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This paper presents and analyzes four finite difference methods for linear shape from shading problem. Comparisons of accuracy, solvability, stability and convergence of these methods indicate that the weighted semi-implicit method and the box method are better than the other ones because they are easily calculated, more accurate, faster in convergence and unconditionally stable.

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Abstract

This paper presents and analyzes four finite difference methods for linear shape from shading problem. Comparisons of accuracy, solvability, stability and convergence of these methods indicate that the weighted semi-implicit method and the box method are better than the other ones because they are easily calculated, more accurate, faster in convergence and unconditionally stable.

Keywords: shape from shading, differential equations, finite difference, stability, convergence

1 Introduction

The basic problem in shape from shading is to recover surface values of an object surface from its variation in brightness. If the surface is modeled by a function $Z(x, y)$, then it satisfies the following *image irradiance equation*

$$E_0 \cdot \rho(x, y) \cdot \frac{1 + p_s p + q_s q}{\sqrt{1 + p_s^2 + q_s^2}\sqrt{1 + p^2 + q^2}} = E(x, y) \quad (1)$$

over a compact image domain Ω, where $E(x, y)$ is the image brightness of the object formed by an orthographic (parallel) projection of reflected light onto the xy-image plane, and is measured with respect to the known illumination direction $(p_s, q_s, -1)$. E_0 is the known illumination intensity. $\rho(x, y)$ is the albedo of the surface material at that surface point. $(p, q) = (p(x, y), q(x, y))$ is the surface gradient with $p = \partial Z/\partial x$ and $q = \partial Z/\partial y$. The above nonlinear, first-order partial differential equation has been studied with a variety of different techniques (see, e.g., Horn [1, 3]; Horn and Brooks [2]; Klette et al. [4]). Recently, Zhang et al. [10] pointed out that all shape from shading algorithms produce generally poor results. Therefore, new shape from shading methods should be developed to provide more accurate, and realistic results.

Linear shape from shading problems arose in the study of the maria of the moon [1] and in a local shape from shading analysis [7]. If a small portion of an object surface, having reflectivity properties approximated by a linear reflectance map, is illuminated by a light source of unit intensity from direction $(p_s, q_s, -1)$, then the corresponding image function satisfies the following *linear image irradiance equation:*

$$\frac{1 + p_s p + q_s q}{\sqrt{1 + p_s^2 + q_s^2}} = E(x, y). \quad (2)$$

Defining $F(x, y) = E(x, y)\sqrt{1 + p_s^2 + q_s^2} - 1$, we can rewrite (2) as

$$p_s \frac{\partial Z}{\partial x}(x, y) + q_s \frac{\partial Z}{\partial y}(x, y) = F(x, y). \quad (3)$$

Kozera and Klette [5, 6] presented some algorithms based on explicit finite difference method. Ulich [9] also discussed two explicit and one implicit finite difference algorithms for (3). So far it has not yet been studied which finite difference algorithms for (3) are better. In this paper, We consider (3) over a rectangle domain

$$\Omega = \{(x, y) \in R^2 : \quad 0 \le x \le a, \quad 0 \le y \le b\}$$

with the initial condition

$$Z(x, 0) = \phi(x), \ 0 \le x \le a$$

and boundary conditions

$$Z(0, y) = \psi_0(y), \ Z(a, y) = \psi_1(y), \ 0 \le y \le b,$$

where the given functions $\phi(x), \psi_0(y)$ and $\psi_1(y)$ satisfy $\phi \in C([0, a]) \cap C^2((0, a))$, $\psi_0, \psi_1 \in C([0, b]) \cap C^2((0, b))$, $\phi(0) = \psi_0(0), \phi(a) = \psi_1(a)$, and $(p_s, q_s) \neq (0, 0)$. Throughout this paper we assume that the above Cauchy's problem is *well-posed* over a rectangle domain Ω, that is, there exists a unique solution $Z(x, y)$ to the corresponding partial differential equation satisfying the boundary conditions and depending continuously on the given initial condition, and we also suppose that the solution $Z(x, y)$ is sufficiently smooth, at least $Z(x, y) \in C^2(\bar{\Omega})$.

The organization of the the rest of the paper is as follows. In Section 2 we present different discretizations of equation (3): four semi-implicit methods. The initial condition $Z(x,0)$ is used for all these methods, but different boundary conditions $Z(0,y)$ and/or $Z(a,y)$ are required. In Section 3 we discuss the accuracy, solvability, consistency, stability and convergence of these methods. The conclusions are given in Section 4.

2 Finite difference algorithms

Suppose that the rectangular domain Ω is divided into small grids by parallel lines $x = x_i$ $(i = 0, 1, \ldots, M)$ and $y = y_j$ $(j = 0, 1, \ldots, N)$, where $x_i = ih$, $y_j = jk$ and $Mh = a$, $Nk = b$, M and N are integers, h is the grid constant in x-direction (i.e., distance between neighboring grid columns) and k is the grid constant in y-direction. For convenience, we shall denote by $Z(i,j)$ the value $Z(x_i, y_j)$ of solution $Z(x,y)$ on the grid point (x_i, y_j), and by $Z_{i,j}$ an approximation of $Z(i,j)$.
An *explicit* finite difference method for (3) contains only one unknown value of Z at each j level. The unknown value is calculated directly from the known values of Z at the previous levels. Therefore, explicit methods are easy to be computed. The disadvantage of explicit methods is that their accuracy is lower since the order of their truncation errors is usually lower. For an *implicit* method, there are three unknown values of Z at each j level. Implicit methods are more accurate than explicit methods since the order of the truncation errors of implicit methods is higher than that of explicit methods. However, the computation of implicit methods takes much more time than that of explicit methods because implicit methods require to solve a linear algebraic systems for each j. In order to overcome the drawbacks and take the advantages of explicit and implicit methods, we consider the following methods which contain two unknown values of Z at each j level.

2.1 Forward-Backward (FB) method

Approximating $\partial Z/\partial x$ with the forward difference quotient and $\partial Z/\partial y$ with the backward difference quotient gives the following discretization for (3), at any pixel (i,j):

$$p_s \frac{Z(i+1,j) - Z(i,j)}{h} + q_s \frac{Z(i,j) - Z(i,j-1)}{k} + O(h+k) = F(i,j).$$

where $O(h+k) = -hp_s Z_{xx}(\theta_1, y_j)/2 + kq_s Z_{yy}(x_i,\theta_2)/2$, $x_i \le \theta_1 \le x_{i+1}$, $y_{j-1} \le \theta_2 \le y_j$. Dropping the truncation error $O(h+k)$, and rearranging the above equation then yields

$$Z_{i+1,j} = (1-c)Z_{i,j} + cZ_{i,j-1} + \frac{h}{p_s}F_{i,j}, \tag{4}$$
$$i = 0, 1, \ldots, M-1; j = 1, 2, \ldots, N,$$

where the corresponding finite difference initial conditions $Z_{i,0}(i = 0, \ldots, M)$ and boundary conditions $Z_{0,j}(j = 0, \ldots, N)$ are given, $p_s \neq 0$, $c = \frac{q_s h}{p_s k}$. The truncation error of the FB method is $O(h+k)$. Given an linear shape from shading problem (3), the *influence domain* of the FB method with the above boundary condition coincides with the entire domain Ω.

2.2 Backward-Backward (BB) method

Approximating both $\partial Z/\partial x$ and $\partial Z/\partial y$ with the backward difference method, then using the above techniques, we can get the following two-level semi-implicit method

$$\begin{aligned} Z_{i,j} &= \frac{1}{1+d} Z_{i,j-1} + \frac{d}{1+d} Z_{i-1,j} + \\ &+ \frac{k}{q_s(1+d)} F_{i,j}, \\ & i = 1, \ldots, M; j = 1, \ldots, N, \end{aligned} \tag{5}$$

where the initial conditions $Z_{i,0}(i = 0, 1, \ldots, M)$ and boundary conditions $Z_{0,j}$ $(j = 0, 1, \ldots, N)$ are given, $d = \frac{p_s k}{q_s h}$, $d \neq -1$, $q_s \neq 0$. The truncation error of the BB method is $O(h+k)$. The influence domain of the BB method with the above boundary condition is the entire domain Ω.

2.3 Weighted Semi-implicit (WS) method

The weighted semi-implicit method for solving (3) is

$$\begin{aligned} Z_{i,j+1} &= Z_{i,j} + \frac{d}{2+d}(Z_{i-1,j+1} - Z_{i+1,j}) + \\ &+ \frac{k}{q_s(2+d)} F_{i,j}, \\ & i = 1, \ldots, M-1; j = 0, 1, \ldots, N-1, \end{aligned} \tag{6}$$

where $Z_{i,0}(i = 0, 1, \ldots, M)$, the boundary conditions $Z_{0,j}$ and $Z_{M,j}$ $(j = 0, \ldots, N)$ are given, $d = \frac{p_s k}{q_s h}$, $d \neq -2$, $q_s \neq 0$. The truncation error of WS method is $O\left(h + k^2\right)$, and the influence domain of the WS method also coincides with the entire domain Ω.

2.4 Box method

The box method for solving (3) is as follows:

$$\begin{aligned} Z_{i+1,j+1} &= Z_{i,j} + \frac{1-d}{1+d}(Z_{i+1,j} - Z_{i,j+1}) \\ &+ \frac{2k}{q_s(1+d)} F_{i,j} \\ & i = 0, 1, \ldots, M-1; j = 0, 1, \ldots, N-1, \end{aligned} \tag{7}$$

where $Z_{i,0}$ $(i = 0, 1, \ldots, M)$ and $Z_{0,j}$ $(j = 0, 1, \ldots, N)$ are given, $d = \frac{p_s k}{q_s h}$, $d \neq -1$, $q_s \neq 0$. The truncation error of the box method is $O\left(h^2 + k^2\right)$. The influence domain of the box method is the whole domain Ω.

3 Analysis of finite difference methods

A finite difference method is said to be *consistent* with a partial differential equation iff as the grid constants tend to zero, the difference method becomes in the limit the same as the partial differential equation at each point in the solution domain. It is not difficult to prove that the above four finite difference methods are consistent with (3).

Theorem 1 (solvability) *Let $d = \frac{p_s k}{q_s h}$ be a fixed constant, $q_s \neq 0$. Then, the FB method is solvable if $p_s \neq 0$, the BB and box methods are solvable if $d \neq -1$, and the WS method is solvable if $d \neq -2$.*

Proof: The FB method involves two unknown values of $Z_{i+1,j}$ and $Z_{i,j}$ at the jth level, but $Z_{i+1,j}$ can be calculated in the order $i = 1$ to $M-1$, since $Z_{i,j}$ is known when the initial values $Z_{i,0}(i = 0, \ldots, M)$ and boundary conditions $Z_{0,j}(j = 0, \ldots, N)$ are given. That is, the FB method is solvable. The solvability of the BB method can be proved analogously. About the WS method, it also involves two unknown values $Z_{i,j+1}$ and $Z_{i-1,j+1}$ at the $(j+1)$th level. However, commencing with a given boundary value $Z_{0,j}$ and $Z_{M,j}(j = 0, \ldots, N)$, the values of $Z_{i,j+1}$ can be computed in the order $i = 1$ to $M-1$, since $Z_{i-1,j+1}$ is known at each application of the WS method. Using the same method, we can see that the box method is also solvable under the given conditions.

Lemma 1 (Von Neumann criterion of stability [8]) *Given a finite difference method with constant coefficients*

$$L_h Z_{i,j} = F_{i,j}, \tag{8}$$

where L_h is a finite difference operator. Let $Z_{i,j} = g^j e^{I\theta i}$ with $I^2 = -1, \theta = 2\pi l h, l = \pm 1$ (g is called amplification factor*). If we substitute this $Z_{i,j}$ into the homogenous finite difference method associated with* (8)*, eliminating the common factor we can obtain an expression of g. Then the finite difference method* (8) *is* stable *iff there is a constant $K > 0$ (independent of θ, h and k) such that*

$$|g| \leq 1 + Kk \tag{9}$$

for all θ. If k/h is constant, the stability condition (9) *can be replaced with $|g| \leq 1$.*

Theorem 2 (stability) *Let $d = \frac{p_s k}{q_s h}$ be a fixed constant, $q_s \neq 0$.Then,*
(a) *the FB method is stable iff $c \leq 1$, where $c = 1/d$;*
(b) *the BB method is stable iff $d \geq 0$ or $d < -1$;*
(c) *the WS and the box methods are unconditionally stable.*

Proof: (a) Replacing $Z_{i,j}$ in the homogenous method associated with the FB method (4) by $g^j e^{I\theta i}$ for each value of i and j, we have that

$$g^j e^{I\theta(i+1)} = (1-c) g^j e^{I\theta i} + c g^{j-1} e^{I\theta i},$$

which gives the amplification factor as

$$\begin{aligned} g &= \frac{c}{c - (1-\cos\theta) + I.\sin\theta} \\ &= \frac{c}{c - 2\sin^2\frac{\theta}{2} + I.2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}. \end{aligned}$$

Therefore

$$|g|^2 = c^2/[c^2 + 4(1-c)\sin^2\frac{\theta}{2}].$$

We see that $|g| \leq 1$, that is, the FB method is stable iff $c \leq 1$.
(b) The amplification factor for the BB method is given by

$$\begin{aligned} g &= \frac{1}{1 + d(1-\cos\theta) + Id\sin\theta} \\ &= \frac{1}{1 + 2d\sin^2\frac{\theta}{2} + I.2d\sin\frac{\theta}{2}\cos\frac{\theta}{2}}. \end{aligned}$$

The magnitude of g is

$$|g| = 1/[1 + 4d(1+d)\sin^2\frac{\theta}{2}].$$

It is easy to find out that $|g| \leq 1$ iff $d \geq 0$ or $d \leq -1$. Combining this condition with the solvability of the method, we have that the BB method is stable iff $d \geq 0$ or $d < -1$.
(c) The amplification factor of the WS method is

$$g = \frac{2 + d - de^{I\theta}}{2 + d - de^{-I\theta}} = \frac{(2 + d(1-\cos\theta)) - Id\sin\theta}{(2 + d(1-\cos\theta)) + Id\sin\theta}.$$

We then see that $|g| = 1$ for all values of d and θ, that is, the WS method is unconditionally stable. For the box method the amplification factor is given by

$$\begin{aligned} g &= \frac{1 + d + (1-d)e^{I\theta}}{1 - d + (1+d)e^{I\theta}} \\ &= \frac{((1+d) + (1-d)\cos\theta) + I(1-d)\sin\theta}{((1-d) + (1+d)\cos\theta) + I(1+d)\sin\theta}. \end{aligned}$$

It follows that

$$\begin{aligned} |g|^2 &= \frac{|((1+d) + (1-d)\cos\theta) + I(1-d)\sin\theta|}{|((1-d) + (1+d)\cos\theta) + I(1+d)\sin\theta|} \\ &= \frac{(1+d)^2 + 2(1-d^2)\cos\theta + (1-d)^2}{(1-d)^2 + 2(1-d^2)\cos\theta + (1+d)^2} \\ &= 1. \end{aligned}$$

Therefore, $|g| = 1$ for all d and θ, that is, the box method is also unconditionally stable.

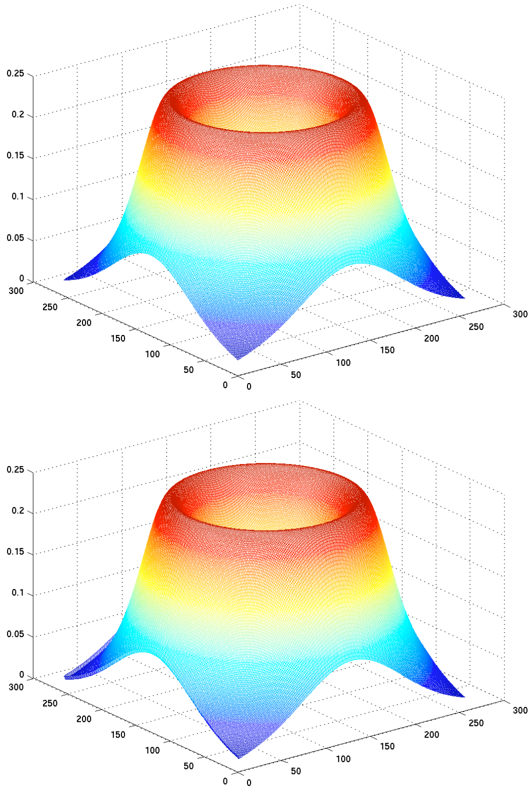

Figure 1. Original (above) and reconstructed (below) surface.

Lemma 2 *Given a finite difference method for a well-posed initial boundary value problem of a partial differential equation. If the method is consistent and stable, then it is* convergent.

Theorem 3 (convergence) *Let* $d = \frac{p_s k}{q_s h}$ *be a fixed constant,* $q_s \neq 0$*.Then,*
(a) *the FB method is convergent if* $c \leq 1$*, where* $c = 1/d$*;*
(b) *the BB method is convergent if* $d \geq 0$ *or* $d < -1$*;*
(c) *the WS method is convergent if* $d \neq -2$*; the box method is convergent if* $d \neq -1$.

4 Experiments and Conclusions

Four finite difference algorithms are analyzed for the linear shape from shading problem. The analysis of different algorithms is achieved by comparing the influence domain, truncation error, consistency, solvability, stability and convergence of each method. The comparison indicates that the box method is more useful. Figure 1 shows the experimental results for the box method. The upper one is the original 3D shape for a volcano-like surface $Z(x, y) = 1/(4(1 + (1 - x^2 - y^2)^2))$ and the lower one is the computed surface map, where $\Omega = [0, 1] \times [0, 1]$, $h = k = 1/255$, $p_s = 0.5$, and $q_s = 1$.

Finnally, we conclude this paper by itemizing a few main result:

- The box method is much more useful by comparing with other finite difference methods for linear shape from shading.
- All methods presented in this paper are supplemented by a full domain of influence, truncation error, consistency, solvability, stability and convergence analysis.
- The influence domain of each method in this paper coincides with the entire Ω.
- In comparison with the results in Kozera and Klette [5, 6] and Ulich [9], the range of the stability and convergence of the FB and BB methods is identified as being larger.

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