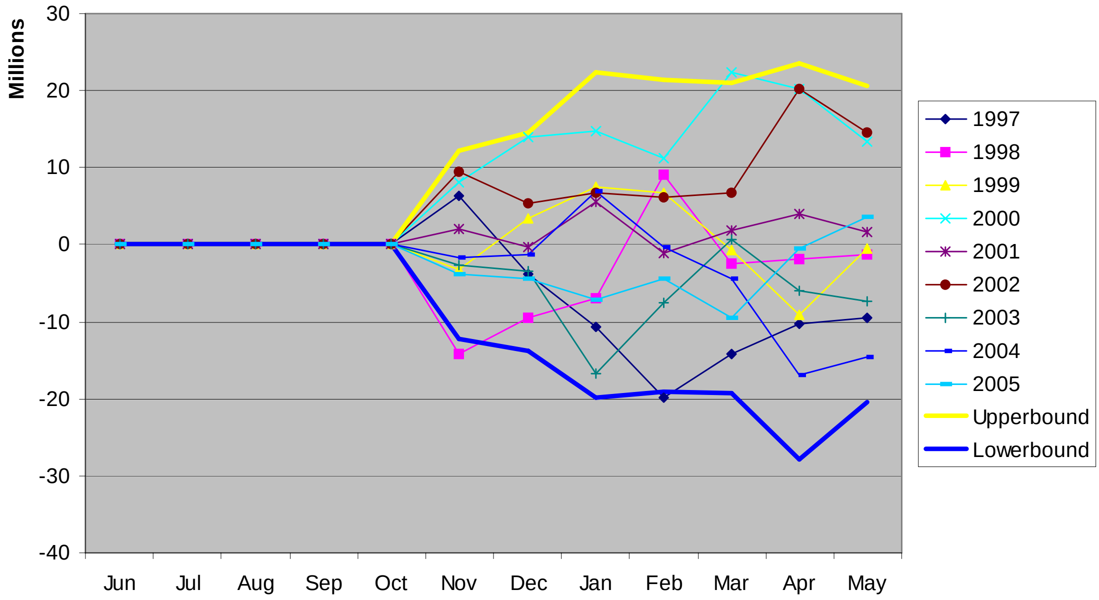

Figure 3.7: Residuals of the real data and the coverage of residuals in a sample of milk scenarios.

We model ε_t by two random variables ϕ_t and ω_t,

$$\varepsilon_t = \phi_t \omega_t,$$

where ϕ_t is the *state* in a Markov chain that is sampled from the finite set Φ_t conditional on ϕ_{t-1}, ω_t is independently sampled from Ω_t, and $\phi_t\omega_t$ is unique. Using two random variables with different properties enables a richness of milk scenarios.

We use a Markov chain model for ϕ_t (following the method used in [39]). A Markov chain is a series of random variables over stages which are called *states*, where the state in each stage is (only) dependent on the state in the previous stage, and the probability of each realization is defined by a transition matrix. Without loss of generality, we define the state in each stage t by z_t with each being sampled from a finite set Z_t. In a mathematical form, in a Markov chain $\{z_1, z_2, \dots\}$,

$$\Pr(z_t \mid z_{t-1}, z_{t-2}, \dots, z_1) = \Pr(z_t \mid z_{t-1}).$$

Suppose Z_t consists of q_t values in each stage t, that is,

$$Z_t = \{z_{t,i},\ i = 1, 2, \ldots, q_t\}.$$

Define the transition matrix by $\Pr_{Z_t|Z_{t-1}}$. $\Pr_{Z_t|Z_{t-1}}$ is a q_{t-1} by q_t matrix, and is defined as

$$\begin{aligned} \Pr_{Z_t|Z_{t-1}}(i,j) = \Pr(z_{t,j} \mid z_{t-1,i}), \quad & j = 1, 2, \ldots, q_t, \\ & i = 1, 2, \ldots, q_{t-1}, \end{aligned}$$

where $\Pr_{Z_t|Z_{t-1}}(i,j)$ is the component in the ith row and jth column. $\Pr_{Z_t|Z_{t-1}}$ may vary over stages. Note that if in each stage t, $\Pr(z_{t,j} \mid z_{t-1,i})$ is independent of $z_{t-1,i}$, then the Markov chain degenerates into a series of independent random variables.

Now we return to our multiplicative model for uncertain milk supply. The milk supply in year h, $h = 1, 2, \ldots, H$, is

$$\begin{aligned} s_t &= (f_t + hg_t)\phi_t\omega_t, & t &= 1, 2, \ldots, T, \\ \phi_t &\sim (\Phi_t, \Pr_{\Phi_t|\Phi_{t-1}}), & t &= 1, 2, \ldots, T, \\ \omega_t &\sim (\Omega_t, \Pr_{\Omega_t}), & t &= 1, 2, \ldots, T, \end{aligned}$$

where ϕ_t is a state sampled from Φ_t with a transition matrix $\Pr_{\Phi_t|\Phi_{t-1}}$, and ω_t is independently sampled from Ω_t with a probability $\Pr_{\Omega_t}$. In each stage t, given ϕ_{t-1}, the probability for a realization of $\phi_t\omega_t$ is

$$\Pr_{\Phi_t|\Phi_{t-1}}(\phi_t \mid \phi_{t-1}) \Pr_{\Omega_t}(\omega_t).$$

which implies that the realization of $\phi_t\omega_t$ is dependent on the state ϕ_{t-1} but not on ω_{t-1}.

We assume milk supply in month $t < \bar{t}$ is the forecast milk supply and we take account for random variations in milk supply in month $t \geq \bar{t}$. We define $(\Phi_t, \Pr_{\Phi_t|\Phi_{t-1}})$ and $(\Omega_t, \Pr_{\Omega_t})$ for $t < \bar{t}$ as

$$\begin{aligned} \Phi_t &= \{1\}, \quad \Pr_{\Phi_t|\Phi_{t-1}}(\phi_t = 1 \mid \phi_{t-1} = 1) = 1, \\ \Omega_t &= \{1\}, \quad \Pr_{\Omega_t}(\omega_t = 1) = 1, \end{aligned}$$

and we define the sets Φ_t and Ω_t for $t \geq \bar{t}$ as

$$\Phi_t = \{b_1, b_2, \dots, b_{M_t}\},$$

$$\Omega_t = \{a_1, a_2, \dots, a_{K_t}\},$$

with $\Pr_{\Phi_t|\Phi_{t-1}}$ and $\Pr_{\Omega_t}$ estimated from the historical data using maximum likelihood estimation, which is described later. This results in $\prod_{t=\bar{t}}^{T} M_t K_t$ milk scenarios.

In processing the data, since we use the same random proportion for milk supply in each region, we sum up the actual milk supply and forecast milk supply over regions and then calculate the proportions. In general, in any stage t these proportions do not equal any $\phi_t\omega_t$, and thus we transform them into $\phi_t\omega_t$'s using a set of *bins*.

A bin defines a range for the proportions and includes one $\phi_t\omega_t$. We construct one bin for each $\phi_t\omega_t$, and thus there are $M_t K_t$ bins in each stage t. These bins are disjoint, and cover the entire feasible space of proportions in the real data. Then a proportion in the real data must fall into one of these bins, say a bin for a particular $\hat{\phi}_t\hat{\omega}_t$, and we replace this proportion by $\hat{\phi}_t\hat{\omega}_t$.

Note that the sizes of the bins can be arbitrarily chosen. For example, we may use a mid-point method to construct the bins. We list $\phi_t\omega_t$'s in order, and obtain the mid-points of each pair of $\phi_t\omega_t$'s (there are $M_t K_t - 1$ mid-points). For the ith $\phi_t\omega_t$, the lower bound and upper bound of the bin are defined by the $(i-1)$th and ith mid-points. The exceptions are the first and the last $\phi_t\omega_t$, for the former the bin only has an upper bound which is the first mid-point, and for the latter the bin only has a lower bound which is the last mid-point.

After transforming the proportions from the real data into observations of $\phi_t\omega_t$, we estimate $\Pr_{\Phi_t|\Phi_{t-1}}$ using maximum likelihood estimation. For each ϕ_{t-1}, we observe ϕ_t given that ϕ_{t-1} has occurred, and count the number of occurrences of each $\phi_t \in \Phi_t$, which is denoted $x_t(\phi_t \mid \phi_{t-1})$. Then the probability of sampling a particular $\hat{\phi}_t$ given a particular $\hat{\phi}_{t-1}$ is

$$\Pr(\hat{\phi}_t \mid \hat{\phi}_{t-1}) = \frac{x_t(\hat{\phi}_t \mid \hat{\phi}_{t-1})}{\sum_{\phi_t \in \Phi_t} x_t(\phi_t \mid \hat{\phi}_{t-1})}.$$

We repeat this calculation for each $\hat{\phi}_t \in \Phi_t$, and for each $\hat{\phi}_{t-1} \in \Phi_{t-1}$. Then we obtain $\Pr(\phi_t \mid \phi_{t-1})$ for each $\phi_t \in \Phi_t$ and $\phi_{t-1} \in \Phi_{t-1}$, and so thus $\Pr_{\Phi_t|\Phi_{t-1}}$. We repeat this process from stage $\bar{t}$ to the last stage to obtain $\Pr_{\Phi_t|\Phi_{t-1}}$ in these stages.

We then estimate $\Pr_{\Omega_t}$ using a similar method. We count the number of occurrences of each ω_t, denoted $y_t(\omega_t)$. Then the probability of sampling a particular $\hat{\omega}_t$ is

$$\Pr(\hat{\omega}_t) = \frac{y_t(\hat{\omega}_t)}{\sum_{\omega_t \in \Omega_t} y_t(\omega_t)},$$

and we repeat this calculation for each $\hat{\omega}_t \in \Omega_t$ to obtain $\Pr_{\Omega_t}$. Then we repeat this process from stage $\bar{t}$ to the last stage to obtain $\Pr_{\Omega_t}$ in each of these stages.

In Figure 3.8, we use an example to illustrate the construction of bins and estimation of $\Pr_{\Phi_t|\Phi_{t-1}}$ and $\Pr_{\Omega_t}$. Suppose in each stage t,

$$\begin{aligned}\Phi_t &= \{\phi_{t,1}, \phi_{t,2}, \phi_{t,3}\},\\ \Omega_t &= \{\omega_{t,1}, \omega_{t,2}, \omega_{t,3}\},\end{aligned}$$

and for i, $i' = 1, 2, 3$ and j, $j' = 1, 2, 3$,

$$\begin{aligned}\phi_{t,i}\omega_{t,j} &< \phi_{t,i'}\omega_{t,j'}, \text{ if } i < i',\\ \phi_{t,i}\omega_{t,j} &< \phi_{t,i}\omega_{t,j'}, \text{ if } j < j'.\end{aligned}$$

The figure displays the transition of states from ϕ_{t-1} to ϕ_t and the number of observations in the transition, x_1, x_2,..., x_9, as well as the sampling of ω_t and the number of observations in the sampling, y_1, y_2,..., y_9. The figure also shows 9 bins, denoted bin 1, bin 2, ..., bin 9, and their bounds which are a pair of horizontal lines, with every three bins between a pair of long lines having the same ϕ_t. Note that the values of random variables are displayed with the smallest at the top and the largest at the bottom.

The 9 bins, with one for each $\phi_{t,i}\omega_{t,j}$, are constructed by using the mid-point method. The horizontal lines indicate the mid-points, except the two lines at the top and the bottom indicating the lower and upper bounds of the feasible space of the observations. Then each of bin 2 to bin 8 is between a pair of mid-points. Bin

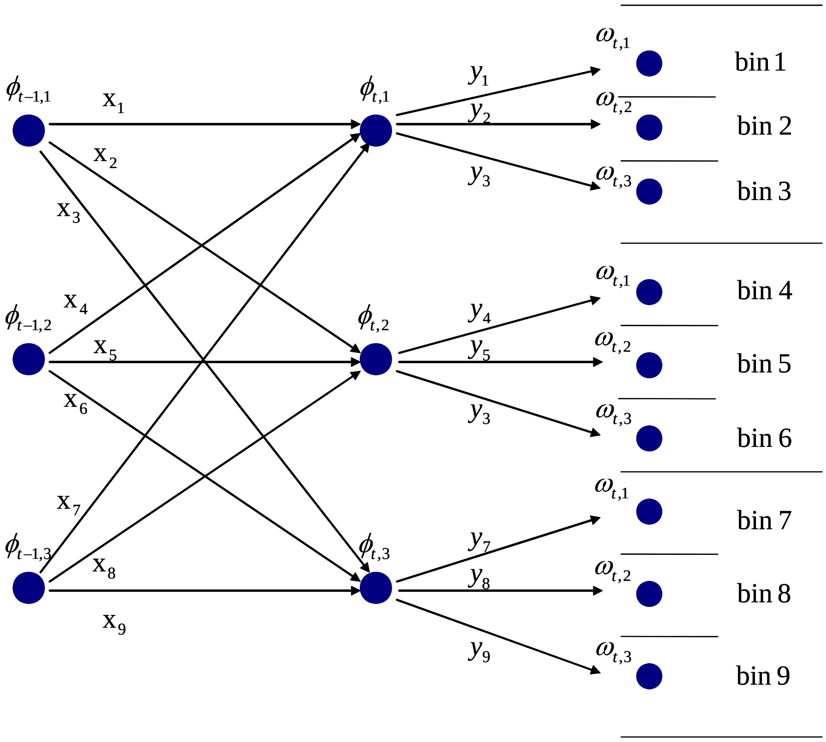

Figure 3.8: Construction of bins and estimation of $\Pr_{\Phi_t|\Phi_{t-1}}$ and $\Pr_{\Omega_t}$

1 has an upper bound at the first mid-point, and a lower bound at the top line. Bin 9 has a lower bound at the last mid-point, and an upper bound at the bottom line. Note that the 9 bins are disjoint and cover the entire feasible space of the observations.

To estimate the transition probability for sampling $\phi_{t,i}$, we need to estimate $\Pr(\phi_{t,i'} \mid \phi_{t-1,i})$ for $i = 1, 2, 3$ and $i' = 1, 2, 3$. For example, given $\phi_{t-1,1}$,

$$\Pr(\phi_{t,1} \mid \phi_{t-1,1}) = \frac{x_1}{x_1 + x_2 + x_3},$$
$$\Pr(\phi_{t,2} \mid \phi_{t-1,1}) = \frac{x_2}{x_1 + x_2 + x_3},$$
$$\Pr(\phi_{t,3} \mid \phi_{t-1,1}) = \frac{x_3}{x_1 + x_2 + x_3}.$$

Then the transition matrix is

$$\begin{bmatrix} \frac{x_1}{x_1+x_2+x_3} & \frac{x_2}{x_1+x_2+x_3} & \frac{x_3}{x_1+x_2+x_3} \\ \frac{x_4}{x_4+x_5+x_6} & \frac{x_5}{x_4+x_5+x_6} & \frac{x_6}{x_4+x_5+x_6} \\ \frac{x_7}{x_7+x_8+x_9} & \frac{x_8}{x_7+x_8+x_9} & \frac{x_9}{x_7+x_8+x_9} \end{bmatrix}.$$

On the other hand, to estimate the probability for sampling $\omega_{t,j}$, we need to estimate $\Pr(\omega_{t,j})$ for $j = 1, 2, 3$. For example,

$$\Pr(\omega_{t,1}) = \frac{y_1 + y_4 + y_7}{y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9}.$$

Then the probability is

$$\begin{aligned} &\frac{y_1 + y_4 + y_7}{y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9} \text{ for } \omega_{t,1}, \\ &\frac{y_2 + y_5 + y_8}{y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9} \text{ for } \omega_{t,2}, \\ &\frac{y_3 + y_6 + y_9}{y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9} \text{ for } \omega_{t,3}. \end{aligned}$$

So far we have assumed different Φ_t, Ω_t, $\Pr_{\Phi_t|\Phi_{t-1}}$ and $\Pr_{\Omega_t}$ in each stage. However, if the size of the set of real data is small, then these estimators are likely to have a large variance. In this circumstance, we may assume identical Φ_t and $\Pr_{\Phi_t|\Phi_{t-1}}$ in each stage. Then the probability of sampling a particular $\hat{\phi}_t$ given a particular $\hat{\phi}_{t-1}$ is

$$\Pr(\hat{\phi}_t \mid \hat{\phi}_{t-1}) = \frac{\sum_{t \geq \bar{t}+1} x_t(\hat{\phi}_t \mid \hat{\phi}_{t-1})}{\sum_{t \geq \bar{t}+1} \sum_{\phi_t \in \Phi_t} x_t(\phi_t \mid \hat{\phi}_{t-1})}$$

where $\hat{\phi}_{t-1}$ are identical in each stage t, and so are $\hat{\phi}_t$. Note that $x_{\bar{t}}(\hat{\phi}_{\bar{t}} \mid \hat{\phi}_{\bar{t}-1})$ are not used since the $\Phi_{t-1} = \{1\}$, otherwise the estimate would be biased. We repeat this for each $\hat{\phi}_t \in \Phi_t$, and repeat for each $\hat{\phi}_{t-1} \in \Phi_{t-1}$. Similarly, we assume identical Ω_t and $\Pr_{\Omega_t}$ in each stage. The probability of sampling a particular $\hat{\omega}_t$ is

$$\Pr(\hat{\omega}_t) = \frac{\sum_{t \geq \bar{t}} y_t(\hat{\omega}_t)}{\sum_{t \geq \bar{t}} \sum_{\omega_t \in \Omega_t} y_t(\omega_t)}$$

where $\hat{\omega}_t$ are identical in each stage t, and we repeat this for each $\hat{\omega}_t \in \Omega_t$.

Note that Φ_t and Ω_t are arbitrarily chosen, and the estimation of $\Pr_{\Phi_t|\Phi_{t-1}}$ and $\Pr_{\Omega_t}$ are dependent on Φ_t and Ω_t. Then to obtain useful $\Pr_{\Phi_t|\Phi_{t-1}}$ and $\Pr_{\Omega_t}$, for example having few zeros, we may need to repeat the estimation of $\Pr_{\Phi_t|\Phi_{t-1}}$ and $\Pr_{\Omega_t}$ with different Φ_t and Ω_t.

To illustrate the model, we give an example of processing some real milk data. We use the same production year as in the previous section: a twelve month production year starts from June, and the first month with unpredictable milk supply is November, the sixth month.

Since we only have nine years of milk supply data in each month, we use the same Φ_t, Ω_t, $\Pr_{\Phi_t|\Phi_{t-1}}$ and $\Pr_{\Omega_t}$ for each $t \geq \bar{t}$. We assume the effect of a state ϕ_t in random variation is stronger than ω_t, and thus we use values representing large variation for ϕ_t and values representing small variation for ω_t. We assume Φ_t and Ω_t for $t \geq \bar{t}$ to be

$$\Phi_t = \{1,\, 1.1,\, 0.9\},$$

$$\Omega_t = \{1,\, 1.02,\, 0.98\}.$$

These sets result in 9 values for $\phi_t \omega_t$. We use the mid-point method to construct the bins. The estimation results in

$$\Pr_{\Phi_t|\Phi_{t-1}}(\phi_t \mid \phi_{t-1}) = \begin{bmatrix} 0.31 & 0.46 & 0.23 \\ 0.25 & 0.30 & 0.45 \\ 0.14 & 0.38 & 0.48 \end{bmatrix}$$

with each row defining the probabilities for the realizations of $\phi_t = 1$, 1.1 and 0.9 given $\phi_{t-1} = 1$, 1.1 or 0.9, and

$$\Pr_{\Omega_t}(\omega_t) = \begin{cases} 0.60, & \text{for } \omega_t = 1, \\ 0.21, & \text{for } \omega_t = 1.02, \\ 0.19, & \text{for } \omega_t = 0.98. \end{cases}$$

In Figure 3.9, we display a matrix of probabilities, with each row defining the probabilities of each realization of $\phi_t \omega_t$ given a realization of $\phi_{t-1}\omega_{t-1}$. The sum of probabilities in each row is one.

With these distributions for random variables, there are 9 outcomes of milk supply in each month from November to May, which results in 4.8 million milk supply scenarios.

For the same region in the example of processing real milk data in the previous section, we sample 100 scenarios and display the coverage along with the forecast milk supply in Figure 3.10. The milk supply from June to October are the forecast milk supply. It shows that the milk supply have small random variation in the months near the end of the year.

	$\phi_1\omega_1$	$\phi_1\omega_2$	$\phi_1\omega_3$	$\phi_2\omega_1$	$\phi_2\omega_2$	$\phi_2\omega_3$	$\phi_3\omega_1$	$\phi_3\omega_2$	$\phi_3\omega_3$
$\phi_1\omega_1$	0.1856	0.0635	0.0586	0.2784	0.0952	0.0879	0.1392	0.0476	0.0440
$\phi_1\omega_2$	0.1856	0.0635	0.0586	0.2784	0.0952	0.0879	0.1392	0.0476	0.0440
$\phi_1\omega_3$	0.1856	0.0635	0.0586	0.2784	0.0952	0.0879	0.1392	0.0476	0.0440
$\phi_2\omega_1$	0.1508	0.0516	0.0476	0.1810	0.0619	0.0571	0.2714	0.0929	0.0857
$\phi_2\omega_2$	0.1508	0.0516	0.0476	0.1810	0.0619	0.0571	0.2714	0.0929	0.0857
$\phi_2\omega_3$	0.1508	0.0516	0.0476	0.1810	0.0619	0.0571	0.2714	0.0929	0.0857
$\phi_3\omega_1$	0.0862	0.0295	0.0272	0.2298	0.0786	0.0726	0.2872	0.0983	0.0907
$\phi_3\omega_2$	0.0862	0.0295	0.0272	0.2298	0.0786	0.0726	0.2872	0.0983	0.0907
$\phi_3\omega_3$	0.0862	0.0295	0.0272	0.2298	0.0786	0.0726	0.2872	0.0983	0.0907

Figure 3.9: Probabilities for each realization of $\phi_t\omega_t$ given each realization of $\phi_{t-1}\omega_{t-1}$.

We then display the coverage of the residuals of this sample along with the residuals of the real data from the forecast milk supply in Figure 3.11. Note that the residuals from June to October are one. We observe that the residuals of the real data have large variations in the last three months, but the bounds of the sample are narrow and thus does not have a good cover in the last three months.

Note that since the multiplicative model assumes uniform random variations over the regions, it cannot capture the variation between regions.

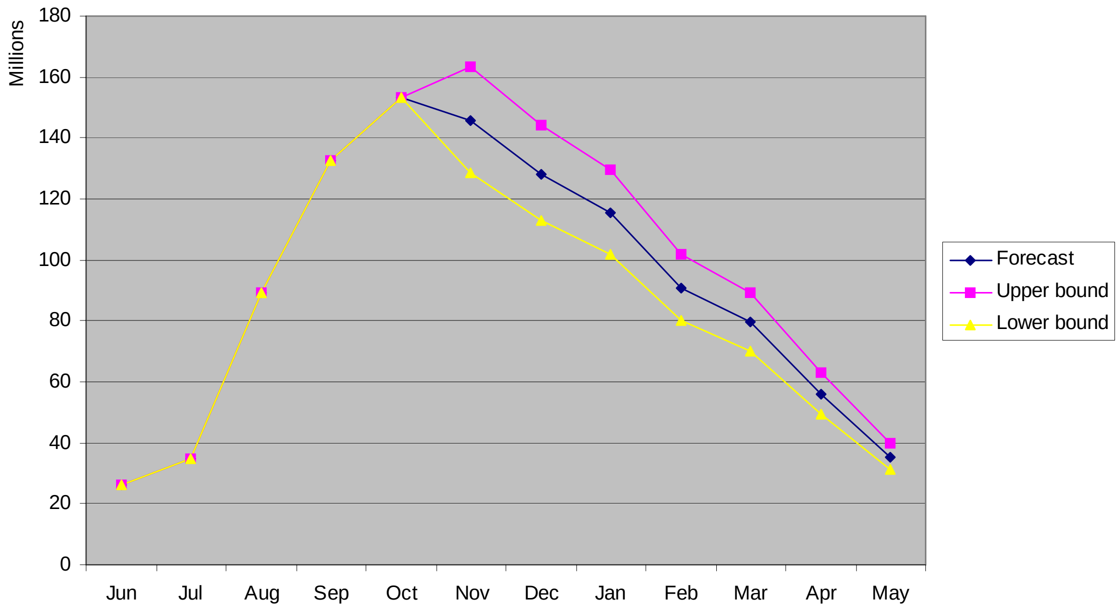

Figure 3.10: Coverage of a sample of milk scenarios from the multiplicative model.

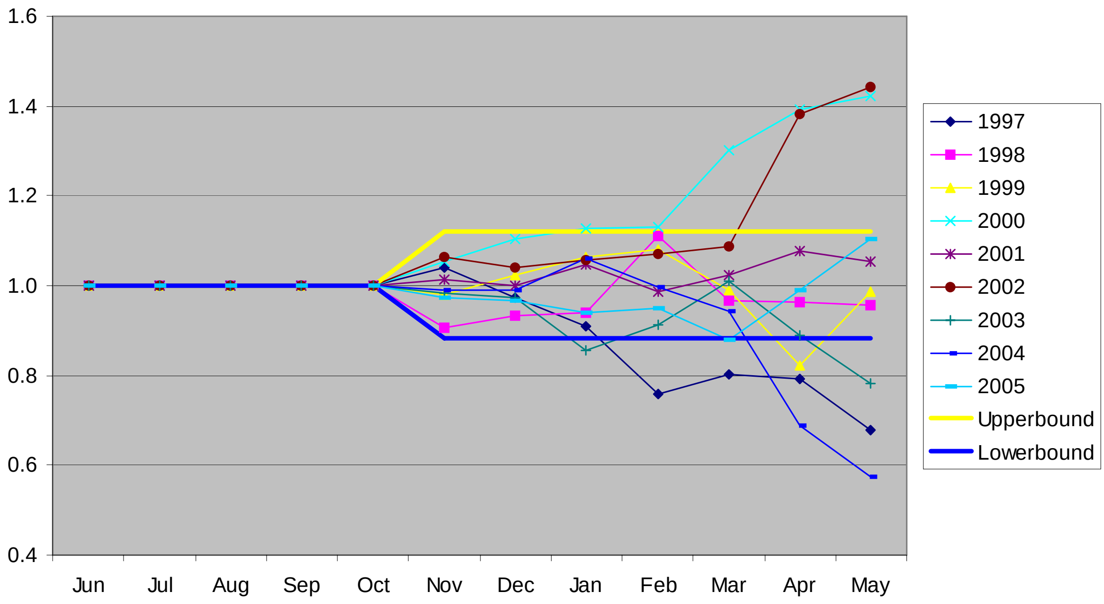

Figure 3.11: Residuals of the real data and the coverage of residuals in a sample of milk scenarios.

Chapter 4

Optimization models

Given that we have derived models for milk supply, in this chapter we present optimization models to solve the optimization problem, assuming (1) forecast milk supply, (2) milk supply from the additive model, and (3) milk supply from the multiplicative model. We list the models in Table 4.1.

Abbreviation	**Optimization model**
DS	Deterministic stationary model
DN	Deterministic non-stationary model
ADO	Adaptive deterministic optimization model
ADOS	Adaptive deterministic optimization sales-projections model
MSPF	Multistage stochastic programming full-scenario model
MSPE	Multistage stochastic programming expected-scenario model
DE	Deterministic equivalent model
PF	Perfect forecast model

Table 4.1: Optimization models.

The deterministic stationary model (denoted DS) and the deterministic non-stationary model (denoted DN) assume a forecast milk supply. The DS model assumes insignificant annual growth in milk supply and unchanged production and market conditions and thus results in a steady-state production plan. This model is used to obtain the start-of-year inventory and the start-of-year contracts used in the other models. The DN model is similar to the DS model, but it doesn't assume a stationary production year. This model is used to obtain a penalty cost on distress trading from Australia.

We study six models of the optimization problem with uncertain milk supply. We list their main characteristics in Table 4.2.

Model	**Formulation**	**Sales decision**	**Scenario set**	Algorithm
ADO	Adaptive	Free decision	Conditional expected value	QP
ADOS	Adaptive	Meet targets	Conditional expected value	QP
MSPF	Recourse	Free decision	Full scenario set	DOASA
MSPE	Recourse	Free decision	Expected value	DOASA
DE	Deterministic equivalent	Free decision	Full scenario set	Intractable
PF	Deterministic	Free decision	Perfect forecast	QP

Table 4.2: Optimization models.

The adaptive deterministic optimization model (denoted ADO) forecasts milk supply in the future months based on the current observed milk supply and solves an optimization problem in each month. Thus this model has an adaptive milk supply forecast and decision horizon. The adaptive deterministic optimization sales-projections model (denoted ADOS) is similar to the ADO model but commits to a set of sales projections which is obtained from the solution to the DN model. These two models make use of quadratic programming (QP) algorithms in solving the optimization problems.

The multistage stochastic programming full-scenario model (denoted MSPF) solves a set of stage problems with each stage problem taking into account the future cost in the future months, in which the future cost is the expected value of the stage problems over all scenarios of milk supply in the next stage. The decision made in each stage depends on the realization of scenario in that stage and thus this is a recourse formulation for stochastic optimization with a full scenario tree. The multistage stochastic programming expected-scenario model (denoted MSPE) is a simple version of the MSPF model with the future cost being the value of the stage problem for the expected scenario of milk supply in the next stage. This model is a recourse formulation for the DN model assuming milk supply from the additive model, but is a recourse formulation for stochastic optimization with a Markov chain assuming milk supply from the multiplicative model. The algorithm DOASA decribed in chapter 5 is used to solve the optimization problems in the two models.

The deterministic equivalent model (denoted DE) is a deterministic optimization problem for the entire year taking into account each scenario of milk supply in each month. In contrast to the MSPF model, this is a deterministic-equivalent formulation (or formally in stochastic programming terminology, split variable formulation) for stochastic optimization with a full scenario tree. (We include this formulation for completeness. In practice this model is too large to be solved by any existing QP software.) The perfect forecast model (denoted PF) assumes a single perfect forecast of milk supply at the beginning of the year and then solves a deterministic optimization problem.

Prior to presenting these models, we define some notation.

Sets and indices

i, i'	$\in$	I	regions.
j	$\in$	J	processes.
k	$\in$	K	international spot markets.
n	$\in$	N	index for independent random variables.
p	$\in$	P	products.
$\hat{p}$	$\in$	$\hat{P}$	perishable products.
$\check{p}$	$\in$	$\check{P}$	powder products.
$\bar{p}$	$\in$	$\bar{P}$	contract products.
$\tilde{p}$	$\in$	$\tilde{P}$	non-contract products.

$t \in T$	months in the decision horizon.
Ω_t	the set of a vector of independent random variables in month t.
Φ_t	the set of Markov state in month t.

Parameters

h	a production year.
$\bar{t}$	the first month with unpredictable milk supply in year h.
$\hat{t}$	the first month of year h.
$\check{t}$	the last month of year h.
$\tilde{t}$	the first month of decision horizon T.
$\varepsilon_{i,\hat{t}-1}$	the start-of-year random error in region i.
σ_{it}	standard deviation in region i and month t.
ρ	serial correlation coefficient of ε_t.
λ_n	the nth eigenvalue.
V_{in}	the ith component in the nth eigenvector.
Pr_{Ω_t}	probability for sampling ω_t from Ω_t.
$\mathrm{Pr}_{\Phi_t \mid \Phi_{t-1}}$	probability for sampling ϕ_t from Φ_t given ϕ_{t-1} from Φ_{t-1}.
$b_{\bar{p}t}$	contract price for product $\bar{p}$ in month t.
c_{jp}	yield of product p in process j.
d_t	the number of days in month t.
f_{it}	base of regional growth in region i and month t.
g_{it}	slope of regional growth in region i and month t.
α_p	marginal cost for inland transportation for product p.
β_{kp}	marginal cost for overseas transportation to market k for product p.
γ_{pt}	penalty cost on distress trading of product p in month t.
μ	penalty cost on milk supply dummy variable.
$\pi_{ii'}$	marginal cost for milk transportation from region i to region i'.
τ_p	marginal cost for production of product p.
κ_p	marginal cost for storage of product p.
$\varsigma_{\bar{p}}$	marginal transportation cost to deliver contract of product $\bar{p}$.

$\hat{q}_{ii'}$ maximum percentage of milk transported from region i to region i'.

$\hat{r}_{ij}$ daily process capacity for process j in region i.

$v_{p,\hat{t}-1}$ start-of-year inventory for product p.

$\hat{v}$ storage capacity of the refrigerated store.

$\check{v}$ storage capacity of the dry store.

$\hat{x}_{kpt}$ maximum sales for product p at market k in month t.

$\check{x}_{kpt}$ minimum demand for product p at market k in month t.

$\tilde{x}_{kpt}$ sales of product p at market k in month t in a set of sales-projections.

$\hat{z}_{\bar{p}}$ maximum for contract for product $\bar{p}$.

$z_{\bar{p},\hat{t}-3}$ start-of-year contract for product $\bar{p}$ to be delivered in month $\hat{t}$.

$z_{\bar{p},\hat{t}-2}$ start-of-year contract for product $\bar{p}$ to be delivered in month $\hat{t}+1$.

$z_{\bar{p},\hat{t}-1}$ start-of-year contract for product $\bar{p}$ to be delivered in month $\hat{t}+2$.

$\tilde{z}_{\bar{p}t}$ contract for product $\bar{p}$ in month t in a set of sales-projections.

Variables

ε_{it} a random error in region i and month t.

η_{it} a random variable in region i and month t.

ω_t a vector of independent random variables in month t.

ω_{tn} the nth random variable in ω_t in month t.

ϕ_t a Markov state in month t.

a_{kpt} market price for product p at market k in month t, as a function of sales.

$q_{ii't}$ milk transported from region i to region i' in month t.

r_{ijt} milk processed each day in process j in region i in month t.

s_{it} milk supply in region i and month t.

s'_{it} milk supply dummy variable in region i and month t.

u_{pt} production of product p in month t.

v_{pt} inventory of product p in month t.

v'_p product p in distress trading to meet the inventory target at the end of the year.

x_{kpt}	sales of product p from production at market k in month t.
x'_{kpt}	product p in distress trading to meet the minimum demand at market k in month t.
$y_{\bar{p}t}$	sales of product $\bar{p}$ from production by contract in month t.
$y'_{\bar{p}t}$	product $\bar{p}$ in distress trading to meet the contract in month t.
$z_{\bar{p}t}$	contract for product $\bar{p}$ made in month t.
Θ_{t+1}	future cost in the stage problem in month t.
Δ_t	the minimum cost in the stage problem in month t.

Note that since we have assumed a linear price-demand curve for sales in the spot market, the market price $a_{kpt}(\bullet)$ is a linear function of sales. The sales are the total of the products from production x_{kpt} and the products in distress trading x'_{kpt}. Thus the market price is $a_{kpt}(x_{kpt} + x'_{kpt})$ and the revenue is $a_{kpt}(x_{kpt} + x'_{kpt})(x_{kpt} + x'_{kpt})$, which gives rise to a quadratic objective function.

4.1 Optimization models assuming forecast milk supply

We present two optimization models assuming that the forecast milk supply occurs, the deterministic stationary model (denoted DS) and the deterministic non-stationary model (denoted DN).

4.1.1 Deterministic stationary model

The DS model assumes that growth in milk supply is insignificant and production and market conditions are unchanged over years, and thus results in a steady-state production plan. The start-of-year inventory is set to be the inventory at the end of the year. Then the inventory target at the end of the year is met and thus distress trading to meet the target is not needed. The start-of-year contracts are the contracts made in the last three months of the year. Thus the solution to this model gives the start-of-year inventory and the start-of-year contracts. We use a high penalty cost on distress trading so that no distress trading occurs in the solution.

Now we present the formulation of the model. The model consists of an objective which is minimizing the annual cost, and constraints for milk supply, use of milk in production, manufacturing of products, inventory, sales to spot markets, sales by contract and domains of decision variables.

$$\begin{aligned}\min \quad & -\sum_{t\in T}\sum_{k\in K}\sum_{p\in P} a_{kpt}(x_{kpt}+x'_{kpt})(x_{kpt}+x'_{kpt}) - \sum_{t\in T}\sum_{\bar{p}\in\bar{P}} b_{\bar{p}t}(y_{\bar{p}t}+y'_{\bar{p}t}) \\ & + \sum_{t\in T}\sum_{i\in I}\sum_{i'\in I} \pi_{ii'}q_{ii't} + \sum_{t\in T}\sum_{p\in P}\tau_p u_{pt} + \sum_{t\in T}\sum_{p\in P}\kappa_p v_{pt} \\ & + \sum_{t\in T}\sum_{\bar{p}\in\bar{P}} \varsigma_{\bar{p}t}y_{\bar{p}t} + \sum_{t\in T}\sum_{p\in P}\sum_{k\in K}\alpha_p x_{kpt} + \sum_{t\in T}\sum_{p\in P}\sum_{k\in K}\beta_{kp}x_{kpt} \\ & + \sum_{t\in T}\sum_{k\in K}\sum_{p\in P}\gamma_{pt}x'_{kpt} + \sum_{t\in T}\sum_{\bar{p}\in\bar{P}}\gamma_{\bar{p}t}y'_{\bar{p}t}, \end{aligned} \tag{4.1}$$

s.t. Milk supply:

$$s_{it} = f_{it} + hg_{it}, \quad \forall i \in I, t \in T, \tag{4.2}$$

Maximum input to process per day:

$$r_{ijt} \leq \hat{r}_{ij}, \quad \forall i \in I, j \in J, t \in T, \tag{4.3}$$

Milk flow:

$$s_{it} + \sum_{i'\in I} q_{i'it} = \sum_{j\in J} d_t r_{ijt} + \sum_{i'\in I} q_{ii't}, \quad \forall i \in I, t \in T, \tag{4.4}$$

Maximum allowance of milk transportation beween regions:

$$q_{ii't} \leq \hat{q}_{ii'}s_{it}, \quad \forall i \in I, i' \in I, t \in T, \tag{4.5}$$

Yield of products:

$$u_{pt} = \sum_{i\in I}\sum_{j\in J} d_t c_{jp} r_{ijt}, \quad \forall p \in P, t \in T, \tag{4.6}$$

Product flow for contract products:

$$v_{\bar{p}t} + \sum_{k \in K} x_{k\bar{p}t} + y_{\bar{p}t} = u_{\bar{p}t} + v_{\bar{p},t-1}, \quad \forall \bar{p} \in \bar{P}, t \in T, \tag{4.7}$$

Product flow for non-contract products:

$$v_{\tilde{p}t} + \sum_{k \in K} x_{k\tilde{p}t} = u_{\tilde{p}t} + v_{\tilde{p},t-1}, \quad \forall \tilde{p} \in \tilde{P}, t \in T, \tag{4.8}$$

Maximum inventory level for perishable products:

$$\sum_{\hat{p} \in \hat{P}} v_{pt} \leq \hat{v}, \quad \forall t \in T, \tag{4.9}$$

Maximum inventory level for powder products:

$$\sum_{\check{p} \in \check{P}} v_{pt} \leq \check{v}, \quad \forall t \in T, \tag{4.10}$$

Start-of-year inventory:

$$v_{p,\hat{t}-1} = v_{p\check{t}}, \quad \forall p \in P, \tag{4.11}$$

Sales meet minimum demands in spot markets:

$$x_{kpt} + x'_{kpt} \geq \check{x}_{kpt}, \quad \forall k \in K, p \in P, t \in T, \tag{4.12}$$

Maximum sales in spot market:

$$x_{kpt} + x'_{kpt} \leq \hat{x}_{kpt}, \quad \forall k \in K, p \in P, t \in T, \tag{4.13}$$

Meet contract:

$$y_{\bar{p}t} + y'_{\bar{p}t} = z_{\bar{p},t-3}, \quad \forall \bar{p} \in \bar{P}, t \in T, \tag{4.14}$$

Maximum for contract:

$$z_{\bar{p}t} \leq \hat{z}_{\bar{p}}, \quad \forall \bar{p} \in \bar{P}, t \in T, \tag{4.15}$$

Start-of-year contract:

$$z_{\bar{p},\hat{t}-3} = z_{\bar{p},\check{t}-2}, z_{\bar{p},\hat{t}-2} = z_{\bar{p},\check{t}-1}, z_{\bar{p},\hat{t}-1} = z_{\bar{p},\check{t}}, \quad \forall \bar{p} \in \bar{P}, \tag{4.16}$$

Decision variable domain:

$$\text{All decision variables in Roman typeface } \geq 0. \tag{4.17}$$

We describe the formulation of the model as follows.

1. Expression (4.1) defines that the objective is to minimize the annual cost, which consists of (minus) the revenues from sales to spot markets and sales by contracts, and the costs of milk transportation, milk processing, inventory storage, inland transportation and overseas transportation of products, and the cost of distress trading to meet minimum demands and contracts.

2. Constraint (4.2) defines the forecast milk supply in region i and month t. The year h corresponds to the year we wish to investigate. Observe that s_{it} is a decision variable in this model, but it is fixed as the forecast milk supply.

3. Constraints (4.3) to (4.5) describe the use of milk in production. Constraint (4.3) states that the amount of daily-processed milk is under the daily process capacity of process j in region i and month t. Constraint (4.4) gives the balance of milk flow in production, which is that in region i and month t, the total of milk supply and incoming milk are equal to the total of milk processed and outgoing milk. Constraint (4.5) gives the upper bound on the outgoing milk from region i to region i' in month t .

4. Constraints (4.6) to (4.8) represent the manufacturing of products. Constraint (4.6) calculates the total production of product p in month t. Constraint (4.7) presents the balance of product flow for contract products, that is, for product $\bar{p}$ in month t, the total of inventory, sales to spot markets and sales by contracts is equal to the total of production and past inventory. Constraint (4.8) presents the balance of product flow for non-contract products, that is, for product $\tilde{p}$ in month t, the total of inventory and sales to spot markets is equal to the total of production and past inventory.

5. Constraints (4.9) and (4.11) describe the inventory. Constraints (4.9) and (4.10) state that in month t, the total inventory of perishable products cannot exceed the refrigerated storage capacity, and the total inventory of powder products cannot exceed the dry-storage capacity. Constraint (4.11) states that for product p, the start-of-year inventory must equal the inventory at

the end of the year.

6. Constraints (4.12) to (4.13) describe the sales to spot markets. Constraint (4.12) states that in month t, the total sales of product p by production and in distress trading in spot market k meet the minimum demand. Constraint (4.13) states that in month t, the total sales of product p in spot market k cannot exceed the market capacity.

7. Constraints (4.14) and (4.16) describe the sales by contracts. Constraint (4.14) states that in month t, a contract for product $\bar{p}$ is delivered by the total of products from production and in distress trading. Constraint (4.15) defines the upper bound on the contract for product $\bar{p}$ made in month t. Constraint (4.16) states that for product p, the start-of-year contracts are the contracts made in the last three months of the year.

8. Constraint (4.17) states that all decision variables in Roman typeface are non-negative, which implies that the other decision variables (in Greek) are unconstrained.

4.1.2 Deterministic non-stationary model

The DN model does not assume a stationary year and results in a production plan for a particular year. The start-of-year inventory and start-of-year contracts are known which are those in the solution to the DS model. The inventory of the last month needs to meet the inventory target at the end of the year which is the start-of-year inventory, that is, the inventory in the last month is at least as much as the start-of-year inventory. Then distress trading to meet the target may be needed. We use this model to search for the penalty cost on distress trading which is the lowest value such that no distress trading occurs in the solution.

We present the formulation as follows.

$$\begin{aligned}
\min \quad & -\sum_{t\in T}\sum_{k\in K}\sum_{p\in P} a_{kpt}(x_{kpt}+x'_{kpt})(x_{kpt}+x'_{kpt}) - \sum_{t\in T}\sum_{\bar{p}\in\bar{P}} b_{\bar{p}t}(y_{\bar{p}t}+y'_{\bar{p}t}) \\
& + \sum_{t\in T}\sum_{i\in I}\sum_{i'\in I}\pi_{ii'}q_{ii't} + \sum_{t\in T}\sum_{p\in P}\tau_p u_{pt} + \sum_{t\in T}\sum_{p\in P}\kappa_p v_{pt} \\
& + \sum_{t\in T}\sum_{\bar{p}\in\bar{P}}\varsigma_{\bar{p}t}y_{\bar{p}t} + \sum_{t\in T}\sum_{k\in K}\sum_{p\in P}\alpha_p x_{kpt} + \sum_{t\in T}\sum_{k\in K}\sum_{p\in P}\beta_{kp}x_{kpt} \\
& + \sum_{t\in T}\sum_{k\in K}\sum_{p\in P}\gamma_{pt}x'_{kpt} + \sum_{t\in T}\sum_{\bar{p}\in\bar{P}}\gamma_{\bar{p}t}y'_{\bar{p}t} + \sum_{p\in P}\gamma_{p,\check{t}}v'_p, \qquad (4.18)
\end{aligned}$$

s.t. constraints (4.2)-(4.10), (4.12)-(4.15), (4.17), and

Meet the inventory target at the end of the year:

$$v_{p\check{t}} + v'_p \geq v_{p,\hat{t}-1}, \quad \forall p \in P, \qquad (4.19)$$

1. Expression (4.18) defines that the annual cost includes the penalty cost for distress trading to meet the inventory target at the end of the year.
2. Constraint (4.19) states that the total of inventory in last month $\check{t}$ and products in distress trading meets the inventory target at the end of the year.

4.2 Optimization models assuming milk supply from the additive model

Now we describe the optimization models which assume milk supply from the additive model, which are the adaptive deterministic optimization model (denoted ADO), the adaptive deterministic optimization sales-projections model (denoted ADOS), the multistage stochastic programming full-scenario model (denoted MSPF), the multistage stochastic programming expected-scenario model (denoted MSPE), the deterministic equivalent model (denoted DE) and the perfect forecast model (denoted PF). All of these models use the start-of-year inventory and the start-of-year contracts obtained in the solution to the DS model and the penalty cost on distress trading obtained in the solution to the DN model.

4.2.1 Adaptive deterministic optimization model

The ADO model has a decision horizon starting from a given month to the end of the year. Each of these months is called a stage. In each ADO stage problem, the milk supply in the future months is forecasted based on the current observed milk supply. Each stage minimizes the total cost over the decision horizon.

We present the problem in stage $\tilde{t}$ as follows.

$$\begin{aligned}
&\Delta_{\tilde{t}}(z_{\bar{p},\tilde{t}-3}, z_{\bar{p},\tilde{t}-2}, z_{\bar{p},\tilde{t}-1}, v_{p,\tilde{t}-1}, \varepsilon_{i,\tilde{t}-1}, \omega_{\tilde{t}}) = \\
\min \quad & -\sum_{t\in T}\sum_{k\in K}\sum_{p\in P} a_{kpt}(x_{kpt}+x'_{kpt})(x_{kpt}+x'_{kpt}) - \sum_{t\in T}\sum_{\bar{p}\in\bar{P}} b_{\bar{p}t}(y_{\bar{p}t}+y'_{\bar{p}t}) \\
&+\sum_{t\in T}\sum_{i\in I}\sum_{i'\in I}\pi_{ii'}q_{ii't} + \sum_{t\in T}\sum_{p\in P}\tau_p u_{pt} + \sum_{t\in T}\sum_{p\in P}\kappa_p v_{pt} \\
&+\sum_{t\in T}\sum_{\bar{p}\in\bar{P}}\varsigma_{\bar{p}t}y_{\bar{p}t} + \sum_{t\in T}\sum_{k\in K}\sum_{p\in P}\alpha_p x_{kpt} + \sum_{t\in T}\sum_{k\in K}\sum_{p\in P}\beta_{kp}x_{kpt} \\
&+\sum_{t\in T}\sum_{k\in K}\sum_{p\in P}\gamma_{pt}x'_{kpt} + \sum_{t\in T}\sum_{\bar{p}\in\bar{P}}\gamma_{\bar{p}t}y'_{\bar{p}t} + \sum_{p\in P}\gamma_{p,\tilde{t}}v'_p + \sum_{t\in T}\sum_{i\in I}\mu s'_{it}, \qquad (4.20)
\end{aligned}$$

s.t. constraints (4.3)-(4.10), (4.19), (4.12)-(4.15), (4.17), and

Milk supply:

$$s_{it} = f_{it} + hg_{it} + \sigma_{it}\varepsilon_{it} + s'_{it}, \quad \forall i \in I, t \in T, \qquad (4.21)$$

Calculate random error ε_{it}:

$$\varepsilon_{it} = \rho\varepsilon_{i,t-1} + \eta_{it}, \quad \forall i \in I, t \in T, \qquad (4.22)$$

Calculate random variables η_{it}:

$$\eta_{it} = \sum_{n\in N}(1-\rho^2)^{\frac{1}{2}} V_{in}\lambda_n^{\frac{1}{2}}\omega_{tn}, \quad \forall i \in I, t \in T, \qquad (4.23)$$

Random variables ω_{tn} in future months:

$$\omega_{tn} = 0, \quad \forall\, t > \tilde{t}, n \in N, \qquad (4.24)$$

1. Expression (4.20) defines that the objective of the stage problem is to minimize the total cost in decision horizon $T = \{\tilde{t}, \tilde{t}+1, \ldots, \check{t}\}$ given the past contracts $z_{\bar{p},\tilde{t}-3}$, $z_{\bar{p},\tilde{t}-2}$ and $z_{\bar{p},\tilde{t}-1}$, past inventory $v_{p,\tilde{t}-1}$, past random error $\varepsilon_{i,\tilde{t}-1}$ and a vector of random variables $\omega_{\tilde{t}}$ in the current stage (and thus current milk supply), which is the (minus) revenue and the cost in production, transportation, storage, distress trading and dummy milk supply in decision horizon T.

2. Constraints (4.21) to (4.24) define the milk supply using the additive model for milk supply defined in chapter 3. Constraint (4.21) states that milk supply in region i in month t is the sum of the forecast milk supply $f_{it} + hg_{it}$, the random variation $\sigma_{it}\varepsilon_{it}$ and dummy variable s'_{it}. Constraint (4.22) states that ε_{it} in region i and month t is calculated from $\varepsilon_{i,t-1}$ and η_{it}. Here $\varepsilon_{i,\hat{t}-1}$ is known. Constraint (4.23) calculates η_{it} in region i and month t, given ω_{tn}, $n \in N$, which are known. Constraint (4.24) states that all ω_{tn} in month $t > \tilde{t}$ are zero, and thus milk supply in these months is the milk supply forecast based on the milk supply in month $\tilde{t}$. Note that s_{it}, s'_{it}, ε_{it} and η_{it} are decision variables, although they are effectively determined by the realization of the stochastic process.

4.2.2 Adaptive deterministic optimization sales-projections model

It is not an unusual practice for a company to have a set of sales projections at the beginning of a year and then committing to the set of sales projections in production planning as time moves on. We use the ADOS model to capture this feature. The ADOS model has a similar structure to the ADO model but commits to a set of sales projections that set to be the sales in spot markets and the contracts from the solution to the DN model.

We present the problem at stage $\tilde{t}$ as follows.

Expression (4.20),

s.t. constraints (4.21)-(4.24), (4.3)-(4.10), (4.19), (4.12)-(4.15), (4.17), and

Meet sales in spot markets in the set of sales projections:

$$x_{kpt} + x'_{kpt} \geq \tilde{x}_{kpt}, \quad \forall k \in K, p \in P, t \in T, \tag{4.25}$$

Make contracts in the set of sales projections:

$$z_{\bar{p}t} = \tilde{z}_{\bar{p}t}, \quad \forall \bar{p} \in \bar{P}, t \in T. \tag{4.26}$$

1. Constraint (4.25) states that for product p, the total sales of products from production and in distress trading in market k in month t are above the sales in the set of sales-projections.
2. Constraint (4.26) states that for product p, the contract made in month t is the contract in the set of sales-projections.

4.2.3 Multistage stochastic programming full-scenario model

The MSPF model solves a set of stage problems. Each stage problem takes into account the future cost, which is the expected cost of the stage problems for all scenarios of milk supply in the next stage.

We present the formulation of the stage t problem as follows.

$$
\begin{aligned}
&\Delta_t(z_{\bar{p},t-3}, z_{\bar{p},t-2}, z_{\bar{p},t-1}, v_{p,t-1}, \varepsilon_{i,t-1}, \omega_t) = \\
\min\quad & -\sum_{k\in K}\sum_{p\in P} a_{kpt}(x_{kpt}+x'_{kpt})(x_{kpt}+x'_{kpt}) - \sum_{\bar{p}\in\bar{P}} b_{\bar{p}t}(y_{\bar{p}t}+y'_{\bar{p}t}) \\
& + \sum_{i\in I}\sum_{i'\in I}\pi_{ii'}q_{ii't} + \sum_{p\in P}\tau_p u_{pt} + \sum_{p\in P}\kappa_p v_{pt} \\
& + \sum_{\bar{p}\in\bar{P}}\varsigma_{\bar{p}t}y_{\bar{p}t} + \sum_{k\in K}\sum_{p\in P}\alpha_p x_{kpt} + \sum_{k\in K}\sum_{p\in P}\beta_{kp}x_{kpt} \\
& + \sum_{k\in K}\sum_{p\in P}\gamma_{pt}x'_{kpt} + \sum_{\bar{p}\in\bar{P}}\gamma_{\bar{p}t}y'_{\bar{p}t} + \sum_{i\in I}\mu s'_{it} \\
& + \Theta_{t+1}(z_{\bar{p},t-2}, z_{\bar{p},t-1}, z_{\bar{p},t}, v_{p,t}, \varepsilon_{i,t}), \quad \text{if } t<\check{t}, \\
& + \sum_{p\in P}\gamma_{pt}v'_p, \quad \text{if } t=\check{t}, \qquad (4.27)
\end{aligned}
$$

s.t. constraints (4.21)-(4.23), (4.3)-(4.10), (4.19), (4.12)-(4.15), (4.17), and

The future cost:

$$
\begin{aligned}
&\Theta_{t+1}(z_{\bar{p},t-2}, z_{\bar{p},t-1}, z_{\bar{p},t}, v_{p,t}, \varepsilon_{i,t}) = \\
&\qquad \sum_{\omega_{t+1}\in\Omega_{t+1}} \Pr_{\Omega_{t+1}}(\omega_{t+1})\Delta_{t+1}(z_{\bar{p},t-2}, z_{\bar{p},t-1}, z_{\bar{p},t}, v_{p,t}, \varepsilon_{i,t}, \omega_{t+1}). \qquad (4.28)
\end{aligned}
$$

1. Expression (4.27) defines that the objective is to minimize the cost given the past contracts, past inventory, past random error and the realized random variables, which takes into account the future cost $\Theta_{t+1}(z_{\bar{p},t-2}, z_{\bar{p},t-1}, z_{\bar{p},t}, v_{p,t}, \varepsilon_{i,t})$ for $t<\check{t}$ or the cost of distress trading to meet the inventory target at the end of the year.

2. Constraint (4.28) defines the future cost given the past contracts, the contracts made, the inventory and the random error in this stage. The future cost is the average of the costs of the stage problems for all $\omega_{t+1}\in\Omega_{t+1}$ in stage $t+1$.

4.2.4 Multistage stochastic programming expected-scenario model

The MSPE model is a simple version of the MSPF model. This model solves a set of stage problems with the future cost being the value of the stage problem for the expected scenario of milk supply in the next stage. The stage problem in this model is the stage problem in the MSPF model with the set Ω_t re-defined as $\{0\}$ for all t.

Note that in this model, the optimization problem is deterministic and thus this does not use stochastic programming techniques. However, as we will describe later, for the multiplicative model, a model with the same idea solves a stochastic problem and thus stochastic programming is used. Thus we choose to use the same name for these two models.

4.2.5 Deterministic equivalent model

A stochastic problem with finite distribution can be treated as an equivalent deterministic problem that can be solved using large-scale optimization techniques. This model is called the deterministic equivalent model (denoted DE).

We present additional sets, parameters and variables and then the formulation.

Sets and indices

ψ	$\in$	Ψ	set of scenarios, i.e., $(\omega_{\hat{t}}, \omega_{\hat{t}+1}, \ldots, \omega_{\check{t}})$.
$\hat{\psi}_t$	$\in$	$\hat{\Psi}_t$	an *upper* set of Ψ defining the set of scenarios up to month t, i.e., $(\omega_{\hat{t}}, \omega_{\hat{t}+1}, \ldots, \omega_t)$.
$\check{\psi}_t$	$\in$	$\check{\Psi}_t$	a *lower* set of Ψ defining the set of scenarios from month t, i.e., $(\omega_t, \omega_{t+1}, \ldots, \omega_{\check{t}})$.
$\bar{\psi}_t$	$\in$	$\bar{\Psi}_t$	a single-element set containing a particular scenario in $\check{\Psi}_t$.

Parameters

$\varepsilon_{i,\hat{t}-1,\psi}$	the start-of-year random error in region i in scenario ψ.
$\Pr_{\Psi}(\psi)$	probability of sampling ψ from Ψ.

Variables

$\varepsilon_{it\psi}$	a random error in region i in month t in scenario ψ.
$\eta_{it\psi}$	a random variable in region i in month t in scenario ψ.
$\omega_{t\psi}$	a vector of random variables sampled from $(\Omega_t, \Pr_{\Omega_t})$ in month t in scenario ψ.
$\omega_{tn\psi}$	the nth random variable in $\omega_{t\psi}$.
$a_{kpt\psi}$	market price for product p at market k in month t in scenario ψ, as a function of sales.
$q_{ii't\psi}$	milk transported from region i to region i' in month t in scenario ψ.
$r_{ijt\psi}$	milk processed each day in process j in region i and month t in scenario ψ.
$s_{it\psi}$	milk supply in region i and month t in scenario ψ.
$s'_{it\psi}$	milk supply dummy variable in region i and month t in scenario ψ.
$u_{pt\psi}$	production of product p in month t in scenario ψ.
$v_{pt\psi}$	inventory of product p in month t in scenario ψ.
$v'_{p\psi}$	product p in distress trading to meet the inventory target at the end of the year in scenario ψ.
$x_{kpt\psi}$	sales of product p from production at market k in month t in scenario ψ.
$x'_{kpt\psi}$	product p in distress trading to meet the minimum demand at market k in month t in scenario ψ.
$y_{\bar{p}t\psi}$	sales of product $\bar{p}$ from production by contract in month t in scenario ψ.
$y'_{\bar{p}t\psi}$	product $\bar{p}$ in distress trading to meet the contract in month t in scenario ψ.

$z_{\bar{p}t\psi}$ contract of product $\bar{p}$ made in month t in scenario ψ.

$\chi_{t\psi}$ a decision variable in month t in scenario ψ.

$\chi_{t\hat{\psi}_t\check{\psi}_{t+1}}$ a decision variable in month t in scenario $\hat{\psi}_t$ and scenario $\check{\psi}_{t+1}$.

Formulation

$$\begin{aligned}
\min \quad & -\sum_{\psi\in\Psi}\Pr_\Psi(\psi)\left(\sum_{t\in T}\sum_{k\in K}\sum_{p\in P}a_{kpt\psi}(x_{kpt\psi}+x'_{kpt\psi})(x_{kpt\psi}+x'_{kpt\psi})\right.\\
& -\sum_{t\in T}\sum_{\bar{p}\in\bar{P}}b_{\bar{p}t}(y_{\bar{p}t\psi}+y'_{\bar{p}t\psi})+\sum_{t\in T}\sum_{i\in I}\sum_{i'\in I}\pi_{ii'}q_{ii't\psi}+\sum_{t\in T}\sum_{p\in P}\tau_p u_{pt\psi}\\
& +\sum_{t\in T}\sum_{p\in P}\kappa_p v_{pt\psi}+\sum_{t\in T}\sum_{\bar{p}\in\bar{P}}\varsigma_{\bar{p}t}y_{\bar{p}t\psi}+\sum_{t\in T}\sum_{k\in K}\sum_{p\in P}\alpha_p x_{kpt\psi}\\
& +\sum_{t\in T}\sum_{k\in K}\sum_{p\in P}\beta_{kp}x_{kpt\psi}+\sum_{t\in T}\sum_{k\in K}\sum_{p\in P}\gamma_{pt}x'_{kpt\psi}+\sum_{t\in T}\sum_{\bar{p}\in\bar{P}}\gamma_{\bar{p}t}y'_{\bar{p}t\psi}\\
& \left.+\sum_{p\in P}\gamma_{p,\check{t}}v'_{p\psi}+\sum_{t\in T}\sum_{i\in I}\mu s'_{it\psi}\right),
\end{aligned} \tag{4.29}$$

s.t. Milk supply:

$$s_{it\psi}=f_{it}+hg_{it}+\sigma_{it}\varepsilon_{it\psi}+s'_{it\psi},\quad \forall i\in I,t\in T,\psi\in\Psi, \tag{4.30}$$

Calculate random error $\varepsilon_{it\psi}$:

$$\varepsilon_{it\psi}=\rho\varepsilon_{i,t-1,\psi}+\eta_{it\psi},\quad \forall i\in I,t\in T,\psi\in\Psi, \tag{4.31}$$

Calculate random variable $\eta_{it\psi}$:

$$\eta_{it\psi}=\sum_{n\in N}(1-\rho^2)^{\frac{1}{2}}V_{in}\lambda_n^{\frac{1}{2}}\omega_{tn\psi},\quad \forall i\in I,t\in T,\psi\in\Psi, \tag{4.32}$$

Maximum input to process per day:

$$r_{ijt\psi}\le\hat{r}_{ij},\quad \forall i\in I,j\in J,t\in T,\psi\in\Psi, \tag{4.33}$$

Milk flow:

$$s_{it\psi}+\sum_{i'\in I}q_{i'it\psi}=\sum_{j\in J}d_t r_{ijt\psi}+w_{it\psi}+\sum_{i'\in I}q_{ii't\psi},$$
$$\forall i\in I,t\in T,\psi\in\Psi, \tag{4.34}$$

Maximum allowance for milk transportation:

$$q_{ii't\psi} \leq \hat{q}_{ii'} s_{it\psi}, \quad \forall i \in I, i' \in I, t \in T, \psi \in \Psi, \tag{4.35}$$

Yield of products:

$$u_{pt\psi} = \sum_{i \in I} \sum_{j \in J} d_t c_{jp} r_{ijt\psi}, \quad \forall p \in P, t \in T, \psi \in \Psi, \tag{4.36}$$

Product flow for contract products:

$$v_{\bar{p}t\psi} + \sum_{k \in K} x_{k\bar{p}t\psi} + y_{\bar{p}t\psi} = u_{\bar{p}t\psi} + v_{\bar{p},t-1,\psi}, \quad \forall \bar{p} \in \bar{P}, t \in T, \psi \in \Psi, \tag{4.37}$$

Product flow for non-contract products:

$$v_{\tilde{p}t\psi} + \sum_{k \in K} x_{k\tilde{p}t\psi} = u_{\tilde{p}t\psi} + v_{\tilde{p},t-1,\psi}, \quad \forall \tilde{p} \in \tilde{P}, t \in T, \psi \in \Psi, \tag{4.38}$$

Maximum inventory for perishable products:

$$\sum_{\hat{p} \in \hat{P}} v_{pt\psi} \leq \hat{v}, \quad \forall t \in T, \psi \in \Psi, \tag{4.39}$$

Maximum inventory for powder products:

$$\sum_{\check{p} \in \check{P}} v_{pt\psi} \leq \check{v}, \quad \forall t \in T, \psi \in \Psi, \tag{4.40}$$

Meet inventory target at the end of year:

$$v_{p,\check{t},\psi} + v'_{p\psi} \geq v_{p,\hat{t}-1}, \quad \forall p \in P, \psi \in \Psi, \tag{4.41}$$

Meet minimum demand in spot market:

$$x_{kpt\psi} + x'_{kpt\psi} \geq \check{x}_{kpt}, \quad \forall k \in K, p \in P, t \in T, \psi \in \Psi, \tag{4.42}$$

Maximum sales to spot markets:

$$x_{kpt\psi} + x'_{kpt\psi} \leq \hat{x}_{kpt}, \quad \forall k \in K, p \in P, t \in T, \psi \in \Psi, \tag{4.43}$$

Meet contract:

$$y_{\bar{p}t\psi} + y'_{\bar{p}t\psi} = z_{\bar{p},t-3,\psi}, \quad \forall \bar{p} \in \bar{P}, t \in T, \psi \in \Psi, \tag{4.44}$$

Maximum contract:

$$z_{\bar{p}t\psi} \leq \hat{z}_{\bar{p}}, \quad \forall \bar{p} \in \bar{P}, t \in T, \psi \in \Psi, \tag{4.45}$$

Non-anticipative contraints for decision variables:

$$\chi_{t\hat{\psi}_t\check{\psi}_{t+1}} = \chi_{t\hat{\psi}_t\bar{\psi}_{t+1}}, \quad \forall t < \check{t}, \hat{\psi}_t \in \hat{\Psi}_t, \check{\psi}_{t+1} \in \check{\Psi}_{t+1}, \bar{\psi}_{t+1} \in \bar{\Psi}_{t+1}, \tag{4.46}$$

Domain of decision variables:

All decision variables in Roman typeface ≥ 0. (4.47)

We give the description of the formulation as the follows.

1. Since for $t < \check{t}$,

$$\{\psi \mid \psi \in \Psi\} = \{(\hat{\psi}_t, \check{\psi}_{t+1}) \mid \hat{\psi}_t \in \hat{\Psi}_t, \check{\psi}_{t+1} \in \check{\Psi}_{t+1}\},$$

   a variable indexed by $\psi \in \Psi$ is same as by $\hat{\psi}_t \in \hat{\Psi}_t$ and $\check{\psi}_{t+1} \in \check{\Psi}_{t+1}$. $\bar{\Psi}_t$ contains a scenario in $\check{\Psi}_t$ which is arbitrarily chosen. All decision variables are indexed by $\psi \in \Psi$, and thus they are scenario-dependent.

2. Expression (4.29) defines that the objective is to minimize the expected annual cost, which is the probabilistic sum of the annual costs in all scenarios $\psi \in \Psi$.

3. For each scenario ψ, constraints (4.30) to (4.32) define the milk supply curve, constraints (4.33) to (4.35) describe the use of milk supply in production, constraints (4.36) to (4.38) describe the manufacturing of products, constraints (4.39) and (4.41) describe the inventory, constraints (4.42) and (4.43) describe the sales to spot markets, and constraints (4.44) and (4.45) describe the sales by contracts.

4. Constraint (4.46) is a *non-anticipative* constraint. Since in any month we cannot anticipate the realization of scenarios in the future months, we must have the same solution in month t for the scenarios that have the same random variables up to month t. Thus constraint (4.46) states that for each $t < \check{t}$,

the decision variables in month t in each $\hat{\psi}_t \in \hat{\Psi}_t$ have the same value for all $\check{\psi}_{t+1} \in \check{\Psi}_{t+1}$.

5. Constraint (4.47) states that all decision variables in Roman type face are non-negative, which implies that the other decision variables (in Greek) are unconstrained.

4.2.6 Perfect forecast model

The PF model assumes a perfect forecast of milk supply at the beginning of the year and then solves a deterministic problem for the entire year.

We present the formulation as follows.

$$\begin{aligned} \min \quad & -\sum_{t\in T}\sum_{k\in K}\sum_{p\in P} a_{kpt}(x_{kpt}+x'_{kpt})(x_{kpt}+x'_{kpt}) - \sum_{t\in T}\sum_{\bar{p}\in\bar{P}} b_{\bar{p}t}(y_{\bar{p}t}+y'_{\bar{p}t}) \\ & +\sum_{t\in T}\sum_{i\in I}\sum_{i'\in I}\pi_{ii'}q_{ii't} + \sum_{t\in T}\sum_{p\in P}\tau_p u_{pt} + \sum_{t\in T}\sum_{p\in P}\kappa_p v_{pt} \\ & +\sum_{t\in T}\sum_{\bar{p}\in\bar{P}}\varsigma_{\bar{p}t}y_{\bar{p}t} + \sum_{t\in T}\sum_{k\in K}\sum_{p\in P}\alpha_p x_{kpt} + \sum_{t\in T}\sum_{k\in K}\sum_{p\in P}\beta_{kp}x_{kpt} \\ & +\sum_{t\in T}\sum_{k\in K}\sum_{p\in P}\gamma_{pt}x'_{kpt} + \sum_{t\in T}\sum_{\bar{p}\in\bar{P}}\gamma_{\bar{p}t}y'_{\bar{p}t} + \sum_{p\in P}\gamma_{p,\check{t}}v'_p + \sum_{t\in T}\sum_{i\in I}\mu s'_{it}, \end{aligned} \tag{4.48}$$

s.t. constraints (4.21)-(4.23), (4.3)-(4.10), (4.19), (4.12)-(4.15) and (4.17).

Expression (4.20) defines that the objective is to minimize the annual cost in the production year based on a realized milk scenario defined in constraints (4.21)-(4.23).

4.3 Optimization models assuming milk supply from the multiplicative model

In this section, we present the optimization models assuming milk supply from the multiplicative. First of all, a Markov state ϕ_t is sampled, the vector of independent random variables ω_t has only one random variable, and the sets Φ_t and Ω_t have the same number of values.

4.3.1 Adaptive deterministic optimization model

We present the stage $\tilde{t}$ problem of the ADO model as follows.

$$
\begin{aligned}
& \Delta_{\tilde{t}}(z_{\bar{p},\tilde{t}-3}, z_{\bar{p},\tilde{t}-2}, z_{\bar{p},\tilde{t}-1}, v_{p,\tilde{t}-1}, \varepsilon_{i,\tilde{t}-1}, \phi_{\tilde{t}}, \omega_{\tilde{t}}) = \\
\min \quad & -\sum_{t\in T}\sum_{k\in K}\sum_{p\in P} a_{kpt}(x_{kpt}+x'_{kpt})(x_{kpt}+x'_{kpt}) - \sum_{t\in T}\sum_{\bar{p}\in\bar{P}} b_{\bar{p}t}(y_{\bar{p}t}+y'_{\bar{p}t}) \\
& + \sum_{t\in T}\sum_{i\in I}\sum_{i'\in I} \pi_{ii'} q_{ii't} + \sum_{t\in T}\sum_{p\in P} \tau_p u_{pt} + \sum_{t\in T}\sum_{p\in P} \kappa_p v_{pt} \\
& + \sum_{t\in T}\sum_{\bar{p}\in\bar{P}} \varsigma_{\bar{p}t} y_{\bar{p}t} + \sum_{t\in T}\sum_{k\in K}\sum_{p\in P} \alpha_p x_{kpt} + \sum_{t\in T}\sum_{k\in K}\sum_{p\in P} \beta_{kp} x_{kpt} \\
& + \sum_{t\in T}\sum_{k\in K}\sum_{p\in P} \gamma_{pt} x'_{kpt} + \sum_{t\in T}\sum_{\bar{p}\in\bar{P}} \gamma_{\bar{p}t} y'_{\bar{p}t} + \sum_{p\in P} \gamma_{p,\tilde{t}} v'_p + \sum_{t\in T}\sum_{i\in I} \mu s'_{it}, \qquad (4.49)
\end{aligned}
$$

s.t. constraints (4.3)-(4.10), (4.19), (4.12)-(4.15), (4.17), and

Milk supply:

$$s_{it} = \phi_t \omega_t (f_{it} + h g_{it}), \quad \forall i \in I, t \in T, \qquad (4.50)$$

Random variables ϕ_t and ω_t in future months:

$$\phi_t = \phi_{\tilde{t}}, \omega_t = 1, \quad \forall\, t > \tilde{t}. \qquad (4.51)$$

Expression (4.49) defines that the objective is to minimize the cost in decision horizon T, given the past contracts $z_{\bar{p},\tilde{t}-3}$, $z_{\bar{p},\tilde{t}-2}$ and $z_{\bar{p},\tilde{t}-1}$, the past inventory $v_{p,\tilde{t}-1}$, the past random error $\varepsilon_{i,\tilde{t}-1}$ and the random variables $\phi_{\tilde{t}}$ and $\omega_{\tilde{t}}$ (and thus the current milk supply). Constraint (4.50) defines the milk supply. Constraint (4.51) defines the milk supply in the future months forecast based on the current milk supply. Note that in each of these months, the Markov state ϕ is the same as that in the current month, but the independent random variable ω is set to 1.

4.3.2 Adaptive deterministic optimization sales-projections model

The stage $\tilde{t}$ problem of the ADOS model is given as follows.

$$
\begin{aligned}
& \text{Expression (4.49),} \\
\text{s.t.} \quad & \text{constraints (4.50)-(4.51), (4.3)-(4.10), (4.19), (4.12)-(4.15), (4.17), and} \\
& \text{Meet sales in spot markets in the set of sales projections:} \\
& x_{kpt} + x'_{kpt} \geq \tilde{x}_{kpt}, \quad \forall k \in K, p \in P, t \in T, \qquad (4.52) \\
& \text{Make contracts in the set of sales projections:} \\
& z_{\bar{p}t} = \tilde{z}_{\bar{p}t}, \quad \forall \bar{p} \in \bar{P}, t \in T. \qquad (4.53)
\end{aligned}
$$

Constraint (4.52) states that for product p, the total sales of products from production and in distress trading are above the sales in market k in month t in the set of sales-projections. Constraint (4.53) states that for product p, the contracts to be made in month t are the contracts in the set of sales-projections.

4.3.3 Multistage stochastic programming full-scenario model

The stage t problem of the MSPF model is as follows.

$$
\begin{aligned}
& \Delta_t(z_{\bar{p},t-3}, z_{\bar{p},t-2}, z_{\bar{p},t-1}, v_{t-1}, \varepsilon_{i,t-1}, \phi_t, \omega_t) = \\
\min \quad & -\sum_{k\in K}\sum_{p\in P} a_{kpt}(x_{kpt} + x'_{kpt})(x_{kpt} + x'_{kpt}) - \sum_{\bar{p}\in\bar{P}} b_{\bar{p}t}(y_{\bar{p}t} + y'_{\bar{p}t}) \\
& + \sum_{i\in I}\sum_{i'\in I} \pi_{ii'} q_{ii't} + \sum_{p\in P} \tau_p u_{pt} + \sum_{p\in P} \kappa_p v_{pt} \\
& + \sum_{t\in T}\sum_{\bar{p}\in\bar{P}} \varsigma_{\bar{p}t} y_{\bar{p}t} + \sum_{k\in K}\sum_{p\in P} \alpha_p x_{kpt} + \sum_{k\in K}\sum_{p\in P} \beta_{kp} x_{kpt} \\
& + \sum_{k\in K}\sum_{p\in P} \gamma_{pt} x'_{kpt} + \sum_{\bar{p}\in\bar{P}} \gamma_{\bar{p}t} y'_{\bar{p}t} + \sum_{i\in I} \mu s'_{it} \\
& + \Theta_{t+1}(z_{\bar{p},t-2}, z_{\bar{p},t-1}, z_{\bar{p},t}, v_{p,t}, \varepsilon_{i,t}, \phi_t), \quad \text{if } t < \check{t}, \\
& + \sum_{p\in P} \gamma_{pt} v'_p, \quad \text{if } t = \check{t}, \qquad (4.54)
\end{aligned}
$$

s.t. constraints (4.50), (4.3)-(4.10), (4.19), (4.12)-(4.15), (4.17), and

The future cost:

$$\Theta_{t+1}(z_{\bar{p},t-2}, z_{\bar{p},t-1}, z_{\bar{p},t}, v_t, \varepsilon_{i,t}, \phi_t) = \sum_{\phi_{t+1} \in \Phi_{t+1}} \sum_{\omega_{t+1} \in \Omega_{t+1}} \big(\Pr_{\Phi_{t+1}|\Phi_t}(\phi_{t+1} \mid \phi_t)$$

$$\Pr_{\Omega_{t+1}}(\omega_{t+1}) \Delta_{t+1}(z_{\bar{p},t-2}, z_{\bar{p},t-1}, z_{\bar{p},t}, v_t, \varepsilon_{i,t}, \phi_{t+1}, \omega_{t+1}) \big), \quad (4.55)$$

Expression (4.54) defines that the objective is to minimize the cost given the past contracts, the past inventory and the random errors. Constraint (4.55) defines the future cost $\Theta_{t+1}(z_{\bar{p},t-2}, z_{\bar{p},t-1}, z_{\bar{p},t}, v_t, \varepsilon_{i,t}, \phi_t)$, which is dependent on ϕ_t since $\Pr_{\Phi_{t+1}|\Phi_t}(\phi_{t+1} \mid \phi_t)$ is dependent on ϕ_t.

4.3.4 Multistage stochastic programming expected-scenario model

Since ϕ_t is dependent on ϕ_{t-1}, the expected value in stage t for each ϕ_{t-1} may not be the same. We want to use the same set of values for the random variables in each stage, and so we retain the set Φ_t for ϕ_t, and re-define the set Ω_t to $\{1\}$ for ω_t. Thus the future cost in the stage t problem is the expected value of the stage problems for $\phi_{t+1} \in \Phi_{t+1}$ and $\omega_{t+1} = 1$.

4.3.5 Deterministic equivalent model

The DE model is based on that assuming milk supply from the additive model. We re-define the following sets and parameters.

Sets and indices

ψ	$\in$	Ψ	set of scenarios, i.e., $(\phi_{\hat{t}}, \omega_{\hat{t}}, \phi_{\hat{t}+1}, \omega_{\hat{t}+1}, \ldots, \phi_{\check{t}}, \omega_{\check{t}})$.
$\hat{\psi}_t$	$\in$	$\hat{\Psi}_t$	an upper set of Ψ defining the set of scenarios up to month t, i.e., $(\phi_{\hat{t}}, \omega_{\hat{t}}, \phi_{\hat{t}+1}, \omega_{\hat{t}+1}, \ldots, \phi_t, \omega_t)$.

$\check{\psi}_t \in \check{\Psi}_t$ a lower set of Ψ defining the set of scenarios from month t, i.e., $(\phi_t, \omega_t, \omega_{t+1}, \phi_{t+1}, \ldots, \phi_{\check{t}}, \omega_{\check{t}})$.

$\bar{\psi}_t \in \bar{\Psi}_t$ a single-element set containing a particular scenario in $\check{\Psi}_t$.

Variables

$\phi_{t\psi}$ a random variable sampled from $(\Phi_t, \mathrm{Pr}_{\Phi_t|\Phi_{t-1}})$ in month t in scenario ψ.

Formulation

Expression (4.29),

s.t. constraints (4.33)-(4.47), and

Milk supply:

$$s_{it\psi} = \phi_{t\psi}\omega_{t\psi}(f_{it} + hg_{it}), \quad \forall i \in I, t \in T, \psi \in \Psi, \tag{4.56}$$

Constraint (4.56) defines the milk supply in region i and month t in scenario ψ.

4.3.6 Perfect forecast model

The formulation of the PF model is as follows.

Expression (4.48),

s.t. constraints (4.50), (4.3)-(4.10), (4.19), (4.12)-(4.15) and (4.17).

Chapter 5

Multistage stochastic programming

To solve the multistage stochastic programming models (the MSPF model and the MSPE model), we develop an algorithm, the Dynamic Outer-Approximation Sampling Algorithm (DOASA), which uses multistage Benders decomposition. Since we will use linear price-demand curves which gives rise to a quadratic problem, we will present the algorithm for quadratic programming.

For quadratic programming with independent random variables, we describe the multistage Benders decomposition and DOASA, and then give a mathematical proof of the almost-sure convergence of DOASA for the linear case and discuss that for quadratic programming. Then we give a similar discussion on quadratic programming with Markov states and independent random variables.

5.1 Multistage Benders decomposition for quadratic programming

We define the properties of the multistage stochastic programming models assuming milk supply from the additive model which has independent random variables as follows:

(A1) Random variables appear only on the right-hand side of the constraints in each stage.

(A2) The set Ω_t of random outcomes in each stage $t = 2, 3, \ldots, T$ is discrete and finite, that is, $\Omega_t = \{\omega_{ti} | \; i = 1, \ldots, q_t < \infty\}$ with probabilities $p_{ti} > 0$ for all i.

(A3) Random variables in different stages are independent.

(A4) The feasible region of the quadratic program in each stage is non-empty and bounded.

Then a multistage stochastic programming model for a quadratic problem can be written in the following form:

Solve the problem defined by

$$\begin{aligned} [QP_1] \quad & \Delta_1 = \min\nolimits_{x_1} \tfrac{1}{2} x_1^\intercal D_1 x_1 + c_1^\intercal x_1 + \Theta_2(x_1) \\ \text{subject to} \quad & A_1 x_1 = b_1, \\ & x_1 \geq 0, \end{aligned}$$

where for all $t = 2, \ldots, T$,

$$\Theta_t(x_{t-1}) = \sum_{i=1}^{q_t} p_{ti} \Delta_t(x_{t-1}, \omega_{ti}),$$

$\Delta_t(x_{t-1}, \omega_{ti})$ is defined by the problem

$$\begin{aligned} [QP_t] \quad & \Delta_t(x_{t-1}, \omega_t) = \min\nolimits_{x_t} \tfrac{1}{2} x_t^\intercal D_t x_t + c_t^\intercal x_t + \Theta_{t+1}(x_t) \\ \text{subject to} \quad & A_t x_t = \omega_t - B_{t-1} x_{t-1}, \\ & x_t \geq 0, \end{aligned}$$

and we set $\Theta_{T+1} \equiv 0$.

The problem $[QP_t]$ depends on the choice of ω_t and x_{t-1}, and so we could write $[QP_t(x_{t-1}, \omega_t)]$, but we choose to suppress this dependence in the notation. By Assumption (A3), $[QP_t]$ is independent of $\omega_{t-1}, \omega_{t-2} \ldots$.

The functions $\Theta_t(x_{t-1})$ in each stage can be approximated by the maximum of a collection of linear functions, each of which is called a *cut*, since $\Theta_t(x_{t-1})$ is convex on x_{t-1} which is proved by induction as follows.

Let $f(x_t) = \frac{1}{2}x_t^\intercal D_t x_t + c_t^\intercal x_t + \Theta_{t+1}(x_t)$. Consider $t = T$ and a particular ω_{Ti}, let x'_T be the optimal solution to the problem $\Delta_T(x'_{T-1}, \omega_{Ti})$, and let x''_T be that to $\Delta_T(x''_{T-1}, \omega_{Ti})$. Then $\lambda x'_T + (1-\lambda)x''_T$ is a feasible solution to the problem $\Delta_T(\lambda x'_{T-1} + (1-\lambda)x''_{T-1}, \omega_{Ti})$, since $\lambda x'_T + (1-\lambda)x''_T \geq 0$ and

$$\begin{aligned} A_T(\lambda x'_T + (1-\lambda)x''_T) &= \lambda A_T x'_T + (1-\lambda)A_T x''_T \\ &= \lambda(\omega_T - B_{T-1}x'_{T-1}) + (1-\lambda)(\omega_T - B_{T-1}x''_{T-1}) \\ &= \omega_T - B_{T-1}(\lambda x'_{T-1} + (1-\lambda)x''_{T-1}). \end{aligned}$$

This implies that there is an optimal solution, say x^*_T, to $\Delta_T(\lambda x'_{T-1} + (1-\lambda)x''_{T-1}, \omega_{Ti})$ so that

$$f(x^*_T) \leq f(\lambda x'_T + (1-\lambda)x''_T).$$

Since $f(x_t)$ is convex,

$$f(\lambda x'_T + (1-\lambda)x''_T) \leq \lambda f(x'_T) + (1-\lambda)f(x''_T).$$

This results in $f(x^*_T) \leq \lambda f(x'_T) + (1-\lambda)f(x''_T)$, and thus

$$\Delta_T(\lambda x'_{T-1} + (1-\lambda)x''_{T-1}, \omega_{Ti}) \leq \lambda\Delta_T(x'_{T-1}, \omega_{Ti}) + (1-\lambda)\Delta_T(x''_{T-1}, \omega_{Ti}),$$

which means $\Delta_T(x_{T-1}, \omega_{Ti})$ is convex in x_{T-1}. This applies to all $\omega_{Ti},\ i = 1, \ldots, q_t$ and thus $\Theta_T(x_{T-1})$ is convex.

For $t = T-1, T-2, \ldots, 1$, it is easy to see by induction that since $\Theta_{t+1}(x_t)$ and f are convex, $\Delta_t(x_{t-1}, \omega_{ti})$ is convex in x_{t-1} for each ω_{ti}, and so $\Theta_t(x_{t-1})$ is also. This completes the proof.

In each iteration $k = 1, 2, \ldots$, the algorithm computes a set of feasible solutions $\{x_t^k : t = 1, 2, \ldots, T-1\}$, and a set of cuts, one for each stage $t = 1, 2, \ldots, T-1$. This gives rise to a sequence of approximate problems $[AP_t^k]$, $k = 1, 2, \ldots$, for each stage. These are defined as follows:

For $t = 1$, we solve the quadratic program

$$\begin{aligned}
[AP_1^k] \quad & C_1^k = \min_{x_1,\theta_2} \tfrac{1}{2} x_1^\intercal D_1 x_1 + c_1^\intercal x_1 + \theta_2 \\
\text{subject to} \quad & A_1 x_1 = b_1, \\
& \theta_2 + (\beta_2^j)^\top x_1 \geq \alpha_{2,j}, \qquad j = 0, 1, \ldots, k-1, \\
& x_1 \geq 0,
\end{aligned}$$

and, for $t = 2, \ldots, T-1$, we solve

$$\begin{aligned}
[AP_t^k] \quad & C_t^k(x_{t-1}^k, \omega_t) = \min_{x_t,\theta_{t+1}} \tfrac{1}{2} x_t^\intercal D_t x_t + c_t^\top x_t + \theta_{t+1} \\
\text{subject to} \quad & A_t x_t = \omega_t - B_{t-1} x_{t-1}^k, \\
& \theta_{t+1} + (\beta_{t+1}^j)^\top x_t \geq \alpha_{t+1,j}, \qquad j = 0, 1, \ldots, k-1, \\
& x_t \geq 0.
\end{aligned}$$

Finally for every k, we set $[AP_T^k]$=$[QP_T]$. The problems $[AP_t^k]$ are approximations of $[QP_t]$ in the sense that $\Theta_{t+1}(x_t)$ is approximated (below) by the polyhedral function

$$\max_{j=0,\ldots,k-1} \{\alpha_{t+1,j} - (\beta_{t+1}^j)^\top x_t\}.$$

which is illustrated in Figure 5.1. This means that any solution to $[AP_t^k]$ has a value that is a lower bound on the optimal value of $[QP_t]$.

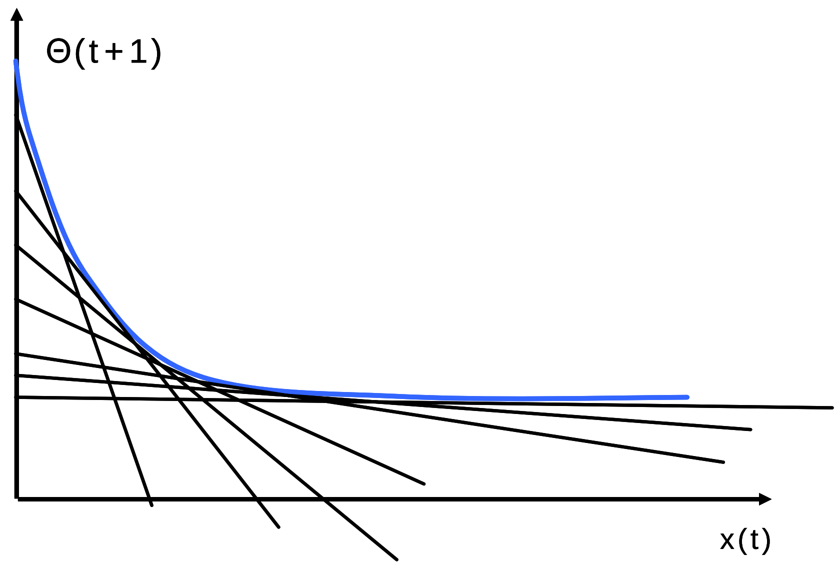

Figure 5.1: Cuts approximating the future cost.

For all stages, the first cut ($j = 0$) is set as the trivial cut $\theta_{t+1} \geq -\infty$. We use the notation $\mathcal{C}_t^k(x_{t-1})$ to denote $\sum_{i=1}^{q_t} p_{ti} C_t^k(x_{t-1}, \omega_t)$. In the last stage, T, we have $[AP_T^k] = [QP_T]$, and so for every x_{T-1} and ω_T

$$C_T^k(x_{T-1}, \omega_T) = \Delta_T(x_{T-1}, \omega_T), \quad k = 1, 2, \ldots .$$

Since cuts are added from one iteration to the next, and no cuts are taken out, the optimal values of $[AP_t^k]$ form a monotonic sequence, i.e. for $k = 1, 2, \ldots$

$$C_t^{k+1}(x_{t-1}, \omega_t) \geq C_t^k(x_{t-1}, \omega_t), \qquad t = 2, 3, \ldots, T,$$

and

$$C_1^{k+1} \geq C_1^k.$$

Observe that under Assumption (A4),

$$\{x_t \mid A_t x_t = \omega_t - B_{t-1} x_{t-1}^k, \quad x_t \geq 0\}$$

is nonempty and bounded. So $[AP_t^k]$ always has a nonempty feasible set and hence an optimal solution. Thus we have a vector of Lagrange multipliers π_t for the equality constraints and a vector of Lagrange multipliers ρ_t for the cut constraints, which are used to calculate the cuts at x_{t-1}^k as follows.

Initially set the iteration count $k = 0$. At any subsequent iteration k the coefficients of the cuts at each stage $t = 1, 2, \ldots, T-1$, are calculated as follows.

Cut Calculation Algorithm (CCA)

1. Solve $[AP_t^k]$ for all $\omega_{ti} \in \Omega_t$ for the Lagrange multipliers $(\pi_t^i(x_{t-1}^k, \omega_{ti}), \rho_t^i(x_{t-1}^k, \omega_{ti}))$.

2. The cut at stage $t-1$ has the formula

$$\theta_t \geq \alpha_{t,k} - (\beta_t^k)^\top x_{t-1},$$

where

$$
\begin{aligned}
\beta_t^k &= \sum_{i=1}^{q_t} p_{ti}\ B_{t-1}^\top\ \pi_t^i(x_{t-1}^k, \omega_{ti}), \quad \text{for } 2 \le t \le T,\\
\alpha_{t,k} &= \sum_{i=1}^{q_t} p_{ti}\left[\omega_{ti}^\top\ \pi_t^i(x_{t-1}^k, \omega_{ti}) + (\alpha_{t+1}^{k-1})^\top \rho_t^i(x_{t-1}^k, \omega_{ti})\right], \quad \text{for } 2 \le t \le T-1,\\
\alpha_{T,k} &= \sum_{i=1}^{q_T} p_{Ti}\ \omega_{Ti}^\top\ \pi_T^i(x_{T-1}^k, \omega_{Ti}).
\end{aligned}
$$

Observe that $\alpha_{t,k}$ is a scalar, whereas α_{t+1}^{k-1} denotes a $(k-1)$-dimensional vector. This means that the dimensions of α_{t+1}^{k-1} and $\rho_t^i(x_{t-1}^k, \omega_{ti})$ are increasing as the iteration count k increases, and thus the collection of (α_t, β_t) may be infinite.

5.2 Dynamic Outer-Approximation Sampling Algorithm

The DOASA algorithm has the following steps.

Step 0: (Initialization) Set $k = 1$.

Step 1: (Forward pass)

Sample a single outcome ω_t of the random variable in each stage $t = 2, 3, \dots, T-1$, to give a single scenario $\{\omega_t^k\}$. For each stage $t = 1, 2, \dots, T-1$, compute the primal solution (x_t^k, θ_{t+1}^k) of the problem $[AP_t^k]$.

Step 2: (Cut Generation)

For each stage $t = T, T-1, \dots, 2$, apply CCA to generate a cut at x_{t-1}^k with Ω_t.

Step 3: Set $k = k + 1$ and go to Step 1.

The set of scenarios $\{x_t^k\}$ contains $\prod_{t=2}^{T-1} q_t$ scenarios. Define $\omega(j)$ as the jth scenario in the set. Then DOASA requires the following property in the sampling methods used to obtain $\{\omega_t^k\}$:

Forward Pass Sampling Property (FPSP):

For each $j = 1, 2, \dots, \prod_{t=2}^{T-1} q_t$, with probability 1

$$\left|\{k : \{\omega_t^k \mid t = 2, 3, \dots, T-1\} = \omega(j)\}\right| = \infty.$$

FPSP states that each scenario $\omega(j)$ is traversed infinitely many times with probability 1 in the forward pass. There are many sampling methods satisfying this property. For example, consider independently sampling a single outcome in each stage with a positive probability for each ω_{ti} in the forward pass. Then by the Borel-Cantelli lemma (see [42]) this method satisfies the property. Another sampling method that satisfies FPSP is to repeat an exhaustive enumeration of each scenario $\omega(j)$, $j = 1, 2, \dots, \prod_{t=2}^{T-1} q_t$, although such a method would be prohibitively expensive in all but the smallest examples.

5.3 Convergence for linear case

Let D_t be a zero matrix for $t = 1, 2, \dots, T$, and then $[QP_t]$ becomes a linear problem $[LP_t]$, which is a special case of the quadratic problems.

For linear problems, the Lagrange multipliers are termed the *duals*. In linear programming, a dual is the optimal solution to the dual problem of $[AP_t^k]$ (see [6]). By Assumption (A1), the set of extreme points of the dual of $[AP_t^k]$ is independent of the outcomes of the random quantities, which allows us to construct a valid cut at each stage based on an assembled collection $\mathcal{D}_t^k$ of extreme-point dual solutions from different samples. Then we can have an extended CCA which uses the extreme-point dual solutions in $\mathcal{D}_t^k$ and a sample $\Omega_t^k \subseteq \Omega_t$ in calculating the cuts (see [71] or [89] for the detail of this CCA). For brevity we refer to the extended CCA as CCA in this section. However, DOASA needs a Backward Pass Sampling Property in sampling Ω_t^k which is defined as

Backward Pass Sampling Property (BPSP):

For each $t = 2, 3, \dots, T$ and $i = 1, 2, \dots, q_t$, with probability 1

$$\left|\left\{k : \omega_{ti} \in \Omega_t^k\right\}\right| = \infty.$$

BPSP states that each scenario outcome ω_{ti} is visited infinitely many times with probability 1 in the backward pass. For example, the two example sampling methods satisfying FPSP both satisfy this property.

5.3.1 Previous results

Previous published results in [16] and [71] give proofs for the almost sure convergence of the algorithm CUPPS and MSBD respectively, which are also based on multistage Benders decomposition and sampling. The proofs in both of these papers require an important but unstated assumption. Here we state this assumption formally and discuss it.

Let the iterations of the algorithm be indexed by $\mathcal{N} = \{1, 2, \dots\}$ and suppose $t \in \{1, \dots, T-1\}$. Let $\{\omega_t^n, x_t^n\}_{n\in\mathcal{N}}$ be the sequence generated by the sampling algorithm at stage t.

Assumption 1: For any infinite subsequence $\{x_t^k\}_{k\in\mathcal{K}}$ of $\{x_t^n\}_{n\in\mathcal{N}}$ there exists a convergent subsequence $\{x_t^j\}_{j\in\mathcal{J}}$ that is independent of $\{\omega_{t+1}^j\}_{j\in\mathcal{J}}$.

Remark 4.1 in [16] correctly claims that if $\mathcal{N}$ is infinite then with probability one $\mathcal{N}$ has an infinite subset $\mathcal{N}_{ti}$ corresponding to draws of outcome ω_{ti} for any $i = 1, \dots, q_t$ and $t = 2, \dots, T$. This follows by an application of the Borel-Cantelli lemma, because each ω_t^n in $\{\omega_t^n\}_{n\in\mathcal{N}}$ is independently sampled and $\Pr[\omega_t^n = \omega_{ti}] > 0$.

However, the situation becomes more subtle in the proof of Lemma 5.2 in [16]. Here the authors claim that for any infinite subset $\mathcal{K}$ of $\mathcal{N}$, there exists an infinite subset $\mathcal{J}$ with a convergent subsequence $\{x_{T-1}^j\}_{j\in\mathcal{J}}$ such that with probability one there exists an infinite subset $\mathcal{J}_i$ of $\mathcal{J}$ corresponding to draws of each sample ω_{Ti} for $i = 1, \dots, q_T$. The convergent subsequence $\{x_{T-1}^j\}_{j\in\mathcal{J}}$ in this lemma is constructed using the assumed compactness of the set X in which x_{T-1} lies. Of course, compactness guarantees a convergent subsequence $\{x^j_{T-1}\}_{j\in\mathcal{J}}$ of $\{x^k_{T-1}\}_{k\in\mathcal{K}}$, but it cannot be deduced from this and Remark 4.1 in [16] that there are infinite number of ω_{Ti} in $\{\omega^j_T\}_{j\in\mathcal{J}}$ for every $i = 1,\dots,q_T$. (The problem here is that for every *convergent* subsequence it might be the case that there are only finitely many ω_{Ti} for some $i = 1,\dots,q_T$, and this possibility needs to be ruled out somehow.)

In claiming the independence of the sampling procedure from the convergence of the subsequence, the authors of [16] are making an implicit assumption (Assumption 1), which is needed to make the proof of Lemma 5.2 valid. The proof in [71] is based on Lemma 5.2 in [16], and so it is also flawed in the absence of Assumption 1.

In the following sections we give a direct proof of the almost-sure convergence that does not rely on Assumption 1. The new proof formalizes the assertion in [23] that convergence follows from resampling. It also clarifies the role that extreme-point dual solutions play in the almost-sure convergence of these sampling algorithms.

5.3.2 Finiteness of set of distinct cuts

The collection of distinct values of $(\beta^k_t, \alpha_{t,k})$ is provably finite, as we show in the following lemma.

Lemma 1 *For each $t = 2, 3, \dots, T$, define the set*

$$\mathcal{G}^k_t = \{(\beta^j_t, \alpha_{t,j}) : j = 1, 2, \dots, k-1\}.$$

Then for any sequence $\mathcal{G}^k_t$, $k = 1, 2, \dots$ generated by the repeated application of CCA there exists m_t such that for all k

$$\left|\mathcal{G}^k_t\right| \le m_t.$$

Furthermore, there exists k_t, so that if $k > k_t$ then $\mathcal{G}^k_t = \mathcal{G}^{k_t}_t$.

Proof. Consider any realization of the sequence $\mathcal{G}^k_t$, $k = 1, 2, \dots$ generated by the repeated application of CCA. We use induction on t to construct m_t such that

$\left|\mathcal{G}_t^k\right| \leq m_t$. The second part of the lemma follows immediately. First at T, $\rho_T = 0$ and π_T is an extreme point of $\{\pi \mid A_T^\top \pi \leq c_T\}$ of which there are at most m_{T+1}, say. Then the cut coefficients

$$\alpha_{T,k} = \sum_{i=1}^{q_T} p_{Ti}\ \omega_{Ti}^\top\ \pi_T^i(x_{T-1}^k),$$
$$\beta_T^k = \sum_{i=1}^{q_T} p_{Ti}\ B_{T-1}^\top\ \pi_T^i(x_{T-1}^k),$$

each can only take at most $m_{T+1}^{q_T}$ values, and thus if $m_T = m_{T+1}^{2q_T}$, then for all k

$$\left|\mathcal{G}_T^k\right| \leq m_T.$$

Now suppose at t that there exists m_{t+1} such that for all k

$$\left|\mathcal{G}_{t+1}^k\right| \leq m_{t+1}.$$

It follows that there exists k_{t+1}, so that if $k > k_{t+1}$ then $\mathcal{G}_{t+1}^k = \mathcal{G}_{t+1}^{k_{t+1}}$ and the cut at iteration $k > k_{t+1}$ is a repeat of some cut in the existing cuts. Consider the feasible region of the dual of $[AP_t^k]$, namely

$$\mathcal{H}_t^k = \{(\pi_t, \rho_t) \mid A_t^\top \pi_t + \sum_{j=1}^{k-1} \beta_{t+1}^j \rho_t^j \leq c_t, \quad \sum_{j=1}^{k-1} \rho_t^j = 1, \quad \rho_t \geq 0\}.$$

If $k > k_{t+1}$ then any extreme point (π_t^k, ρ_t^k) of $\mathcal{H}_t^k$ corresponds to an extreme point (π, ρ) of $\mathcal{H}_t^{k_{t+1}}$ with the same dual objective value, obtained by choosing $\pi = \pi_t^k$ and basic columns β_{t+1}^j for $j < k_{t+1}$ that match the basic columns β_{t+1}^j, $k_{t+1} \leq j < k$. This is because each latter column β_{t+1}^j and its cost coefficient $\alpha_{t+1,j}$ is a duplicate of some $(\beta, \alpha) \in \mathcal{G}_{t+1}^{k_{t+1}}$. Since there are a finite number, say υ_t, of extreme point solutions to $\mathcal{H}_t^{k_{t+1}}$, there are at most υ_t distinct values of

$$\left[\omega_{ti}^\top\ \pi_t^i(x_{t-1}^k) + (\alpha_{t+1}^{k-1})^\top \rho_t^i(x_{t-1}^k)\right]$$

and so $(\upsilon_t)^{q_t}$ distinct values of

$$\alpha_{t,k} = \sum_{i=1}^{q_t} p_{ti} \left[\omega_{ti}^\top\ \pi_t^i(x_{t-1}^k) + (\alpha_{t+1}^{k-1})^\top \rho_t^i(x_{t-1}^k)\right].$$

Similarly,

$$\beta_t^k = \sum_{i=1}^{q_t} p_{ti}\ B_{t-1}^\top\ \pi_t^i(x_{t-1}^k),$$

can take at most $(\upsilon_t)^{q_t}$ values and so if $m_t = (\upsilon_t)^{2q_t}$ then

$$\left|\mathcal{G}_t^k\right| \leq m_t,$$

which proves the result. ■

Lemma 1 states that in any realization of the algorithm there will exist finite m_t and k_t independent of k. Observe however that in the case that Ω_t^k is randomly sampled, m_t and k_t are random variables with distribution determined by the sampling distribution. So they could be arbitrarily large.

5.3.3 Single-scenario Multistage Benders Decomposition

To demonstrate the convergence of DOASA it is helpful to first understand the convergence of an algorithm that uses a single scenario. This algorithm will construct a cut for every $x_t^k, \quad t = 1, \ldots, T-1, \quad k = 1, 2, \ldots$ that is visited by simulating the solution forward over a single sample scenario $\omega(j)$ that remains the same throughout the course of the algorithm. We call this algorithm SSMBD.

SSMBD

Step 0: (Initialization) Set iteration counter $k = 1$. Select at each stage $t = 2, 3, \ldots, T-1$, a single outcome ω_t of the random variable to give a single scenario.

Step 1: (Forward pass)

For each stage $t = 1, 2, \ldots, T-1$, solve $[AP_t^k]$ to yield the primal solution (x_t^k, θ_{t+1}^k).

Step 2: (Cut Generation)

For each stage $t = T, T-1, \ldots, 2$, apply CCA to generate a cut at x_{t-1}^k with a sample Ω_t^k.

Step 3: Set $k = k + 1$ and go to Step 1.

We can apply Lemma 1 to give the following result.

Lemma 2 *Under every realization of iterations, SSMBD converges in a finite number of iterations to a policy giving* $\lim_k C_1^k$ *which is at most equal to the optimal expected cost of [LP_1].*

Proof. Under every realization of iterations, by Lemma 1, for $t \in \{2, \ldots, T\}$, there exists k_t, so that if $k > k_t$ then $\mathcal{G}_t^k = \mathcal{G}_t^{k_t}$ and thus there is no further change in the cuts defining $\mathcal{C}_t^k(x_{t-1})$, that is, for every x_{t-1}

$$\max_{j=0,\ldots,k-1}\{\alpha_{t,j} - (\beta_t^j)^\top x_{t-1}\} = \max_{j=0,\ldots,k_t-1}\{\alpha_{t,j} - (\beta_t^j)^\top x_{t-1}\}.$$

Thus all solutions (x_1^k, θ_2^k) to $[AP_1^k]$ are the same for $k > k_2$, as are all solutions (x_t^k, θ_{t+1}^k) to $[AP_t^k]$, $t = 2, 3, \ldots, T$, so the SSMBD algorithm terminates after iteration k_2.

Since any solution to $[AP_1^k]$ has a value that is a lower bound on the optimal value of $[LP_1]$, this will be true at termination of SSMBD. ■

The solution obtained from SSMBD defines a set of cuts at each stage. If for every k, $\Omega_t^k = \Omega_t$ then this set of cuts at termination will be the same every time the algorithm is run (assuming the single scenario $\omega(j)$ remains fixed). On the other hand if Ω_t^k is a random sample then the set of cuts will also be random, and defined by the sampling distribution. Every time the algorithm is run (with different random number seeds) we should expect to obtain a different sequence of cuts.

The cuts at stage 1 define a lower bound on the expected cost of any policy. Every time the algorithm is run, with a possibly different $\omega(j)$, this lower bound will be (possibly) different. However every realization of this value will be a lower bound on the expected cost of any policy, and so the maximum of these values will be the best lower bound of those available.

The solution obtained from SSMBD is not the same as the optimal solution to the mathematical program obtained by using a single scenario and solving a deterministic problem. The latter solution would define a single set of actions, one for each stage t, that may not be feasible for some scenarios in the original problem. On the other hand, the SSMBD solution is a set of (possibly) random cuts defining a policy that is feasible for the original problem. The simulation of this policy using a randomly sampled forward pass, yields a random value having an expectation that is greater than or equal to the optimal expected cost of the underlying stochastic program. A simulation of the policy with the single scenario used in SSMBD gives the cost of the policy when implemented in the single scenario. The observed value of this simulation depends on the outcomes in the single scenario. This means that it may be significantly lower or significantly higher than the true expected cost of the policy.

5.3.4 Multiscenario Multistage Benders Decomposition

We now consider a multiple-scenario version of SSMDB called MSMBD. In this version a finite set of N scenarios is sampled in advance. The algorithm then constructs an optimal solution corresponding to a scenario tree consisting of these scenarios.

MSMBD

Step 0: (Initialization) Set $k = 1$. For $s = 1$ to N, select at each stage $t = 2, 3, \ldots, T-1$, a single outcome ω_{st} of the random variable to give a set of N scenarios.

Step 1: (Forward pass)

For each scenario s, and stage $t = 1, 2, \ldots, T-1$, compute the primal solution $(x_{st}^k, \theta_{s,t+1}^k)$ of the problem $[AP_t^k]$.

Step 2: (Cut Generation)

For each stage $t = T, T-1, \ldots, 2$, apply CCA to generate N cuts at the states $x^k_{s,t-1}$ with samples $\Omega^k_{s,t}$, $s = 1, 2, \ldots, N$.

Step 3: Set $k = k + 1$ and go to Step 1.

Lemma 3 *In every realization of iterations, MSMBD converges in a finite number of iterations to a policy giving value* $\lim_k C^k_1$ *which is at most equal to the optimal expected cost of [LP_1].*

Proof. The proof is similar to that for SSMBD. For each $s = 1, 2, \ldots, N$, since $k = 1, 2, \ldots$ and one cut is constructed in each iteration k, then by Lemma 1, for $t \in \{2, \ldots, T\}$, there exists $k_{s,t}$, so that if $k > k_{s,t}$ then $\mathcal{G}^k_t = \mathcal{G}^{k_{s,t}}_t$ and thus there is no further change in the cuts defining $\mathcal{C}^k_{s,t}(x_{s,t-1})$, that is, for every $x_{s,t-1}$

$$\max_{j=0,\ldots,k-1}\{\alpha_{t,j} - (\beta^j_t)^\top x_{s,t-1}\} = \max_{j=0,\ldots,k_{s,t}-1}\{\alpha_{t,j} - (\beta^j_t)^\top x_{s,t-1}\}.$$

For $t \in \{2, \ldots, T\}$, if we choose $k_t = \max^N_{s=1}\{k_{s,t}\}$, then for each $k > k_t$ there is no change in the cuts defining $\mathcal{C}^k_t(x_{t-1})$, that is, for every x_{t-1}

$$\max_{j=0,\ldots,k-1}\{\alpha_{t,j} - (\beta^j_t)^\top x_{t-1}\} = \max_{j=0,\ldots,k_t-1}\{\alpha_{t,j} - (\beta^j_t)^\top x_{t-1}\}.$$

Thus all solutions (x^k_1, θ^k_2) to $[AP^k_1]$ are the same for $k > k_2$, as are all solutions (x^k_t, θ^k_{t+1}) to $[AP^k_t]$, $t = 2, 3, \ldots, T$, so the MSMDB algorithm terminates after iteration k_2.

It is easy to see that for every k the optimal value of $[AP^k_1]$ is a lower bound on the optimal expected cost of $[LP_1]$. ■

The algorithm MSMBD works with N scenarios that do not change over the course of the algorithm. All of the remarks that were made for SSMBD apply in this case also. In particular we observe that a termination criterion that uses the N scenarios to simulate the candidate policy defined by the cuts might give a misleading indication of convergence. Lemma 3 demonstrates that MSMBD will terminate at some policy that gives a lower bound on the optimal expected cost of $[LP_1]$. Simulating this policy using a set of randomly sampled scenarios will give a statistical estimate of an upper bound on the optimal expected cost of $[LP_1]$. If the sample of N scenarios is small then we might expect termination of MSMBD at a poor policy. In this case the standard termination criterion using the statistical estimate of the upper bound might fail to be met, even though no further improvement in the policy is possible by continuing to iterate MSMBD.

A special case of MSMBD uses the universe of $N = \prod_{t=2}^{T-1} q_t$ scenarios.

Lemma 4 *Under BPSP, MSMBD with the universe of scenarios converges with probability 1 to an optimal solution to $[LP_1]$ in a finite number of iterations.*

Proof. From Lemma 3 in every realization of iterations MSMBD will converge in a finite number of steps to a policy that has $\lim_k C_1^k$ giving a lower bound on the true expected cost. Now consider a realization of MSMBD iterations, and denote the limiting policy by $(\bar{x}_1, \bar{x}_2(\omega_2), \bar{x}_3(\omega_2, \omega_3), \dots)$, which is obtained at iteration $\bar{k}$, say. For any scenario $\omega_2, \omega_3, \dots, \omega_T$, we denote $\bar{x}_t(\omega_2, \dots, \omega_t)$ by $\bar{x}_t(\omega)$. We claim that for every $k > \bar{k}$, and any scenario ω,

$$\mathcal{C}_T^k(\bar{x}_{T-1}(\omega)) = \Theta_T(\bar{x}_{T-1}(\omega)), \tag{5.1}$$

with probability 1, which implies $C_T^k(\bar{x}_{T-1}(\omega), \omega_T) = \Delta_T(\bar{x}_{T-1}(\omega), \omega_T)$ for all ω_T. Otherwise for some particular outcome $\hat{\omega}_T$, we have $\hat{\omega}_T \notin \Omega_t^k$, for every $k > \bar{k}$, with positive probability which violates BPSP.

Now we claim that if $k > \bar{k}$ then for every scenario ω

$$\mathcal{C}_{T-1}^k(\bar{x}_{T-2}(\omega)) = \Theta_{T-1}(\bar{x}_{T-2}(\omega)). \tag{5.2}$$

Otherwise for some particular outcome $\hat{\omega}_{T-1}$,

$$C_{T-1}^k(\bar{x}_{T-2}(\omega), \hat{\omega}_{T-1}) < \Delta_{T-1}(\bar{x}_{T-2}(\omega), \hat{\omega}_{T-1}). \tag{5.3}$$

But

$$\begin{aligned} C_{T-1}^k(\bar{x}_{T-2}(\omega), \hat{\omega}_{T-1}) = \quad & \min\nolimits_{x_{T-1}, \theta_T} c_{T-1}^\top x_{T-1} + \theta_T \\ \text{subject to} \quad & A_{T-1} x_{T-1} = \hat{\omega}_{T-1} - B_{T-2} \bar{x}_{T-2}(\omega), \\ & \theta_T + (\beta_T^j)^\top x_{T-1} \geq \alpha_{T,j}, \qquad j = 0, \dots, k-1, \\ & x_{T-1} \geq 0, \end{aligned}$$

which has optimal solution

$$(x_{T-1}^*, \theta_T^*) = (\bar{x}_{T-1}(\omega), \max_{j=0,\dots,k-1}\{\alpha_{T,j} - (\beta_T^j)^\top \bar{x}_{T-1}(\omega)\})$$

with $\omega_{T-1} = \hat{\omega}_{T-1}$.

If $\theta_T^* < \mathcal{C}_T^k(x_{T-1}^*)$, then for any $k > \bar{k}$

$$\max_{j=0,\dots,k-1}\{\alpha_{T,j} - (\beta_T^j)^\top \bar{x}_{T-1}(\omega)\} < \mathcal{C}_T^k(x_{T-1}^*) = \Theta_T(\bar{x}_{T-1}(\omega)) \tag{5.4}$$

by (5.1). But by BPSP we have with probability 1 that for each ω_T there is some $k(\omega_T) > \bar{k}$ with $\omega_T \in \Omega_T^{k(\omega_T)}$. If we let $\hat{k}$ denote the maximum of the $k(\omega_T)$ then the height of the cut at $\bar{x}_{T-1}(\omega)$ evaluated at iteration $\hat{k}$ is $\Theta_T(\bar{x}_{T-1}(\omega))$ contradicting (5.4) (see Figure 1).Thus we have

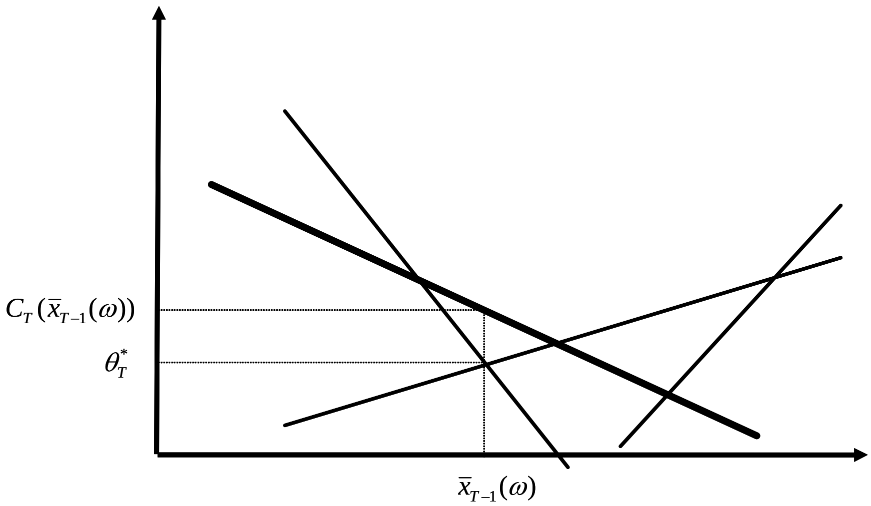

Figure 5.2: A new cut shown in bold would be created if $\theta_T^* < \mathcal{C}_T^k(x_{T-1}^*)$.

$$\theta_T^* = \mathcal{C}_T^k(x_{T-1}^*) = \Theta_T(x_{T-1}^*)$$

and

$$C_{T-1}^k(\bar{x}_{T-2}(\omega), \hat{\omega}_{T-1}) = c_{T-1}^\top x_{T-1}^* + \Theta_T(x_{T-1}^*) = \Delta_{T-1}(\bar{x}_{T-2}(\omega), \hat{\omega}_{T-1})$$

contradicting (5.3), thereby demonstrating (5.2). Observe that since $\hat{\omega}_{T-1}$ was arbitrary this shows that $\bar{x}_{T-1}(\omega)$ solves $[LP_{T-1}(\bar{x}_{T-2}(\omega), \omega_{T-1})]$ for any ω_{T-1}.

In a similar way, it is easy to show by induction that $\bar{x}_{t-1}(\omega)$ solves $[LP_{t-1}(\bar{x}_{t-2}(\omega), \omega_{t-1})]$ thus demonstrating that $(\bar{x}_1, \bar{x}_2(\omega_2), \bar{x}_3(\omega_2, \omega_3), \dots)$ is an optimal policy. ■

5.3.5 DOASA

We now return to DOASA, in which a single scenario is re-sampled in each forward pass, in contrast to the methods above when these are sampled once and then fixed.

Theorem 5 *Under FPSP and BPSP, DOASA converges with probability 1 to an optimal solution to [LP_1] in a finite number of iterations.*

Proof. By FPSP, each scenario in the finite collection of $N = \prod_{t=2}^{T-1} q_t$ scenarios will occur an infinite number of times in the course of the algorithm with probability 1. Thus with probability 1, DOASA will contain a sequence of iterations that are equivalent to MSMBD applied to the universe of scenarios. We may then apply Lemma 4 which shows that with probability 1, DOASA will converge in a finite number of steps to an optimal solution to $[LP_1]$ in a finite number of iterations. ∎

5.3.6 Discussion

The proof of almost-sure convergence above assumes the sampling procedures satisfy FPSP and BPSP. The proof of convergence in [71] makes some different assumptions, namely the Cut Sampling Property and the Sample Intersection Property. The Cut Sampling Property (CSP) states that there are only a finite number of iterations in the algorithm where Ω_t^k is empty. Since we are investigating convergence as $k \to \infty$, CSP is effectively the same as assuming that Ω_t^k is nonempty for all k.

The Sample Intersection Property (SIP) states that for any t, each $\omega_{ti} \in \Omega_t$ and each k (given $\Omega_t^k \neq \emptyset$),

$$\Pr[(\omega_{ti} \in \Omega_t^k) \cap (\omega_t^k = \omega_{ti})] > 0.$$

SIP is sufficient to guarantee FPSP and BPSP if it is accompanied by independent sampling in the forward pass and the backward pass. We state this formally.

Lemma 6 *Given independent sampling in the forward pass, SIP implies FPSP. Given independent sampling in the backward pass, SIP implies BPSP.*

Proof. By SIP, for each $\omega_{ti} \in \Omega_t$ and each k (given $\Omega_t^k \neq \emptyset$),

$$\Pr[\omega_t^k = \omega_{ti}] > 0, \tag{5.5}$$

$$\Pr[\omega_{ti} \in \Omega_t^k] > 0. \tag{5.6}$$

By (5.5) and independent sampling in the forward pass, for any scenario $\omega(j)$ with $\omega_{ti} \in \omega(j)$, $t = 2, 3, \dots, T-1$,

$$\Pr[\{\omega_t^k\} = \omega(j)] = \prod_{t=2}^{T-1} \Pr[\omega_t^k = \omega_{ti}] > 0.$$

Then with independent sampling in the forward pass, by the Borel-Cantelli lemma, there are infinite traversals of each scenario $\omega(j)$, $j = 1, 2, \dots, \prod_{t=2}^{T-1} q_t$ with probability 1, and thus FPSP is satisfied.

With (5.6) and independent sampling in the backward pass, by the Borel-Cantelli lemma, there are infinite visits to each ω_{ti} with probability 1, and thus BPSP is satisfied. ■

Independent sampling is necessary in Lemma 6. If independent sampling in the forward pass is not assured, then FPSP is not guaranteed. For example, suppose for $t = 2, 3, \dots, T-1$, $\Omega_t = \{\omega_1, \omega_2\}$ and we choose ω_t^k with

$$\Pr[\omega_t^1 = \omega_1] = \Pr[\omega_t^1 = \omega_2] = \frac{1}{2},$$
$$\omega_t^k = \omega_t^1, \quad k \geq 2.$$

Then for ω_1,

$$\Pr[\omega_t^k = \omega_1] = \Pr[\omega_t^1 = \omega_1] > 0,$$

and it is easy to show that $\Pr[\omega_t^k = \omega_2] > 0$ for each k, and thus this sampling method satisfies (5.5). But obviously some of the scenarios will never be visited, and thus the sampling method does not satisfy FPSP.

Similarly if independent sampling in the backward pass is not assured, then BPSP is not guaranteed. For example, suppose for $t = 2, 3, \dots, T$, $\Omega_t = \{\omega_1, \omega_2\}$, and we choose Ω_t^k with

$$\Pr[\Omega_t^1 = \{\omega_1\}] = \Pr[\Omega_t^1 = \{\omega_2\}] = \frac{1}{2},$$
$$\Omega_t^k = \Omega_t^1, \quad k \geq 2.$$

Then for example for ω_1 and $k \geq 2$,

$$\Pr[\Omega_t^k = \{\omega_1\}] = \Pr[\Omega_t^1 = \{\omega_1\}] > 0.$$

Similarly $\Pr[\Omega_t^k = \{\omega_2\}] > 0$ for each k, and thus this sampling method satisfies (5.6), but does not satisfy BPSP.

The CUPPS algorithm ([16]) comprises independent sampling in the forward pass, and cuts computed using $\Omega_t^k = \{\omega_t^k\}$. This is easily seen to satisfy SIP, and FPSP and BPSP, even though the backward pass is not sampled independently, but constructed from the forward pass. However, SIP is not necessary for FPSP and BPSP to hold. Consider a version of CUPPS in which cuts are computed using $\Omega_t^k = \Omega_t \setminus \{\omega_t^k\}$. This does not satisfy SIP, but it does satisfy FPSP and BPSP. The algorithms SDDP[1], AND, and ReSa all use independent sampling in the forward pass, and set $\Omega_t^k = \Omega_t$. In this case BPSP is trivially true, and FPSP follows by the Borel-Cantelli lemma. These algorithms also satisfy SIP trivially.

Lemma 8 in [71] asserts that CSP, SIP and independent sampling in the forward pass are sufficient for almost sure convergence. As discussed above there is an implicit independence assumption in the proof of Lemma 8. It is tempting to suppose that independent sampling in the forward pass and SIP give BPSP, which would make Lemma 8 true. However this is not true in general as we have shown. Thus, in the absence of independent sampling in the backward pass, Lemma 8 in [71] remains unproven.

[1]We are assuming here that SDDP re-samples in its forward pass. Some commercial implementations of SDDP do not re-sample and so are more akin to MSMBD than DOASA.

5.4 Convergence for quadratic programming

In solving our problem which has quadratic objectives, we could approximate the linear price-demand curves by stepwise functions, which gives a problem with piecewise linear objectives. Since these objectives have finite number of pieces, this problem resembles the linear objective case, and thus DOASA would converge in solving this problem. However, since this approach may require many pieces in the stepwise functions approximating the price-demand curves to obtain a good solution to our problem, we solve our problem with quadratic objectives directly.

In quadratic programming, the optimal solution satisfies the Karush-Kuhn-Tucker conditions. For instance, the optimal solution to $[AP_t^k]$ satisfies the following equations

$$
\begin{aligned}
A_t x_t - \omega_t^k + B_{t-1} x_{t-1}^k &= 0, \\
\theta_{t+1} + (\beta_{t+1}^j)^\top x_t &\geq \alpha_{t+1,j}, \qquad j = 0, 1, \ldots, k-1, \\
x_t &\geq 0, \\
D_t x_t + c_t + A_t^\top \pi_t^k - \sum\nolimits_{j=0}^{k-1} \rho_t^j \beta_{t+1}^j &= 0, \\
1 - \sum\nolimits_{j=0}^{k-1} \rho_t^j &= 0, \\
\rho_t^j &\geq 0, \quad j = 0, 1, \ldots, k-1, \\
\rho_t^j (\theta_{t+1} + (\beta_{t+1}^j)^\top x_t - \alpha_{t+1}^j) &= 0, \quad j = 0, 1, \ldots, k-1,
\end{aligned}
$$

where π_t^k is a vector of Lagrange multipliers, and ρ_t^j, $j = 0, 1, \ldots, k-1$, is a Lagrange multiplier. This can be solved using simplex method for quadratic programming (see[112]), or primal-dual interior-point methods (see [113]), for example the Barrier optimizer in CPLEX.

Note that the Lagrange multipliers π_t^k and ρ_t^j, $j = 0, 1, \ldots, k-1$, are dependent on ω_t^k and x_{t-1}^k. Thus all stage problems for $\omega_t \in \Omega_t$ are solved to calculate the subgradients to generate a cut. Since the set of subgradients is infinite, the finiteness of the set of distinct cuts is not guaranteed and thus the proof of convergence of DOASA for linear programming problems cannot be applied.

Although we have no mathematical proof of the almost-sure convergence of DOASA for quadratic programming problems, we can check its convergence in practice using a convergence criterion described in [53]. If the lower bound obtained in cut generation is contained by a 95% confidence interval of the sample average in a large sample simulation[2] using the cuts, then we claim that the algorithm has converged.

5.5 Quadratic programming with Markov states and independent random variables

We define the properties of the multistage stochastic programming models assuming milk supply from the multiplicative model which has Markov states and independent random variables as follows:

(A1) Random variables appear only on the right-hand side of the constraints in each stage.

(A2) A set Φ_t of random outcomes in each stage $t = 2, 3, \ldots, T$ are discrete and finite, and are dependent on those in the previous stage, that is, $\Phi_t = \{\phi_{tn} | \, n = 1, \ldots, m_t < \infty\}$ with probabilities $p_t(\phi_{tn} \mid \phi_{t-1}) > 0$ for all n given any ϕ_{t-1}.

(A3) A set Ω_t of random outcomes in each stage $t = 2, 3, \ldots, T$ are discrete, finite and independent, that is, $\Omega_t = \{\omega_{ti} | \; i = 1, \ldots, q_t < \infty\}$ with probabilities $p_t(\omega_{ti}) > 0$ for all i.

(A4) The random variables are in multiplicative form.

(A5) The feasible region of the quadratic program in each stage is non-empty and bounded.

[2] The lower bound of a 95% confidence interval is computed by subtracting two times the standard error from the sample average, and the upper bound is by adding that to the sample average.

Then a multistage stochastic programming model for a quadratic problem can be written in the following form:

Solve the problem defined by

$$\begin{aligned} [QP_1] \quad & \Delta_1 = \min_{x_1} \tfrac{1}{2} x_1^\intercal D_1 x_1 + c_1^\intercal x_1 + \Theta_2(x_1) \\ \text{subject to} \quad & A_1 x_1 = b_1, \\ & x_1 \geq 0, \end{aligned}$$

where for all $t = 2, \dots, T$,

$$\Theta_t(x_{t-1}, \phi_{t-1}) = \sum_{i=1}^{q_t} \sum_{n=1}^{m_t} p_t(\phi_{tn} \mid \phi_{t-1}) p_t(\omega_{ti}) \Delta_t(x_{t-1}, \phi_{tn}, \omega_{ti}),$$

$\Delta_t(x_{t-1}, \phi_{tn}, \omega_{ti})$ is defined by the problem

$$\begin{aligned} [QP_t] \quad & \Delta_t(x_{t-1}, \phi_t, \omega_t) = \min_{x_t} \tfrac{1}{2} x_t^\intercal D_t x_t + c_t^\intercal x_t + \Theta_{t+1}(x_t, \phi_t) \\ \text{subject to} \quad & A_t x_t = \phi_t \cdot \omega_t - B_{t-1} x_{t-1}, \\ & x_t \geq 0, \end{aligned}$$

where $\phi_t \cdot \omega_t$ means the two vectors are multiplied component-wise, and we set $\Theta_{T+1} \equiv 0$.

The problem $[QP_t]$ depends on the choice of ϕ_t, ω_t and x_{t-1}, and so we could write $[QP_t(x_{t-1}, \phi_t, \omega_t)]$, but we choose to suppress this dependence in the notation. By Assumptions (A2) and (A3), $[QP_t]$ is dependent on ϕ_{t-1} but independent of $\omega_{t-1}, \omega_{t-2} \dots$.

The functions $\Theta_t(x_{t-1}, \phi_{t-1})$ in each stage are approximated by the maximum of a collection of cuts. In each iteration $k = 1, 2, \dots,$ the algorithm computes a set of feasible solutions $\{x_t^k : t = 1, 2, \dots, T-1\}$, and a set of cuts, one for each stage $t = 1, 2, \dots, T-1$, which is dependent on ϕ_t. This gives rise to a sequence of approximate problems $[AP_t^k]$, $k = 1, 2, \dots$, for each stage, which are defined as follows:

For $t = 1$, we solve the quadratic program

$$\begin{aligned}
[AP_1^k] \quad & C_1^k = \min_{x_1,\theta_2} \tfrac{1}{2} x_1^\intercal D_1 x_1 + c_1^\intercal x_1 + \theta_2 \\
\text{subject to} \quad & A_1 x_1 = b_1, \\
& \theta_2 + (\beta_2^j)^\top x_1 \geq \alpha_{2,j}, \qquad j = 0, 1, \dots, k-1, \\
& x_1 \geq 0,
\end{aligned}$$

and, for $t = 2, \dots, T-1$, we solve

$$\begin{aligned}
[AP_t^k] \quad & C_t^k(x_{t-1}, \phi_t, \omega_t) = \min_{x_t,\theta_{t+1}} \tfrac{1}{2} x_t^\intercal D_t x_t + c_t^\top x_t + \theta_{t+1} \\
\text{subject to} \quad & A_t x_t = \phi_t^k \cdot \omega_t^k - B_{t-1} x_{t-1}^k, \\
& \theta_{t+1} + (\beta_{t+1}^j(\phi_t^k))^\top x_t \geq \alpha_{t+1,j}(\phi_t^k), \qquad j \in N_t(\phi_t^k), \\
& x_t \geq 0.
\end{aligned}$$

where $N_t(\phi_t^k)$ defines the set of iterations such that a cut constructed in iteration h for $h = 0, 1, \dots, k-1$ is in the set only if $\phi_t^h = \phi_t^k$. Finally for every k, we set $[AP_T^k]$=$[QP_T]$. The problems $[AP_t^k]$ are approximations of $[QP_t]$ in the sense that $\Theta_{t+1}(x_t, \phi_t)$ is approximated (below) by the polyhedral function

$$\max_{j \in N_t(\phi_t^k)} \{\alpha_{t+1,j}(\phi_t^k) - (\beta_{t+1}^j(\phi_t^k))^\top x_t\}.$$

which has been illustrated in Figure 5.1, and thus any solution to $[AP_t^k]$ has a value that is a lower bound on the optimal value of $[QP_t]$.

For all stages, the first cut ($j = 0$) is set as the trivial cut $\theta_{t+1} \geq -\infty$. We use the notation $\mathcal{C}_t^k(x_{t-1}, \phi_{t-1})$ to denote $\sum_{i=1}^{q_t} \sum_{n=1}^{m_t} p_t(\phi_{tn} \mid \phi_{t-1}) p_t(\omega_{ti}) C_t^k(x_{t-1}, \phi_t, \omega_t)$. In the last stage, T, we have $[AP_T^k] = [QP_T]$, and so for every x_{T-1}, ϕ_T and ω_T

$$C_T^k(x_{T-1}, \phi_T, \omega_T) = \Delta_T(x_{T-1}, \phi_T, \omega_T), \quad k = 1, 2, \dots .$$

Since cuts are added by iterations, and no cuts are taken out, the optimal values of $[AP_t^k]$ for the same ϕ_{t-1}^k form a monotonic sequence, i.e. for $j \in N_{t-1}(\phi_{t-1}^k)$,

$$C_t^j(x_{t-1}, \phi_t, \omega_t) \geq C_t^{j-1}(x_{t-1}, \phi_t, \omega_t), \qquad t = 2, 3, \dots, T,$$

and

$$C_1^j \geq C_1^{j-1}.$$

Observe that under Assumption (A5),

$$\{x_t \mid A_t x_t = \phi_t \cdot \omega_t - B_{t-1} x_{t-1}^k, \quad x_t \geq 0\}$$

is nonempty and bounded. So $[AP_t^k]$ always has a nonempty feasible set and hence an optimal solution. Thus we have a vector of Lagrange multipliers π_t for the equality constraints and a vector of Lagrange multipliers ρ_t for the cut constraints, which are used to calculate the cuts at x_{t-1}^k for ϕ_{t-1}^k as follows.

Initially set the iteration count $k = 0$. At any subsequent iteration k the coefficients of the cuts at each stage $t = 1, 2, \ldots, T-1$, are calculated as follows.

Cut Calculation Algorithm (CCA)

1. Given ϕ_{t-1}, solve $[AP_t^k]$ for all $\phi_{tn} \in \Phi_t$ and $\omega_{ti} \in \Omega_t$ for the Lagrange multipliers $(\pi_t(x_{t-1}^k, \phi_{tn}, \omega_{ti}), \rho_t(x_{t-1}^k, \phi_{tn}, \omega_{ti}))$.

2. The cut at stage $t-1$ has the formula

$$\theta_t \geq \alpha_{t,k}(\phi_{t-1}) - (\beta_t^k(\phi_{t-1}))^\top x_{t-1},$$

where

$$\begin{aligned}
\beta_t^k(\phi_{t-1}) &= \sum_{i=1}^{q_t} \sum_{n=1}^{m_t} p_t(\phi_{tn} \mid \phi_{t-1}) p_t(\omega_{ti}) \; B_{t-1}^\top \; \pi_t(x_{t-1}^k, \phi_{tn}, \omega_{ti}), \quad \text{for } 2 \leq t \leq T, \\
\alpha_{t,k}(\phi_{t-1}) &= \sum_{i=1}^{q_t} \sum_{n=1}^{m_t} p_t(\phi_{tn} \mid \phi_{t-1}) p_t(\omega_{ti}) [(\phi_t \cdot \omega_t)^\top \; \pi_t(x_{t-1}^k, \phi_{tn}, \omega_{ti}) \\
&\qquad + (\alpha_{t+1}^{k-1}(\phi_{t-1}))^\top \rho_t(x_{t-1}^k, \phi_{tn}, \omega_{ti})], \quad \text{for } 2 \leq t \leq T-1, \\
\alpha_{T,k}(\phi_{T-1}) &= \sum_{i=1}^{q_t} \sum_{n=1}^{m_t} p_T(\phi_{Tn} \mid \phi_{T-1}) p_T(\omega_{Ti}) \; (\phi_T \cdot \omega_T)^\top \; \pi_T(x_{T-1}^k, \phi_{Tn}, \omega_{Ti}).
\end{aligned}$$

Observe that $\alpha_{t,k}(\phi_{t-1})$ is a scalar, whereas $\alpha_{t+1}^{k-1}(\phi_{t-1})$ denotes a vector with the length of $N_{t-1}(\phi_{t-1})$, which grows as the iteration count k increases. This means that the dimensions of $\alpha_{t+1}^{k-1}(\phi_{t-1})$ and $\rho_t(x_{t-1}^k, \phi_{tn}, \omega_{ti})$ are increasing as the iteration count k increases, and thus the collection of (α_t, β_t) may be infinite.

With Markov states and independent random variables, the DOASA algorithm is adapted and has the following steps.

Step 0: (Initialization) Set $k = 1$.

Step 1: (Forward pass)

Sample a single outcome ϕ_t and ω_t of the random variable in each stage $t = 2, 3, \ldots, T-1$, to give a single scenario $\{\phi_t^k, \omega_t^k\}$. For each stage $t = 1, 2, \ldots, T-1$, compute the primal solution (x_t^k, θ_{t+1}^k) of the problem $[AP_t^k]$.

Step 2: (Cut Generation)

For each stage $t = T, T-1, \ldots, 2$, apply CCA to generate a cut at x_{t-1}^k for ϕ_{t-1}^k with Ω_t.

Step 3: Set $k = k + 1$ and go to Step 1.

In solving our multistage stochastic quadratic programming models, the algorithm has converged based on the same convergence criterion.

Chapter 6

Implementation

In this chapter we describe the implementation facilities, the implementation of the optimization models and DOASA, and present the convergence check of DOASA for the multistage stochastic programming models (the MSPF model and the MSPE model).

6.1 Implementation facilities

The models are implemented in a mathematical modelling language AMPL in text format files. The data is import from spreadsheet tables by the AMPL ODBC driver. The models are solved by the QP Simplex algorithm in CPLEX solver of version 10.0. This is performed on a computer with a Window XP Professional platform, a Core Dual 2.4GHz CPU and a 2GB RAM. The solutions of the optimization models for the forecast milk supply are output to spreadsheet tables. For the optimization models for uncertain milk supply, the set of cut coefficients for the MSPF model and the MSPE model are output to text format files in AMPL-readable format, which are used in the simulation. The simulation results are written to text format files.

6.2 Implementation of optimization models

We solve our multistage stochastic programming models (the MSPF model and the MSPE model) using DOASA. Hence we replace the future cost Θ in each stage problem by θ, the minimum of the approximation by the cuts, and replace constraint (4.28) by the collection of cut constraints. The state variables x_{t-1} in these constraints are the inventory variables, the contracts, and the random error observed in the previous month.

To compute these cuts, we need the Lagrange multipliers of the constraints with the past contract variables, past inventory variables and past random errors on the right hand side. For past contract variables $z_{\bar{p},t-3}$, $z_{\bar{p},t-2}$ and $z_{\bar{p},t-1}$, we add the following constraints to the problem:

$$\bar{z}_{\bar{p},t-1} = z_{\bar{p},t-1}, \quad \forall \bar{p} \in \bar{P}, \tag{6.1}$$

$$\bar{z}_{\bar{p},t-2} = z_{\bar{p},t-2}, \quad \forall \bar{p} \in \bar{P}, \tag{6.2}$$

$$\bar{z}_{\bar{p},t-3} = z_{\bar{p},t-3}, \quad \forall \bar{p} \in \bar{P}, \tag{6.3}$$

and replace the past contract variables by the dummy variables $\bar{z}$ in other constraints of the problem, including the cut constraints. Then we use the Lagrange multipliers of these constraints for past contract variables, and the coefficients in B_{t-1} in cut calculation are -1. For past inventory variables $v_{p,t-1}$, we use the Lagrange multipliers of constraints (4.7) and (4.8) for contract products and non-contract products respectively, and the coefficients in B_{t-1} are -1. For past random errors $\varepsilon_{i,t-1}$, we use the Lagrange multipliers of constraint (4.22) and the coefficients in B_{t-1} are $-\rho$.

For example, suppose in iteration k, we solve a stage t problem given $z^k_{\bar{p},t-3}$, $z^k_{\bar{p},t-2}$, $z^k_{\bar{p},t-1}$, $v^k_{p,t-1}$, $\varepsilon^k_{i,t-1}$ and ω_t. Let $\gamma^k_{pt}(\omega_t)$ for $p \in \bar{P}$ be the Lagrange multiplier of constraint (4.7), and $\gamma^k_{pt}(\omega_t)$ for $p \in \tilde{P}$ be the Lagrange multiplier of constraint (4.8). Let $\delta^k_{\bar{p}t}(\omega_t)$, $\lambda^k_{\bar{p}t}(\omega_t)$ and $\mu^k_{\bar{p}t}(\omega_t)$ be the Lagrange multipliers of constraints (6.1), (6.2) and (6.3) respectively. Let $\pi^k_{it}(\omega_t)$ be the Lagrange multiplier of constraint (4.22). Then after solving all stage t problems, the cut generated for the stage $t-1$

problem is

$$
\begin{aligned}
\theta_t \geq & \sum_{\omega_t \in \Omega_t} \mathrm{Pr}_{\Omega_t}(\omega_t) \Delta_t(z^k_{\bar{p},t-3}, z^k_{\bar{p},t-2}, z^k_{\bar{p},t-1}, v^k_{p,t-1}, \varepsilon^k_{i,t-1}, \omega_t) \\
& + \sum_{\omega_t \in \Omega_t} \sum_{p \in P} \mathrm{Pr}_{\Omega_t}(\omega_t) \gamma^k_{pt}(\omega_t)(v_{p,t-1} - v^k_{p,t-1}) \\
& + \sum_{\omega_t \in \Omega_t} \sum_{\bar{p} \in \bar{P}} \mathrm{Pr}_{\Omega_t}(\omega_t) \delta^k_{\bar{p}t}(\omega_t)(z_{\bar{p},t-1} - z^k_{\bar{p},t-1}) \\
& + \sum_{\omega_t \in \Omega_t} \sum_{\bar{p} \in \bar{P}} \mathrm{Pr}_{\Omega_t}(\omega_t) \lambda^k_{\bar{p}t}(\omega_t)(z_{\bar{p},t-2} - z^k_{\bar{p},t-2}) \\
& + \sum_{\omega_t \in \Omega_t} \sum_{\bar{p} \in \bar{P}} \mathrm{Pr}_{\Omega_t}(\omega_t) \mu^k_{\bar{p}t}(\omega_t)(z_{\bar{p},t-3} - z^k_{\bar{p},t-3}) \\
& + \sum_{\omega_t \in \Omega_t} \sum_{i \in I} \mathrm{Pr}_{\Omega_t}(\omega_t) \rho \pi^k_{it}(\omega_t)(\varepsilon_{i,t-1} - \varepsilon^k_{i,t-1}).
\end{aligned}
$$

For inequality constraints, except the constraints with distress trading and the cut constraints, we add non-negative penalty variables to ensure feasibility. For inequality constraints with a greater-equal sign, we add a penalty variable to the left hand side, and for those with a less-equal sign, we add a minus penalty variable to the left hand side. These penalty variables are decision variables and we set penalty costs on them, so that they are positive only for ensuring feasibility.

Since the MSPE model solves fewer stage problems in cut generation and its policy generated is feasible for the MSPF model, we can use some cuts of the former at the beginning of cut generation of the latter to speed up the cut generation. In practice we found this is very useful in reducing computational effort in cut generation of the MSPF model.

6.3 Implementation of DOASA

In the cut generation, we can bias sampling to improve convergence, as long as the method satisfies the forward pass sampling property (FPSP). For example, using higher probabilities for rare scenarios with big impacts will enable their effects to be captured by the cuts. Antithetic sampling (i.e. sampling a pair of opposite values for the random variable, see [75]) is useful as a variance reduction technique.

We can also fix some scenarios in some iterations, for example, fix a scenario that results in the lowest milk supply throughout the year for twenty iterations, so that more cuts for these scenarios are generated.

Notwithstanding these observations, in simulation we need unbiased sampling so that the sample average is unbiased. The simplest approach is to sample milk scenarios from the probability distribution that we have used to generate the set of milk scenarios. Importance sampling is also an option (see [40]), in which we use higher probabilities in sampling for the scenarios that rarely occur and then calculate the sample average with proper weights on the cost for these scenarios to scale down their effect. Antithetic sampling may also be used.

The DOASA algorithm does not specify a stopping criterion. Stopping criteria of sampling-based algorithms for solving multistage stochastic problems are discussed in [80] and [53]. In [53], a non-statistical stopping criterion suggests the termination of an algorithm if the lower bound hasn't improved for more than a pre-defined percentage of the one in the previous iteration for some successive iterations, and a statistical stopping criteria suggests the termination of an algorithm if a 95% confidence interval of the sample average in a large-sample simulation contains the lower bound in cut generation which can be used as a convergence criterion. In implementation, we use both stopping criteria for the termination of DOASA. First we terminate DOASA when the lower bound hasn't improved by 0.1% of its previous iterate for 100 iterations. Then we perform a large-sample simulation and then test the convergence criterion. If the algorithm hasn't converged, we run DOASA for some more iterations and then test the convergence criterion again. We repeat this process until the criterion is satisfied.

In solving the stage problems with a large number of cuts, the barrier method in the CPLEX solver has primal infeasibility problems, but the dual QP simplex method works fine. Thus we use the dual QP simplex method.

6.4 Convergence check of DOASA

The convergence criterion for DOASA for the MSPF model and the MSPE model is that if the 95% confidence interval of the sample average (in cost) in simulation contains the lower bound obtained in cut generation. We use DOASA to generate cuts for the test problem described in the next chapter, and check the convergence in simulation with 1,000 milk scenarios.

For milk supply from the additive model, we display the number of iterations and the CPU time measured in seconds at the convergence of DOASA for the MSPF model and the MSPE model in Table 6.1.

	Iterations	Time
MSPF	1,500	45,955
MSPE	600	5,082

Table 6.1: Number of iterations and CPU time for the additive model.

To preserve the confidentiality of data, we give a value of -100 to the lower bound and we index the results relative to this value. For the MSPF model, at the termination of DOASA, the 95% confidence interval of the upper bound in simulation with a sample of 1,000 is $(-100.1, -99.8)$. The 95% confidence interval contains the lower bound, and thus DOASA has converged. For the MSPE model, the annual cost with the expected milk scenario (the only scenario in simulation) is -100. This is the same as the lower bound in cut generation of -100, and thus DOASA has converged. DOASA requires 900 more iterations and 11 hours and 20 minutes more to converge in the cut generation of the MSPF model.

On the other hand, for milk supply from the multiplicative model, we present the number of iterations and the CPU time at the convergence of DOASA for the two models in Table 6.2.

For the MSPF model, the 95% confidence interval of the sample average in simulation is $(-100.6, -99.6)$. This contains the lower bound obtained in cut generation, -100. Thus DOASA has converged. For the MSPE model, the 95% confi-

	Iterations	Time
MSPF	$2,700$	$77,213$
MSPE	$2,100$	$28,530$

Table 6.2: Number of iterations and CPU time for the multiplicative model.

dence interval of the sample average with the expected milk scenarios in simulation is $(-100.5, -99.5)$. This contains the upper bound in cut generation, -100, and thus DOASA has converged. DOASA for the MSPF model requires 600 iterations, and 13 hours and 20 minutes more to converge.

Note that the DE model for either of the additive model and the multiplicative model is impossible to solve, due to the huge scale of the problem, which has 22 billion constraints and 21 billion decision variables.

Chapter 7

Policy simulation

In this chapter, we present the results of testing policies in simulation that are defined by solutions of the optimization models. First we define the policies, and describe the data that we used. We use the DS model and DN model to compute the start-of-year inventory, the start-of-year contracts and the penalty cost on distress trading. Using these values we compute results for the policies in simulation assuming milk supply from the additive model, namely additive policies, and assuming milk supply from the multiplicative model, namely multiplicative policies. We compare the additive policy and the multiplicative policy that use the MSPF model. We then assume extra storage capacity can be obtained with a penalty cost and evaluate the additive policies. Finally we assess the cost of not accounting for elastic price in making decisions.

7.1 Policies

The policies are defined by solutions of the optimization models described in chapter 4. We list the policies in Table 7.1, and describe how they are computed.

An ADO policy is obtained by solving the stage problems of the ADO model in each successive month until the end of the year. In each month, after milk supply is realized, a stage problem in the ADO model for the current month is solved, in which milk supply for the future months in the shortening decision horizon is forecast.

Abbreviation	Policy
ADO	Adaptive deterministic optimization policy
ADOS	Adaptive deterministic optimization sales-projections policy
MSPF	Multistage stochastic programming full-scenario policy
MSPE	Multistage stochastic programming expected-scenario policy
PF	Perfect forecast policy

Table 7.1: Policies.

The solution of the current month is implemented, the earnings are computed, and contracts, inventory and random errors are passed to the next month. At the end of the year, a sum of the earnings in each month gives the annual earnings. An ADOS policy is obtained in a similar way, except that it solves a stage problem in the ADOS model in each month.

Prior to computing an MSPF policy, we use DOASA to generate cuts from the MSPF model. In simulation, for a particular milk scenario, we repeat the following process in each month throughout the year. In each month, after milk supply is realized (note that the one in the first month is known beforehand), the corresponding stage problem of the MSPF model with the cuts is solved. The solution in the current month is implemented and the earnings are calculated. Contracts, inventory and random errors are passed to the next month. At the end of year, we sum up the earnings in each month to obtain the annual earnings. We compute an MSPE policy in a similar way, except that in solving the stage problems of the MSPF model we use cuts generated from the MSPE model by DOASA.

Note that we may use the MSPF model in different ways to compute a policy. For example, in each month, after milk supply is realized, we may run DOASA to re-generate a new set of cuts for the decision horizon from the current month to the end of year. Then we solve the stage problem in the current month with the new cuts, and then implement the solution. This requires eleven sets of cuts for one milk scenario, and thus a huge amount of time in simulation. However, in experiment, we have observed no significant advantage of this policy over the MSPF policy.

A PF policy is obtained by solving the PF model for the year assuming milk supply to be realized in each month is known at the beginning of the year, and then evaluating the solution assuming this perfect information. The expected earnings of a PF policy give an upper bound on the value of an optimal policy.

We evaluate the policies in simulation. We sample 1,000 milk scenarios of the year, and compute the above policies for each milk scenario to obtain their annual earnings. For a particular policy, we compute the average of the annual earnings. To compare two policies, we compute the difference of their annual earnings for each milk scenario and then compute the average of the differences. The PF policy is anticipative and so is not implementable, but we use its sample average as a benchmark. Note that we have used different samples in testing and obtained similar results, but we will report the result of one sample in this chapter.

7.2 Data

As we have described in chapter 3, a production year has twelve months starting from June, and the first month with unpredictable milk supply is November.

There are six regions in New Zealand, which are the upper North Island, central North Island, lower North Island, upper South Island, central South Island and lower South Island. Given the historical milk supply for nine years from the 1997 production year to the 2005 production year, we model the 2006 production year. For the model for uncertain milk supply, we use the set of milk scenarios in the example in processing the real data in chapter 3.

Each factory operates five processes, which are the casein mix, cheese mix, milk protein concentrate mix, skim milk powder mix, and whole milk powder mix. Each factory produces five main products which are casein and caseinate (denoted casein), cheese, milk protein concentrate (MPC), skim milk powder (SMP), whole milk powder (WMP), and four by-products which are buttermilk power (BMP), butter and anhydrous milk fat (butter), lactose, and whey protein concentrate (whey). Production is measured in metric tonnes, and costs (as well as prices and earnings) are in New Zealand dollars ($).

There are four international spot markets, which are America, Europe, Japan and Oceania. We use a linear price-demand curve for each product in each market and month. A linear price-demand curve is of the form of $p = a - bq$, where p and q are the market price and demand, a and b are the intercept and negative slope. To derive the price-demand curves for the markets in each month, we use the demand data provided by Fonterra as the *reference demands* and a global base price data provided by Fonterra as the *reference price*. The market price at a zero demand is 105% of the reference price and that at the reference demand is 95% of the reference price (see Figure 7.1 for an illustration). As a result, the price-demand curves at different markets have the same intercepts but different negative slopes in each month. Fonterra has segmented the demand in order of the importance of customers, and we use the first segment as the minimum demand. The market capacity is such that sales at the market capacity result in a zero market price, and thus the sales can be above the reference demand.

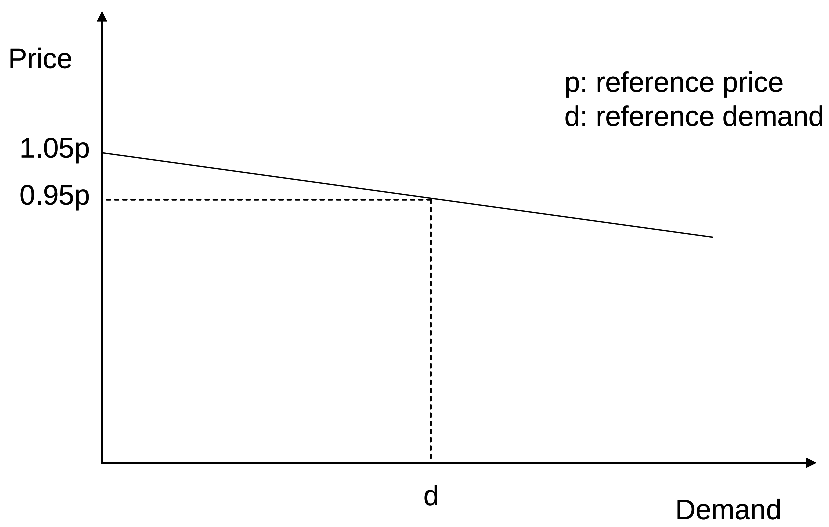

Figure 7.1: Linear price-demand curve.

We assume the contract products are butter, SMP and WMP. The contract price in each month is assumed to be the intercept of the price-demand curve for the spot market, so that in each month selling products by contract always generates a higher revenue. We obtain the start-of-year contracts and the start-of-year inventory by solving the DS model.

We assume the penalty cost on distress trading to be a multiple of the contract price in each month. It is the lowest value such that no distress trading occurs in the solution to the DN model.

Since we use a linear price-demand curve for the sales in a spot market, the revenue generated is a quadratic function of sales, which gives rise to a quadratic objective function. The objective in the optimization models minimizes cost, but we will use earnings in reporting the results.

The MSPF model assuming milk supply from the additive model has 380 variables with 24 state variables, 270 constraints and 9 scenarios in each stage. With 12 stages this gives 4.8 million scenarios in total, resulting in 22 billion variables and 21 billion constraints in the DE model which is impossible to solve. The MSPF model assuming milk supply from the multiplicative model has 368 variables with 20 state variables, 252 constraints and 9 scenarios in each stage. With 12 stages, this gives 4.8 million scenarios, and results in 21 billion variables and 20 billion constraints in the DE model which is also impossible to solve.

7.3 Start-of-year inventory, start-of-year contracts and penalty cost on distress trading

We solve the DS model with a high penalty cost so that no distress trading occurs, and so obtain a start-of-year inventory position and start-of-year contract position from the solution. The start-of-year inventory is displayed in Table 7.2 and the start-of-year contracts are displayed in Table 7.3. These would be the values obtained in a steady-state deterministic solution for a forecast milk supply. We solve the DN model to search for the lowest penalty cost so that no distress trading occurs. The value chosen is roughly five times the contract price for each product in each month. The sales in spot markets and contracts in the solution are the sales-projections for the ADOS model.

BMP	Butter	Casein	Cheese	Lactose	MPC	SMP	Whey	WMP
$5,098$	$180,873$	705	$18,992$	139	$2,391$	$3,682$	167	$224,321$

Table 7.2: Start-of-year inventory (metric tonnes).

	Butter	SMP	WMP
The first month	0	0	$233,764$
The second month	$183,162$	0	$17,904$
The third month	$15,959$	$5,714$	$107,914$

Table 7.3: Start-of-year contracts to be delivered in the first three months (metric tonnes).

7.4 Additive policy simulation

For the policies assuming milk supply from the additive model, namely the additive policies, we perform in-sample simulation and synthetic out-of-sample simulation. In an in-sample simulation, milk supply scenarios are sampled from a set of milk scenarios constructed by using two principal components with each random variable sampled from a discrete distribution, which has been described in chapter 3. Note that the cut generation for the MSPF model and MSPE model uses a sample from this set of milk scenarios.

An out-of-sample simulation would ideally use a historical data set that had not been used in the model development. Because of a lack of suitable data we use instead a synthetic time series in which milk supply scenarios are sampled from a set of milk scenarios constructed by using all six principal components with each random variable sampled from a standard normal distribution, which has been described in the same chapter. We call this a *synthetic out-of-sample simulation.*

7.4.1 In-sample simulation

To preserve the confidentiality of data, we index all results relative to the ADO policy which we give a value of 100. In the simulation, the PF policy has a sample average of 103.4, and is used as a benchmark for the other policies[1].

We compare the ADO policy and the ADOS policy to assess the cost of committing to a set of sales-projections at the beginning of the year, which are the sales in spot markets and contracts in the solution to the DN model. The ADOS policy has a sample average objective value of 99.0. The distribution of earnings in simulation is displayed in Figure 7.2. This figure (as well as those displayed later) is a modified histogram of the earnings in simulation, with the conventional columns replaced by points indicating the heights and then the points fitted by a smooth curve, and with the counts on the vertical axis and the earnings on the horizontal axis. The figure shows that both policies have large downside risks (long thick lower tail). With 95% confidence, the ADO policy achieves higher expected earnings than the ADOS policy by 0.87% to 1.10% of the benchmark value. This indicates that committing to a set of sales-projections at the beginning of the year could be very expensive.

We now compare the MSPF policy and the ADO policy. The MSPF policy has a sample average of 102.6. We display the distribution of earnings in simulation in Figure 7.3. It shows that the MSPF policy has a smaller downside risk. With 95% confidence, the MSPF policy results in higher expected earnings than the ADO policy by between 2.34% and 2.71% of the benchmark value, and thus the MSPF policy has an advantage over the ADO policy.

Next we compare the result for the MSPF policy and the MSPE policy to assess the trade-off between less computational effort and a sub-optimal policy.

[1]The earnings reported in the results have excluded some costs, such as the transportation costs of milk to factories, the transportation costs of products from factories to storage, and thus the earnings have been overestimated. Given the fact that these costs are roughly constant, the advantage relative to the benchmark value in comparing two policies should be larger than that reported in these results.

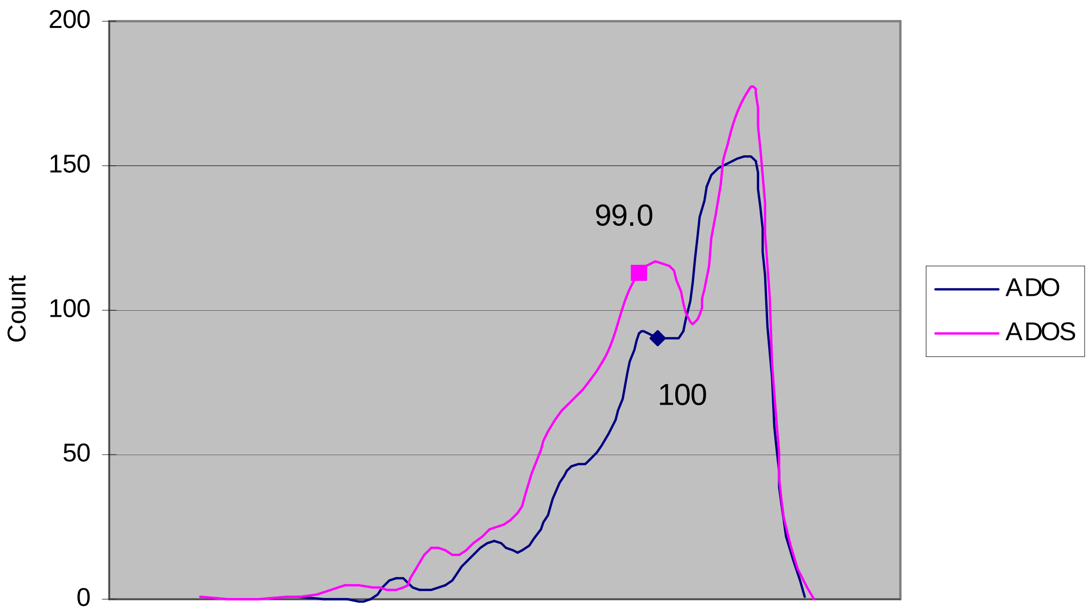

Figure 7.2: Distribution of earnings of ADO and ADOS in simulation.

The sample average of the MSPE policy is 99.9. With 95% confidence, the MSPF policy is better than the MSPE policy by 2.39% to 2.77% of the benchmark value. Figure 7.4 displays the distributions of their earnings in simulation, which shows a large downside risk in the MSPE policy.

The cut generation for the MSPE policy saves 11 hours and 20 minutes in CPU time (shown in Table 6.1 in chapter 6), but this is insignificant compared with the loss in expected earnings. Note that the MSPE policy even has lower earnings in expectation than the ADO policy, and thus the MSPE policy has no advantage over the ADO policy. Hence, the trade-off is not worthy.

Finally, we compare the MSPF policy with the best possible policy, the PF policy. The PF policy has a sample average of 103.4. As shown in Figure 7.5, these two policies have similar distributions of earnings. Observe that the PF policy (an unattainable solution) is 3.4 better in expectation than the ADO policy. Almost 76% of this difference (2.6) can be captured using the MSPF policy.

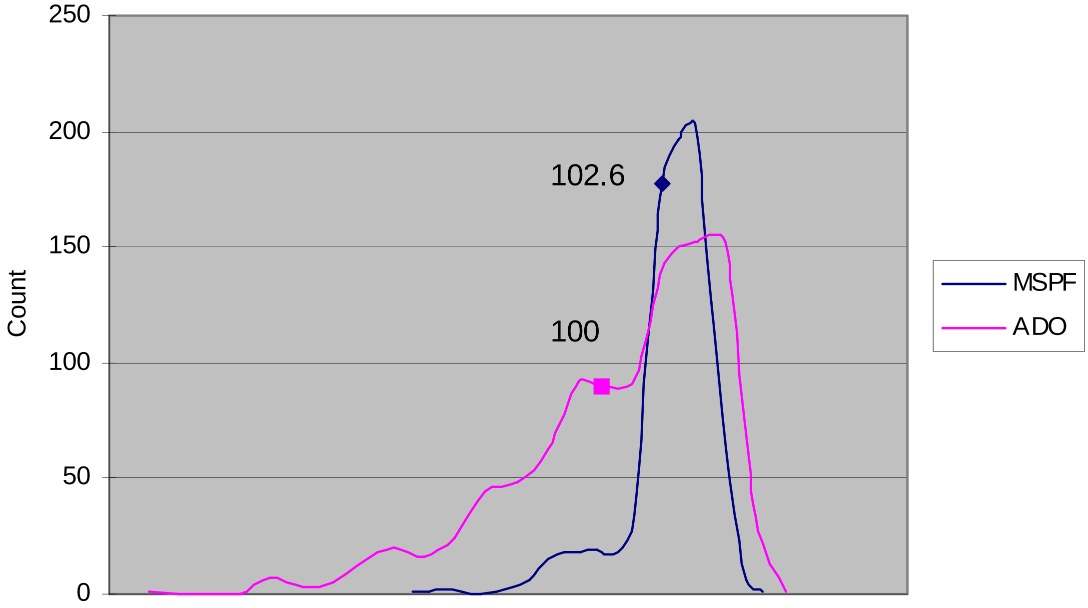

Figure 7.3: Distribution of earnings of MSPF and ADO in simulation.

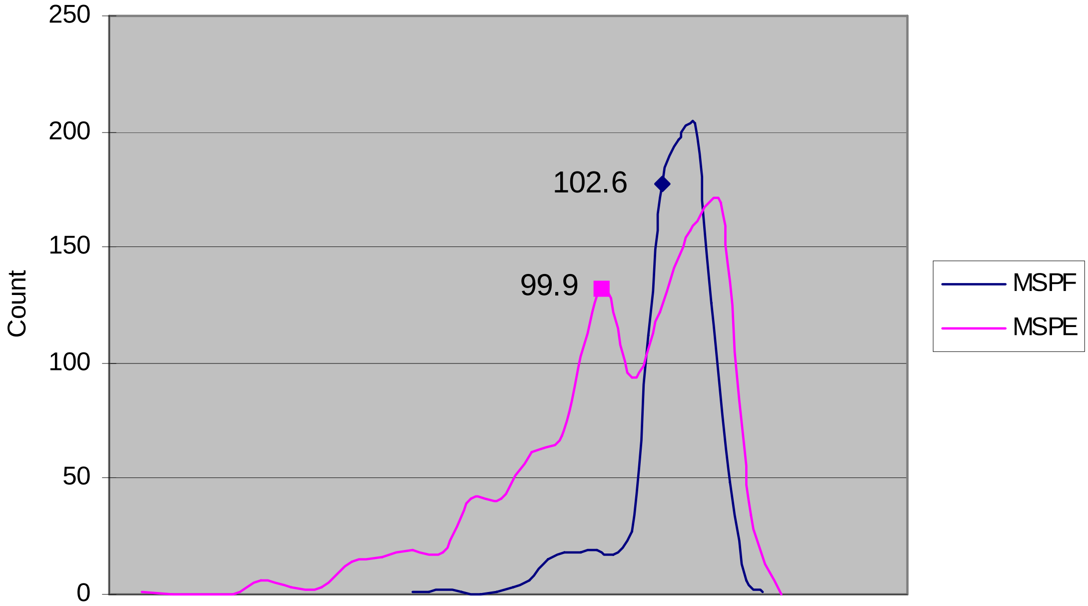

Figure 7.4: Distribution of earnings of MSPF and MSPE in simulation.

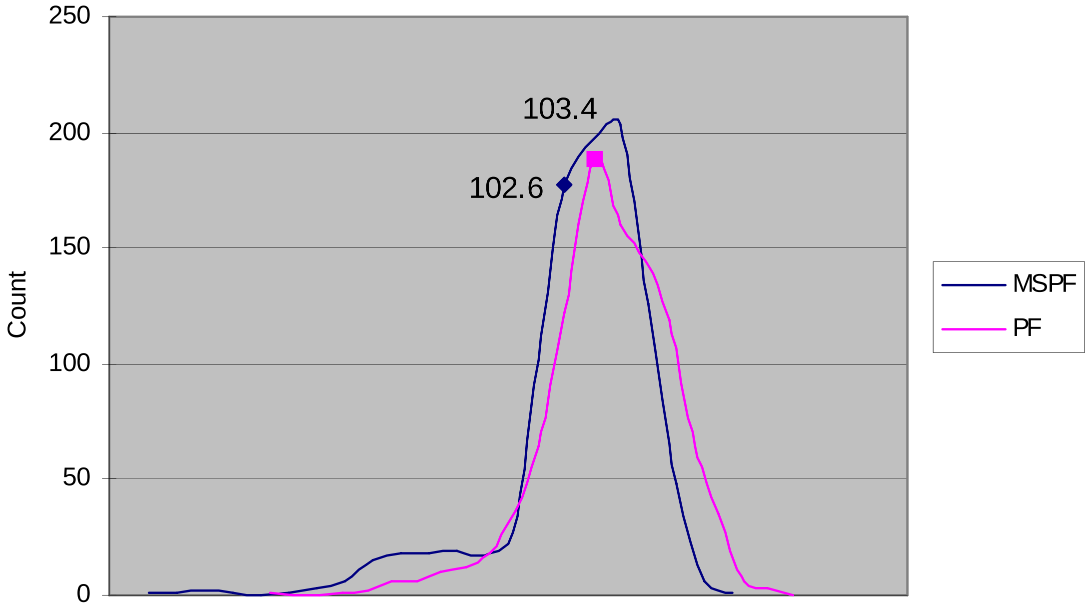

Figure 7.5: Distribution of earnings of MSPF and PF in simulation.

7.4.2 Advantage of the MSPF policy over the ADO policy

In the previous result, the MSPF policy is the best attainable policy using multi-stage stochastic programming, while the ADO policy is a deterministic policy. To understand the advantage of the MSPF policy over the ADO policy, we investigate the difference of these two policies.

We choose two milk scenarios, and display them for a particular region in Figure 7.6. The blue curve is the forecast milk supply scenario, and the pink curve is a scenario that has lower milk supply than forecast in month six and seven.

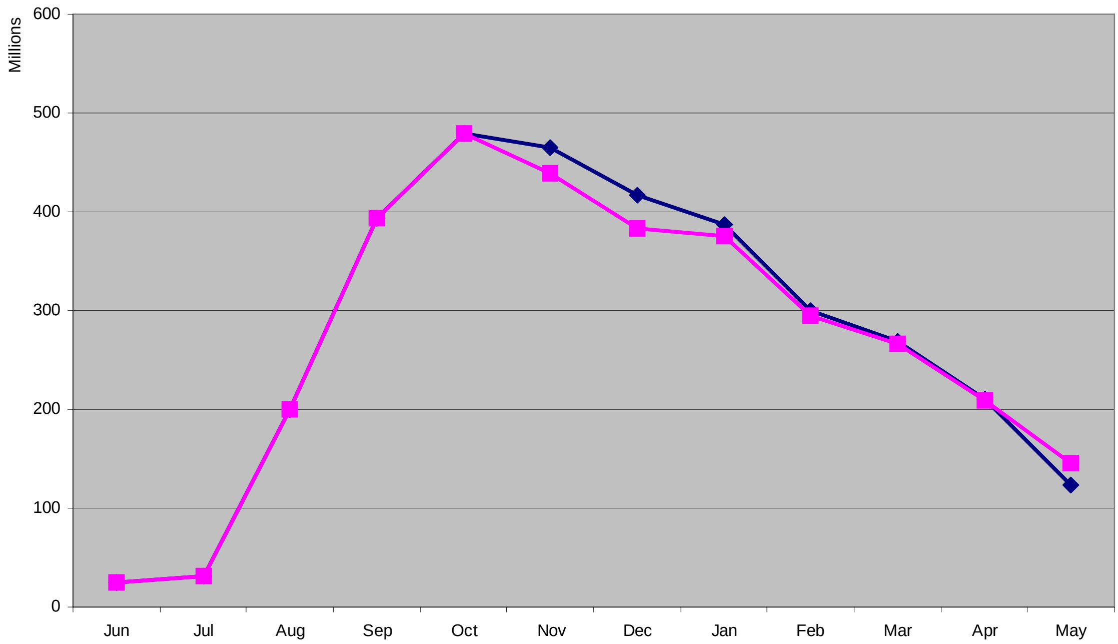

Figure 7.6: Two milk supply scenarios in a particular region.

For the forecast milk supply scenario, we observe that the contract decisions between the two policies have large differences, which are marked by blue in Figure 7.7. We observe no distress trading for either policy. Since the ADO policy has a perfect forecast of milk supply in this scenario, it has higher earnings than the MSPF policy as expected.

On the other hand, for the other milk scenario, we display the contract decisions and distress trading in Figure 7.8. We observe that the contract decisions also have

MSP	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Jan	Feb
Butter	47,154	58,873	55,697	303	0	0	0	0	267
SMP	0	110,465	60,178	53,226	659	84,402	656	456	225
WMP	0	303,842	147,216	151,795	153,640	0	0	2,040	17,656

No distress trading

ADO	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Jan	Feb
Butter	47,156	58,870	57,569	0	0	0	0	0	0
SMP	0	110,469	63,290	56,946	0	73,190	0	0	0
WMP	0	303,846	147,495	152,533	153,375	0	0	60,273	0

No distress trading

Figure 7.7: Contract decisions and distress trading for the forecast milk supply scenario.

large differences which are marked by pink in the figure, but the ADO policy has more distress trading, by about 9,000 tonnes in total. Hence the MSPF policy has higher earnings.

The large differences in contract decisions and distress trading indicate that the MSPF policy is conservative in making contracts, and on average, the gains for the MSPF policy from conservative contracting outweigh any losses that this policy might sustain in the more predictable scenarios.

To test the sensitivity of the model to the penalty costs on distress trading, we perform the following experiment: suppose the penalty cost of distress trading is low, say 120% of the contract price of each product in each month. Since with a lower penalty cost the policies may change, we run DOASA to generate a new set of cuts from the MSPF model, and re-assess the policies in simulation.

The distribution of earnings of the MSPF and ADO policies with a low penalty cost is displayed in Figure 7.9. The sample average of the MSPF policy is 100.1. With 95% confidence, the MSPF policy is better than the ADO policy by only between 0.05% and 0.06% of the benchmark value.

The result shows that the main advantage of the MSPF policy over the ADO policy comes from considering distress trading at high costs. In this circumstance

MSP	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Jan	Feb	Mar	Apr	May
Butter	47,154	58,873	55,697	303	0	0	0	460	0			
SMP	0	110,465	60,178	53,226	659	72,215	632	1,117	7,682			
WMP	0	303,842	147,216	151,795	153,640	0	0	111	18,809			

Butter						31						
Lactose												139
SMP						71						
Whey												46
WMP												

ADO	Jun	Jul	Aug	Sep	Oct	Nov	Dec	Jan	Feb			
Butter	47,156	58,870	57,569	0	0	0	0	0	771			
SMP	0	110,469	63,290	56,946	0	75,050	5,752	0	0			
WMP	0	303,846	147,495	152,533	153,375	0	0	46,227	9,410			

Butter						1,297	0					
Lactose												117
SMP						1,841	3,919					
Whey												
WMP						2,208	0					

Figure 7.8: Contract decisions and distress trading for a scenario with lower milk supply.

the MSPF policy achieves higher expected earnings by being conservative in making contracts.

7.4.3 Synthetic out-of-sample simulation

The tests described above simulate the policies using the same set of milk scenarios used in cut generation (i.e., generated from two principal components as described in chapter 3), and so the results may suffer from in-sample bias. To test this, we assess the MSPF policy against the ADO policy in a set of synthetic milk scenarios generated from all six principal components with each random variable sampled from a continuous normal distribution (as described in the discussion on out-of-sample testing in the same chapter).

With synthetic scenarios constructed in this way, the MSPF policy has a sample average of 102.7 and the PF policy has a sample average of 103.5. The distribution of earnings for the MSPF policy and ADO policy are presented in Figure 7.10. The large lower tail of the ADO policy indicates a large downside risk of this policy. With 95% confidence, the MSPF policy is better than the ADO policy by 2.42%

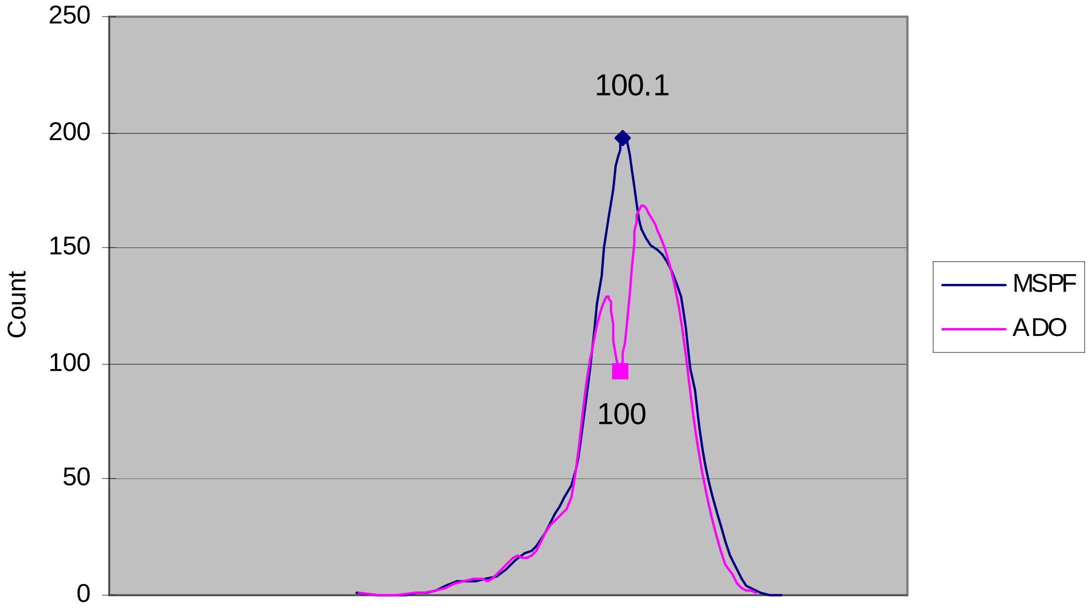

Figure 7.9: Distribution of earnings of MSPF and ADO with a low penalty cost.

to 2.78% of the benchmark value. This shows that most of the variation in the synthetic milk supply scenarios has been captured by the two principal components. (The performance of the MSPF policy would have been much worse if the scenario tree construed with the two principal components had not captured the major variations in the synthetic milk supply scenarios.)

The distribution of earnings for the MSPF policy and PF policy is displayed in Figure 7.11. Almost 77% of this difference (2.7) can be captured using the MSPF policy.

The synthetic out-of-sample result confirms the result from the previous in-sample simulations, and thus the advantage of the MSPF policy is not attributed to in-sample bias.

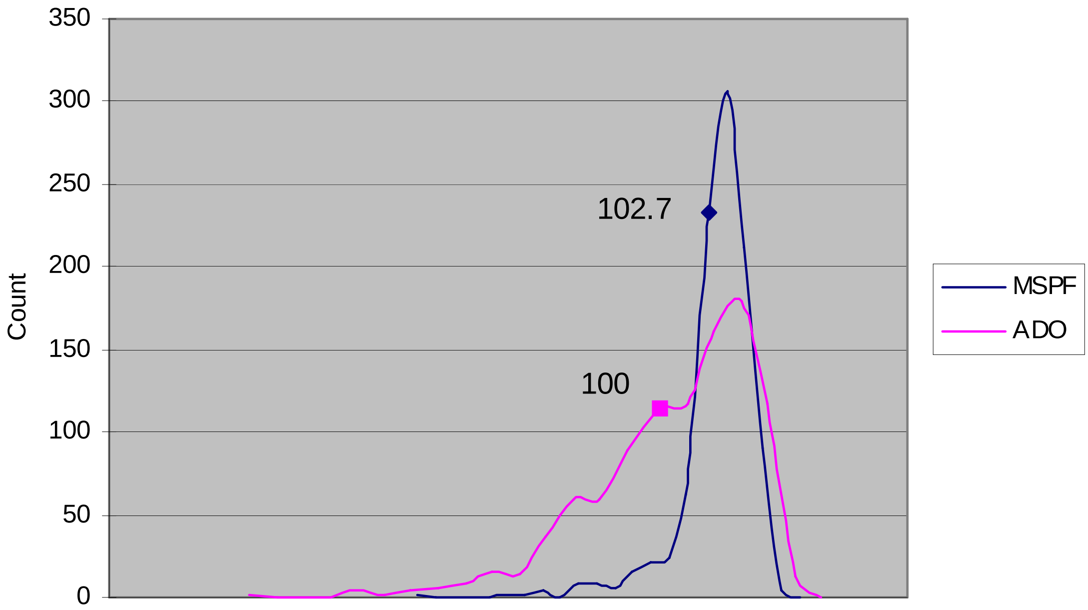

Figure 7.10: Distribution of earnings of MSPF and ADO in synthetic out-of-sample simulation.

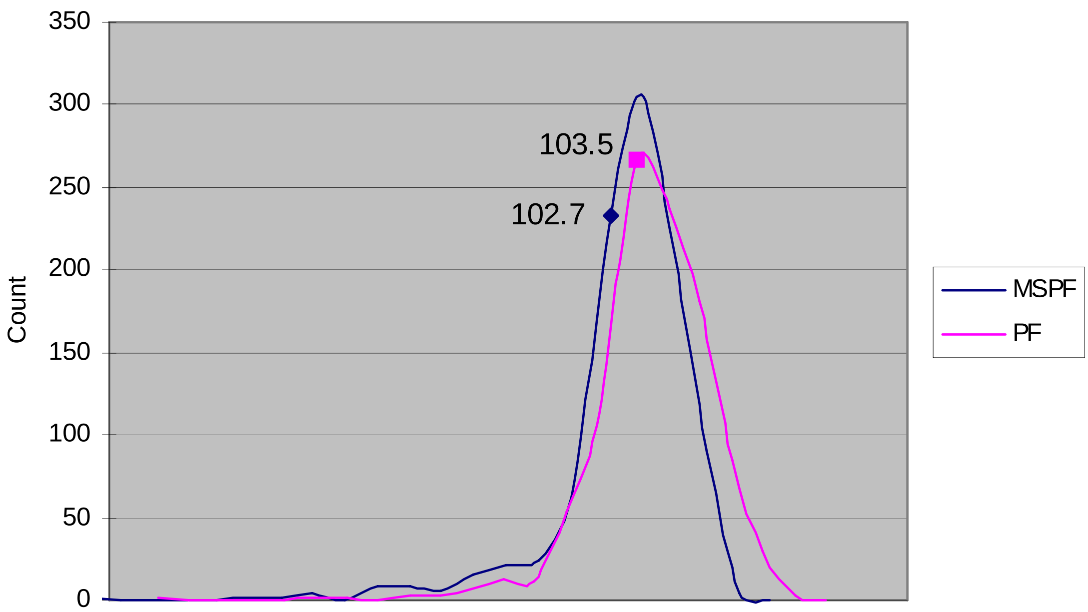

Figure 7.11: Distribution of earnings of MSPF and PF in synthetic out-of-sample simulation.

7.5 Multiplicative policy simulation

We perform in-sample simulation experiments for the policies assuming milk supply from the multiplicative model, namely the multiplicative policies. The PF policy has a sample average of 111.7, which is the benchmark for the other policies.

We display the distribution of earnings of the ADO policy and the ADOS policy in Figure 7.12. The ADOS policy has a sample average of 100.7. With 95% confidence, the ADOS policy has higher expected earnings than the ADO policy by 0.15% to 1.03% of the benchmark value. This indicates that committing a set of sales-projections set at the beginning of the year gives better earnings in expectation, which is a different result from that in the additive policy simulation.

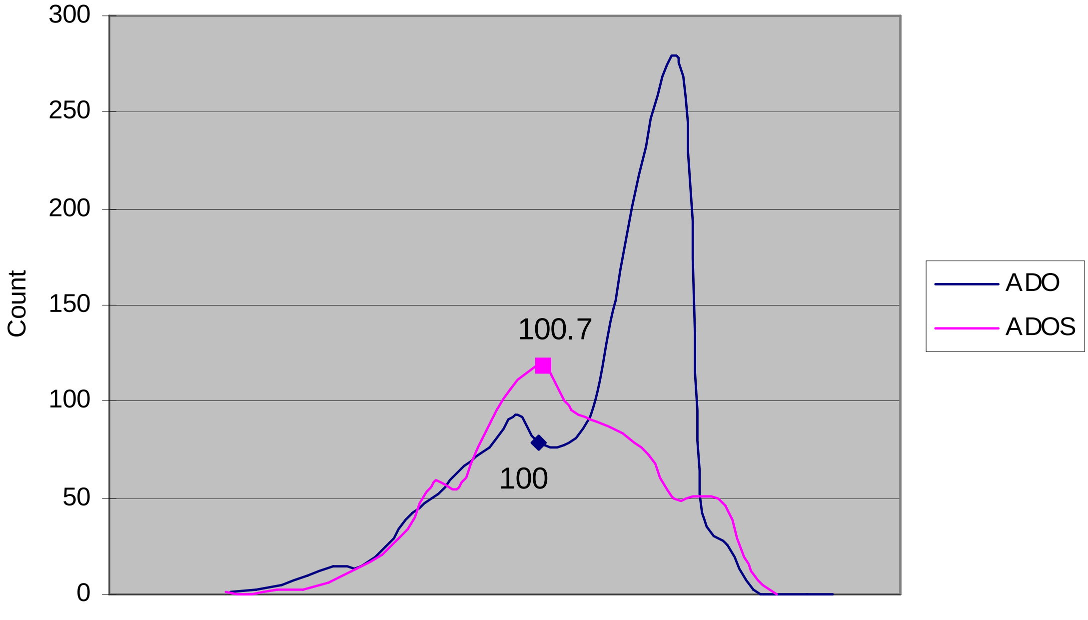

Figure 7.12: Distribution of earnings of ADO and ADOS in simulation.

Note that the milk scenarios in the multiplicative model has small variations, as shown in Figure 7.13 which is a re-produced figure in the example of processing real data in chapter 3. Compared to the ADOS policy, the ADO policy can make adaptive contract decisions based on the milk supply that has realized. This would give a benefit if the milk supply realized in the future months roughly matches what the ADO policy has forecast, but may result in a loss if the milk supply realized in the future months has large deviation which requires distress trading to meet the contracts. With milk scenarios with low uncertainty, the adaptability may not be beneficial, which is what this result has shown. However, the differences are very small.

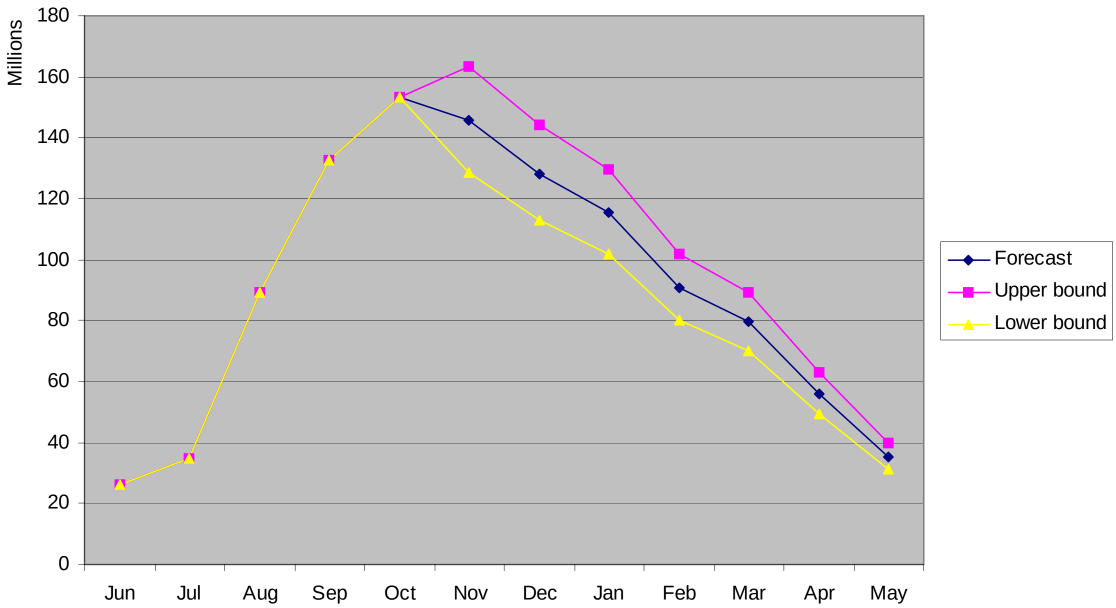

Figure 7.13: Coverage of a sample of milk scenarios from the multiplicative model.

We now compare the result of the MSPF policy and the ADO policy. The MSPF policy has a sample average of 110.7. With 95% confidence, the MSPF policy is better than the ADO policy by between 9.18% and 10% of the benchmark value. The distributions of earnings are displayed in Figure 7.14. The MSPF policy has a shorter lower tail, and thus a smaller downside risk.

The MSPE policy also has a sample average of 110.7, and with 95% confidence, the MSPF policy only results in higher earnings in expectation by only up to 0.04% of the benchmark value. Figure 7.15 shows that these two policies have similar distribution of earnings. Note that the cut generation for the MSPE policy is shorter than that for the MSPF policy by over 13 hours and 20 minutes (see

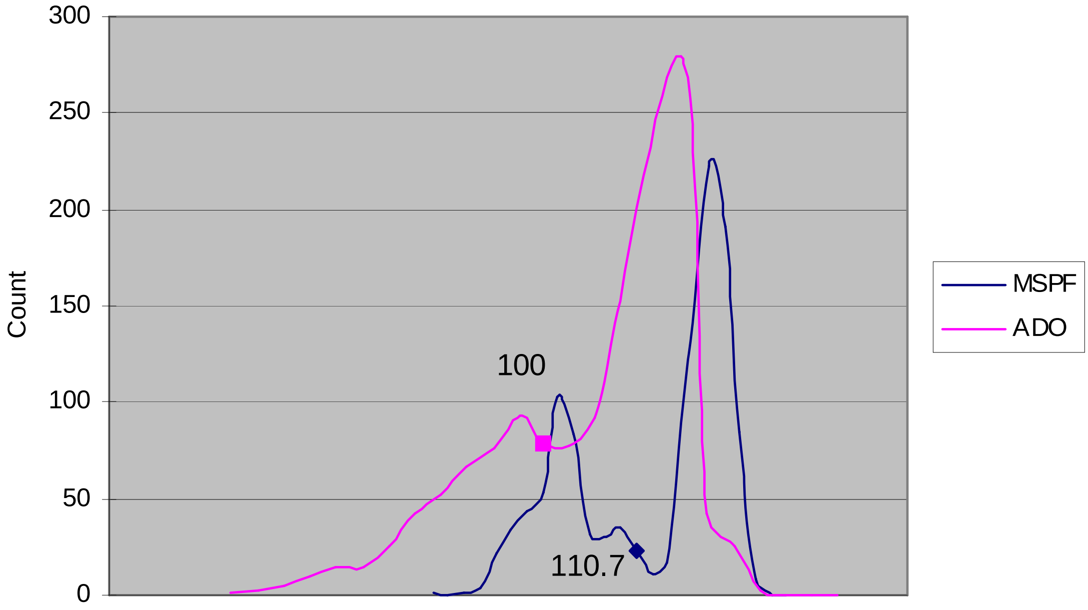

Figure 7.14: Distribution of earnings of MSPF and ADO in simulation.

Table 6.2 in chapter 6).

Finally, we compare the MSPF policy and the PF policy. As shown in Figure 7.16, these two policies have similar distributions of earnings. Observe that the MSPF policy captures 91% (10.7) of the difference between the PF policy and the ADO policy (11.7).

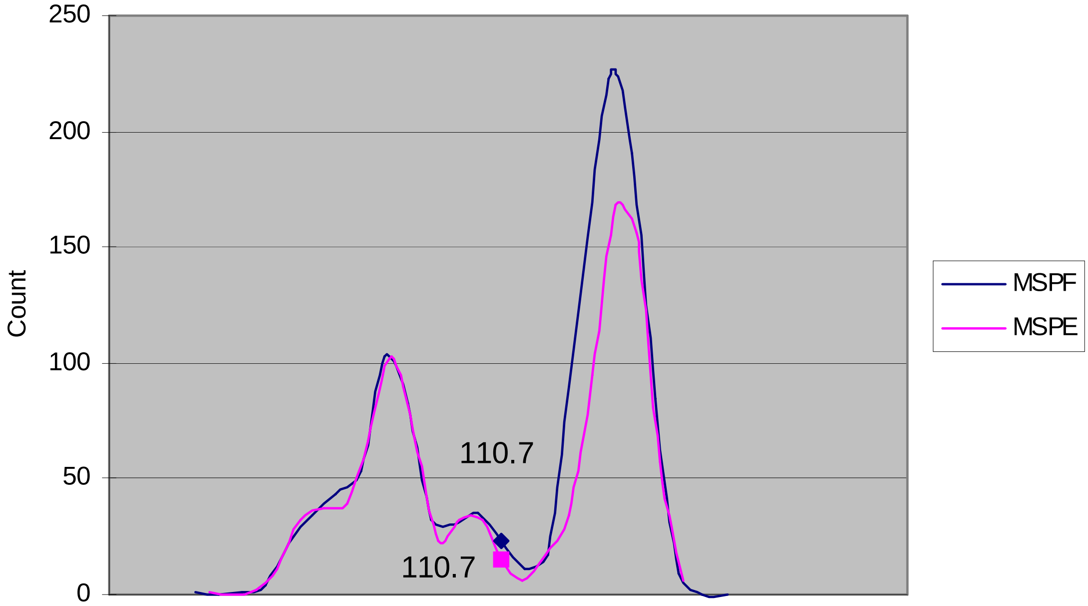

Figure 7.15: Distribution of earnings of MSPF and MSPE in simulation.

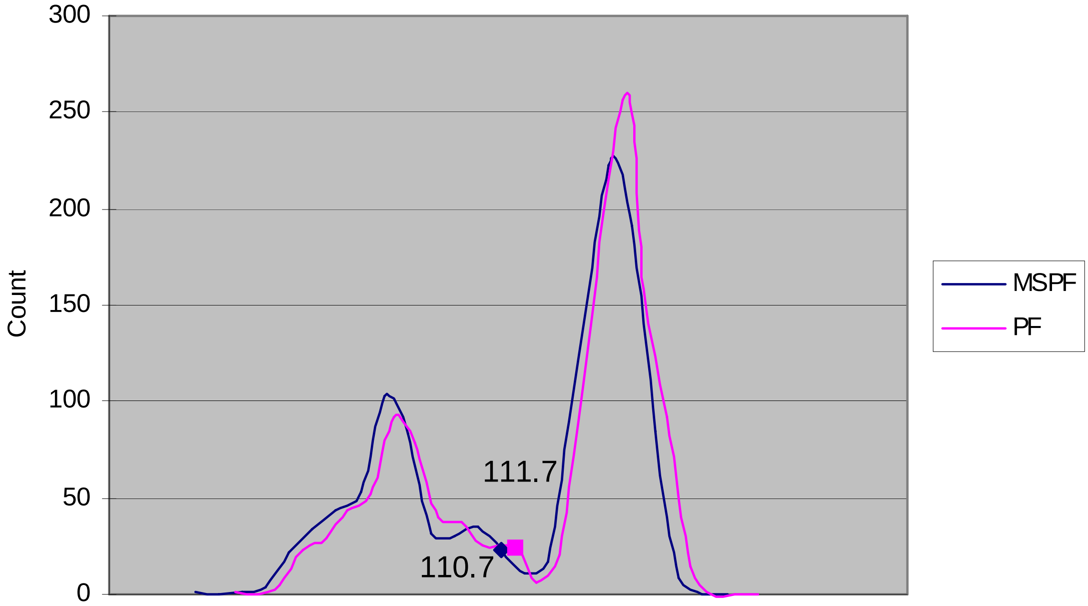

Figure 7.16: Distribution of earnings of MSPF and PF in simulation.

7.6 Additive and multiplicative MSPF policies

Each of the additive MSPF policy and the multiplicative MSPF policy uses a set of cuts generated from the corresponding MSPF model. To compare these two policies, we evaluate them in a synthetic out-of-sample simulation. We sample 1,000 milk scenarios from the additive model. Note that each milk scenario defines the milk supply in each region. We compute the total milk supply and the total forecast milk supply over the regions in each month. Then we compute the ratio of these two to give the value of the random error in the multiplicative model, and then compute the values of the Markov state and the independent random variable. We run DOASA to generate $2,700$ cuts from the corresponding MSPF model for both policies. The two policies are also evaluated against the ADO policy, with a benchmark from the PF policy.

In simulation, the PF policy has a sample average of 103.5. The sample average of the additive MSPF policy is 102.7, and that of the multiplicative MSPF policy is 102.5. We display the distribution of earnings of the ADO policy, the additive MSPF policy and the multiplicative MSPF policy in Figure 7.17.

Compared to the ADO policy, with 95% confidence, the additive MSPF policy has higher expected earnings by between 2.43% and 2.79% of the benchmark value, and the multiplicative MSPF policy has higher expected earnings by between 2.27% and 2.6% of the benchmark value. Thus both MSPF policies have advantages over the ADO policy. Note that even though the multiplicative MSPF policy uses cuts generated based on the total milk supply, it is better than the ADO policy in simulation with (regional) milk supply.

We observe that with 95% confidence the additive MSPF policy is better than the multiplicative MSPF policy by 0.13% to 0.21% of the benchmark value. This is attributed to the fact that the multiplicative MSPF policy does not take into account the variation of milk supply in each region. Accounting for this in our policies yields approximately 0.2 in value. (Note that the additive MSPF policy is expected to be better than the multiplicative MSPF since the scenarios from the

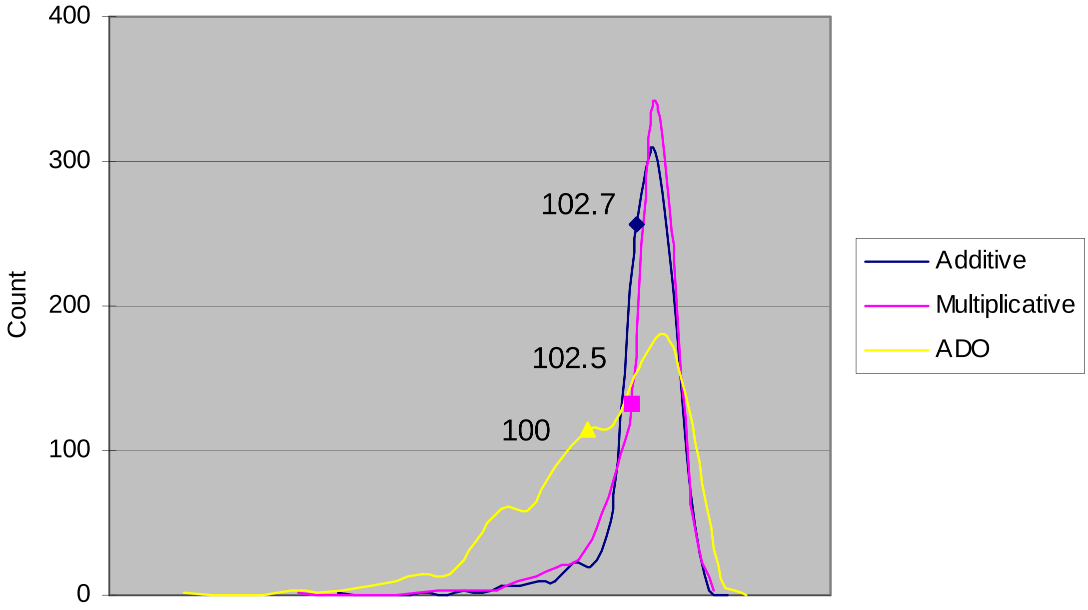

Figure 7.17: Distribution of earnings of ADO, additive MSPF and multiplicative MSPF in simulation.

additive model are used in simulation, and the test result shows by how much.)

7.7 Policy allowing extra storage capacity

In the previous results, we have assumed a fixed storage capacity. However, we have observed that the total inventory of powder products is at the storage capacity in some milk scenarios. In these circumstances, the *shadow price* for storage capacity tells us how much will be gained in earnings if the storage capacity increases by one tonne. For example, if the maximum shadow price is $14,000, then the earnings can be up to $14,000 more in some milk scenarios by getting one more tonne in storage. To model the ability to increase storage, we assume that extra storage capacity for powder products can be obtained, say from third party, with a penalty cost of $300 per tonne, and we evaluate the additive MSPF policy against the ADO policy in a synthetic out-of-sample simulation.

We add a decision variable v'_t, the extra storage capacity measured in metric tonnes, into the constraint for the maximum inventory for powder products in the stage problems,

Maximum inventory level for powder products:

$$\sum_{\check{p}\in\check{P}} v_{pt} - v'_t \leq \check{v}, \quad \forall t \in T,$$

and add the cost to the objective functions. We use DOASA to generate a new set of cuts for the additive MSPF policy ($1,500$ cuts).

When indexed against the ADO policy in the previous synthetic out-of-sample simulation, the ADO policy has expected earnings of 100.4 and the MSPF policy has expected earnings of 103.1. This shows that extra storage capacity (with a penalty cost) improves the earnings of both policies.

With a sample average of 100 for the ADO policy, the sample average of the MSPF policy is 102.6 and that of the PF policy is 103.5. With 95% confidence, the MSPF policy is better than the ADO policy by 2.38% to 2.73% of the benchmark. The distribution of earnings is displayed in Figure 7.18. The MSPF policy captures 76% of the benefit of the PF policy over the ADO policy. These results are only slightly lower than the previous results, which implies that the MSPF policy cannot be replaced by using extra inventory in the ADO policy.

Note that the penalty cost of $300 per tonne for storage is much higher than the real marginal cost of obtaining extra capacity, and thus the earnings of the policies in expectation would be even higher. In this circumstance, the advantage of the MSPF policy relative to the benchmark value would be lower.

7.8 Fixed-price MSPF policy

In the final section of this chapter, we will investigate the cost of not accounting for price variation. To illustrate this, we compare three policies, the ADO policy and the MSPF policy with linear price-demand curves, and an MSPF policy with fixed prices. For the MSPF policy with fixed prices, we generate 1,500 cuts from the

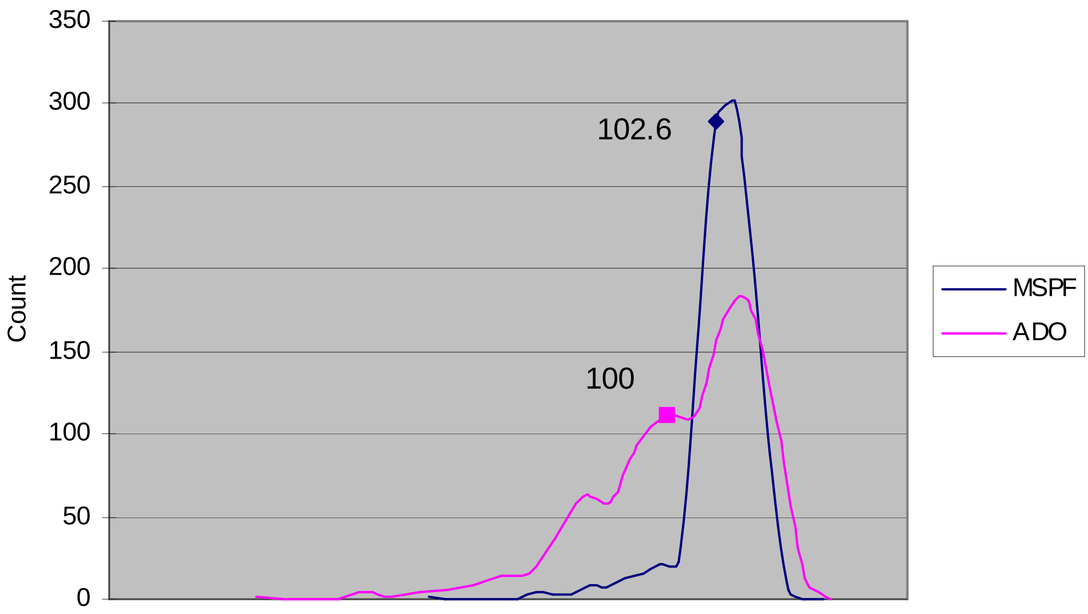

Figure 7.18: Distribution of earnings allowing extra storage capacity in simulation.

MSPF model but assuming that the market prices are fixed at the reference prices. Then we evaluate these three policies in a synthetic out-of-sample simulation with linear price-demand curves.

The linear-price-demand MSPF policy has expected earnings of 102.7, the fixed-price MSPF policy earns 98.9, and the PF policy earns 103.5. With 95% confidence, the linear-price-demand MSPF policy is better than the fixed-price MSPF policy by 3.55% and 3.79% of the benchmark. The distribution of earnings in Figure 7.19 shows that the fixed-price MSPF policy not only has lower earnings in many scenarios, but also has a higher downside risk. Note that the fixed-price MSPF policy also has lower earnings than the ADO policy. These results show that not accounting for price variation in the market will incur a loss in earnings.

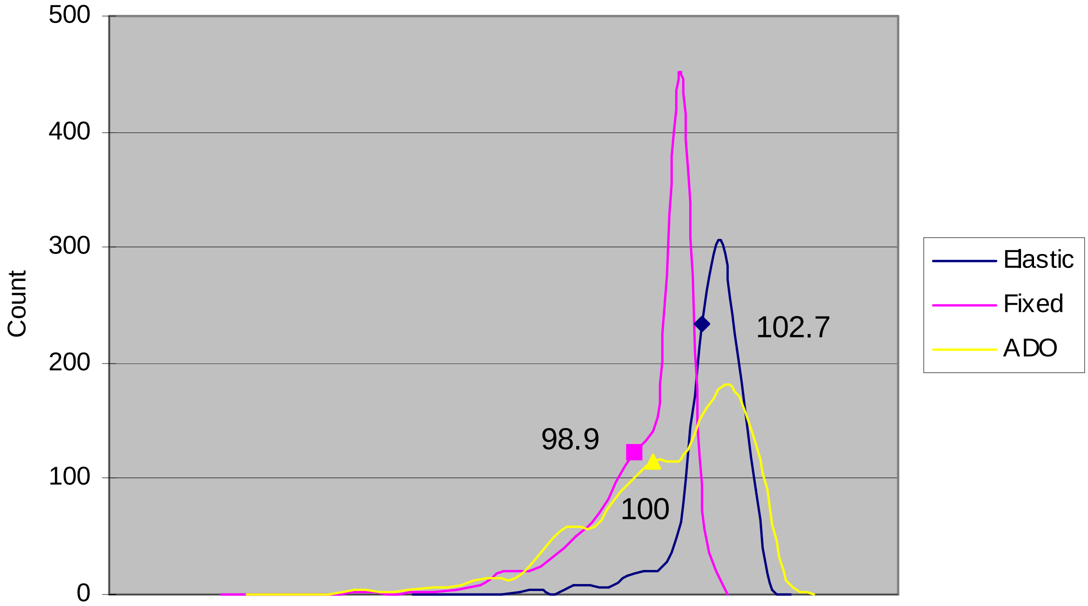

Figure 7.19: Distribution of earnings of ADO, linear-price-demand MSPF and fixed-price MSPF in simulation.

Part II

Game-theoretic inventory model for the European dairy market

Chapter 8

European dairy market

In Part I we have assumed that Fonterra is a monopoly supplier. In this part of the thesis we assume that there are other agents selling products in the markets, and their sales have an impact on the market prices. The uncertainty in market prices needs to be properly assessed in production planning, so that Fonterra's expected price matches the market price.

One of the main target markets for Fonterra is the European dairy market where they compete with companies from, for example, Australia, the European Union and USA. The European Union Commission (denoted EUC) regulates the market and this has an impact on Fonterra's decision making. Under current regulations which includes intervention system and tariffs, the opportunities for strategic selling are very limited. However the EUC continually faces pressure to reform the regulations in favour of free-trade. Eventually such reform will give enough freedom to the players to incentivize strategic selling. To understand how this might evolve, we develop a game-theoretic inventory model to assess Fonterra's strategy in the European dairy market. We will describe the European dairy market in this chapter, and present the model in the next chapter.

In the European dairy market, the EUC aims to maintain the market price during fluctuations of short-term supply, that is, it reduces products in the market if the market price is low and sells products to the market if the market price is high.

The intervention system is a core mechanism for the regulations. Private storage aid, tariffs for import controls and export refunds are also important mechanisms.

The intervention system is open from March 1st to August 31st in each year, when production in the European Union (denoted EU) is at a peak. The EUC observes the market, and buys products from the market if the market price is lower than a pre-determined price, the *intervention price*. As a result, the market price is not much lower than the intervention price. Thus, even though the intervention price is not a threshold for the market price, it acts as a floor price in the European market. The intervention price for a production year is set and announced by the EUC and thus is common knowledge at the beginning of the production year. It is typically higher than the prices in the world market.

The intervention system applies to butter and SMP. The EUC sets limits on the amount of butter and SMP in intervention. If these limits are reached, intervention purchase will be by tender to the dairy producers in the EU, in which the EUC buys products at the lowest price that the producers bid, and thus the intervention stocks may go above the limits. When the market prices are good, products in the intervention stockpile are put onto the market for sales.

Private storage aid is a scheme used for a similar purpose. This scheme subsidises the producers in the EU in the storage of dairy products, which incentivizes them to put less products onto the market. The scheme is available for both butter and SMP, and is open from March 15 to August 15.

Since the market price responds to the sales in the market, the intervention price defines a maximum market size. We illustrate this effect in Figure 8.1. If the intervention price is low, then the European market capacity is large, so suppliers can sell all products to the European market, and the market price is above the intervention price. If the intervention price is high, then the European market capacity is small, and the total sales in the European market are only up to the market capacity, where the market price is the intervention price. Observe that the unsold EU production (blue) is held as storage for later sales.

Figure 8.1: Intervention price.

Since the prices in the European market are substantially higher than those in the world market, the EUC imposes tariffs on the imports from the non-EU countries. The import tariffs are set by the EUC and agreed by the World Trade Organization, and they are common knowledge at the beginning of a production year. There are three types of tariffs. The market access allows unlimited volume for any product but has a high tariff, which effectively prevents any imports. The minimum access has a low tariff but is subject to a limited volume per annum which is called quota, and it is available to each country. The current access has a even lower tariff and is subject to a quota, but it is only available to some countries that have historical trading relationships with the EU, such as Australia and New Zealand.

We illustrate the effect of tariffs and quotas in Figure 8.2. When the intervention price is low, if there were no tariff or quota, then Fonterra and other agents would sell to the European market as much as possible, but in fact due to the tariffs and quotas, Fonterra and other agents choose lower sales in the European market. On the other hand, when the intervention price is high, if there were no tariff or quota, then Fonterra and other agents would sell to the European market as much as possible, but with large production the total sales may exceed the market capacity, and thus the EUC would have to buy products from the market. In practice this never happens due to the tariffs and quotas.

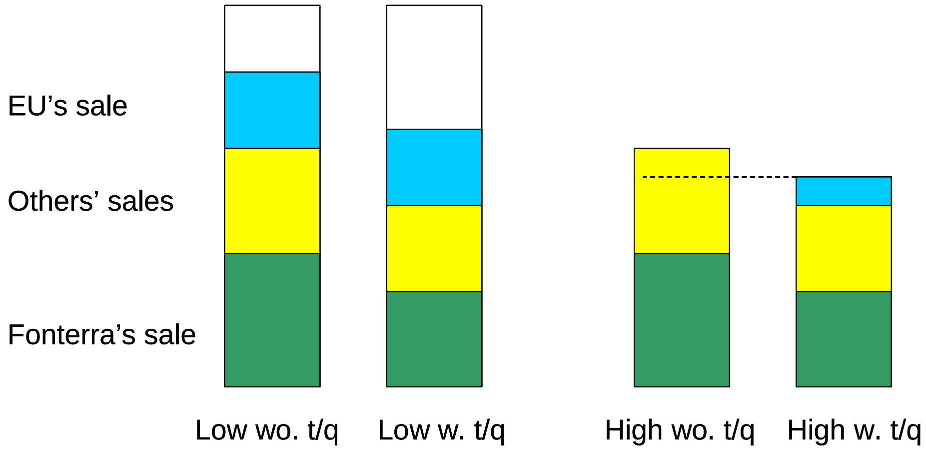

Figure 8.2: Tariffs and quotas.

When the EUC decides to sell the intervention stocks to the world market to reduce the stockpile, it sells to exporters by bidding. Since the world market prices are much lower than the price in the European market, the EUC subsidises the exporters with refunds. This is called the restitution system. The rate of refunds is set by the EUC, or in a tender system to the exporters where the exporter who asks for the lowest rate gets the refund. There are different rates of refunds for products in four categories; butter, SMP, cheese and others. There are limits imposed by the World Trade Organization on the volume of the products receiving export refunds and the budget of refunds per annum.

Chapter 9

Model and algorithm

In this chapter we describe a game-theoretic inventory model for the European dairy market. We will first describe the assumptions that we have made. Then we present the formulation of the model, which consists of an optimization problem for each leader and an optimization problem for each follower. This results in an equilibrium problem with equilibrium constraints (EPEC). To solve the model, we implement an algorithm that uses a *grid search* and *sequential best response*.

9.1 Assumptions

We assume the players are Fonterra, Australia and EU, where EU is modelled as a cartel of the European Union Commission and the companies in the European Union that produce, sell and export dairy products. Note that the follower EU delivers sales as a monopoly rather than under perfect competition of the companies it represents. Fonterra and Australia make decisions simultaneously. Since under the intervention system, the European Union Commission observes the market price and then makes a sales decision, we assume EU to be a follower in decision making. Note that there are other countries in the market, but we assume that they are non-strategic players, that is, their sales are fixed.

The decision horizon is multistage in a production year which is from June to May of the next calendar year. Since the production process will make the players' sales strategies difficult to analyse in all but simple cases, we choose to ignore the production process and its costs and assume that each player has fixed production in each stage. This has some validity for a dairy cooperative like Fonterra, who are committed to producing from all the milk that they collect. Thus each player makes decisions solely on inventory and sales to the European market (denoted UM) and the world market (denoted WM). We assume that each player makes a policy at the beginning of the year that defines the inventory and sales in each stage and then commits to the policy throughout the year without review.

The intervention and the private storage aid are only open at the peak months of production of the European Union countries in each production year. However, if production at the off-peak months were largely increased, the European Union Commission would use these mechanisms to maintain the market price. Thus we assume in our model that the intervention and private storage aid is open for the entire production year. We assume the intervention price is a floor on the market price in UM, and it is a constant throughout the year.

The intervention stocks are the inventory for EU. Since the effect of the private storage aid is similar to the intervention, the stocks in this scheme are assumed to be part of the inventory. Since we do not know the limit on the amount of intervention stocks purchased by tender, nor the limit on the stocks in the private storage aid, we impose no limit on EU's inventory. We assume each player has a start-of-year inventory at the beginning of the first stage, and an inventory target at the last stage.

Although the European Union Commission purchases products into intervention only if the market price falls to the intervention price level, since EU represents the companies in the European Union that can store products even if the intervention price is high, we assume that EU is also able to do so.

We assume that for each market and stage there is a linear price-demand curve. Note that these price-demand curves represent the residual demand given that the non-strategic players' sales are fixed. There is a floor price for each market and stage, which is the intervention price for UM and zero for WM. A floor price defines a market capacity, which is the sales for the floor price defined by the price-demand curve.

The minimum access tariff and the current access tariff are effectively used. We assume that the quota for a leader is the total of the two quotas, and the tariff is the average of the total payment to tariffs from sales at the total quota. Since in reality Fonterra and Australia's total sales do not exceed the market capacity in UM, for validity of the solution, we assume that the total sales of Fonterra and Australia in each stage are below the market capacity, which implies that the market prices for their total sales are above the intervention price.

The refund scheme, under which the EUC provides incentives to exporters is not represented in our model, since the exporters and the EUC are assumed to be the same agent. On the other hand, the volume and budget limits imposed by the World Trade Organization are modelled.

Each player has an inventory holding cost, and each player has a marginal transportation cost for selling products to the markets. Since the players have fixed production, the players can make decisions on sales and inventory for each product separately, and thus we treat each product separately.

9.2 Notation

We present the notation as follows.

Sets and indices

i, i'	$\in$	I	set of leaders.
j, j'	$\in$	J	set of followers.
k	$\in$	K	set of markets.
t	$\in$	T	stages.

Parameters

$c_{it},\ c_{jt}$ inventory holding cost of leader i/follower j in stage t.

$d_{ikt},\ d_{jkt}$ marginal transportation cost to market k of leader i/follower j in stage t.

$g_{it},\ g_{jt}$ production of leader i/follower j in stage t.

m_{jkt} rate of refund on sales to market k in stage t for follower j.

$\hat{p}_k$ floor price at market k.

q_{ik} annual quota for sales in market k by leader i.

q_{jk} annual volume limit for sales in market k by follower j.

r_{jk} annual budget limit for refund on sales in market k by follower j.

s_{ikt} tariff on sales in market k by leader i in stage t.

$\hat{t}$ the last stage.

$u_{i,0},\ u_{j,0}$ start-of-year inventory of leader i/follower j in stage t.

$\hat{u}_i,\ \hat{u}_j$ inventory target of leader i/follower j.

Variables

p_{kt} market price at market k in stage t, as a linear function of sales.

$u_{it},\ u_{jt}$ inventory of leader i/follower j in stage t.

$u_{J,t}$ the vector of inventory of the followers in stage t.

$x_{ikt},\ x_{jkt}$ sales in market k by leader i/follower j in stage t.

$x_{J,kt}$ the vector of sales in market k by the followers in stage t.

α_{jt} Lagrange multiplier for a constraint of product flow for follower j in stage t.

β_{kt} Lagrange multiplier for a constraint of market floor price in market k in stage t.

γ_{jk} Lagrange multiplier for a constraint of annual volume limit on sales in market k by follower j.

λ_{jk} Lagrange multiplier for a constraint of annual budget limit on refund for sales in market k by follower j.

μ_j Lagrange multiplier for a constraint of meeting inventory target by follower j.

9.3 Formulation

In the game-theoretic inventory model, each leader solves an optimization problem assuming that the other leaders' strategies are fixed, and the optimal responses of the followers to the leaders' strategies are known, and each follower solves an optimization problem given that the leaders' strategies and the other followers' strategies are fixed. The game-theoretic inventory model solves the leaders' optimization problems simultaneously. If each leader's strategy is optimal given the others' fixed strategies, then the set of strategies is a Nash equilibrium for the game.

We present the optimization problem for a leader $i \in I$ as follows.

$$\begin{aligned}
\max_{x_{ikt}, u_{it}} \quad & \sum_{kt} \left(p_{kt} \left(\textstyle\sum_{i'} x_{i'kt} + \sum_j x_{jkt}(x_{I,kt}) \right) - s_{ikt} \right) x_{ikt} - \sum_{kt} c_{it} u_{it} \\
& - \sum_{kt} d_{ikt} x_{ikt}, \qquad (9.1) \\
\text{s.t.} \quad & \text{Product flow:} \\
& \sum_k x_{ikt} + u_{it} - u_{i,t-1} - g_{it} = 0, \quad \forall t \in T, \qquad (9.2) \\
& \text{Market floor price:} \\
& p_{kt} \left(\textstyle\sum_{i'} x_{i'kt} \right) - \hat{p}_k \geq 0, \quad \forall k \in K, t \in T, \qquad (9.3) \\
& \text{Annual quota for sales:} \\
& q_{ik} - \textstyle\sum_t x_{ikt} \geq 0, \quad \forall k \in K, \qquad (9.4) \\
& \text{Meeting inventory target:} \\
& u_{i\hat{t}} - \hat{u}_i \geq 0, \qquad (9.5) \\
& \text{Decision variable domain:} \\
& x_{ikt}, u_{it} \geq 0, \quad \forall k \in K, t \in T. \qquad (9.6)
\end{aligned}$$

The decision variables are the sales x_{ikt} and inventory u_{it}. Expression (9.1) defines that the objective is to maximize the payoff, in which the other leaders' sales are fixed and the followers' sales are optimal responses to the set of the leaders' sales $x_{I,kt}$. Note that p_{kt} is a linear function of sales which gives rise to a quadratic optimization problem. Constraint (9.2) defines the product flow, in which the sales and inventory are balanced with the past inventory and the production in each stage. Constraint (9.3) defines that the market price for the total sales of the leaders are above the market floor price. This implies that the leader needs to ensure its sales do not exceed the residual market capacity given the other leaders' sales. Constraint (9.4) states that the total sales in a market are below the annual quota. Constraint (9.5) states that the inventory in the last stage is above the inventory target, and constraint (9.6) defines the domain of the decision variables.

The optimization problem for a follower $j \in J$ is as follows.

$$\max_{x_{jkt}, u_{jt}} \quad \sum_{kt} \left(p_{kt} \left(\sum_i x_{ikt} + \sum_{j'} x_{j'kt} \right) \right) x_{jkt} - \sum_{kt} c_{jt} u_{jt} - \sum_{kt} d_{jt} x_{jkt}, \tag{9.7}$$

s.t. Product flow:

$$\sum_k x_{jkt} + u_{jt} - u_{j,t-1} - g_{jt} = 0, \quad \forall t \in T, \tag{9.8}$$

Market floor price:

$$p_{kt} \left(\sum_i x_{ikt} + \sum_{j'} x_{j'kt} \right) - \hat{p}_k \geq 0, \quad \forall k \in K, t \in T, \tag{9.9}$$

Annual volume limit for sales:

$$q_{jk} - \sum_t x_{jkt} \geq 0, \quad \forall k \in K, \tag{9.10}$$

Annual budget limit for refund on sales:

$$r_{jk} - \sum_t m_{jkt} x_{jkt} \geq 0, \quad \forall k \in K, \tag{9.11}$$

Meeting inventory target:

$$u_{j\hat{t}} - \hat{u}_j \geq 0, \tag{9.12}$$

Decision variable domain:

$$x_{jkt}, u_{jt} \geq 0, \quad \forall k \in K, t \in T. \tag{9.13}$$

The decision variables are the sales x_{jkt} and inventory u_{jt}. Expression (9.7) defines that the objective is to maximize the payoff, given that the leaders' sales and the other followers' sales are fixed. Constraint (9.8) describes the product flow, in which the sales and inventory are equal to the past inventory and the production in each stage. Constraint (9.9) states that the market prices for the total sales of the leaders and followers are above the market floor price, which implies that the follower needs to ensure its sales do not exceed the residual market capacity given the other players' sales. Constraint (9.10) states that the total sales in a market are below the annual volume limit, and constraint (9.11) states that the refund for the total sales in a market are below the annual budget limit. Constraint (9.12) states that the inventory in the last stage meets the inventory target, and constraint (9.13) defines the domain of the decision variables.

9.4 Equilibrium problem with equilibrium constraints

Since each of the followers' optimization problems can be reformulated as a set of equilibrium constraints, our model can be reformulated as an equilibrium problem with equilibrium constraints (denoted EPEC) (see [55]). The EPEC is a set of mathematical programs with equilibrium constraints (denoted MPECs) with one for each leader. The MPEC for a leader is its optimization problem with the set of equilibrium constraints for the followers' optimization problems. In the EPEC, the set of MPECs are simultaneously solved for a solution.

The formulation of the MPEC for a leader $i \in I$ is as follows.

$$\begin{aligned}
\max_{x_{ikt}, u_{it}, x_{J,kt}, u_{J,t}} \; & \sum_{kt} \left(p_{kt} \left(\textstyle\sum_{i'} x_{i'kt} + \sum_j x_{jkt} \right) - s_{ik} \right) x_{ikt} - \sum_{kt} c_{it} u_{it} \\
& - \sum_{kt} d_{ikt} x_{ikt}, && (9.14) \\
\text{s.t.} \quad & \text{constraints (9.2), (9.4)-(9.6), and} \\
& \text{Equilibrium constraints for the followers' problems:} \\
& \sum_k x_{jkt} + u_{jt} - u_{j,t-1} - g_{jt} = 0 \perp \alpha_{jt} \text{ free}, \quad \forall j \in J, t \in T, && (9.15) \\
& p_{kt} \left(\textstyle\sum_{i'} x_{i'kt} + \sum_j x_{jkt} \right) - \hat{p}_k \geq 0 \perp \beta_{kt} \geq 0, \quad \forall k \in K, t \in T, && (9.16) \\
& q_{jk} - \textstyle\sum_t x_{jkt} \geq 0 \perp \gamma_{jk} \geq 0, \quad \forall j \in J, k \in K, && (9.17)
\end{aligned}$$

$$r_{jk} - \sum_t m_{jkt} x_{jkt} \geq 0 \perp \lambda_{jk} \geq 0, \quad \forall j \in J, k \in K, \tag{9.18}$$

$$u_{j\hat{t}} - \hat{u}_j \geq 0 \perp \mu_j \geq 0, \quad \forall j \in J, \tag{9.19}$$

$$d_{jkt} - \left(\frac{\partial}{\partial x_{jkt}} p_{kt} \left(\sum_{i'} x_{i'kt} + \sum_j x_{jkt} \right) \right) (x_{jkt} + \beta_{kt}) -$$

$$p_{kt} \left(\sum_{i'} x_{i'kt} + \sum_j x_{jkt} \right) + \alpha_{jt} + \gamma_{jk} + m_{jkt} \lambda_{jk} \geq 0 \perp$$

$$x_{jkt} \geq 0, \quad \forall j \in J, k \in K, t \in T, \tag{9.20}$$

$$\alpha_{jt} - \alpha_{j,t+1} + c_{jt} \geq 0 \perp u_{jt} \geq 0, \quad \forall j \in J, t < \hat{t}, \tag{9.21}$$

$$\alpha_{jt} - \mu_j + c_{jt} \geq 0 \perp u_{jt} \geq 0, \quad \forall j \in J, t = \hat{t}, \tag{9.22}$$

The decision variables are the sales x_{ikt} and inventory u_{it} of the leader and the set of sales $x_{J,kt}$ and inventory $u_{J,t}$ of the followers. Expression (9.14) defines that the objective is to maximize the payoff, in which the sales of the other players are fixed. Constraints (9.15) to (9.19) are the equilibrium constraints for the constraints in the followers' problems, and constraints (9.20) to (9.22) are the equilibrium constraints for the derivatives of the objectives over the decision variables in the followers' problems. In constraint (9.15), the left hand side of $\perp$ is an equality constraint, and the Lagrange multiplier is a free variable, that is, its domain is unrestricted. In each of constraints (9.16) to (9.22), at least one of the constraints in the two sides of $\perp$ is an equality in a solution to the MPEC.

There is large amount of literature on using EPECs, MPECs, and MCPs (mixed complementarity problems), particularly in the electricity industry. For example, a recent paper [15] solves a leader-follower game using an EPEC, and [54] solves a competing-firms game using an MCP. However, due to the non-convexity of the feasible region, any MPEC may have local optima, and so a candidate solution for the EPEC found by mathematical programming software may not be an equilibrium for the game. A recent development in solving MPECs is the sequential quadratic programming approach (SQP) presented in [32]. But it can only find a stationary point and thus does not guarantee to find a global maximum point to an MPEC.

In our computational experiments in chapter 10, we will show that the EPECs are difficult to solve in all but simple cases. Hence we have not used the EPECs approach to solve our model.

9.5 Algorithm

To solve the model, we implement an algorithm that uses a *grid search* followed by a *sequential best response*. In a grid search, we divide the strategy space of each leader into a discrete grid. We then fix the strategy of each leader i', $i' \neq i$, and solve the followers' optimization problems for each point in i's grid to search for the optimal point for leader i. We use a coarse grid, for example with an increment of 1, and a fine grid, for example with an increment of 0.01. *Sequential best response* means that we repeatedly solve each leader's optimization problem in turn, keeping the others' strategies fixed until the set of leaders' strategies converges (see [33]). This algorithm aims to find one equilibrium of the game at termination.

Define X_i as a grid in the strategy space of leader i, $i \in I$, and define $x_i \in X_i$ as a point of X_i. Define k as the iteration count. Define z_i as the payoff of leader i after solving the followers' optimization problems. Define x^*_{ik} as the optimal point on X_i and z^*_{ik} as the optimal payoff for leader i in iteration k. Then the algorithm performs the following steps.

Coarse grid search

1. For each $i \in I$, initialize a coarse grid X_i for the entire strategy space of i.
2. Set $k = 0$, initialize $x^*_{i0} = 0$ for $i \in I$.
3. For each $i \in I$,
   (a) set $z^*_{ik} = -\infty$, and fix $x_{i'}$ as $x^*_{i',k-1}$ or $x^*_{i',k}$ if $x^*_{i',k}$ exists, $i' \in I$, $i' \neq i$,
   (b) for each $x_i \in X_i$,

i. solve the followers' problems to give z_i,

ii. if $z_i > z^*_{ik}$, then let $z^*_{ik} = z_i$ and $x^*_{ik} = x_i$.

4. If $x^*_{ik} \neq x^*_{i,k-1}$ for any $i \in I$, then set $k = k + 1$ and return to step 3.

Fine grid search

5. For each $i \in I$, initialize a fine grid X_i for a neighbourhood around x^*_{ik}.
6. Repeat step 3 and step 4.

Coarse grid verification

7. For each $i \in I$, initialize a coarse grid X_i for the entire strategy space of i.
8. For each $i \in I$,

   (a) fix $x_{i'}$ as $x^*_{i',k}$, $i' \in I$, $i' \neq i$,

   (b) for each $x_i \in X_i$,

      i. solve the followers' problems,

      ii. if $z_i > z^*_{ik}$ but $x_i \neq x^*_{ik}$, then go back to step 1.

The coarse grid search uses sequential best response with a coarse grid and results in an equilibrium in the coarse grid. Starting from this equilibrium, the fine grid search uses sequential best response with a fine grid and results in an equilibrium in the fine grid. Given this equilibrium, the coarse grid verification searches for the global optimal strategy for each leader with the other leaders' strategies fixed at the equilibrium solution, and compare this to the leader's equilibrium strategy to examine the validity of the equilibrium. If the algorithm cannot find an equilibrium, we need to run the algorithm for different sized grids.

Note that this algorithm has two main limitations. An optimal solution obtained in a discrete grid may not be a true optimum in a continuous strategy space, and thus an equilibrium obtained may not be a true equilibrium. On the other hand, since the dimension of a grid depends on the leader's strategy space, the CPU time grows exponentially as the dimension of the grid increases, so this approach is only feasible for problems having low dimension.

Chapter 10

Computational experiments

In this chapter, we will perform computational experiments on the game using two sets of data. At first, we present the result for a game with a set of real data, and investigate the impact of intervention price, tariffs and quotas and inventory holding costs on the leaders' strategies in equilibrium. We observe that the result of this game is very simple. We consider that the game could be far more complicated, say if the current situation of the market is changed, and to illustrate this, we use a set of fictitious data. To illustrate the nature of this game, we will present a simplified version of the game where the leaders act together as a single leader and present its optimal strategies. Then we show that the strategies are different when the leaders compete, where simultaneous optimization leads to an equilibrium. We describe five equilibria, and illustrate the equilibria using the leaders' best responses. Then we use sequential best response to illustrate how the leaders' strategies evolve given their starting strategies. Finally we investigate the impact of inventory holding cost on equilibrium.

10.1 A game with real data

In this section, we describe the real data, and present the result of the game as well as the impact of intervention price, tariffs and quotas and inventory holding costs on the leaders' strategies.

10.1.1 Data

In our experiment there are three players with Fonterra and Australia being leaders and EU being a follower in decision making. There are two markets, the European market (denoted UM) and the world market (denoted WM). We assume the decision horizon is two-stage, with stage 1 being the first half of the year and stage 2 being the second half. For simplicity, we restrict attention to only one product, butter, which is one of the main products in the dairy market and is regulated by the intervention system.

We obtain data from two sources, the United States Department of Agriculture (USDA)[1], and Datum[2]. We use the data from USDA for annual exports, consumptions and ending stocks, and the data from Datum for the other data. The data from USDA is for some selected countries only, which are (in the order of continents they are in) Canada, Mexico, USA, Brazil, European Union (excluding Bulgaria and Romania), Russia, Ukraine, India, Japan, Taiwan, Australia and New Zealand. We use the data for New Zealand as identical to that for Fonterra. The data for New Zealand are for the year ending in May which coincides with Fonterra's production year, the data for Australia are for the year ending in April, and the data for the European Union are for the year ending in December.

The prices in the data for the European Union are measured in Euros. These are converted to NZ dollars with an exchange rate of 1.9, which is the average of the exchanges rates in 2006/7. The world market prices in the data are measured in US dollars. These are converted to NZ dollars with an exchange rate of 1.45, which is also the average in 2006/7. Henceforth all prices and costs are expressed in New Zealand dollars.

We assume that for each country outside the European Union, domestic consumption is its first priority, and thus its exports are what it will sell to the rest

[1]The home page of USDA is http://www.usda.gov/.

[2]Datum is the information service of DairyCo, which is the dairy sector company of British dairy farmers. The home page of Datum is http://www.mdcdatum.org.uk/.

of the world. Hence we use the exports as the productions of the leaders in the model. The annual export from New Zealand in 2006 is 400 thousand tonnes, and the annual export from Australia is 70 thousand tonnes. We have no exact data about the exports in each half of a year. We cannot estimate them either, since the countries may use inventory to smooth the supply to the market, even though milk supply has peak and off-peak months. Hence we assume that the export in each stage is half of the annual export. This gives 200 thousand tonnes per stage for Fonterra, and 35 thousand tonnes per stage for Australia. On the other hand, the annual production from the European Union is 2,055 thousand tonnes (average of productions in 2006 and 2007). We use this as the production of EU, and assume that the production in each stage to be half of this quantity, which gives roughly 1,027 thousand tonnes per stage. Note that in the optimization models in Part I, the solution of the DN model gives an annual export of 367 thousand tonnes, which is 90% of the real data.

We observe that each country has an ending stock in each year. We use the ending stock in year 2005 as the start-of-year inventory for Fonterra and Australia, which is 23 thousand tonnes for Fonterra, and 16 thousand tonnes for Australia. On the other hand, the data from Datum gives 188 thousand tonnes for the total stock in intervention and private storage aid on June 15, 2006. This is used as EU's start-of-year inventory. We assume the inventory target for each player is its start-of-year inventory.

The average wholesale price in the European Union in year 2006 is \$4,849 per tonne, and the average wholesale price in the world is \$2,797 per tonne. The annual domestic consumption in the European Union is 1,950 thousand tonnes, and we use this as the demand in UM. The total annual exports from the selected countries including the European Union is 778 thousand tonnes, and we use this as the demand in WM. We have no information about the consumptions in each half of the year, but we assume the demands are roughly equal. Thus the demand in UM is 975 thousand tonnes in each stage, and that in WM is 389 thousand tonnes in each stage.

We use the wholesale prices and demands to derive price-demand curves for the markets. We assume that there is a reference price in each market. The market price for sales at the demand is 95% of the reference price, which is the wholesale price that we have, and the market price for zero sales is 105% of the reference price. For UM, the reference price is 5,104. 105% of this price gives the intercept in the price-demand curve which is 5,359, and 10% of this price over the demand gives the negative slope which is roughly 0.5 per thousand tonnes. For WM, the reference price is 2,944, which gives an intercept of 3,091 and a negative slope of roughly 0.75 per thousand tonnes.

The intervention price for the year 2006 is \$4,931 per tonne, which was set on July 1, 2006. Note that this price is higher than the average wholesale price 4,849 in the year, but by only a small difference. This implies that as we have described in the chapter of the European dairy market, the intervention price is not a threshold on the price in the market, but it does maintain the market price. Hence the modeling assumption that the intervention price is a floor price is reasonable.

The current access tariff for New Zealand is 1,651 per tonne and the annual quota is 77 thousand tonnes. The current access tariff for Australia for butter is not available. The minimum access tariff for each country is 1,801 per tonne and the annual quota is 10 thousand tonnes. For Fonterra, the annual quota is the total of the two quotas, which is 87 thousand tonnes, and the tariff is the average of the total tariff paid for sales at the total quota, which is 1,681. For Australia, we use the minimum access tariff and quota.

In the restitution system, the annual volume limit on the exports of the European Union is 393 thousands tonnes, and the annual budget limit is 1,801 million dollars, which are used for EU in the model.

We assume that Fonterra, Australia and EU have the same marginal transportation costs. The transportation cost to UM is assumed to be that used in the optimization models in Part I, which is \$300 per tonne, and the cost to WM is the average of the cost to the other three markets, America, Japan and Oceania, which is \$360 per tonne. We also assume that they have the same inventory holding cost in each stage. This is six times the monthly storage cost in the optimization models in Part I (since it is for six months), which is \$210 per tonne.

In the result we measure the sales in units of thousands of tonnes, and measure the payoffs in millions of NZ dollars.

10.1.2 Result

We solve the game and display the players' strategies in Table 10.1. The leaders' sales in UM in stage 1 are at their quotas. They have no inventory in stage 1 to save inventory holding costs, and their inventory in stage 2 is at the inventory target. EU takes the residual capacities in UM in both stages, and thus the market prices are at the intervention price 4,931. EU has an inventory higher than the inventory target in stage 2, since its total sales to WM are at the annual volume limit. The market prices in WM are $2,755$ and $2,853$. Fonterra has a payoff of $\$1,019$ million, Australia has a payoff of \$170 million, and EU has a payoff of $\$8,344$ million.

	Fonterra's sales		Australia's sales		EU's sales	
	stage 1	stage 2	stage 1	stage 2	stage 1	stage 2
UM	87	0	10	0	759	856
WM	136	177	41	19	271	122

Table 10.1: The players' sales (in thousand tonnes).

Since the leaders' total sales in UM are at their quota, a varying intervention price has no impact on the leaders' sales in UM, unless the intervention price is above 5,310, where the market capacity in UM in stage 1 would be lower than the total quota of the leaders. However, observe in the solution that EU is not able to sell all products to UM, so a lower intervention price will allow EU to sell more products in UM, resulting in a higher payoff. The leaders' payoffs are also higher, since EU sells less in WM resulting a higher market price in WM. If the intervention price is below 4,757, then EU can sell all products to UM and thus its payoff is at its maximum which is $9,214 million, and the leaders also have their maximum payoffs which are $1,047 million and $179 million. This implies that all players prefer a low intervention price.

We observe that the market prices in UM and WM differ by 2,176 and 2,078 in each stage. These are higher than the tariffs for the leaders, which are 1,681 for Fonterra and 1,801 for Australia. This implies that if the tariffs were lifted, the leaders would not transfer sales in UM to WM in either stage. Setting the tariffs to zero gives the same solution, but Fonterra's payoff increases from $1,019 million to $1,165 million (by $146 million), and Australia's payoff increases to $188 million.

Now suppose the quotas are lifted but the tariffs remain. As shown in Table 10.2, the leaders only sell products to UM. They have no inventory thereby saving holding cost. EU takes the residual capacity in UM and the market prices in UM are at the intervention price. Fonterra's payoff is $1,173 million (an increase of $154 million), and Australia's payoff is $195 million.

	Fonterra's sales		Australia's sales	
	stage 1	stage 2	stage 1	stage 2
UM	223	177	51	19
WM	0	0	0	0

Table 10.2: Leaders' sales (in thousand tonnes) if quotas are lifted.

If both quotas and tariffs are lifted, then the leaders have the same strategies as those when only the quotas are lifted, and their payoffs increase to $1,848 million (by $829 million) and $321 million. These results show that quotas and tariffs have seriously eroded Fonterra's payoff.

We observe that in the result, the market prices in WM are 2,755 and 2,853 in each stage. If the inventory holding cost for EU is low, then EU will have a large inventory to sell more in WM in stage 2 resulting in closer market prices in WM in the two stages. The leaders' sales remain unchanged, even if the leaders holding costs are also low. In these circumstances, the leaders have higher payoffs.

However, if the holding costs of the leaders are zero, then there are many equilibria. In each of these equilibria, the leaders' sales are at the quotas in UM, and they have different sales in UM in two stages. For example, in one equilibrium, Fonterra sells 32 in stage 1 and 55 in stage 2, and Australia sells 0 and 10, and in another equilibrium, Fonterra sells 66 in stage 1 and 21 in stage 2, and Australia sells 10 and 0. In response, EU takes the residual market capacity in UM. In WM, Fonterra's sales are 156.5 and Australia's are 30 in each stage, and the market price is 2,804 in each stage. The leaders have the same payoffs in each equilibrium.

Note that if we fix Australia's strategy as the equilibrium solution, and solve an MPEC for Fonterra, then we obtain a solution in Table 10.3, where the sales of Fonterra and EU are displayed. In this solution, Fonterra's sales are not at the quota level, but its payoff is \$992 million, which is lower than the payoff \$1,019 million that we have found using our algorithm. This shows that the MPEC solution is not a global optimal solution to Fonterra, and thus it shows that, as we have stated in chapter 9, treating even this simple game as an EPEC and applying mathematical programming techniques such as GAMS/PATH will not yield an equilibrium.

	Fonterra's sales		EU's sales	
	stage 1	stage 2	stage 1	stage 2
UM	4.5	36.5	841.5	819.5
WM	174	185	193.75	199.25

Table 10.3: The sales of Fonterra and EU in an MPEC solution.

10.2 A game with fictitious data

The game with real data in the previous section gives a single intuitive equilibrium. This emerges because both leaders are too small to have a major effect on the follower's strategy. When the leaders have more market power, the game structure is actually far more complicated than the result has shown. To illustrate this, we solve the game with some fictitious data.

10.2.1 Data

In our fictitious game, we assume that Fonterra's exports are 1,830 thousand tonnes in stage 1 and 923 thousand tonnes in stage 2. This might result from the acquisition of other producers, or collaboration in a cartel with others. In a similar vein, we assume Australia forms a cartel with some large producers in the world, say America, and Australia acts for the cartel. The cartel's exports are lower than the Fonterra group, say 1,445 thousand tonnes in stage 1 and 695 tonnes in stage 2. Henceforth we shall refer to the Australian cartel as Australia and the Fonterra group the name Fonterra. On the other hand, we assume that the European Union lifts the milk production quotas on its member countries (which currently limits their production), which results in large increases in production for EU, to 2,403 thousand tonnes in stage 1 and 1,157 thousand tonnes in stage 2. Notwithstanding this increase, the greater increase in capacity of Fonterra and Australia gives them more market power than previously, which they exploit in equilibrium.

We relax the inventory target at the end of stage 2 to be zero, so that there are more options in the players' sales. We assume a smaller inventory holding cost of $72 per tonne. We use the same marginal transportation costs as those in the game with the real data.

We assume a linear price-demand curve in each stage in UM and WM. The intercepts in the price-demand curves for WM are 3,042 for stage 1 and 2,908 for stage 2. Then we assume the intercepts for UM are 3,750 higher than those for WM, which gives 6,792 for stage 1 and 6,658 for stage 2. On the other hand, we use the same negative slopes for both UM as those in the real data, which are 0.5 per thousand tonnes. We also use this for the price-demand curves for WM.

We use the intervention price in the real data as the intervention price in this game, which is $4,931 per tonne. To give more options to the players' sales decisions, we assume there are large quotas and zero tariffs for the leaders, and large volume limit and budget limit on EU's exports.

In the results, the sales will be measured in units of thousands of tonnes, and the payoffs will be in million of NZ dollars.

10.2.2 A single-leader-follower game

To illustrate the nature of the leader-follower game, we examine a simplified version of the game in which the leaders Fonterra and Australia act together as a single leader. We observe five optimal strategies for the leader (henceforth denoted Fonterra) for different intervention prices. Hereby we define a strategy by its general characteristics but we do not quantify it. Thus at different intervention prices a strategy may have different sales decisions. One common characteristic of these strategies is that Fonterra sells all products to UM due to the high market price, and thus sells nothing to WM. Then given its sales in UM in stage 1, Fonterra's sales in UM in stage 2 are fixed, and thus its strategy can be represented by its sales in UM in stage 1.

When the intervention price is lower than 4,515, Fonterra's optimal strategy is an *unconstrained* strategy. We illustrate this strategy in Figure 10.1, where the UM capacities in the two stages are represented by rectangles, each containing a green rectangle denoting Fonterra's sales, and a blue one representing EU's sales. The coloured arrows represent transfers of butter though inventory.

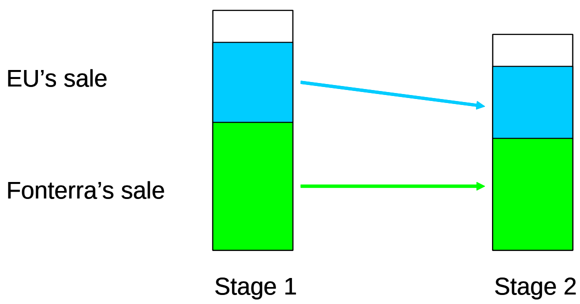

Figure 10.1: Unconstrained strategy.

In this strategy, Fonterra sells all products to UM in both stages and has a small amount of inventory. In response, EU sells all products to UM in both stages and has inventory. Note that in each stage the total sales are below the market capacity and thus the market price is above the intervention price. The sales in this strategy are constant for each intervention price, since the market price is not constrained by the intervention price.

When the intervention price reaches 4,515, Fonterra changes to a *large-inventory* strategy, which is illustrated in Figure 10.2. In UM, Fonterra has small sales in stage 1, and has a large inventory for large sales in stage 2. EU takes the residual UM capacity in stage 2 and thus sells the remaining production in stage 2 to WM. The market price in stage 2 is at the intervention price. Thus Fonterra has small sales for a high market price in stage 1 and has large sales in stage 2 for the intervention price, thereby incurring a large storage cost. This implies that Fonterra exploits the intervention price in stage 2.

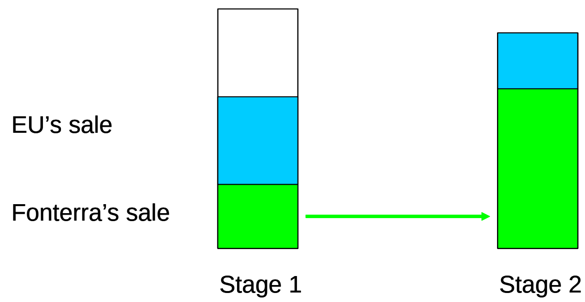

Figure 10.2: Large-inventory strategy.

We observe Fonterra's payoff curves with respect to its sales in UM in stage 1 with the intervention price at 4,514 and 4,515. These two payoff curves have two local maxima, as shown in Figure 10.3. (Other parts of the curves are ignored since they are lower.) The unconstrained strategy gives the maximum at B, and the large-inventory strategy gives the maximum at A (having small sales). As the intervention price increases to 4,515, the global maximum switches from B to A, and thus Fonterra's optimal strategy changes to the large-inventory strategy.

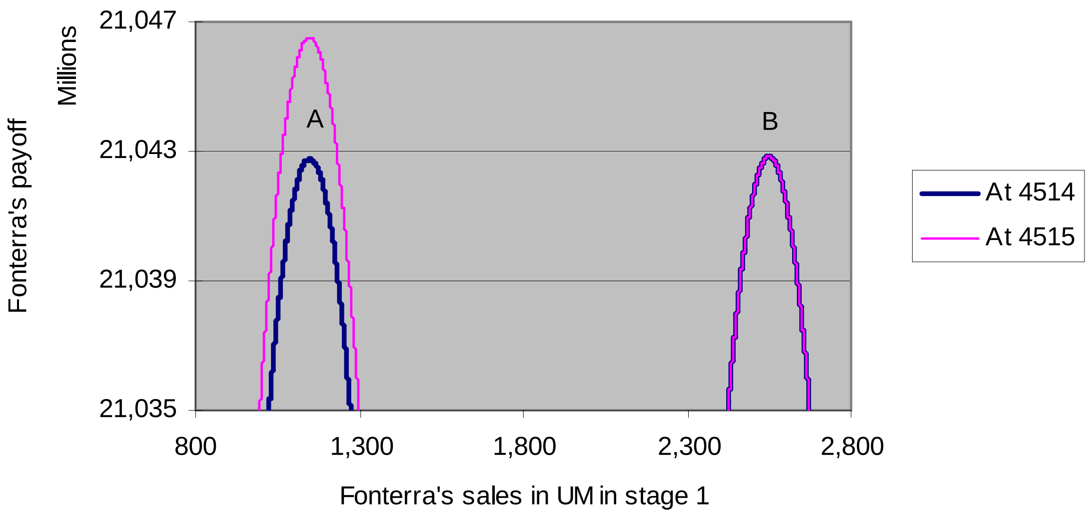

Figure 10.3: Fonterra's payoff curves for intervention price at 4,514 and 4,515.

The large-inventory strategy remains Fonterra's optimal strategy, until the intervention price reaches 4,844, where Fonterra changes to a *zero-inventory* strategy. As shown in Figure 10.4, Fonterra has a zero inventory, and sells its entire production in UM in each stage. Due to the high intervention price and thus a low market capacity, EU takes the residual UM capacity in both stages, and thus the market prices are at the intervention price.

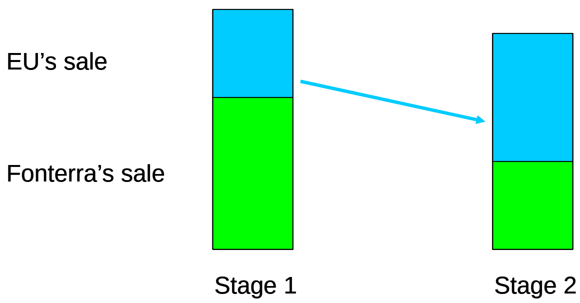

Figure 10.4: Zero-inventory strategy.

We observe that when the intervention price increases from 4,843 to 4,844, Fonterra's sales in UM in stage 1 switch between two local maxima. We display the payoff curves in Figure 10.5. (Other parts are ignored.) The large-inventory strategy gives the maximum at A (having small sales), and the zero-inventory strategy gives the maximum at B (having maximum sales). As the intervention price increases from 4,843 to 4,844, the global maximum changes from A to B, which induces the change of Fonterra's optimal strategy.

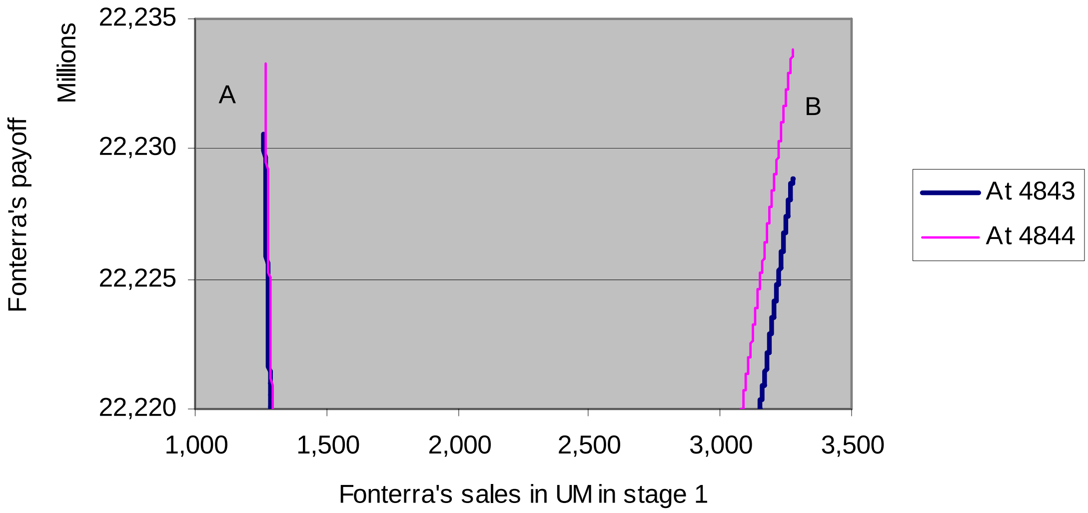

Figure 10.5: Fonterra's payoff curves for intervention price at 4,843 and 4,844.

The zero-inventory strategy is Fonterra's optimal strategy for the intervention price between 4,844 and 5,155. When the intervention price is between 5,155 and 5,501, Fonterra's optimal strategy is a *forced-inventory* strategy, which is illustrated in Figure 10.6. Due to the low market capacity of UM in stage 1, Fonterra has to store in stage 1. EU takes the residual UM capacity in stage 2 and the market prices in both stages are at the intervention price.

When the intervention price is at 5,501, the total market capacity in UM in two stages is equal to Fonterra's total production. As illustrated in Figure 10.7, Fonterra takes the entire market in UM in both stages, and is forced to sell products to WM. We call this a *forced-sales* strategy. This strategy is Fonterra's optimal strategy for an intervention price from this point to the maximum 6,658, which is the intercept of price-demand curve for UM in stage 2.

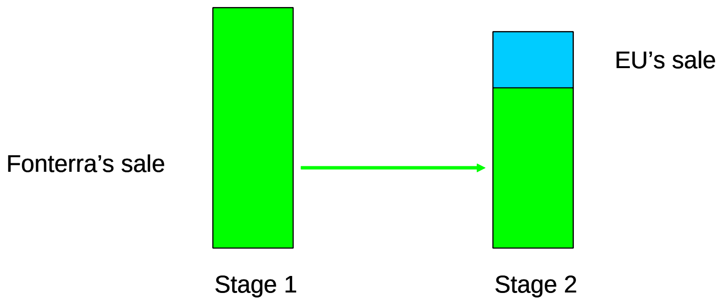

Figure 10.6: Forced-inventory strategy.

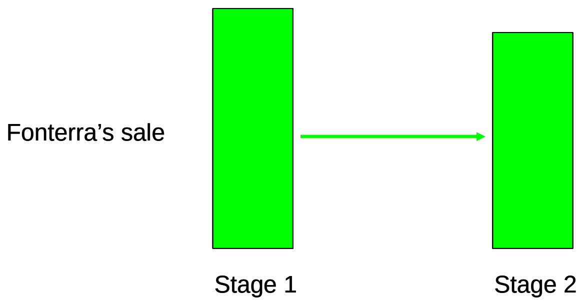

Figure 10.7: Forced-sales strategy.

We observe that the switches of Fonterra's strategies have a large impact on EU. When Fonterra changes its optimal strategy to the large-inventory strategy at 4,515, some of EU's sales in UM in stage 2 are pushed out and thus sold to WM at a low market price. Thus there is a jump in EU's payoff curve at this intervention price, which is point A in Figure 10.8. Then as Fonterra changes from the large-inventory strategy to the zero-inventory strategy at 4,844, which is point B in the figure, EU is able to capture the residual market capacities in UM by selling more products, and thus its payoff jumps up.

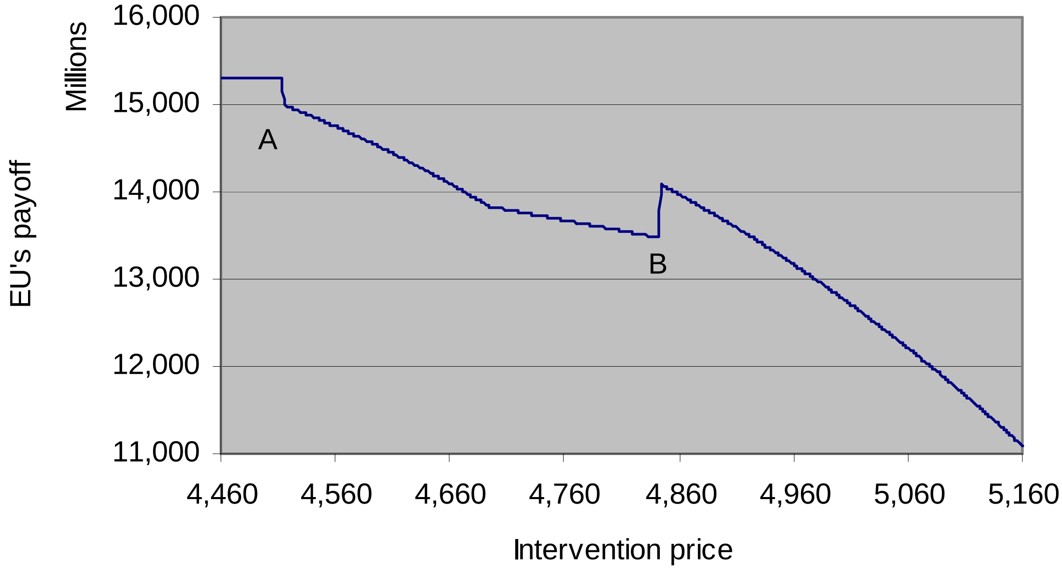

Figure 10.8: EU's payoff curve with two jumps.

10.2.3 A two-leader-follower game

The previous result assumes Fonterra and Australia act together as a single leader, but in reality they compete. To illustrate the effect of competition, we compute Fonterra's payoff with respect to its sales in UM in stage 1 if Australia plays an unconstrained strategy. This is displayed in Figure 10.9, which shows that Fonterra's optimal strategy is to have large sales of 1,445, which is its unconstrained strategy.

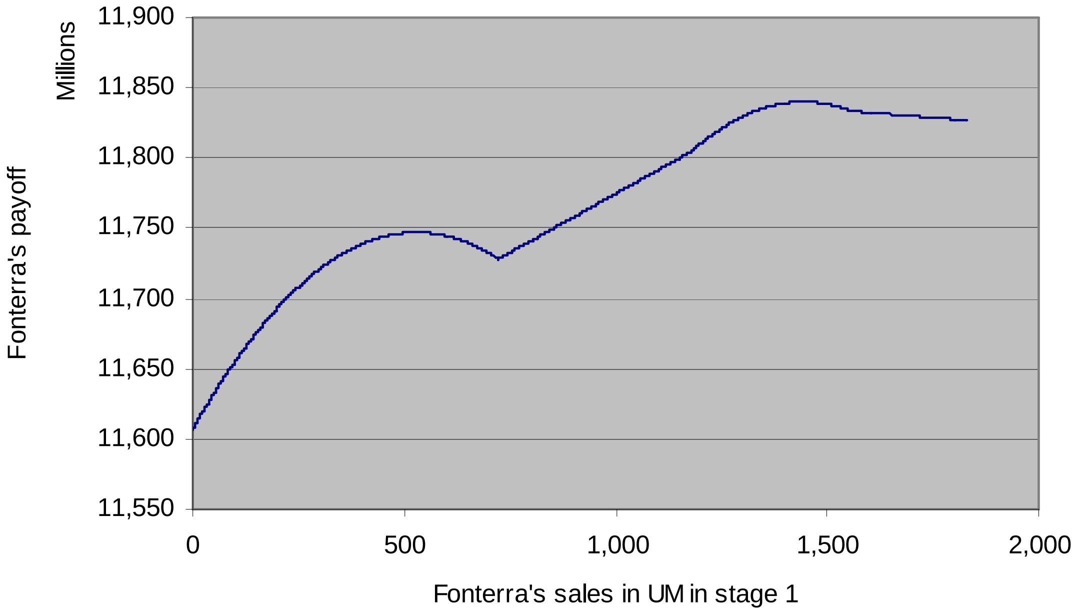

Figure 10.9: Fonterra's payoff if Australia plays an unconstrained strategy.

On the other hand, in Figure 10.10 we display Fonterra's payoff with respect to its sales in UM in stage 1 if Australia plays a large-inventory strategy, of which the two local maxima are displayed in detail in Figure 10.11. The later figure shows that Fonterra's optimal strategy is to have small sales in stage 1 and thus a large inventory, which is its large-inventory strategy.

These show that a leader may have a different optimal strategy if the other leader plays a different strategy, and thus each leader should take into account the other leader's strategy when making decisions. Since the leaders are seeking their optimal strategies simultaneously, a Nash equilibrium is sought. At equilibrium,

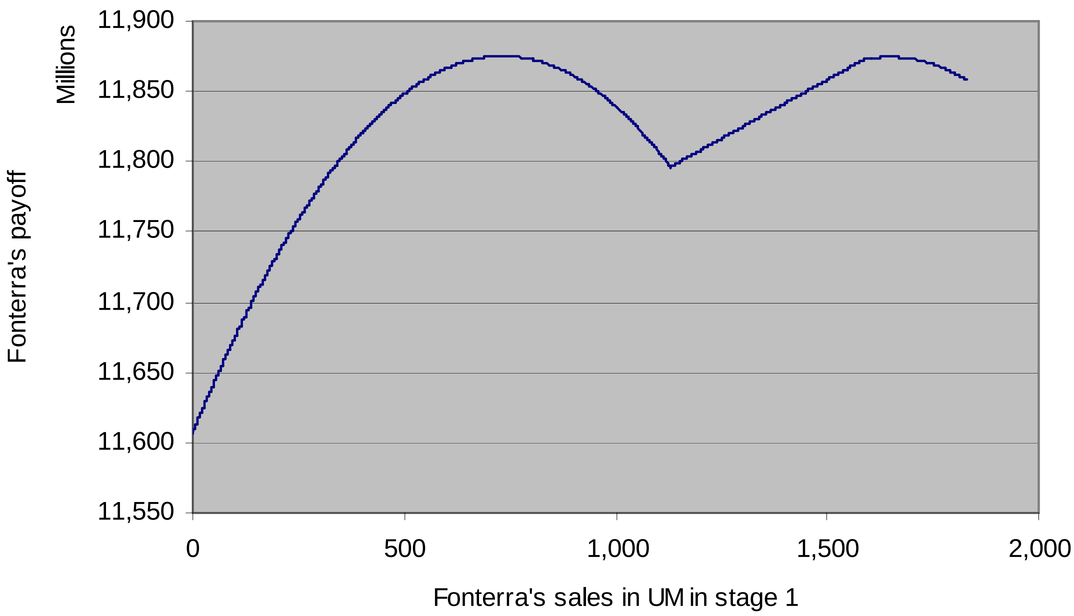

Figure 10.10: Fonterra's payoff if Australia plays a large-inventory strategy.

the leaders have different strategies from those in the optimal solution described in the previous section.

For example, for an intervention price at 4,564, if the leaders compete, there are two equilibria. In Table 10.4, we display their sales in UM in two stages (sales in WM are zero) in the optimal solution when the leaders act as a cartel, and those at the equilibria when they compete.

	Fonterra's sales	Australia's sales
Cartel solution	1,098 and 3,795	
Equilibrium A	732 and 2,021	732 and 1,408
Equilibrium B	1,445 and 1,308	1,139 and 1,001

Table 10.4: Optimal solution and equilibrium.

At both equilibria, the total sales of the leaders in each stage are different from that in the cartel solution, and thus the leaders have used different strategies. The total payoff of the leaders in competition is lower than the payoff in the cartel

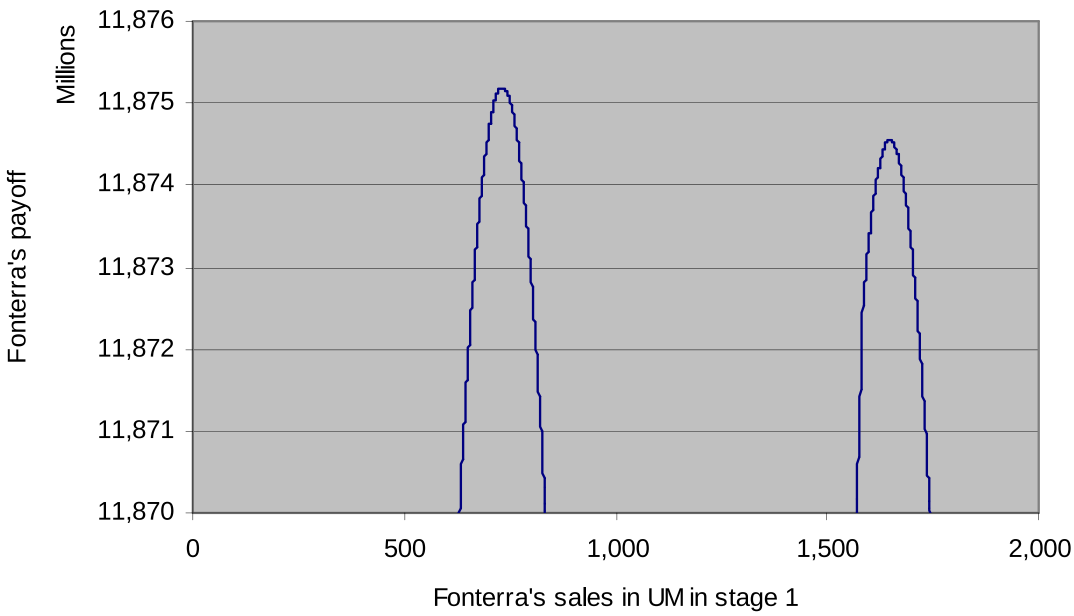

Figure 10.11: Local maxima in Fonterra's payoff.

solution, which implies competition hurts the total benefit of the leaders.

In the next section, we will describe the leaders' strategies at the equilibria for different intervention prices.

10.2.4 Equilibrium

We observe five equilibria at different intervention prices, in which the leaders use one of the strategies described in the single-leader-follower game, which are the unconstrained strategy, large-inventory strategy, zero-inventory strategy, forced-inventory strategy and forced-sales strategy. Note that the leaders sell all products to UM, and thus their strategies can be represented by their sales in UM in stage 1.

The first equilibrium exists when the intervention price is below 4,578. At this equilibrium, each leader plays an unconstrained strategy. We illustrate this equilibrium in Figure 10.12.

In the figure, the UM capacities in the two stages are represented by rectan-

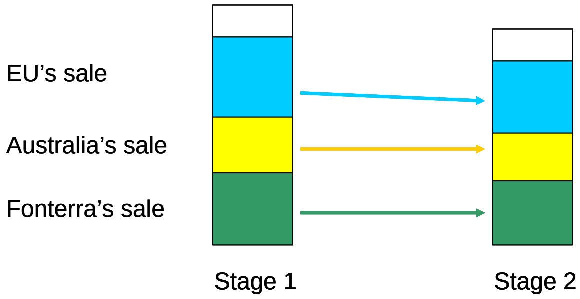

Figure 10.12: An equilibrium at which each leader plays an unconstrainted strategy.

gles, Fonterra's sales and inventory are denoted by green rectangles and arrow, Australia's sales and inventory are denoted by yellow rectangles and arrow, and EU's sales and inventory are denoted by blue rectangles and arrow. In each stage, the total sales are below the market capacity and the market price is above the intervention price. The players' sales do not change with a varying intervention price.

A second equilibrium which is displayed in Figure 10.13 appears when the intervention price is at 4,564 but disappears at 4,901. In this equilibrium, both leaders play a large-inventory strategy. In response, EU takes the residual UM capacity in stage 2 and thus the market price in stage 2 is at the intervention price.

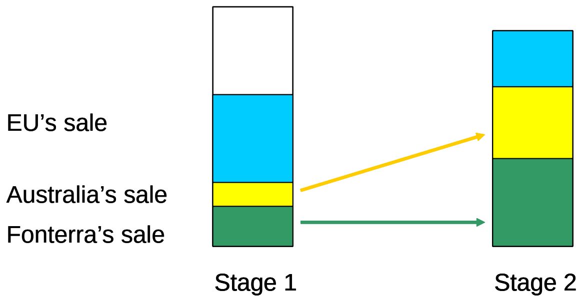

Figure 10.13: An equilibrium where both leaders play a large-inventory strategy.

A third equilibrium emerges for an intervention price between 4,576 and 5,155. As shown in Figure 10.14, each leader plays a zero-inventory strategy by selling its production in UM in each stage. EU sells products in UM in both stages.

A fourth equilibrium appears when the intervention price is above 5,155 and

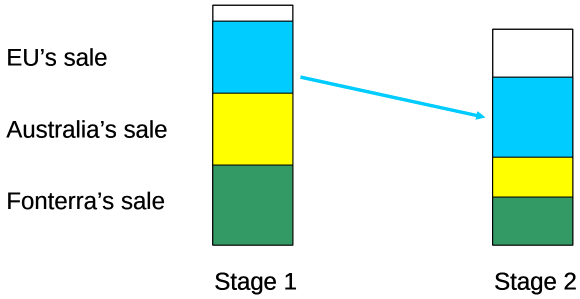

Figure 10.14: An equilibrium where each leader plays a zero-inventory strategy.

below 5,501, which is illustrated in Figure 10.15, where the leaders play a forced-inventory strategy. In stage 1, the leaders' total production exceeds the market capacity, and thus they have some inventory to sell in UM in stage 2. EU takes the residual UM capacity in stage 2, and sells products in WM in both stages.

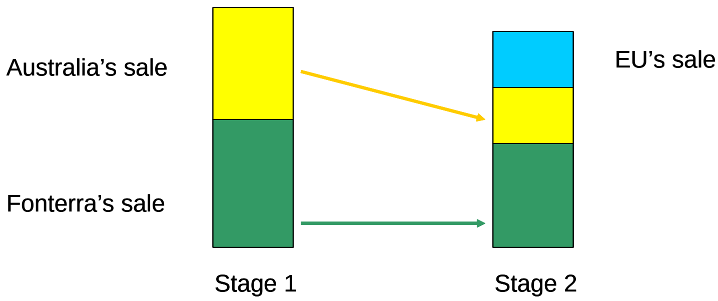

Figure 10.15: An equilibrium where each leader plays a forced-inventory strategy.

The last equilibrium appears when the intervention price is at and above 5,501, which is illustrated in Figure 10.16. The total market capacity in UM is lower than the total production of the leaders, and thus each leader plays a forced-sales strategy selling some productions to WM. EU has to sell all products to WM.

We observe that there is one equilibrium when the intervention price is below 4,564 and above 4,901, two equilibria between 4,564 and 4,576 and between 4,678 and 4,901, and three equilibria between 4,576 and 4,578. The equilibria at one particular intervention price can be illustrated using a plot of *best responses* of the leaders, which is described in the next section.

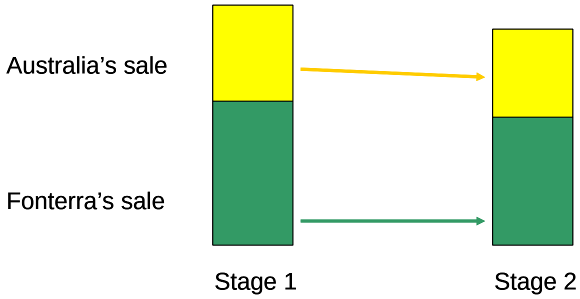

Figure 10.16: An equilibrium where each leader plays a forced-sales strategy.

10.2.5 Best response

The equilibria for an intervention price can be illustrated by plotting the *best responses* of the leaders. In the best response, each leader's optimal strategy is a function of the other leader's strategy. If the best responses intersect, say at some points, then each of these points is an equilibrium, otherwise there is no equilibrium. We display the best responses for three intervention prices, where one, two and three equilibria exist respectively.

When the intervention price is at 4,550, we obtain the best responses in Figure 10.17. Fonterra's best response defines its optimal strategy given Australia's sales in UM in stage 1 between 0 and 1,445, and Australia's best response defines its optimal strategy given Fonterra's sales in UM in stage 1 between 0 and 1,830. Note that the maximum sales are their production in stage 1.

We observe that the best responses are discontinuous and they intersect at point A. At point A, Fonterra's strategy is 1,445 and Australia's strategy is 1,139, which are their unconstrained strategies. There is only one equilibrium for this intervention price.

If the intervention price is 4,564, as shown in Figure 10.18, the best responses intersect at point A and point B. At point A, the leaders play unconstrained strategies, and at point B, they play large-inventory strategies having small sales in UM in stage 1. Hence, there are two equilibria for this intervention price.

When the intervention price is at 4,578, we observe three equilibria, since the

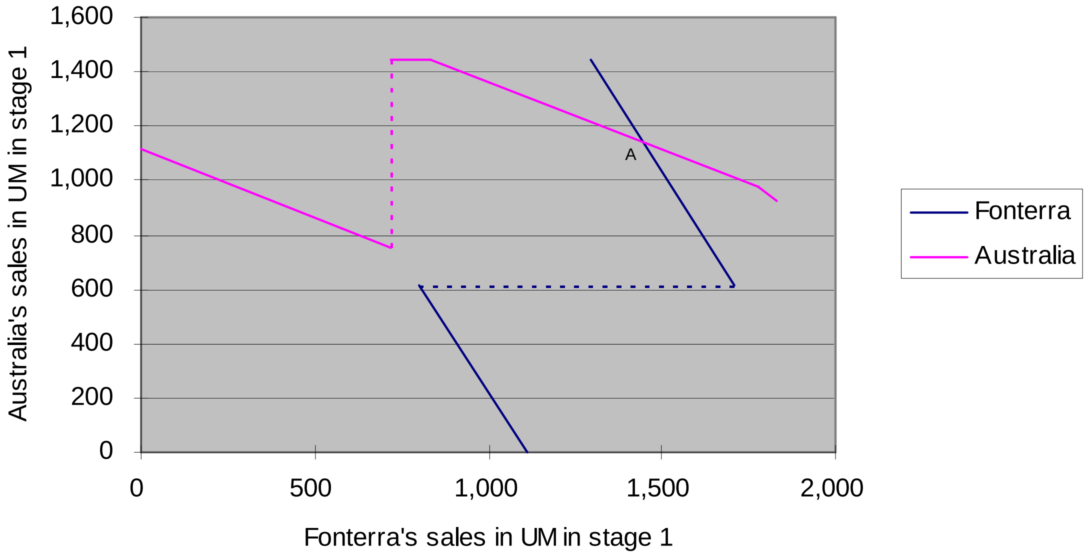

Figure 10.17: Best responses if the intervention price is at 4,550.

best responses shown in Figure 10.19 intersect at three points. The new equilibrium is at point C, where both leaders are selling their production and thus have zero inventory, which are their zero-inventory strategies.

Note that since in the unconstrained strategy and the zero-inventory strategy, the leaders' sales decisions do not change with a varying intervention price, the positions of point A and point C are fixed. But in the large-inventory strategy, the leaders' sales change as the intervention price changes, and thus the position of point B varies.

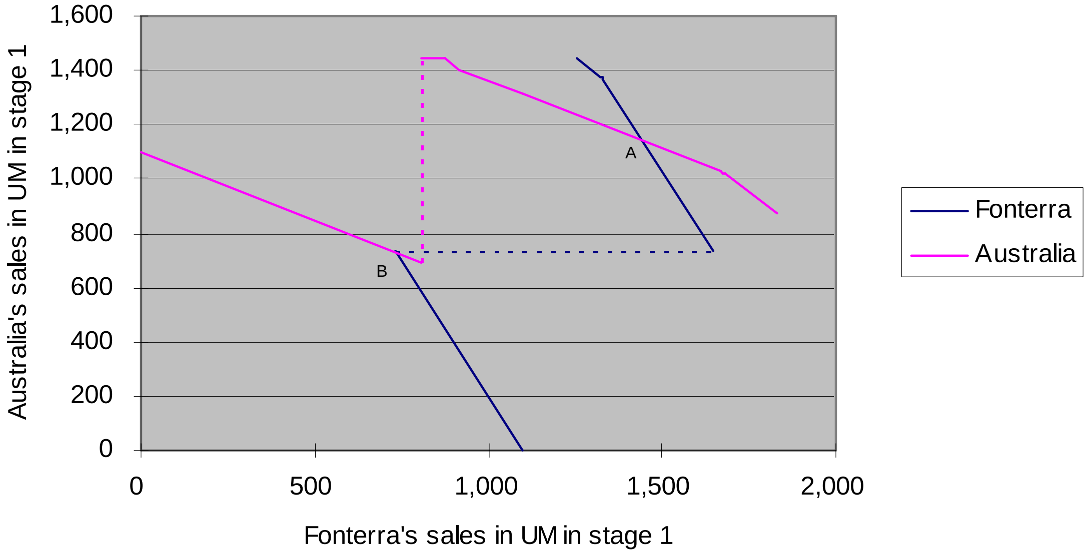

Figure 10.18: Best responses if the intervention price is at 4,564.

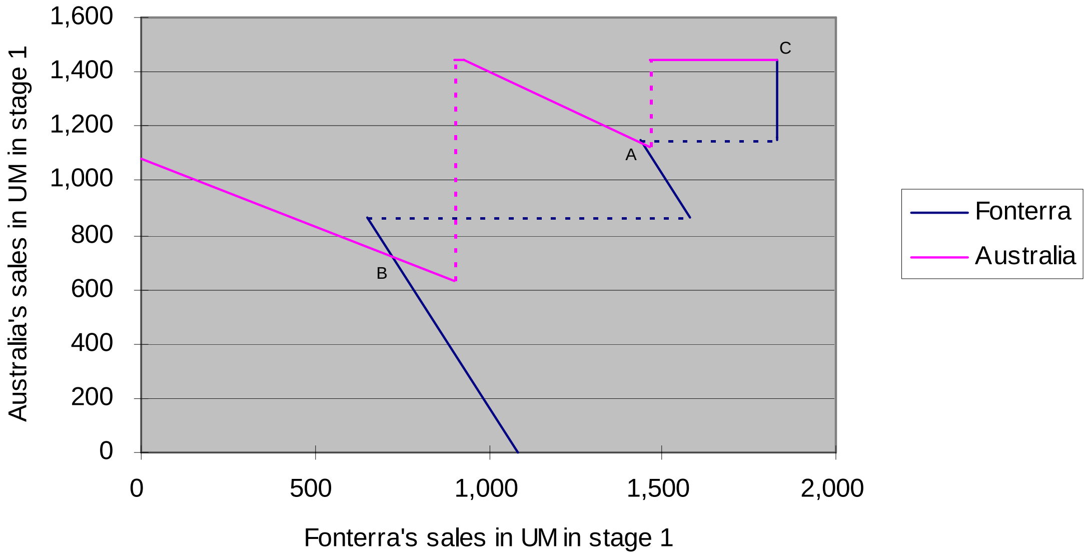

Figure 10.19: Best responses if the intervention price is at 4,578.

10.2.6 Sequential best response

With the best responses, we can not only identify an equilibrium, but also illustrate why a pair of strategies is not an equilibrium. For example, Figure 10.20 shows the best responses for intervention price at 4,579, just one above the intervention price in the previous example. The best responses intersect at point B and C, but not at point A.

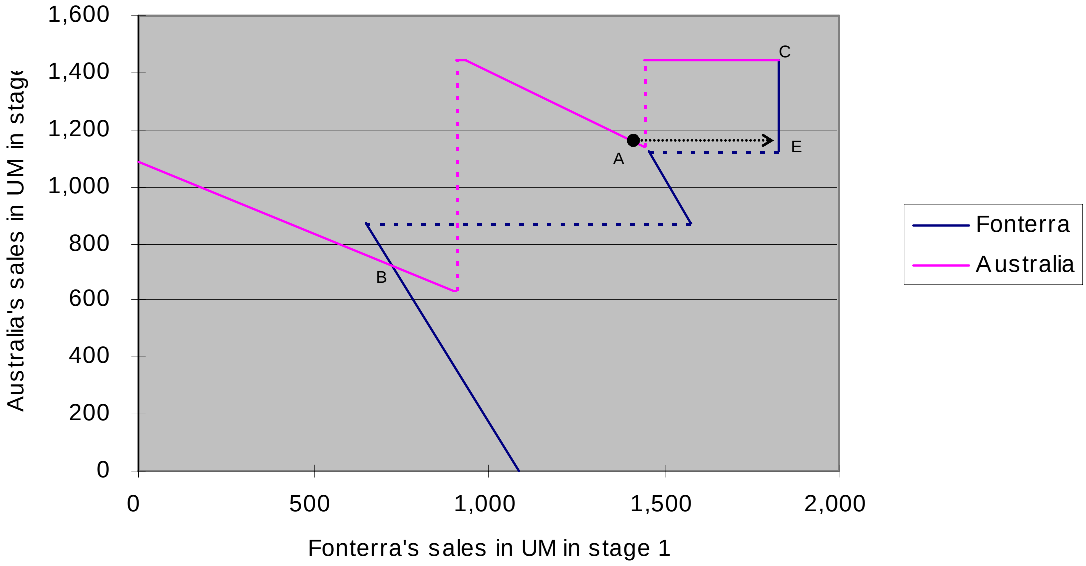

Figure 10.20: Sequential best response with the intervention price at 4,579.

If Fonterra's sales are at point A, then Australia's best response are the sales at point A. But then in response, Fonterra's sales will change to point E, and then Australia's sales will change to point C. Hence, the pair of strategies at point A is not an equilibrium for this intervention price.

Using sequential best response, we can also identify which equilibrium the leaders' strategies will converge to given a starting value of a leader's strategy, say X. For example, in the best responses of the leaders at the intervention price of 4,564, as shown in Figure 10.21, we divide the feasible space of Fonterra's sales in UM in stage 1 into three ranges by point M and point N. Point M is where Fonterra's sales are discontinuous. Note that Fonterra's sales at point M are slightly lower than its sales at the equilibrium at point B. Point N is where Australia's best response is discontinuous.

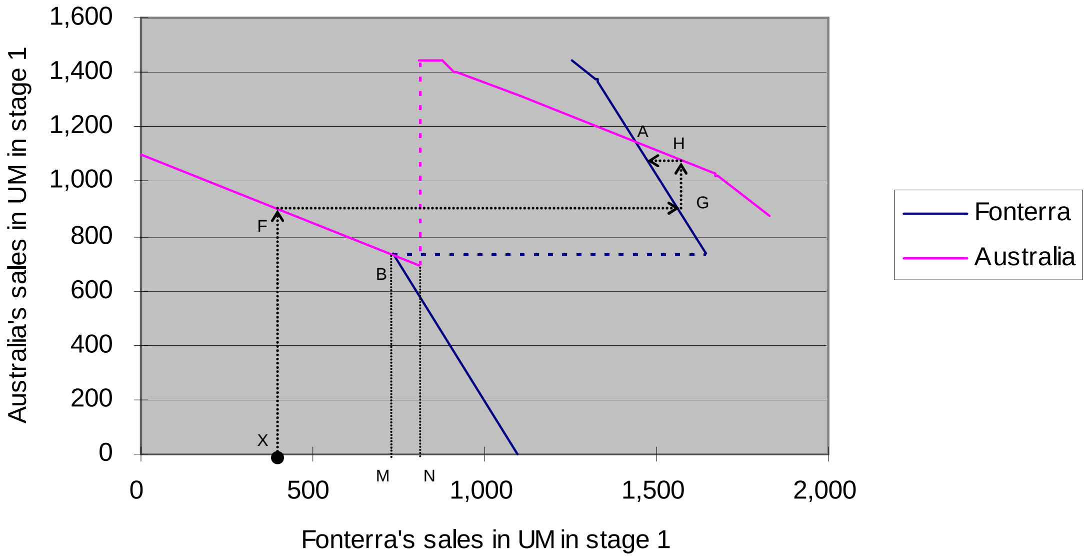

Figure 10.21: Sequential best response given that Fonterra has low starting sales.

If Fonterra's starting sales X are lower than point M, as shown in the figure, Australia would respond with sales at point F. In response, Fonterra's sales would be at point G, and then Australia's sales would be at point H, and so on. It is easy to see that their sales would converge to the equilibrium at point A.

But suppose Fonterra starts with sales between point M and N, as shown in Figure 10.22, which is an enlarged picture of the previous figure. In response, Australia would have sales at point R, and then Fonterra would have sales at point S. Australia then would have sales at point T, and so on. This would converge to the equilibrium at point B.

However, if Fonterra's starting sales are higher than point N, then as shown in Figure 10.23, Australia would have sales at point U. Then in response Fonterra would have sales at point V, and then Australia would have sales at point W, and so on. The sequential best response would lead to the equilibrium at point A.

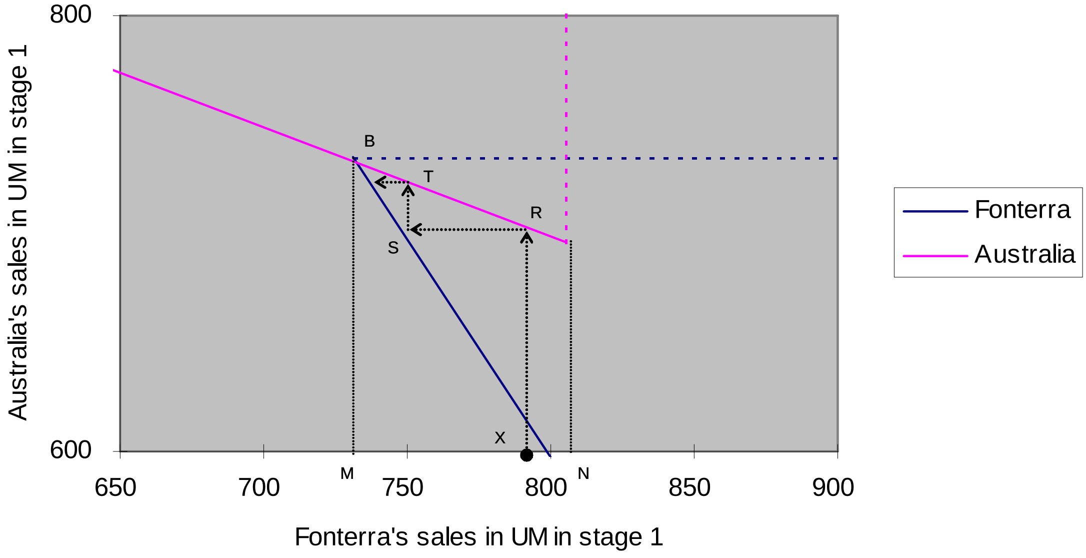

Figure 10.22: Sequential best response given that Fonterra has medium starting sales.

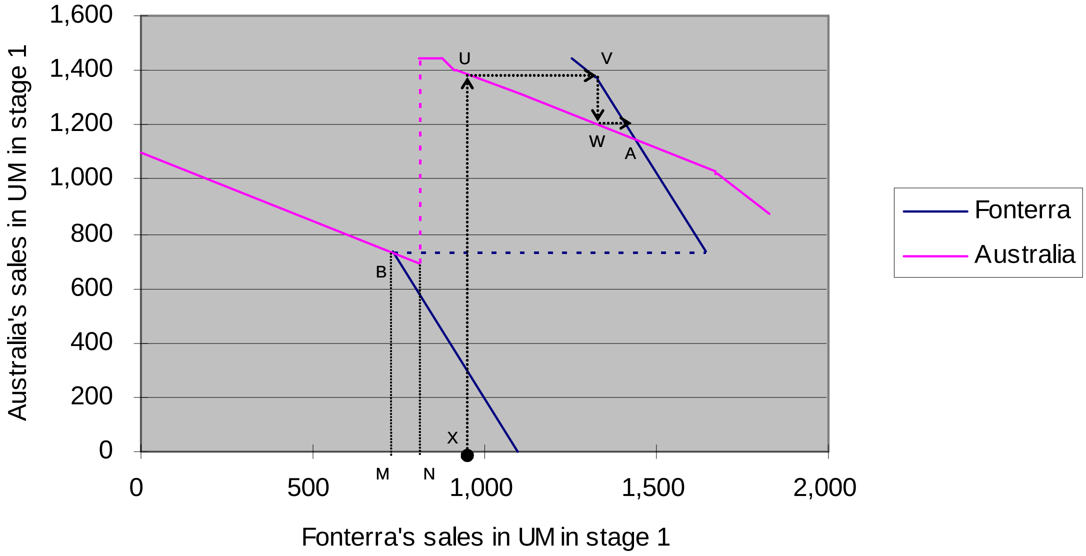

Figure 10.23: Sequential best response given that Fonterra has high starting sales.

10.2.7 Inventory holding cost

In the game with real data, we have shown that the impact of inventory holding cost on equilibrium is simple. In this section, we will show that in this game, the impact of inventory holding cost on the equilibrium is more complicated. Suppose that the inventory holding costs of the players are low, say zero. We observe that the first three equilibria described in the game with the original costs now exist at different intervention prices. We present the range of intervention prices supporting these three equilibria for the two holding costs in Table 10.5, where the equilibria are defined by the strategies that both leaders play.

	Original	Zero
Unconstrained	(0, 4,578)	(0, 4,573)
Large-inventory	(4,564, 4,901)	(4,533, 5,155)
Zero-inventory	(4,576, 5,155)	(4,901, 5,155)

Table 10.5: Range of intervention prices supporting equilibria.

With zero holding costs, the equilibrium with the leaders playing unconstrained strategies disappears at a lower intervention price. The equilibrium where leaders play large-inventory strategies appears at a lower intervention price but disappears at a higher intervention price. The equilibrium with leaders playing zero-inventory strategies emerges at a higher intervention price. Note that the sales are different between the same strategies with the two holding costs.

Chapter 11

Conclusion

In this chapter, we give a summary of the thesis, describe the benefits of the research to Fonterra and how it can be integrated into Fonterra's system, and outline the limitations and suggestions for future research.

In Part I of this thesis, we have developed a set of optimization models to assess the effect of uncertain milk supply on Fonterra's earnings, assuming that Fonterra is a monopoly supplier in the market. The important features that our models include are uncertain milk supply, inventory, price-demand curves in the spot markets and contracting. We are the first to model a dairy industry supply chain with these features using multi-stage stochastic programming.

We have derived a forecast milk supply for a production year that assumes linear regional growth in milk supply over years, and based on this we have derived two models for uncertain milk supply. The first model is an additive model, where the forecast milk supply and the random error are in an additive form. We have used an autoregressive model for the random error, which is generated from a vector of independent random variables. We construct a discrete distribution for sampling the independent random variables by minimizing the Wasserstein metric.

The second model is a multiplicative model, where the forecast milk supply and the random error are in a multiplicative form. The random error is modelled by a Markov state and an independent random variable.

For uncertain milk supply from each of the two models, we have presented six optimization models, which are the ADO model, the ADOS model, the MSPF model, the MSPE model, the DE model and the PF model. We have described an algorithm DOASA to solve the multistage stochastic programming models (the MSPF model and the MSPE model). We have given a mathematical proof of the almost-sure convergence of DOASA for linear objective functions.

In an in-sample simulation for the additive policies,

1. the ADO policy has a significant advantage over the ADOS policy, and thus it is very expensive to commit to a set of sales-projections at the beginning of the year,
2. the MSPF policy has a significant advantage over the MSPE policy,
3. the MSPF policy has an advantage over the ADO policy and the MSPF policy has captured most of the difference in earnings in expectation between the best possible policy and the ADO policy.

We have shown that the main advantage of our policy comes from considering distress trading at high costs, and in this circumstance it achieves higher expected earnings by being conservative in making contracts. A synthetic out-of-sample simulation for these two policies confirms this result.

In an in-sample simulation for the multiplicative policies,

1. the ADO policy has lower earnings in expectation than the ADOS policy, which implies committing a set of sales-projections at the beginning of the year is beneficial with low uncertainty in milk supply scenarios generated in the multiplicative model,
2. the MSPF policy has an insignificant advantage over the MSPE policy,
3. the MSPF policy has an advantage over the ADO policy and has captured most of the gain by the best possible policy over the ADO policy.

The additive model for uncertain milk supply takes into account regional correlation in milk supply. On the other hand, the multiplicative model for uncertain milk supply assumes univariate random variation over the regions, and thus it does not take account of the variation in each region. In processing the real data, the milk scenarios generated from this model have small variation. We have shown that the additive MSPF policy is better than the multiplicative MSPF policy, which implies that accounting for variation in milk supply between regions is beneficial.

We have shown that allowing extra storage capacity gives gains to both the ADO policy and the additive MSPF policy, and the MSPF policy maintains its advantage over the ADO policy which implies that the advantages of the MSPF policy cannot be replicated by allowing more inventory in the ADO policy. We have assessed the fixed-price MSPF policy against the MSPF policy with linear price-demand curves and the ADO policy and shown that not accounting for price variation in decision making incurs a loss in earnings.

We have extended the portfolio of stochastic programming models with demand curves to a multistage setting. This has produced promising results in policy simulation, which supports our hypothesis that our policy from solving a multistage stochastic programming model is an improvement over a deterministic policy.

In Part II of the thesis, we have described some game-theoretic inventory models for the European dairy market. We have described the European dairy market outlining the important regulations on the market which are the intervention system, and tariffs and quotas, and have presented a game-theoretic inventory model for the European dairy market. In the model, we have assumed that Fonterra and Australia are leaders and the European Union is a follower, and each player has fixed production and has sales to the European market and the world market. We have taken into account the intervention system and the tariffs and quotas, and we have assumed linear price-demand curves for sales.

We have discussed an approach using EPECs, which are difficult to solve in all but simple cases. To our knowledge, there is no algorithm for solving a general EPEC which can guarantee the solution to be a Nash equilibrium. We have implemented an algorithm that uses a *grid search* followed by a *sequential best response* to solve the model, which gives an equilibrium at discrete strategy space at termination.

We have performed computational experiments to investigate players' strategies in different game settings. In a game with real data, we have presented the result and investigated the impact of intervention price, tariffs and quotas, and inventory holding cost on the equilibrium. We have shown that all players prefer a lower intervention price, and that the tariffs and quotas have seriously eroded Fonterra's payoff and only zero inventory holding cost will change Fonterra's strategy in equilibrium.

We have used fictitious data to illustrate that the game can be far more complicated than the one with real data. In a simplified version of this game, where the leaders act as a single leader, we have presented five strategies, and the switches between optimal strategies at different intervention prices. In the game where the leaders compete, we have described five equilibria. We have shown that there are different numbers of equilibria at changing intervention prices, and illustrated them using the best responses of the leaders. We have also computed a sequential best response to illustrate how the leaders' strategies will converge to one of the equilibria given a leader's starting strategy. We have also shown that a zero inventory holding cost will change the range of intervention prices supporting the equilibria.

These results in computational experiments have shown that strategic considerations are important to Fonterra in the European dairy market. Of course these results are limited by the simplifying assumptions made in the model, and the insights gained from this study are accompanied by this caveat. It appears from the results that

1. The leaders would prefer low tariffs and large quotas. Any reform by the European Union Commission to reduce the impact of tariffs and quotas will benefit the leaders.

2. The European Union, as a cartel, would prefer a low intervention price as it may sell more in the European dairy market for high market prices. The reform will lower the intervention price and thus will benefit the European Union.

3. If the leaders have small market powers, as those in the simple game, then the players' strategies in equilibrium are simple. But if the leaders have large market powers, as those in the fictitious game, then the result is much more complicated. Multiple equilibria may exist, and the strategies and equilibria are sensitive to the market conditions such as the intervention price and the players' conditions such as their inventory holding costs.

The supply chain model captures the important features in Fonterra's production planning, such as uncertain milk supply, coordination of production over multiple regions, processes and products, inventory to smooth variation in demands, sales over multiple spot markets with price-demand curves, contracting that has an impact on inventory for several months, and a penalty cost that imitates the implicit cost to avoid distress trading. Thus it can be used as a benchmark model of a real problem for Fonterra. It is a monthly-based multistage problem assuming fixed production capacity and thus can be used as a tool for tactical production planning. It can be easily extended to a model with a shorter or a longer decision horizon (for example, eighteen months, the tactical planning horizon used in Fonterra), manufacturing sites and storage places in each region, and some special plants and highly profitable special products, and thus it is very flexible to be adapted to a real problem.

Note that there are other possible strategies in contracting that have not been included in this model. For example, we have assumed that contracts are made in three months in advance, but contracts may have different leading times, say one month to three months. Delivery of contracts may be delayed through negotiation with the customers, say by one month. Contract sizes may also be flexible, say within a range in volume but not a fixed quantity. These strategies in contracting may be looked at in future research.

The forecast milk supply is derived by assuming linear regional growth over years and its estimation takes account of the regional correlation in random variation. Note that we use this assumption just for convenience, and thus other models could be used. For example, we observe in some regions the growths are quadratic.

The additive model for milk supply uses an autoregressive model to capture the serial correlation in random variation. The random variables are independently sampled, which makes the additive model easy to be implemented. A sample of milk scenarios shows good coverage of residuals on the residuals of the historical data, and thus this model can be used to improve the modelling of milk supply and thus prediction of future milk supply for Fonterra. This would reduce the variation of milk supply scenarios, and thus improve the earnings of policies. However, we have only nine years of historical data in deriving the models for uncertain milk supply, and thus more data, say twenty years of real data, are needed for better estimates in the model.

If we obtain a better forecast model with more data which results in smaller variation in milk supply scenarios, then we may assess the difference of earnings of policies evaluated in the milk scenarios form the two models. This difference gives the value of reducing variance in forecasts, and thus indicates that if investment for a better model for forecast is worthy. Furthermore, we may compare the earnings of policies using our forecast model to those using Fonterra's clever forecasts to estimate the value of our forecast model.

We have developed a set of optimization models assuming uncertain milk supply, and we have assessed the policies by solving these models in simulation experiments. The main result is that a policy from solving a multistage stochastic programming model, the MSPF policy, is an improvement over the deterministic policy that Fonterra currently implements, the ADO policy. This policy captures most of the difference in earnings between the best possible policy (which is impossible to attain) and the ADO policy.

The benefits of the MSPF policy accrue from avoiding distress trading, which we have modelled inexactly with a single price multiplier. The exact cost may be lower than the estimates that we have used. As we have shown, if the penalty cost is low, then the MSPF policy does not have an advantage over the ADO policy.

In practice, distress trading would be only one of many actions that can be taken by Fonterra to mitigate the losses incurred by not meeting contracts or targets. For example, it may negotiate with the customer for a delay of contract or a smaller contract, which incurs a lower loss in earnings than using distress trading. A more detailed model of the recourse decisions available to Fonterra will provide a better estimate of the potential savings from a stochastic programming model.

We have shown that Fonterra can achieve higher earnings by obtaining extra (expensive) storage places. This suggests that Fonterra may be better off by building extra storage capacity, which gives a lower ongoing cost. To model this, we may extend our model by allowing such a decision to be made at the beginning of the year in our policies, and evaluate them in simulation to assess the benefits.

Currently in Fonterra's production planning, price variation is not taken into account. We have shown that accounting for prices that depend on sales levels is important. This may improve Fonterra's earnings, given that a better estimation of price-demand curves are obtainable. On the other hand, we have assumed that price-demand curves are known and fixed, but they may be uncertain, say due to fluctuation of exchange rates and random shocks in demand. Accounting for this in our model gives more uncertainty in production planning and may give more benefit to the MSPF policy.

In the markets, SMP and WMP are substitutes to each other, that is, customers may purchase either product and thus the market prices for both products depend on the sales of both products. This would give a different form of price-demand curve. For example, suppose the sales of SMP are m and the sales of WMP are n. Then the price-demand curve for SMP is $p = a - bm - cn$, and that for WMP is $q = d - em - fn$. We may assume the price for a product is more sensitive to the sales of this product than to the sales of the other, which means $b > c$ and $e < f$. The problem can be solved using current mathematical software, say the QP Simplex method in CPLEX.

We have developed the DOASA algorithm in solving the multistage stochastic programming models. We have given a mathematical proof of the almost-sure convergence of DOASA for the linear problem, and have shown that it converges in solving our quadratic problems. This gives support on the applicability of the algorithm, and would increase confidence in the MSPF policy generated by the algorithm. The MSPF policy can be generated within 24 hours for our huge scale problem that is equivalent to a deterministic equivalent problem of billions of constraints and variables. Thus the MSPF policy is a good option for Fonterra in production planning. However, the performance of the algorithm depends on the dimension of the state variables. For example, if we use all nine products as contract products, then the number of state variables would increase from 24 to 42 and the algorithm does not converge within the same number of iterations.

Note that the cut generation and simulation experiment for a policy from a multistage stochastic programming model is somewhat lengthy, and thus some methods are needed to speed up the processes. For example, we observe that some cuts in cut generation are inactive as new cuts are added in, and thus can be reduced by re-defining to a trivial cut to reduce computational time. To do this, at the convergence of DOASA, for each cut we may compare its intercept and slope to the other cuts. If both values are lower than those of the other cuts, then this cut is inactive and thus can be reduced. Note that if the state variables have negative values, then prior to cut generation we can set the lowest value of the state variables to zero and index the others against it. This method could be tested in future research.

The game-theoretic model for the European dairy market has taken account of the important mechanisms in the regulations of the European Union Commission, which are the intervention system, and the tariffs and quotas. The model has some important features, such as two leaders and a follower in decision making, and price-demand curves for sales in the market. Although somewhat unrealistic, the model could be used as a pilot model for future improvement.

We have developed an algorithm to solve the model for an equilibrium of the game. This algorithm results in an equilibrium, which has an advantage over a common approach which solves an EPEC but does not guarantee a solution to be an equilibrium. However, as we have described, this algorithm has two main limitations. The equilibrium solution obtained in the discrete grids may not be an equilibrium in continuous strategy spaces, and the algorithm is only feasible for low dimension.

The computational experiments have shown that all players prefer a low intervention price, and Fonterra can be much better off if the tariffs and quotas are relaxed. Hence the reform on regulations by the EUC would be beneficial to Fonterra.

In the European market, the companies in the European Union sell products at the same time as Fonterra and Australia, but the EUC then observes the market price and buys back products if the market price is at the intervention price. Based on this fact, we have assumed EU, which represents the EUC and these companies, to be a follower in decision making in the game-theoretic model for the European dairy market. This assumption would be reasonable if intervention purchase occurs. However, we observe that the market prices sometimes are above the intervention price, and thus the EUC takes no action. This implies that the player EU would be a simultaneous player with Fonterra and Australia. Hence it would be more realistic to relax this assumption when this occurs. This will be looked at in future research.

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