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## Abstract

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# Height from Gradient with Surface Curvature and Area Constraints

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## Abstract

*This paper presents a new algorithm for height from gradients and analyzes its performance. To derive this algorithm, we combine the integrability constraint and the surface curvature and area constraints into a single functional, which is then minimized. Therefore, the changes of height maps will be more regular. The Frankot-Chellappa-algorithm is a special case of our algorithm in the sense that it uses a subset of constraints only.*

**Keywords:** height from gradients, height map, surface gradients, Fourier transform.

## 1. Introduction

Shading-based 3D shape recovery techniques, e.g. shape from shading (SFS), photometric stereo method (PSM), normally provide gradient values (i.e the discrete gradient vector field) for a discrete set of visible points on object surfaces. In order to achieve the relative height or depth values of the surface, these surface gradients have to be integrated.

Suppose that the surface function $Z(x, y)$ of a scene object is formed by an orthographic (parallel) projection of the surface into the $xy$-image plane, and defined in the image plane over a compact region $\Omega$. The gradient values of this surface at discrete points $(x, y) \in \Omega$

$$p(x, y) = \frac{\partial Z(x, y)}{\partial x} = Z_x$$

and

$$q(x, y) = \frac{\partial Z(x, y)}{\partial y} = Z_y$$

are only available as input data, for instance, in the form of a *needle diagram*.

Essentially there are two main classes of integration techniques for finding $Z(x, y)$ from $p(x, y)$ and $q(x, y)$: *local integration techniques* and *global integration techniques* (for a review, see Klette and Schlüns [6]). Local integration methods [1], [3], [10] are conceptually simple and based on the following curve integrals:

$$Z(x, y) = Z(x_0, y_0) + \int_\gamma p(x, y)dx + q(x, y)dy, \quad (1)$$

where $\gamma$ is an arbitrarily specified integration path from $(x_0, y_0)$ to $(x, y) \in \Omega$. Starting with initial height values, the methods propagate height values according to a local approximation rule (e.g., based on the 4-neighborhood) using the given gradient data. Such a calculation of relative height values can be repeated by using different scan algorithms. Finally, resulting height values can be determined by averaging operations. However, initial height values have to be provided. The locality of the computations propagates errors, i.e. this approach strongly depends on data accuracy. Therefore, local integration techniques perform badly when the data are noisy.

Global integration techniques (Horn and Brooks [4], Frankot and Chellappa [2], Horn [5], Wei and Klette [8, 9]) are based on minimizing the following functional (cost function):

$$W = \iint_\Omega [|Z_x - p|^2 + |Z_y - q|^2] dxdy. \quad (2)$$

Comparing with the local methods, the *Frankot-Chellappa algorithm*, based on the results of the paper [2] and presented in Klette et. al [7], leads to better results for the task of calculating height from gradient. Nevertheless, errors at locations of very low albedo result in reconstruction errors. Also,the algorithm is very sensitive to the abrupt changes in orientation, i.e. there are large errors at the object boundary.

The organization of the rest of the paper is as follows. In Section 2 we present our new algorithm for height from gradient. The experimental results and conclusions are given in Section 4 and 5, respectively.

## 2. Height from gradient

In the following, we apply the Fourier transform theory to derive a new algorithm for solving the height from gradient. In order to improve the accuracy and robustness, and to strengthen the relation between the estimated surface and the original image, the functional to be minimized is as follows:

$$\begin{aligned} W &= \iint_\Omega \left[ |Z_x - p|^2 + |Z_y - q|^2 \right] dxdy \\ &+\lambda \iint_\Omega \left( |Z_x|^2 + |Z_y|^2 \right) dxdy \\ &+\mu \iint_\Omega \left( |Z_{xx}|^2 + 2|Z_{xy}|^2 + |Z_{yy}|^2 \right) dxdy \end{aligned} \quad (3)$$

where the subscripts indicate partial derivatives. In the above cost function, the second term of the right-hand is a small deflection approximation of the surface area, and the third term is a small deflection approximation of the surface curvature (i.e it is a measure of quadratic variation in the surface slopes). The non-negative parameters $\lambda$ and $\mu$ establish a trade-off between the constraints, i.e it is used to adjust the weighting between them. The above new cost function reflects the relations among $Z(x,y), p(x,y)$ and $q(x,y)$ more effectively, and make the best use of the information provided by the surface gradient.

The following objective is to solve the unknown $Z(x,y)$ subject to an optimization process which minimizes the cost function $W$. Instead of using the calculus of variations to derive the Euler equations for the solution to (3), we use the Fourier transform theory. Suppose that the Fourier transform of the surface function $Z(x,y)$ is

$$Z_F(u,v) = \iint_\Omega Z(x,y) e^{-j(ux+vy)} dxdy, \quad (4)$$

and the inverse Fourier transform is

$$Z(x,y) = \frac{1}{2\pi} \iint_\Omega Z_F(u,v) e^{j(ux+vy)} dudv, \quad (5)$$

where *j* is the imaginary unit. According to the differentiation properties of the Fourier transform, we have

$$\begin{aligned} Z_x(x,y) &\leftrightarrow juZ_F(u,v), \\ Z_y(x,y) &\leftrightarrow jvZ_F(u,v), \\ Z_{xx}(x,y) &\leftrightarrow -u^2 Z_F(u,v), \\ Z_{yy}(x,y) &\leftrightarrow -v^2 Z_F(u,v). \end{aligned}$$

Let $P(u,v)$ and $Q(u,v)$ be the Fourier transforms of $p(x,y)$ and $q(x,y)$, respectively. Taking the Fourier transform in (3) and using the above differentiation properties and the following Parseval's formula

$$\iint_\Omega |Z(x,y)|^2 dxdy = \frac{1}{2\pi} \iint_\Omega |Z_F(u,v)|^2 dudv,$$

we obtain

$$\begin{aligned} & \frac{1}{2\pi} \iint_\Omega \left[ |juZ_F(u,v) - P(u,v)|^2 + \right. \\ + & \left. |jvZ_F(u,v) - Q(u,v)|^2 \right] dudv + \\ + & \frac{\lambda}{2\pi} \iint_\Omega \left[ |juZ_F(u,v)|^2 + |jvZ_F(u,v)|^2 \right] dudv \\ + & \frac{\mu}{2\pi} \iint_\Omega \left[ \left|-u^2 Z_F(u,v)\right|^2 + 2\left|-uvZ_F(u,v)\right|^2 + \right. \\ + & \left. \left|-v^2 Z_F(u,v)\right|^2 \right] dudv \to minimum, \end{aligned}$$

The left side of the above expression can be expanded as

$$\begin{aligned} & \frac{1}{2\pi} \iint_\Omega \left[ u^2 Z_F Z_F^* - juZ_F P^* + juZ_F^* P + PP^* \right. \\ & \left. + v^2 Z_F Z_F^* - jvZ_F Q^* + jvZ_F^* Q + QQ^* \right] dudv \\ & + \frac{\lambda}{2\pi} \iint_\Omega \left( u^2 + v^2 \right) Z_F Z_F^* dudv \\ & + \frac{\mu}{2\pi} \iint_\Omega \left( u^4 + 2u^2 v^2 + v^4 \right) Z_F Z_F^* dudv, \end{aligned}$$

where $*$ denotes the conjugate. Differentiating the above expression with respect to $Z_F$ and $Z_F^*$, we can deduce the following minimal conditions for the cost function (3)

$$C_{uv} Z_F + juP + jvQ = 0,$$

$$C_{uv} Z_F^* - juP^* - jvQ^* = 0.$$

where $C_{uv} = (1+\lambda)(u^2+v^2) + \mu(u^2+v^2)^2$. Adding the above two equations together, then subtracting the first one from the second one, this results in the following equations

$$C_{uv}(Z_F + Z_F^*) + ju(P - P^*) + jv(Q - Q^*) = 0,$$

and

$$C_{uv}(Z_F - Z_F^*) + ju(P + P^*) + jv(Q + Q^*) = 0.$$

Solving the above equations except for $(u,v) \neq (0,0)$, we obtain

$$Z_F(u,v) = \frac{-juP(u,v) - jvQ(u,v)}{(1+\lambda)(u^2+v^2) + \mu(u^2+v^2)^2} \quad (6)$$

where $(u,v) \neq (0,0)$. The main result is summarized in the following theorem.

**Theorem 1** *The cost function (3) is minimized by taking the Fourier transform of surface $Z(x,y)$ as in the formula (6).*

The Frankot-Chellappa algorithm [2] as formulated in [7], is a special case when parameter $\lambda = 0$ and $\mu = 0$ in (3). Therefore, let $\lambda = 0$ and $\mu = 0$ in (6), we obtain that the

**Figure 1. Image triplet of Beethoven statue.**

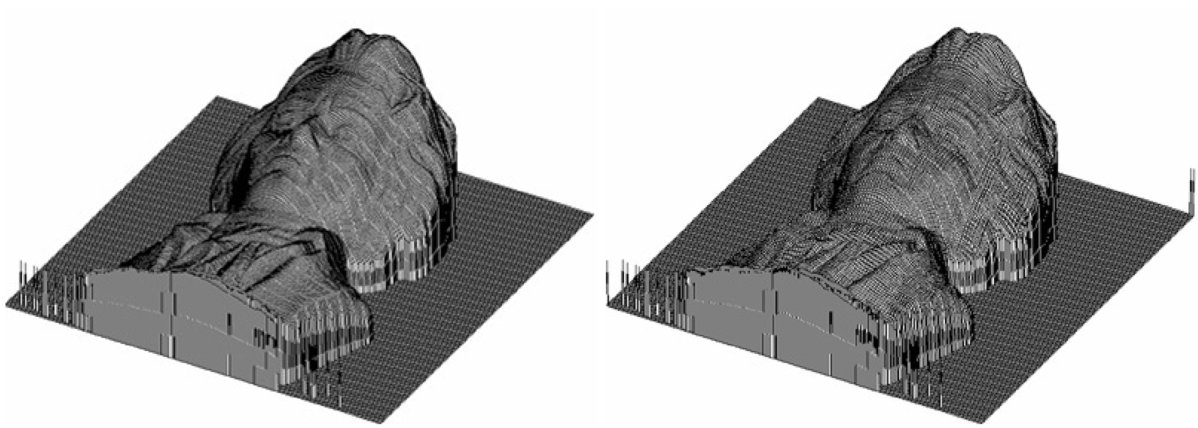

**Figure 2. Recovered surfaces: left - using the Frankot-Chellappa integration method, right - using our new method.**

objective functional (2) is minimized by taking the Fourier transform of the surface $Z(x, y)$ as

$$Z_F(u, v) = \frac{-1}{u^2 + v^2}\left[juP(u, v) + jvQ(u, v)\right], \qquad (7)$$

where $(u, v) \neq (0, 0)$. The formula (7) can also be derived using the above process directly. If so, the process deriving (7) is much simpler than the one used by Frankot-Chellappa in [2]. On the other hand, our new algorithm is capable of dealing with additional constraints.

The following algorithm shows our proposed method for the task of calculating depth from gradients, which use the transformation as specified in Theorem 1 after having the Fourier transforms of the given gradient field. Then an inverse Fourier transform leads to the desired depth map, which allows us to reconstruct object surfaces in 3D space within a subsequent computation step of a general back projection approach.

If the gradient vectors of any length are used as input to the algorithm, then the reconstructed surface is distorted. To avoid this, the value $max_{pq} = 4$ was used in the experiments that are described in the next section.

**Algorithm 1** New algorithm for height from gradient

**input** gradients $p(x,y), q(x,y)$, $\lambda$ and $\mu$
**for** $0 \leq x, y \leq N-1$ **do**
  **if** ($|p(x,y)| < max_{pq}$ & $|q(x,y)| < max_{pq}$) **then**
    P1(x,y)=p(x,y);  P2(x,y)=0;
    Q1(x,y)=q(x,y);  Q2(x,y)=0;
  **else**
    P1(x,y)=0;  P2(x,y)=0;
    Q1(x,y)=0;  Q2(x,y)=0;
  **end if**
**end for**
Calculate the Fourier transforms of P1(x,y) and P2(x,y): P1(u,v), P2(u,v);
Calculate the Fourier transforms of Q1(x,y) and Q2(x,y): Q1(u,v), Q2(u,v);
**for** $0 \leq u, v \leq N-1$ **do**
  **if** ($u \neq 0$ & $v \neq 0$) **then**
    $\Lambda = (1+\lambda)(u^2+v^2) + \mu(u^2+v^2)^2$;
    $\Delta 1 = uP2(u,v) + vQ2(u,v)$;
    $\Delta 2 = -uP1(u,v) - vQ1(u,v)$;
    $H1(u,v) = \Delta 1/\Lambda$;
    $H2(u,v) = \Delta 2/\Lambda$;
  **else**
    $H1(0,0) =$ something; $H2(0,0) = 0$;
  **end if**
**end for**
Calculate the inverse Fourier transforms of H1(u,v) and H2(u,v): H1(x,y), H2(x,y);
**for** $0 \leq x, y \leq N-1$ **do**
  $Z(x,y) = H1(x,y) +$ BackgroundValue;
  {e.g. LSE optimization }
**end for**

## 3. Experimental results

This short note only allows us a very brief report on experimental results. Figure 1 shows three captured images of a Beethoven plaster statue. The gradients were generated using photometric stereo method with three light sources (3S PSM). 3S PSM shape recovery results have been discussed in [7] for this statue.

Figure 2 illustrates both recovered surfaces. The left surface was calculated using the Frankot-Chellappa algorithm and the right one was done using our new algorithm with $\lambda = 0.5, \mu = 0.5$.

## 4. Conclusions

We designed a new algorithm for height from gradient. The new cost function reflects the relations among Z(x,y), p(x,y) and q(x,y) more effectively. In addition, the deriving of the algorithm is much simpler than the one used by Frankot-Chellappa [2]. The new algorithm is capable of dealing with additional constraints. The appropriateness of the approach has been illustrated through experiments using real objects, and these experiments will be reported in a longer paper.

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