Degenerate elliptic second-order differential operators with bounded complex-valued coefficients

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Abstract

This thesis considers core properties for degenerate elliptic second-order differential operators in divergence form with bounded complex-valued coefficients. The main contribution of the thesis is in two parts. In one dimension we characterise when the space of test functions is a core for these operators. In higher dimensions we provide sufficient conditions.

For the first part we consider an operator of the form

$$A = -\frac{d}{dx}\, c\, \frac{d}{dx} + m\, \frac{d}{dx} + w\, I$$

in $L_2(\mathbb{R})$, where c is a bounded Lipschitz continuous complex-valued function which takes values in a sector. We determine for which $p \in [1, \infty)$ the quasi-contraction semigroup generated by $-A$ extends consistently to a quasi-contraction semigroup on $L_p(\mathbb{R})$. For those values of p we characterise when the space of test functions $C_c^\infty(\mathbb{R})$ is a core for the generator on $L_p(\mathbb{R})$.

For the second part let $c_{kl} \in W^{2,\infty}(\mathbb{R}^d, \mathbb{C})$ for all $k, l \in \{1, \ldots, d\}$. Let Σ_θ be the sector with vertex 0 and semi-angle θ in the complex plane. Suppose $(C(x)\, \xi, \xi) \in \Sigma_\theta$ for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$. We consider the divergence form operator

$$A = -\sum_{k,l=1}^{d} \partial_l (c_{kl}\, \partial_k)$$

in $L_2(\mathbb{R}^d)$. We show that for all p in a suitable interval the contraction semigroup generated by $-A$ extends consistently to a contraction semigroup on $L_p(\mathbb{R}^d)$. For those values of p we present a condition on the coefficients such that the space $C_c^\infty(\mathbb{R}^d)$ of test functions is a core for the generator on $L_p(\mathbb{R}^d)$. We also examine the operator A separately in the more special Hilbert space $L_2(\mathbb{R}^d)$ setting. In this setting we provide many more sufficient conditions on the coefficients for $C_c^\infty(\mathbb{R}^d)$ to be a core for A. Furthermore if all the functions in the domain $D(A)$ are smooth enough, we show that $C_c^\infty(\mathbb{R}^d)$ is always a core for A.

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Chapter 3: Degenerate elliptic operators in one dimension

Nature of contribution by PhD candidate	Formulation and proof of theorems
Extent of contribution by PhD candidate (%)	66%

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Chapter 1

Introduction

The theme of this thesis is about core properties for degenerate elliptic second-order differential operators in divergence form with bounded complex-valued coefficients in one and higher dimensions. More specifically we will investigate when the space of test functions is a core for those operators.

This chapter consists of two parts. In the first part I will explain the topic in detail. This will be accompanied by a brief summary of the developments in the field. The second part provides the outline of the thesis. The ideas pursued in subsequent chapters are highlighted. Details about collaborative works and contributions are also given.

1.1 Core properties for degenerate elliptic operators

The Laplace operator

$$-\Delta = -\partial_1^2 - \ldots - \partial_d^2$$

in $L_2(\mathbb{R}^d)$, where $d \in \mathbb{N}$, is undoubtedly the most well-known and ubiquitous type of operator in mathematical analysis. This operator has been studied for centuries and many nice properties have been observed to associate with it. One among those properties is that the maximal domain $D(-\Delta)$ of the Laplace operator consists of all twice weakly differentiable functions in $L_2(\mathbb{R}^d)$. That is,

$$D(-\Delta) = W^{2,2}(\mathbb{R}^d). \tag{1.1}$$

We note that $D(-\Delta)$ is a normed space with the *graph norm* $u \longmapsto \|u\|_2 + \|\Delta u\|_2$ for all $u \in D(-\Delta)$. Therefore (1.1) must be understood in the sense that the graph norm of $D(-\Delta)$ is equivalent to the Sobolev norm on $W^{2,2}(\mathbb{R}^d)$ and the two spaces are equal as vector spaces. Since *the space of test functions* $C_c^\infty(\mathbb{R}^d)$, which consists of all infinitely differentiable functions on $\mathbb{R}^d$ with compact supports, is dense $W^{2,2}(\mathbb{R}^d)$, it is also dense in $D(-\Delta)$ with respect to the graph norm. In other words we say that $C_c^\infty(\mathbb{R}^d)$ is a *core* for the Laplace operator. This result is of our fundamental interest. The subject to study in this thesis is to investigate this core property for a more general class of operators, known as *degenerate elliptic second-order differential operators in divergence form with bounded complex-valued coefficients* or *degenerate elliptic operators* for short, which includes the Laplace operator as a very special case.

Degenerate elliptic operators are extensions of *strongly elliptic operators* which in turn are the most direct generalisation of the Laplace operator. To define strongly elliptic operators, we consider the *second-order differential operator in divergence form* L of the form

$$L = -\sum_{k,l=1}^{d} \partial_l (c_{kl}\, \partial_k),$$

where $c_{kl} \in W^{1,\infty}(\mathbb{R}^d, \mathbb{C})$ for all $k, l \in \{1, \ldots, d\}$. The operator L is called *strongly elliptic* if there exists a $\mu > 0$ such that

$$\operatorname{Re}\big(C(x)\,\xi, \xi\big) \geq \mu\, \|\xi\|^2 \tag{1.2}$$

for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$, where $C(x) = (c_{kl}(x))_{1\leq k,l\leq d}$ for all $x \in \mathbb{R}^d$. Let $\operatorname{Re} C := \frac{1}{2}\,(C + C^*)$, where $C^* = \overline{C^T}$. Then (1.2) is the same as requiring there exists a $\mu > 0$ such that

$$\big((\operatorname{Re} C)(x)\,\xi, \xi\big) \geq \mu\, \|\xi\|^2$$

for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$. That is, all the eigenvalues of $(\operatorname{Re} C)(x)$ must be strictly positive, with the smallest eigenvalue at least the constant μ for all $x \in \mathbb{R}^d$. Most of the properties possessed by the Laplace operator are inherited by strongly elliptic operators, including the core property in particular. The theory of strongly elliptic operators is vast and fairly well-understood. Many important results in this area can be found in treatises on the subject (cf. [GT83], [EE87], [Eva10], [Agm10], [Neč12], etc.) as well as in the large amount of literature devoted to it.

The situation changes drastically when (1.2) is relaxed to

$$\operatorname{Re}\big(C(x)\,\xi, \xi\big) \geq 0 \tag{1.3}$$

for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$, in which case L is said to be *degenerate elliptic*. Again we can read (1.3) in the sense that

$$\big((\operatorname{Re} C)(x)\,\xi, \xi\big) \geq 0$$

for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$. As a consequence the matrix $\operatorname{Re} C$ is positive (semi-)definite when L is degenerate elliptic. But on the contrary to strongly elliptic operators, some or all of the eigenvalues of $\operatorname{Re} C$ may have zeros in this case. The theory of degenerate elliptic operators is an active area of research. Many well-known results for strongly elliptic operators remain unsolved for degenerate elliptic operators. We will shortly point out this kind of distinctive difference between the two classes of operators in terms of the core properties.

Our starting point to formulate the main problems of the thesis is *the first representation theorem*, proved independently by Lions (cf. [Lio61]) and Kato (cf. [Kat80, Theorem VI.2.1]). This theorem presents a convenient realisation of an operator in the setting of the Hilbert spaces via form methods. This approach is in contrast to the classical realisation of an operator via a detailed description of the domain of the operator. The difference lies in the fact that form methods provide more tools and structures to analyse an operator than the classical approach, at least in Hilbert spaces. To be specific let $\theta \in [0, \frac{\pi}{2})$. Define

$$\Sigma_\theta = \{r\, e^{i\,\psi} : r \geq 0 \text{ and } |\psi| \leq \theta\}. \tag{1.4}$$

We consider in $L_2(\mathbb{R}^d)$ the form

$$\mathfrak{a}_0(u,v) = \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} c_{kl}\,(\partial_k u)\,\partial_l \overline{v}$$

on the domain $D(\mathfrak{a}_0) = C_c^\infty(\mathbb{R}^d)$, where c_{kl} are *complex-valued* functions in $W^{1,\infty}(\mathbb{R}^d)$ for all $k,l \in \{1,\ldots,d\}$ which satisfy the condition

$$(C(x)\,\xi,\xi) \in \Sigma_\theta \tag{1.5}$$

for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$. Under those conditions imposed on the coefficients, the form $\mathfrak{a}_0$ can be extended to a so-called closed form $\mathfrak{a}$. The closure $\mathfrak{a}$ of the form $\mathfrak{a}_0$ satisfies all the requirements of the first representation theorem. Using the theorem we can associate an operator A with the form $\mathfrak{a}$ in such a manner that $D(A) \subset D(\mathfrak{a})$ and

$$\mathfrak{a}(u,v) = (Au,v)_{L_2(\mathbb{R}^d)}$$

for all $u \in D(A)$ and $v \in D(\mathfrak{a})$. Formally we can write A in the form

$$A = -\sum_{k,l=1}^{d} \partial_l(c_{kl}\,\partial_k).$$

Since (1.5) implies (1.3), the operator A is degenerate elliptic. Furthermore the same theorem of Lions and Kato gives that $-A$ is the generator of a C_0-semigroup S on $L_2(\mathbb{R}^d)$. Using interpolation, we will see that S can be extended consistently to a C_0-semigroup $S^{(p)}$ on $L_p(\mathbb{R}^d)$ under suitable conditions on the coefficients and for certain $p \in [1,\infty)$. Those values for p depend on the angle θ. A very special case is when c_{kl} is a real-valued function for all $k,l \in \{1,\ldots,d\}$, in which case we have $\theta = 0$ and p can take any value in $[1,\infty)$.

Let $-A_p$ be the generator of $S^{(p)}$. The domain $D(A_p)$ is naturally normed with the *graph norm* $\|\cdot\|_{D(A_p)}$ defined by $\|u\|_{D(A_p)} = \|u\|_p + \|A_p u\|_p$ for all $u \in D(A_p)$. If a subspace D of $D(A_p)$ is dense in $D(A_p)$ with respect to the graph norm, D is said to be a *core* for A_p. If we know that $D(A_p)$ is complete and D is a core for A_p, then $\overline{(D, \|\cdot\|_{D(A_P)})} = D(A_p)$. This explains the significance of a core for an operator, which lies in the fact that the domain of A_p is sometimes too large to work with. However we can still obtain information about A_p by knowing how it acts on a core. Since a core for an operator is often a small and nice subspace of the domain, this is much easier to deal with. With this notion of a core for A_p in mind, we can now state the question of our interest:

"When is $C_c^\infty(\mathbb{R}^d)$ a core for A_p?"

It has been known for a long time that the space of test functions $C_c^\infty(\mathbb{R}^d)$ is a core for A_p if the matrix of coefficients C satisfies the strong ellipticity condition (1.2) with entries belonging to $W^{1,\infty}(\mathbb{R}^d)$ (cf. [ADN59] and [ER97, Theorem 1.5]). Nevertheless if C is known to satisfy the degenerate ellipticity condition (1.3) only, the situation will be very different and usually much more difficult since $C_c^\infty(\mathbb{R}^d)$ is no longer a core in general. A common procedure to investigate the core properties in this situation is to perform approximation arguments to transit from the degenerate case back to the more familiar strongly elliptic case. Next we will describe some previous results obtained by [WD83, Theorem 1], [Ouh05, Theorem 5.2], [CMP98, Theorem 3.5] and [ERS11, Section 4]. The overall contents are as follows.

In 1983 Wong-Dzung considered in [WD83] the operator B_p of the form

$$B_p = -\sum_{k,l=1}^{d} \partial_l(c_{kl}\,\partial_k)$$

on the maximal domain

$$D(B_p) = \{u \in L_p(\mathbb{R}^d) : \text{there exists an } f \in L_p(\mathbb{R}^d) \text{ such that for all } \phi \in C_c^\infty(\mathbb{R}^d)$$
$$-\sum_{k,l=1}^{d} \int_{\mathbb{R}^d} u\,\partial_k(\overline{c_{kl}}\,\partial_l \phi) = \int_{\mathbb{R}^d} f\,\phi\},$$

where $p \in (1,\infty)$, the coefficient matrix $C = (c_{kl})_{1\le k,l\le d}$ satisfies the degenerate ellipticity condition (1.3) and its entries are *real-valued C^2-functions* which are possibly non-symmetric. It was shown that $C_c^\infty(\mathbb{R}^d)$ is a core for B_p. It can be verified that B_p is the same as the operator A_p obtained above via form methods with the same coefficients as those of B_p. In his book [Ouh05] published in 2005, Ouhabaz refined the arguments used by WongDzung in [WD83] to prove that $C_c^\infty(\mathbb{R}^d)$ is core for the operator B_2 in $L_2(\mathbb{R}^d)$ under a weaker assumptions that C still satisfies the degenerate ellipticity condition but the principal coefficients are merely *real-valued functions in $W^{2,\infty}(\mathbb{R}^d)$*.

In another direction, on the unit interval, Campiti, Metafune and Pallara gave a characterisation for when $C_c^\infty(0,1)$ is a core for the operator C_p defined by

$$C_p = -\frac{d}{dx}(c\,\frac{d}{dx}) \tag{1.6}$$

on the domain

$$D(C_p) = \{u \in L_p(0,1) : u \in W^{1,p}_{\mathrm{loc}}(0,1) \text{ and } c\,u' \in W^{1,p}_0(0,1)\},$$

where $p \in [1,\infty)$ and the *real-valued coefficient* $c \in C[0,1]$ satisfies that $c(0) = c(1) = 0$ and $c(x) > 0$ for all $x \in (0,1)$ (cf. [CMP98, Theorem 3.5]). The techniques used to prove the characterisation are intrinsically available in one dimension only. Up to now extensions of this characterisation to higher dimensions remain widely open problems.

Next we will examine the aforementioned results more closely by considering the following example.

Example 1.1. Let $p \in [1,\infty)$ and $\kappa \ge 1$. Let C_p be defined by (1.6) with coefficient $c(x) = \left(\frac{x^2\,(1-x)^2}{1+x^2\,(1-x)^2}\right)^{\frac{\kappa}{2}}$ for all $x \in [0,1]$. Then $c \in W^{\kappa,\infty}(0,1)$. The characterisation [CMP98, Theorem 3.5] gives that $C_c^\infty(0,1)$ is a core for C_p if and only if $\kappa \ge 2 - \frac{1}{p}$.

It can be shown that this example continues to hold on $L_p(\mathbb{R})$ for the same range of p when $c(x) = \left(\frac{x^2\,(1-x)^2}{1+x^2\,(1-x)^2}\right)^{\frac{\kappa}{2}}$ for all $x \in \mathbb{R}$. In this case we can also verify that C_p is in fact the same as the operator A_p obtained via form methods if the coefficient of A_p is the same as that of C_p. Hence the core characterisation also applies to A_p. So at least in one dimension, it tells us in particular that the $W^{1,\infty}(\mathbb{R})$ smoothness of the coefficient does not guarantee $C_c^\infty(\mathbb{R})$ to be a core for A_p, whereas the results of Wong-Dzung [WD83, Theorem 1] and Ouhabaz [Ouh05, Theorem 5.2] states $W^{2,\infty}(\mathbb{R})$ smoothness is sufficient. Generalising to arbitrary dimensions, we will refer to this phenomenon as *the gap in the smoothness of the coefficients* regarding the core properties for A_p. Bridging the gap, i.e. finding the optimal smoothness of the coefficients for $C_c^\infty(\mathbb{R}^d)$ to be a core for A_p, remains unsolved in higher dimensions up to now. A recent paper published in 2011 that slightly touched on this direction is that of ter Elst, Robinson and Sikora [ERS11], in which they considered a mixture of smoothness conditions between $W^{1,\infty}(\mathbb{R}^d)$ and $W^{2,\infty}(\mathbb{R}^d)$ on *real-valued coefficients* for $C_c^\infty(\mathbb{R}^d)$ to be a core for A_p.

In this thesis we will give answers to the main question posted above. We emphasise that here we consider operators with *complex-valued coefficients* in contrast to operators with real-valued coefficients in the aforementioned literature. In one dimension we will provide a characterisation for when $C_c^\infty(\mathbb{R}^d)$ is a core for the operator A_p. This characterisation is an extension of the result in [CMP98] and the pure second-order case in [DE15]. In higher dimensions we will provide many sufficient conditions for when $C_c^\infty(\mathbb{R}^d)$ is a core for A_p. The work is in the spirit of [WD83] and [Ouh05].

Apart from the interests in the core properties for degenerate elliptic second-order differential operators with bounded coefficients, there is a huge literature for sufficient conditions under which the space of test functions is still a core if the coefficients of the operator are real-valued and unbounded either locally or at infinity. The details and many interesting results can be found in [Kat81], [Dav85], [Lis89], [MPPS05], [MPRS10], [CCHL12], [MS14] and references therein.

1.2 Outline of the thesis

In this section we will summarise the content as well as the ideas used in the subsequent chapters. References are given to the original papers which we based our research on. Details about collaborative works are also mentioned.

Chapter 2: Here we collect all the background knowledge required in subsequent chapters. These include the notions of forms, accretive operators, consistent extension of semigroups, first-order and second-order differential operators. Many well-known results from the literature related to these notions are presented here for later use.

Chapter 3: In this chapter we consider the one-dimensional case. The main result in this chapter is an extension of [CMP98, Theorem 3.5] and [DE15, Theorem 1.5] which is on its turn an extension of [CMP98, Theorem 3.5]. Let $\theta \in [0, \frac{\pi}{2})$. Let Σ_θ be defined as in (1.4). Let $c \in W^{1,\infty}(\mathbb{R})$ be a complex-valued function such that $c(x) \in \Sigma_\theta$ for all $x \in \mathbb{R}$. Let $m \in W^{1,\infty}(\mathbb{R})$ and $w \in L_\infty(\mathbb{R})$. Suppose $|m| \leq M \sqrt{\operatorname{Re} c}$ for some $M > 0$. We start with the form $\mathfrak{a}_0$ defined by

$$\mathfrak{a}_0(u, v) = \int_{\mathbb{R}} c\, u'\, \overline{v'} + m\, u'\, \overline{v} + w\, u\, \overline{v}$$

on the domain $D(\mathfrak{a}_0) = C_c^\infty(\mathbb{R})$. Since $\mathfrak{a}_0$ is closable, we let A be the operator associated with the closure of the form $\mathfrak{a}_0$. Then $-A$ generates a quasi-contraction C_0-semigroup S

on $L_2(\mathbb{R})$. For certain $p \in [1,\infty)$, we can extend S consistently to a quasi-contraction C_0-semigroup $S^{(p)}$ on $L_p(\mathbb{R})$. Let $-A_p$ be the generator of $S^{(p)}$. We will give a core characterisation for the operator A_p in terms of the degeneracy of the coefficient c at its zero points.

The pure second-order case, i.e. if $m = w = 0$, is joint work with Tom ter Elst (cf. [DE15]).

Chapter 4: The chapter is motivated by [WD83] and [Ouh05]. Let $d \in \mathbb{N}$ and $\theta \in [0, \frac{\pi}{2})$. Let $c_{kl} \in W^{2,\infty}(\mathbb{R}^d)$ be complex-valued functions for all $k, l \in \{1, \ldots, d\}$. Suppose $(C(x)\,\xi, \xi) \in \Sigma_\theta$ for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$, where $C(x) = (c_{kl}(x))_{1 \le k,l \le d}$ for all $x \in \mathbb{R}^d$ and Σ_θ is defined by (1.4). We start with the form $\mathfrak{a}_0$ defined by

$$\mathfrak{a}_0(u,v) = \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} c_{kl}\,(\partial_k u)\,\partial_l \overline{v}$$

on the domain $D(\mathfrak{a}_0) = C_c^\infty(\mathbb{R}^d)$. Since $\mathfrak{a}_0$ is closable, we let A be the operator associated with the closure $\mathfrak{a}$ of the form $\mathfrak{a}_0$. Then $-A$ generates a quasi-contraction C_0-semigroup S on $L_2(\mathbb{R})$.

Let $C = R + i\,B$, where R and B are real matrices. Suppose B is symmetric. Then we show that it is possible to extend S consistently to a quasi-contraction C_0-semigroup $S^{(p)}$ on $L_p(\mathbb{R})$ for certain $p \in (1,\infty)$. Let $-A_p$ be the generator of $S^{(p)}$. We will show that $C_c^\infty(\mathbb{R}^d)$ is a core for A_p.

The case $p = 2$ is special as we naturally obtain the operator A from the form $\mathfrak{a}$. Consequently the assumption that B is symmetric is not needed. Suppose B is not symmetric. Then we provide many sufficient conditions for when $C_c^\infty(\mathbb{R}^d)$ is a core for A. In particular if $D(A) \subset W^{1,2}(\mathbb{R}^d)$, then $C_c^\infty(\mathbb{R}^d)$ is always a core for A regardless of the symmetry of B.

Chapter 2

Preliminaries

This chapter summarises all the background knowledge required in the subsequent chapters. Forms, accretive operators and semigroups are defined. We will make clear the relations between these notions. Many well-known results will be discussed. The last two sections present applications of previous sections to first-order and second-order differential operators.

2.1 Forms and operators associated with forms

In this section we introduce the powerful tools of form methods. We will consider forms and their associated operators. The central result is the first representation theorem given in Theorem 2.12.

Let H be a Hilbert space. We emphasise that the underlying field is $\mathbb{C}$.

Definition 2.1. Let D be a subspace of H. A function $\mathfrak{a} \colon D \times D \longrightarrow \mathbb{C}$ which is linear in the first variable and anti-linear in the second variable is called a *sesquilinear form*. We also refer to $\mathfrak{a}$ as a *form* for short. The subspace D is called the *domain* of $\mathfrak{a}$. To be specific we also write $D = D(\mathfrak{a})$.

For the rest of the section let $\mathfrak{a}$ be a form with domain $D(\mathfrak{a}) \subset H$. For convenience we will write $\mathfrak{a}(u) = \mathfrak{a}(u, u)$ for all $u \in D(\mathfrak{a})$. It is worth noting that the set of values $\{\mathfrak{a}(u) : u \in D(\mathfrak{a})\}$ determines the form $\mathfrak{a}$ uniquely, thanks to the following proposition.

Proposition 2.2 (Polarisation identity)**.** *The following identity*

$$\mathfrak{a}(u, v) = \frac{1}{4}\big(\mathfrak{a}(u + v) - \mathfrak{a}(u - v) + i\,\mathfrak{a}(u + iv) - i\,\mathfrak{a}(u - iv)\big)$$

holds for all $u, v \in D(\mathfrak{a})$.

For all $\theta \in [0, \frac{\pi}{2})$ we define $\Sigma_\theta = \{re^{i\psi} : r \geq 0 \text{ and } |\psi| \leq \theta\}$, which is a *sector* in the complex plane with *semi-angle* θ. We have the following definitions.

Definition 2.3. The form $\mathfrak{a}$ is called

1. *densely defined* if $D(\mathfrak{a})$ is dense in H.

2. *accretive* if $\operatorname{Re}\mathfrak{a}(u) \geq 0$ for all $u \in D(\mathfrak{a})$. If there exists an $\omega \in \mathbb{R}$ such that $\operatorname{Re}\mathfrak{a}(u) + \omega \|u\|_H^2 \geq 0$ for all $u \in D(\mathfrak{a})$, then $\mathfrak{a}$ is said to be *quasi-accretive*.

3. *sectorial* if there exists a $\theta \in [0, \frac{\pi}{2})$ such that $\mathfrak{a}(u) \in \Sigma_\theta$ for all $u \in D(\mathfrak{a})$.

Definition 2.4. Suppose $\mathfrak{a}$ is sectorial. Then the domain $D(\mathfrak{a})$ is a normed space with norm $\|\cdot\|_{\mathfrak{a}}$ defined by

$$\|u\|_{\mathfrak{a}} = (\operatorname{Re}\mathfrak{a}(u) + \|u\|_H^2)^{1/2}$$

for all $u \in D(\mathfrak{a})$. The form $\mathfrak{a}$ is called *closed* if $(D(\mathfrak{a}), \|\cdot\|_{\mathfrak{a}})$ is complete. A form $\mathfrak{b}$ which has the properties that $D(\mathfrak{a}) \subset D(\mathfrak{b})$ and $\mathfrak{b}(u) = \mathfrak{a}(u)$ for all $u \in D(\mathfrak{a})$ is called an *extension* of the form $\mathfrak{a}$. If there exists a closed form $\mathfrak{b}$ which is an extension of $\mathfrak{a}$, then $\mathfrak{a}$ is called *closable*. In this case we also say that $\mathfrak{b}$ is a *closed extension* of $\mathfrak{a}$.

Note that a closed extension of a form is in general not unique if it exists. If $\mathfrak{a}$ is closable, we will denote by $\bar{\mathfrak{a}}$ the smallest closed extension of $\mathfrak{a}$.

Define the form $\mathfrak{a}^*\colon D(\mathfrak{a}) \times D(\mathfrak{a}) \longrightarrow \mathbb{C}$ by $\mathfrak{a}^*(u, v) = \overline{\mathfrak{a}(v, u)}$. The form $\mathfrak{a}^*$ is called the *adjoint form* of $\mathfrak{a}$. If $\mathfrak{a} = \mathfrak{a}^*$, the form $\mathfrak{a}$ is called *symmetric*. Denote $\Re\mathfrak{a} = \frac{1}{2}(\mathfrak{a} + \mathfrak{a}^*)$ and $\Im\mathfrak{a} = \frac{1}{2i}(\mathfrak{a} - \mathfrak{a}^*)$. Then $\mathfrak{a} = \Re\mathfrak{a} + i\,\Im\mathfrak{a}$. We will refer to $\Re\mathfrak{a}$ as the *real part* of $\mathfrak{a}$ and $\Im\mathfrak{a}$ as the *imaginary part* of $\mathfrak{a}$. Note that both $\Re\mathfrak{a}$ and $\Im\mathfrak{a}$ are symmetric forms. Based on these notions we can now describe some typical properties of sectorial forms.

Proposition 2.5 ([Kat80, Subsection VI.1.2])**.** *Suppose the form $\mathfrak{a}$ is sectorial. Let $u, v \in D(\mathfrak{a})$. Then the following hold.*

1. $|(\Im\mathfrak{a})(u)| \leq \tan\theta\,(\Re\mathfrak{a})(u)$.

2. $|(\Re\mathfrak{a})(u, v)| \leq \big((\Re\mathfrak{a})(u)\big)^{1/2}\big((\Re\mathfrak{a})(v)\big)^{1/2}$.

3. $|(\Im\mathfrak{a})(u, v)| \leq \tan\theta\,\big((\Re\mathfrak{a})(u)\big)^{1/2}\big((\Re\mathfrak{a})(v)\big)^{1/2}$.

4. $|\mathfrak{a}(u, v)| \leq (1 + \tan\theta)\,\big((\Re\mathfrak{a})(u)\big)^{1/2}\big((\Re\mathfrak{a})(v)\big)^{1/2}$.

More generally we can compare two forms with each other.

Theorem 2.6. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be forms in H. Suppose that $\mathfrak{a}$ is symmetric and $D(\mathfrak{a}) = D(\mathfrak{b}) = V$ for some $V \subset H$. Suppose there exists an $M > 0$ such that $|\mathfrak{b}(u)| \leq M\,\mathfrak{a}(u)$ for all $u \in V$. Then the following hold.*

(a) *If $\mathfrak{b}$ is symmetric, then*

$$|\mathfrak{b}(u, v)| \leq M\,\mathfrak{a}(u)^{1/2}\,\mathfrak{a}(v)^{1/2}$$

for all $u, v \in V$.

(b) *If $\mathfrak{b}$ is not symmetric, then*

$$|\mathfrak{b}(u, v)| \leq 2M\,\mathfrak{a}(u)^{1/2}\,\mathfrak{a}(v)^{1/2}$$

for all $u, v \in V$.

Proof. (a) Let $u, v \in V$. By replacing v with $e^{i\psi} v$ for some $\psi \in [0, 2\pi)$, we can assume without loss of generality that $\mathfrak{b}(u,v) \in \mathbb{R}$. Then

$$|\mathfrak{b}(u,v)| = \frac{1}{4}\left|\mathfrak{b}(u+v) - \mathfrak{b}(u-v)\right| \leq \frac{M}{4}\left(\mathfrak{a}(u+v) + \mathfrak{a}(u-v)\right) = \frac{M}{2}\left(\mathfrak{a}(u) + \mathfrak{a}(v)\right), \quad (2.1)$$

where we used Proposition 2.2 in the last step. We consider two cases.

Case 1: Suppose $\mathfrak{a}(u) = 0$.
Then (2.1) gives

$$|\mathfrak{b}(u,v)| \leq \frac{M}{2}\,\mathfrak{a}(v).$$

Replacing v by λv with $\lambda > 0$ gives

$$|\mathfrak{b}(u,v)| \leq \frac{M\lambda}{2}\,\mathfrak{a}(v).$$

Since λ is arbitrary, taking the limit when $\lambda \downarrow 0$ on both sides of the above inequality yields $\mathfrak{b}(u,v) = 0$, which implies the claim.

Case 2: Suppose $\mathfrak{a}(u) \neq 0$.
Replacing u by $\frac{\sqrt{\mathfrak{a}(v)}}{\sqrt{\mathfrak{a}(u)}}\, u$ in (2.1) we obtain the desired conclusion.

(b) Note that $\mathfrak{b} = \Re\mathfrak{b} + i\,\Im\mathfrak{b}$, where $\Re\mathfrak{b}$ and $\Im\mathfrak{b}$ are symmetric. Note that

$$|(\Re\mathfrak{b})(u)| = \Big|\frac{\mathfrak{b}(u) + \mathfrak{b}^*(u)}{2}\Big| \leq |\mathfrak{b}(u)| \leq M\,\mathfrak{a}(u)$$

for all $u \in V$. It follows from (a) that $|(\Re\mathfrak{b})(u,v)| \leq M\,\mathfrak{a}(u)^{1/2}\,\mathfrak{a}(v)^{1/2}$ for all $u, v \in V$. Similarly $|(\Im\mathfrak{b})(u,v)| \leq M\,\mathfrak{a}(u)^{1/2}\,\mathfrak{a}(v)^{1/2}$ for all $u, v \in V$. Hence

$$|\mathfrak{b}(u,v)| \leq |(\Re\mathfrak{b})(u,v)| + |(\Im\mathfrak{b})(u,v)| \leq 2\,M\,\mathfrak{a}(u)^{1/2}\,\mathfrak{a}(v)^{1/2}$$

for all $u, v \in V$. □

Let A be an operator in H, that is, $A\colon D(A) \subset H \longrightarrow H$. The domain $D(A)$ of A is naturally normed with the *graph norm* $\|\cdot\|_{D(A)}$ defined by

$$\|u\|_{D(A)} = \|u\|_H + \|Au\|_H$$

for all $u \in D(A)$. Define

$$\Theta(A) = \{(Au, u)\colon u \in D(A) \text{ and } \|u\|_H = 1\},$$

which is called the *numerical range* of A. We will denote by $\rho(A)$ the resolvent set of A. We have the following definitions.

Definition 2.7. The operator A is called

1. *densely defined* if $D(A)$ is dense in H.
2. *closed* if $(D(A), \|\cdot\|_{D(A)})$ is complete. If there exists a closed operator B in H such that $D(A) \subset D(B)$ and $Bu = Au$ for all $u \in D(A)$, then A is called *closable*. In this case we also say that B is a *closed extension* of A.

3. *accretive* if $\operatorname{Re}(Au, u) \geq 0$ for all $u \in D(A)$. If A is accretive and $1 \in \rho(-A)$, then A is said to be *m-accretive*. If there exists an $\omega \in \mathbb{R}$ such that $\operatorname{Re}(Au, u) + \omega \|u\|_H^2 \geq 0$ for all $u \in D(A)$, then A is called *quasi-accretive*. If there exists an $\omega \in \mathbb{R}$ such that $\operatorname{Re}(Au, u) + \omega \|u\|_H^2 \geq 0$ for all $u \in D(A)$ and $\omega + 1 \in \rho(-A)$, then A is said to be *quasi-m-accretive*.

4. *sectorial* if $\Theta(A) \subset \Sigma_\theta$ for some $\theta \in [0, \frac{\pi}{2})$. If A is sectorial and m-accretive, then A is said to be *m-sectorial*.

Note that a closed extension of an operator is in general not unique if it exists. If A is closable, we will denote by $\overline{A}$ the smallest closed extension of A. Closable operators and accretive operators are related to each other in the following manner.

Proposition 2.8 ([Ouh05, Lemma 1.47]). *Suppose A is densely defined and accretive. Then A is closable.*

The following definition is of our main interest.

Definition 2.9. A subspace $D \subset D(A)$ is called a *core* for A if D is dense in $D(A)$ with respect to the graph norm.

The next theorem provides a useful criterion for proving the core properties for accretive operators. It will be used extensively in Chapter 4.

Theorem 2.10 ([Ouh05, Theorem 1.50]). *Let A be accretive. Assume that S is an m-accretive operator satisfying the following two conditions.*

1. *$D(S) \subset D(A)$.*

2. *There exists a constant $\beta \in \mathbb{R}$ such that* $\operatorname{Re}(Au + \beta\, u, Su) \geq 0$ *for all $u \in D(S)$.*

Then the closure $\overline{A}$ is m-accretive. Furthermore $D(S)$ is a core for $\overline{A}$.

Next we will describe some correspondences between forms and operators. We start first with forms constructed from sectorial operators.

Theorem 2.11 ([Kat80, Theorem VI.1.27]). *Suppose A is sectorial and* $\mathfrak{a}(u, v) = (Au, v)$ *with $D(\mathfrak{a}) = D(A)$. Then $\mathfrak{a}$ is closable.*

Conversely we have the following well-known result.

Theorem 2.12 (The first representation, [Kat80, Theorem VI.2.1]). *Suppose $\mathfrak{a}$ is a densely defined, closed and sectorial form. Let A be defined in the following manner. Let $u, w \in H$. We say that $u \in D(A)$ and $Au = w$ if $u \in D(\mathfrak{a})$ and $\mathfrak{a}(u, v) = (w, v)_H$ for all $v \in D(\mathfrak{a})$. Then A is m-sectorial.*

The operator A in Theorem 2.12 is called the *operator associated with the form* $\mathfrak{a}$. If the form $\mathfrak{a}$ is not closable, we can still associate with it an m-sectorial operator. This is the content of the following theorem.

Theorem 2.13 ([AE12, Theorem 3.2]). *Suppose $\mathfrak{a}$ is a densely defined and sectorial form. Let A be defined in the following manner. Let $u, w \in H$. We say that $u \in D(A)$ and $Au = w$ if there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $D(\mathfrak{a})$ such that $\sup_{n\in\mathbb{N}} \operatorname{Re} \mathfrak{a}(u_n) < \infty$, $\lim_{n\to\infty} u_n = u$ in H and $\lim_{n\to\infty} \mathfrak{a}(u_n, v) = (w, v)_H$ for all $v \in D(\mathfrak{a})$. Then A is m-sectorial.*

It is a remarkable fact that m-sectorial operators are generators of holomorphic semigroups. For background information on semigroups we refer to [EN00], [Nag86], [Paz83], [Gol85] and [Yos80].

Theorem 2.14 ([Kat80, Theorem IX.1.24]). *Let $\theta \in [0, \frac{\pi}{2})$. Suppose A is an m-sectorial operator with $\Theta(A) \subset \Sigma_\theta$. Then $-A$ generates a contraction holomorphic semigroup with angle $\frac{\pi}{2} - \theta$.*

2.2 Accretive operators on Banach spaces

In Section 2.1 we discussed accretive operators on Hilbert spaces. In this section we will extend the notion of accretivity of operators to Banach spaces. We are particularly interested in accretive operators in L_p-spaces.

Let X be a Banach space. Let $(A, D(A))$ be an operator in X. We view the domain $D(A)$ as a normed space with the *graph norm* $\|\cdot\|_{D(A)}$ defined by

$$\|u\|_{D(A)} = \|u\|_X + \|Au\|_X$$

for all $u \in D(A)$. The notion of a core for an operator, which is the definition of our main interest, extends verbatim to this more general context. For the sake of clarity and completeness, we also repeat it here.

Definition 2.15. A subspace $D \subset D(A)$ is called a *core* for A if D is dense in $D(A)$ with respect to the graph norm.

Next we will consider accretive operators. We denote by X^* the dual space of X. Define

$$F(u) = \{f \in X^* : (u, f) = \|u\|_X^2 = \|f\|_{X^*}^2\}$$

for all $u \in X$. By the Hahn-Banach theorem the set $F(u)$ is non-empty for all $u \in X$. A particular case is when $X = L_p(\mathbb{R}^d)$ with $d \in \mathbb{N}$, in which case we have $F(u) = \{\|u\|_p^{2-p}\, |u|^{p-2}\, u\, \mathbb{1}_{[u\neq 0]}\}$ for all $u \in X \setminus \{0\}$. We have the following definition.

Definition 2.16. The operator A is called

1. *densely defined* if $D(A)$ is dense in X.

2. *closed* if $(D(A), \|\cdot\|_{D(A)})$ is complete. If there exists a closed operator $(B, D(B))$ in X such that $D(A) \subset D(B)$ and $Bu = Au$ for all $u \in D(A)$, then A is called *closable*. In this case we also say that B is a *closed extension* of A.

3. *accretive* if for every $u \in D(A)$ there exists an $f \in F(u)$ such that $\operatorname{Re}(Au, f) \geq 0$. If A is accretive and $1 \in \rho(-A)$, then A is called *m-accretive*. If there exists an

$\omega \in \mathbb{R}$ such that for every $u \in D(A)$ there exists an $f \in F(u)$ which satisfies $\mathrm{Re}\,(\omega\, u + Au, f) \geq 0$, then A is called *quasi-accretive*. If there exists an $\omega \in \mathbb{R}$ such that $\omega + 1 \in \rho(-A)$ and for every $u \in D(A)$ there exists an $f \in F(u)$ which satisfies $\mathrm{Re}\,(\omega\, u + Au, f) \geq 0$, then A is said to be *quasi-m-accretive*.

It is easy to verify that this definition coincides with Definition 2.7 if X is a Hilbert space. The importance of m-accretive operators is due to the following well-known theorem.

Theorem 2.17 (Lumer-Phillips theorem, [LP61])**.** *Let A be a densely defined operator in X. Then A is m-accretive if and only if A is the generator of a contraction C_0-semigroup.*

When X is reflexive, the density requirement of the operator's domain in Theorem 2.17 can be removed, as stated by the following theorem.

Theorem 2.18 ([Kat59, Corollary 2], [Kat80, Theorem III.5.29], [EN00, Corollary II.3.20])**.** *Suppose X is reflexive. Let S be an m-accretive operator in X. Then S is densely defined. Moreover, S^* is also densely defined.*

2.3 Consistent semigroups

In this section we will explore the relations between consistent C_0-semigroups on L_p-spaces.

Let $d \in \mathbb{N}$. Let $p_1, p_2 \in [1, \infty)$. Let $S^{(p_1)}$, $S^{(p_2)}$ be C_0-semigroups on $L_{p_1}(\mathbb{R}^d)$ and $L_{p_2}(\mathbb{R}^d)$ respectively.

Definition 2.19. We say that $S^{(p_1)}$ and $S^{(p_2)}$ are *consistent* if $S^{(p_1)}_t u = S^{(p_2)}_t u$ for all $t > 0$ and $u \in L_{p_1}(\mathbb{R}^d) \cap L_{p_2}(\mathbb{R}^d)$.

Theorem 2.20. *Suppose $S^{(p_1)}$ and $S^{(p_2)}$ are bounded C_0-semigroups which are consistent. Let $-A_{p_1}$ and $-A_{p_2}$ be the generators of $S^{(p_1)}$ and $S^{(p_2)}$ respectively. Then $A_{p_1} u = A_{p_2} u$ for all $u \in D(A_{p_1}) \cap D(A_{p_2})$ and*

$$\begin{aligned} D(A_{p_1}) \cap D(A_{p_2}) &= \{u \in D(A_{p_1}) \cap L_{p_2}(\mathbb{R}^d) : A_{p_1} u \in L_{p_2}(\mathbb{R}^d)\} \\ &= \{u \in D(A_{p_2}) \cap L_{p_1}(\mathbb{R}^d) : A_{p_2} u \in L_{p_1}(\mathbb{R}^d)\} \\ &= (I + A_{p_1})^{-1}(L_{p_1}(\mathbb{R}^d) \cap L_{p_2}(\mathbb{R}^d)) \\ &= (I + A_{p_2})^{-1}(L_{p_1}(\mathbb{R}^d) \cap L_{p_2}(\mathbb{R}^d)). \end{aligned} \tag{2.2}$$

Moreover, $D(A_{p_1}) \cap D(A_{p_2})$ is a core for A_{p_1} in $L_{p_1}(\mathbb{R}^d)$ and for A_{p_2} in $L_{p_2}(\mathbb{R}^d)$.

Proof. First let $u \in D(A_{p_1}) \cap D(A_{p_2})$ and $\phi \in L_1(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$. Then

$$(A_{p_1} u, \phi) = \lim_{t \downarrow 0} \frac{1}{t} \big((I - S^{(p_1)}_t) u, \phi\big) = \lim_{t \downarrow 0} \frac{1}{t} \big((I - S^{(p_2)}_t) u, \phi\big) = (A_{p_2} u, \phi).$$

It follows that $A_{p_1} u = A_{p_2} u \in L_{p_1}(\mathbb{R}^d) \cap L_{p_2}(\mathbb{R}^d)$. In particular this implies $D(A_{p_1}) \cap D(A_{p_2}) \subset \{u \in D(A_{p_1}) \cap L_{p_2}(\mathbb{R}^d) : A_{p_1} u \in L_{p_2}(\mathbb{R}^d)\}$.

Conversely let $u \in D(A_{p_1}) \cap L_{p_2}(\mathbb{R}^d)$ be such that $A_{p_1} u \in L_{p_2}(\mathbb{R}^d)$. Then

$$\frac{1}{t} (I - S^{(p_2)}_t) u = \frac{1}{t} (I - S^{(p_1)}_t) u = \frac{1}{t} \int_0^t S^{(p_1)}_s A_{p_1} u \, ds = \frac{1}{t} \int_0^t S^{(p_2)}_s A_{p_1} u \, ds$$

for all $t > 0$. Therefore $\lim_{t\downarrow 0} \frac{1}{t}(I - S_t^{(p_2)})u = A_{p_1}u$ in $L_{p_2}(\mathbb{R}^d)$. It follows that $u \in D(A_{p_2})$. This proves the first two equalities in (2.2).

Secondly, if $u \in D(A_{p_1}) \cap D(A_{p_2})$ then $(I + A_{p_1})u = (I + A_{p_2})u \in L_{p_1}(\mathbb{R}^d) \cap L_{p_2}(\mathbb{R}^d)$. Consequently $D(A_{p_1}) \cap D(A_{p_2}) \subset (I + A_{p_1})^{-1}\big(L_{p_1}(\mathbb{R}^d) \cap L_{p_2}(\mathbb{R}^d)\big)$.

Conversely if $u \in L_{p_1}(\mathbb{R}^d) \cap L_{p_2}(\mathbb{R}^d)$ then

$$\big((I + A_{p_1})^{-1}u, \phi\big) = \int_0^\infty e^{-t}\,(S_t^{(p_1)}u, \phi)\,dt = \int_0^\infty e^{-t}\,(S_t^{(p_2)}u, \phi)\,dt = \big((I + A_{p_2})^{-1}u, \phi\big)$$

for all $\phi \in L_1(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$. Hence $(I + A_{p_1})^{-1}u = (I + A_{p_1})^{-1}u \in D(A_{p_1}) \cap D(A_{p_2})$. The last two equalities in (2.2) now follows.

Lastly we note that $S^{(p_1)}$ leaves $D(A_{p_1})$ invariant and $S^{(p_2)}$ leaves $D(A_{p_2})$ invariant. As a consequence $S_t^{(p_1)}\big(D(A_{p_1}) \cap D(A_{p_2})\big) \subset D(A_{p_1}) \cap D(A_{p_2})$ for all $t > 0$. But $D(A_{p_1}) \cap D(A_{p_2}) = (I + A_{p_1})^{-1}\big(L_{p_1}(\mathbb{R}^d) \cap L_{p_2}(\mathbb{R}^d)\big)$ is dense in $L_{p_1}(\mathbb{R}^d)$. Hence $D(A_{p_1}) \cap D(A_{p_2})$ is a core for A_{p_1} in $L_{p_1}(\mathbb{R}^d)$ by [EN00, Proposition 1.7]. The statement for A_{p_2} is proved similarly. □

Lemma 2.21. *Suppose $S^{(p_1)}$ and $S^{(p_2)}$ are consistent. Suppose further that there exists an $M > 0$ and $\omega \in \mathbb{R}$ such that*

$$\|S_t^{(p_1)}u\|_\infty \le M\,e^{\omega t}\,\|u\|_\infty \tag{2.3}$$

for all $t > 0$ and $u \in L_1(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$. Then $D(A_{p_1}) \cap D(A_{p_2}) \cap L_\infty(\mathbb{R}^d)$ is a core for A_{p_1} in $L_{p_1}(\mathbb{R}^d)$.

Proof. Without loss of generality we assume that $S^{(p_1)}$ and $S^{(p_2)}$ are bounded and $\omega = 0$. By hypothesis $(I + A_{p_1})^{-1} : L_{p_1}(\mathbb{R}^d) \longrightarrow D(A_{p_1})$ is bounded and bijective. Since $L_1(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$ is dense in $L_{p_1}(\mathbb{R}^d)$, we have $(I + A_{p_1})^{-1}\big(L_1(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)\big)$ is dense in $D(A_{p_1})$. We will show that

$$(I + A_{p_1})^{-1}\big(L_1(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)\big) \subset D(A_{p_1}) \cap D(A_{p_2}) \cap L_\infty(\mathbb{R}^d), \tag{2.4}$$

from which the claim follows.

Indeed we have $L_1(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d) \subset L_{p_1}(\mathbb{R}^d) \cap L_{p_2}(\mathbb{R}^d)$. Therefore

$$(I + A_{p_1})^{-1}\big(L_1(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)\big) \subset (I + A_{p_1})^{-1}\big(L_{p_1}(\mathbb{R}^d) \cap L_{p_2}(\mathbb{R}^d)\big) = D(A_{p_1}) \cap D(A_{p_2}),$$

where the last equality follows from (2.2).

Next let $u \in L_1(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$. Then

$$\begin{aligned} \|(I + A_{p_1})^{-1}u\|_\infty &= \Big\|\int_0^\infty e^{-t}\,S_t^{(p_1)}u\,dt\Big\|_\infty \le \int_0^\infty e^{-t}\,\|S_t^{(p_1)}u\|_\infty\,dt \\ &\le M\,\|u\|_\infty \int_0^\infty e^{-t}\,dt = M\,\|u\|_\infty, \end{aligned}$$

where we used (2.3) in the third step. It follows that $(I + A_{p_1})^{-1}\big(L_1(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)\big) \subset L_\infty(\mathbb{R}^d)$.

Hence (2.4) holds. □

2.4 First-order differential operators in L_p-spaces

In this section we present some properties of first-order differential operators with Lipschitz coefficients in L_p-spaces.

Let $d \in \mathbb{N}$. Let $b_l \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R})$ for all $l \in \{1, \ldots, d\}$. For all $p \in (1, \infty)$ consider the first-order differential operator $Y_{p,0}$ of the form

$$Y_{p,0} u = \sum_{l=1}^{d} \partial_l (b_l \, u)$$

on the domain

$$D(Y_{p,0}) = W^{1,p}(\mathbb{R}^d).$$

It is clear that $Y_{p,0} \subset Z^*$ for all $p \in (1, \infty)$, where Z is the operator in $L_q(\mathbb{R}^d)$ defined by

$$Zu = - \sum_{l=1}^{d} \overline{b_l} \, \partial_l u$$

on the domain $D(Z) = C_c^\infty(\mathbb{R}^d)$ and q is the dual exponent of p. Hence $Y_{p,0}$ is closable for all $p \in (1, \infty)$. Let Y_p be the closure of $Y_{p,0}$ in $L_p(\mathbb{R}^d)$ for all $p \in (1, \infty)$.

Theorem 2.22 ([Rob91, Theorem V.4.1])**.** *Let $p \in (1, \infty)$. Then the operator Y_p generates a quasi-contraction C_0-semigroup $S^{(p)}$ on $L_p(\mathbb{R}^d)$. Moreover, $S^{(p_1)}$ and $S^{(p_2)}$ are consistent for all $p_1, p_2 \in (1, \infty)$.*

Let $p \in (1, \infty)$. We have the following core property for Y_p.

Proposition 2.23. *Let $p \in (1, \infty)$. Then the space $C_c^\infty(\mathbb{R}^d)$ is a core for Y_p.*

Proof. Since $W^{1,p}(\mathbb{R}^d) \subset D(Y_p)$, there exists an $M > 0$ such that

$$\|u\|_{D(Y_p)} \leq M \, \|u\|_{W^{1,p}} \tag{2.5}$$

for all $u \in W^{1,p}(\mathbb{R}^d)$.

Let $u \in D(Y_p)$. Let $\varepsilon > 0$. Since $Y_p = \overline{Y_{p,0}}$, there exists a $v \in W^{1,p}(\mathbb{R}^d)$ such that $\|u - v\|_{D(Y_p)} < \frac{\varepsilon}{2}$. Also the space $C_c^\infty(\mathbb{R}^d)$ is dense in $W^{1,p}(\mathbb{R}^d)$. Therefore there exists a $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $\|v - \phi\|_{W^{1,p}} \leq \frac{\varepsilon}{2M}$. It follows that

$$\|u - \phi\|_{D(Y_p)} \leq \|u - v\|_{D(Y_p)} + \|v - \phi\|_{D(Y_p)} \leq \frac{\varepsilon}{2} + M \, \|v - \phi\|_{W^{1,p}} \leq \frac{\varepsilon}{2} + M \, \frac{\varepsilon}{2M} = \varepsilon,$$

where we used (2.5) in the second step. This justifies the claim. □

For each $n \in \mathbb{N}$ let J_n be the usual mollifier with respect to a suitable function in $C_c^\infty(\mathbb{R}^d)$. It is useful that functions in $D(Y_p)$ can be approximated by smooth functions which are formed by using the mollifiers. This is the content of the next proposition.

Proposition 2.24 ([ERS11, Proposition 2.1])**.** *Let $p \in (1, \infty)$. Then for all $u \in D(Y_p)$ we have*

$$\lim_{n \to \infty} Y_p(J_n * u) = Y_p u$$

in $L_p(\mathbb{R}^d)$.

2.5 Second-order differential operators in L_p-spaces

This section deals with second-order differential operators in divergence form in L_p-spaces. In particular we are mostly interested in those operators which are strongly elliptic. We will consider accretive properties, elliptic regularity and semigroup extensions of these operators.

Let $d \in \mathbb{N}$. Let $c_{kl} \in W^{1,\infty}(\mathbb{R}^d)$ and $m_k, w \in L_\infty(\mathbb{R}^d)$ for all $k, l \in \{1, \ldots, d\}$. In what follows we denote $C(x) = (c_{kl}(x))_{1 \le k,l \le d}$ for all $x \in \mathbb{R}^d$.

Definition 2.25. Let $p \in [1, \infty]$. An operator L with a suitable domain in $L_p(\mathbb{R}^d)$ of the form

$$Lu = -\sum_{k,l=1}^{d} \partial_l (c_{kl}\, \partial_k u) + \sum_{k=1}^{d} m_k\, \partial_k u + w\, u$$

is called *strongly elliptic* if there exists a $\mu > 0$ such that

$$\operatorname{Re}\big(C(x)\, \xi, \xi\big) \ge \mu\, \|\xi\|^2 \tag{2.6}$$

for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$. If (2.6) is replaced by

$$\operatorname{Re}\big(C(x)\, \xi, \xi\big) \ge 0$$

for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$, then we say that L is *degenerate elliptic*.

Suppose C satisfies the strong ellipticity condition (2.6). Consider the form

$$\mathfrak{a}(u, v) = \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} c_{kl}\, (\partial_k u)\, \partial_l \overline{v} + \sum_{k=1}^{d} \int_{\mathbb{R}^d} m_k\, (\partial_k u)\, \overline{v} + \int_{\mathbb{R}^d} w\, u\, \overline{v}$$

on the domain $D(\mathfrak{a}) = W^{1,2}(\mathbb{R}^d)$. Then $\mathfrak{a}$ is closed. Since the coefficient matrix C satisfies the strong ellipticity condition, $\mathfrak{a}$ is also quasi-sectorial. Using the first representation theorem, Theorem 2.12, we can associate with the form $\mathfrak{a}$ an quasi-m-sectorial operator A. Note that $W^{2,2}(\mathbb{R}^d) \subset D(A)$ and

$$Au = -\sum_{k,l=1}^{d} \partial_l (c_{kl}\, \partial_k u) + \sum_{k=1}^{d} m_k\, \partial_k u + w\, u$$

for all $u \in W^{2,2}(\mathbb{R}^d)$. Let S be the holomorphic quasi-contraction semigroup generated by $-A$. The following theorem is a consequence of [Aus96, Theorem 4.8].

Theorem 2.26. *For each $t > 0$ let $K_t \in \mathcal{D}'(\mathbb{R}^{2d})$ be the distributional kernel of S_t. Then*

1. *The kernel K_t is a Hölder continuous function for all $t > 0$.*

2. *There exist constants $c, \omega, \kappa, \beta > 0$ such that*

$$|K_t(x, y)| \le \frac{c}{t^{d/2}}\, e^{-\frac{\beta\, |x-y|^2}{t}}\, e^{\omega t},$$

for all $t > 0$ and $x, y \in \mathbb{R}^d$ and

$$|K_t(x,y) - K_t(x+h,y)| + |K_t(x,y) - K_t(x,y+h)| \le \frac{c}{t^{d/2}} \Big(\frac{|h|}{t^{1/2} + |x-y|} \Big)^{\kappa} e^{-\frac{\beta\,|x-y|^2}{t}} e^{\omega t},$$

for all $t > 0$ and $x, y, h \in \mathbb{R}^d$ with $2\,|h| \le t^{1/2} + |x-y|$.

The next theorem guarantees that S can be extended consistently to holomorphic quasi-contraction semigroups on L_p-spaces.

Theorem 2.27 ([Ouh05, Theorem 6.16])**.** *Let T be a bounded holomorphic semigroup on a sector Σ_θ on $L_2(\mathbb{R}^d)$, where $\theta \in [0, \frac{\pi}{2})$. For all $z \in \Sigma_\theta$ let $K_z \in \mathcal{D}'(\mathbb{R}^{2d})$ be the distributional kernel of T_z. Suppose there exist $c, \beta > 0$ such that*

$$|K_t(x,y)| \le c\, t^{-d/2}\, e^{-\frac{\beta\,|x-y|^2}{t}}$$

for all $t > 0$ and for a.e. $(x,y) \in \mathbb{R}^{2d}$. Then for every $\psi \in [0,\theta)$ there exist $c_\psi, \beta_\psi > 0$ such that

$$|K_z(x,y)| \le c_\psi\, (\operatorname{Re} z)^{-d/2}\, e^{-\frac{\beta_\psi\,|x-y|}{|z|}}$$

for all $z \in \Sigma_\psi$ and for a.e. $(x,y) \in \mathbb{R}^{2d}$.

In addition, T extends consistently to a bounded holomorphic semigroup on the sector Σ_θ on $L_p(\mathbb{R}^d)$ for all $p \in [1,\infty)$.

Hence S extends consistently to a holomorphic semigroup $S^{(p)}$ on $L_p(\mathbb{R}^d)$ for all $p \in [1,\infty)$. Let $-A_p$ be the generator of $S^{(p)}$ for all $p \in [1,\infty)$.

Theorem 2.28. *Let $p \in [1,\infty)$. Then $D(A) \cap D(A_p) \cap L_\infty(\mathbb{R}^d)$ is a core for A_p.*

Proof. For each $t > 0$ let $K_t \in \mathcal{D}'(\mathbb{R}^{2d})$ be the distributional kernel of S_t. By Theorem 2.26 there exist constants $c, \omega, \beta > 0$ such that $|K_t(x,y)| \le G_t(x-y)$ for all $t > 0$ and $x, y \in \mathbb{R}^d$, where

$$G_t(x) = \frac{c}{t^{d/2}}\, e^{-\frac{\beta\,|x|^2}{t}}\, e^{\omega t}$$

for all $t > 0$ and $x \in \mathbb{R}^d$.

Let $t > 0$ and $u \in L_1(\mathbb{R}^d) \cap L_\infty(\mathbb{R}^d)$. Then

$$\begin{aligned}
|(S_t^{(p)} u)(x)| = |(S_t u)(x)| &= \Big| \int_{\mathbb{R}^d} K_t(x,y)\, u(y)\, dy \Big| \\
&\le \int_{\mathbb{R}^d} |K_t(x,y)|\, |u(y)|\, dy \le \int_{\mathbb{R}^d} G_t(x-y)\, |u(y)|\, dy \\
&\le \|G_t * |u|\|_\infty \le \|G_t\|_1\, \|u\|_\infty
\end{aligned}$$

for a.e. $x \in \mathbb{R}^d$. We have

$$\|G_t\|_1 = c\, e^{\omega t} \int_{\mathbb{R}^d} t^{-d/2}\, e^{-\frac{\beta\,|x|^2}{t}}\, dx = c\, e^{\omega t} \int_{\mathbb{R}^d} e^{-\beta\,|x|^2}\, dx = M\, e^{\omega t},$$

where

$$M = c \int_{\mathbb{R}^d} e^{-\beta\,|x|^2}\,dx < \infty.$$

It follows that

$$\|S_t^{(p)} u\|_\infty \le M\,e^{\omega t}\,\|u\|_\infty.$$

Since S and $S^{(p)}$ are consistent, the space $D(A) \cap D(A_p) \cap L_\infty(\mathbb{R}^d)$ is a core for A_p by Lemma 2.21. □

It is well-known that elements of $D(A_p)$ possess certain regularity properties when $p \in (1,\infty)$, as stated in the next theorem.

Theorem 2.29 (Elliptic regularity)**.** *Let $p \in (1,\infty)$. Then*

$$D(A_p) = W^{2,p}(\mathbb{R}^d).$$

This theorem is folklore. For an explicit reference see [ER97, Theorem 1.5].

Before ending this section we will consider an integration by parts formula for pure second-order differential operators. For the rest of this section let $c_{kl} \in W^{1,\infty}(\mathbb{R}^d)$ for all $k,l \in \{1,\dots,d\}$. That is, we no longer require that $C = (c_{kl})_{1\le k,l\le d}$ satisfies the strongly elliptic condition (2.6).

Let $p \in (1,\infty)$. Define the pure second-order differential operator $B_p\colon D(B_p) \subset L_p(\mathbb{R}^d) \longrightarrow L_p(\mathbb{R}^d)$ by

$$B_p u = -\sum_{k,l=1}^{d} \partial_l (c_{kl}\,\partial_k u)$$

on the domain

$$\begin{aligned} D(B_p) = \{u \in L_p(\mathbb{R}^d) :\ & \text{there exists an } f \in L_p(\mathbb{R}^d) \text{ such that} \\ & -\sum_{k,l=1}^{d} \int_{\mathbb{R}^d} u\,\partial_k(\overline{c_{kl}}\,\partial_l \phi) = \int_{\mathbb{R}^d} f\,\phi \text{ for all } \phi \in C_c^\infty(\mathbb{R}^d)\}. \end{aligned}$$

Note that $W^{2,p}(\mathbb{R}^d) \subset D(B_p)$.

In Chapters 3 and 4 we will be interested in proving that operators of this type are accretive or quasi-accretive under further restrictions. A convenient tool we would like to use in doing so is the method of integration by parts. The following theorem states that we can indeed perform this method on B_p.

Theorem 2.30. *Let $u \in W^{2,p}(\mathbb{R}^d)$. Then*

$$\begin{aligned} \int_{[u\neq 0]} (B_p u)\,|u|^{p-2}\,\overline{u} = & \int_{[u\neq 0]} |u|^{p-2}\,(C\,\nabla\overline{u}, \nabla\overline{u}) \\ & + (p-2) \int_{[u\neq 0]} |u|^{p-4}\,\big(C\,\mathrm{Re}\,(u\,\nabla\overline{u}), \mathrm{Re}\,(u\,\nabla\overline{u})\big) \\ & - i\,(p-2) \int_{[u\neq 0]} |u|^{p-4}\,\big(C\,\mathrm{Re}\,(u\,\nabla\overline{u}), \mathrm{Im}\,(u\,\nabla\overline{u})\big). \end{aligned} \tag{2.7}$$

Proof. This follows immediately from the proof of [MS08, Proposition 3.5]. □

We emphasise that we do not require $c_{kl} = c_{lk}$ for all $k, l \in \{1, \ldots, d\}$ in the above theorem (cf. [MS08, Theorem 3.1] and [MS08, Proposition 3.5]).

Let $u \in W^{2,p}(\mathbb{R}^d)$. If $p \in [2, \infty)$ then $|u|^{p-2}\,\overline{u} \in W^{1,q}(\mathbb{R}^d)$, where q is the dual exponent of p. In this case (2.7) is a consequence of the integration by parts. Therefore the significance part of Theorem 2.30 is when $p \in (1, 2)$, in which case the smoothness of $|u|^{p-2}\,\overline{u}$ is not obvious due to the singularity of $|u|^{p-2}$ near the zeros of u.

Next we will extend Theorem 2.30 to the case $p = 1$ in one dimension. Let $a, b \in [-\infty, \infty]$ with $a < b$. Let $c \in W^{1,\infty}(\mathbb{R}, \mathbb{C})$.

Proposition 2.31. *Let $u \in W^{2,1}(a, b)$. Then*

$$
\begin{aligned}
-\int_a^b (c\,u')'\,\overline{u}\,(|u|^2 + \delta^2)^{-1/2} = \int_a^b c\,|u'|^2\,(|u|^2 + \delta^2)^{-1/2} - \int_a^b c\,u'\,\overline{u}\,\mathrm{Re}\,(u'\,\overline{u})\,(|u|^2 + \delta^2)^{-3/2} \\
- c\,u'\,\overline{u}\,(|u|^2 + \delta^2)^{-1/2}\Big|_a^b
\end{aligned}
\tag{2.8}
$$

for all $\delta \in \mathbb{R} \setminus \{0\}$.

Proof. First note that $u \in W^{1,\infty}(a, b)$ by the Sobolev imbedding theorem. Let $(u_n)_{n\in\mathbb{N}} \subset C_c^\infty(\mathbb{R})$ be such that $\lim_{n\to\infty} u_n = u$ in $W^{2,1}(a, b)$. Without loss of generality we may assume that $\lim_{n\to\infty} u_n(x) = u(x)$ and $\lim_{n\to\infty} u_n'(x) = u'(x)$ for a.e. $x \in (a, b)$ as well as there exists an $M > 0$ such that $\|u_n\|_{L_\infty(a,b)} \le M\,\|u\|_{L_\infty(a,b)}$ and $\|u_n'\|_{L_\infty(a,b)} \le M\,\|u'\|_{L_\infty(a,b)}$ for all $n \in \mathbb{N}$.

Let $\delta \in \mathbb{R} \setminus \{0\}$. It follows from integration by parts that

$$
\begin{aligned}
-\int_a^b (c\,u_n')'\,\overline{u}_n\,(|u_n|^2 + \delta^2)^{-1/2} &= \int_a^b c\,|u_n'|^2\,(|u_n|^2 + \delta^2)^{-1/2} \\
&\quad - \int_a^b c\,u_n'\,\overline{u}_n\,\mathrm{Re}\,(u_n'\,\overline{u}_n)\,(|u_n|^2 + \delta^2)^{-3/2} \\
&\quad - c\,u_n'\,\overline{u}_n\,(|u_n|^2 + \delta^2)^{-1/2}\Big|_a^b.
\end{aligned}
\tag{2.9}
$$

Since $W^{1,1}(a, b) \subset L_\infty(a, b)$ by the Sobolev imbedding theorem and $\lim_{n\to\infty} u_n = u$ in $W^{2,1}(a, b)$, we have $\lim_{n\to\infty} u_n = u$ in $L_\infty(a, b)$.

Next note that $\lim_{n\to\infty}(|u_n|^2 + \delta^2)^{-1/2}(x) = (|u|^2 + \delta^2)^{-1/2}(x)$ for a.e. $x \in (a, b)$ and $(|u_n|^2 + \delta^2)^{-1/2} \le \delta^{-1}$ for all $n \in \mathbb{N}$. Therefore we have

$$
\begin{aligned}
&\big\|\overline{u}_n\,(|u_n|^2 + \delta^2)^{-1/2} - \overline{u}\,(|u|^2 + \delta^2)^{-1/2}\big\|_{L_\infty(a,b)} \\
&\le \big\|(\overline{u}_n - \overline{u})\,(|u_n|^2 + \delta^2)^{-1/2}\big\|_{L_\infty(a,b)} + \big\|\overline{u}\,\big((|u_n|^2 + \delta^2)^{-1/2} - (|u|^2 + \delta^2)^{-1/2}\big)\big\|_{L_\infty(a,b)} \\
&\le \|\overline{u}_n - \overline{u}\|_{L_\infty(a,b)}\,\frac{1}{|\delta|} + \|u\|_{L_\infty(a,b)}\,\frac{\|u_n - u\|_{L_\infty(a,b)}}{|\delta|^2},
\end{aligned}
$$

where the last step follows from the mean value theorem applied to the function $x \longmapsto (x^2 + \delta^2)^{-1/2}$ on $\mathbb{R}$. Hence $\lim_{n\to\infty} \overline{u}_n\,(|u_n|^2 + \delta^2)^{-1/2} = \overline{u}\,(|u|^2 + \delta^2)^{-1/2}$ in $L_\infty(a, b)$.

On the other hand we have $\lim_{n\to\infty}(c\,u_n')' = (c\,u')'$ in $L_1(a,b)$ as $c \in W^{1,\infty}(\mathbb{R})$, $u \in W^{2,1}(a,b)$ and $\lim_{n\to\infty} u_n = u$ in $W^{2,1}(a,b)$. Therefore

$$\lim_{n\to\infty} \int_a^b (c\,u_n')'\,\overline{u}_n\,(|u_n|^2+\delta^2)^{-1/2} = \int_a^b (c\,u')'\,\overline{u}\,(|u|^2+\delta^2)^{-1/2}.$$

We also have $W^{1,1}(a,b) \subset C[a,b]$ if $a,b \in \mathbb{R}$ by the Sobolev imbedding theorem and $\lim_{x\to\pm\infty} v(x) = 0$ for all $v \in W^{1,1}(\mathbb{R})$ by [Bre11, Corollary 8.9]. Hence

$$\lim_{n\to\infty} c\,u_n'\,\overline{u}_n\,(|u_n|^2+\delta^2)^{-1/2}|_a^b = c\,u'\,\overline{u}\,(|u|^2+\delta^2)^{-1/2}|_a^b.$$

Next we consider the second term on the right hand side of (2.9). We note that $\lim_{n\to\infty} u_n' = u'$ in $L_1(a,b)$ and

$$\begin{aligned}\|u_n'-u'\|_{L_2(a,b)}^2 &= \int_a^b |u_n'-u'|^2 \le \|u_n'-u'\|_{L_\infty(a,b)} \int_a^b |u_n'-u'| \\ &\le (M+1)\,\|u'\|_{L_\infty(a,b)}\,\|u_n'-u'\|_{L_1(a,b)}.\end{aligned}$$

It follows that $\lim_{n\to\infty} u_n' = u'$ in $L_2(a,b)$. Furthermore by assumption $\|u_n\|_{L_\infty(a,b)} \le M\,\|u\|_{L_\infty(a,b)}$ for all $n \in \mathbb{N}$. Therefore $\lim_{n\to\infty} u_n'\,\overline{u_n} = u'\,\overline{u}$ in $L_2(a,b)$. It follows that

$$\lim_{n\to\infty} \int_a^b c\,u_n'\,\overline{u}_n\,\mathrm{Re}\,(u_n'\,\overline{u}_n)\,(|u_n|^2+\delta^2)^{-3/2} = \int_a^b c\,u'\,\overline{u}\,\mathrm{Re}\,(u'\,\overline{u})\,(|u|^2+\delta^2)^{-3/2}$$

by the Lebesgue dominated convergence theorem. The claim now follows. □

Corollary 2.32. *Let $u \in W^{2,1}(a,b)$. Then*

$$\int_a^b |\mathrm{Im}\,(u'\,\overline{u})|^2\,|u|^{-3}\,\mathbb{1}_{[u\neq 0]} < \infty.$$

Proof. By Proposition 2.31 we have

$$\begin{aligned}-\int_a^b u''\,\overline{u}\,(|u|^2+\delta^2)^{-1/2} &= \int_a^b |u'|^2\,(|u|^2+\delta^2)^{-1/2} - \int_a^b u'\,\overline{u}\,\mathrm{Re}\,(u'\,\overline{u})\,(|u|^2+\delta^2)^{-3/2} \\ &\quad - u'\,\overline{u}\,(|u|^2+\delta^2)^{-1/2}|_a^b\end{aligned}$$

for all $\delta \in \mathbb{R}\setminus\{0\}$. Taking the real parts on both sides gives

$$\begin{aligned}\mathrm{Re} \int_a^b u''\,\overline{u}\,(|u|^2+\delta^2)^{-1/2} &= \int_a^b |u'|^2\,(|u|^2+\delta^2)^{-1/2} - \int_a^b \big(\mathrm{Re}\,(u'\,\overline{u})\big)^2\,(|u|^2+\delta^2)^{-3/2} \\ &\quad - \mathrm{Re}\,\big(u'\,\overline{u}\,(|u|^2+\delta^2)^{-1/2}\big)|_a^b \\ &\ge \int_a^b |u'\,\overline{u}|^2\,(|u|^2+\delta^2)^{-3/2} - \int_a^b \big(\mathrm{Re}\,(u'\,\overline{u})\big)^2\,(|u|^2+\delta^2)^{-3/2} \\ &\quad - \mathrm{Re}\,\big(u'\,\overline{u}\,(|u|^2+\delta^2)^{-1/2}\big)|_a^b \\ &= \int_a^b \big(\mathrm{Im}\,(u'\,\overline{u})\big)^2\,(|u|^2+\delta^2)^{-3/2} - \mathrm{Re}\,\big(u'\,\overline{u}\,(|u|^2+\delta^2)^{-1/2}\big)|_a^b.\end{aligned}$$

For the rest of the proof we use the convention that $(u'\,\overline{u}\,|u|^{-1})(x) = 0$ for all $x \in [a,b]$ such that $u(x) = 0$. Since $|u''\,\overline{u}\,(|u|^2+\delta^2)^{-1/2}| \le |u''|$ for all $\delta \in \mathbb{R}\setminus\{0\}$ and $u'' \in L_1(a,b)$, we have

$$\lim_{\delta\to 0}\int_a^b u''\,\overline{u}\,(|u|^2+\delta^2)^{-1/2} = \int_a^b u''\,\overline{u}\,|u|^{-1}$$

by the Lebesgue dominated convergence theorem. It follows from Fatou's lemma that

$$\begin{aligned}
0 &\le \int_a^b \big(\mathrm{Im}\,(u'\,\overline{u})\big)^2\,|u|^{-3}\,\mathbb{1}_{[u\neq 0]} \le \liminf_{\delta\to 0}\int_a^b \big(\mathrm{Im}\,(u'\,\overline{u})\big)^2\,(|u|^2+\delta^2)^{-3/2} \\
&\le \liminf_{\delta\to 0}\Big(\mathrm{Re}\int_a^b u''\,\overline{u}\,(|u|^2+\delta^2)^{-1/2} + \mathrm{Re}\,\big(u'\,\overline{u}\,(|u|^2+\delta^2)^{-1/2}\big)\big|_a^b\Big) \\
&= \mathrm{Re}\int_a^b u''\,\overline{u}\,|u|^{-1} + \mathrm{Re}\,\big(u'\,\overline{u}\,|u|^{-1}\big)\big|_a^b < \infty.
\end{aligned}$$

This verifies the claim. □

Proposition 2.33. *Suppose $c(x) \in [0,\infty)$ for all $x \in \mathbb{R}$. Let $u \in W^{2,1}(a,b)$. Then the limit*

$$L = \lim_{\delta\to 0}\int_a^b c\,|u'|^2\,\delta^2\,(|u|^2+\delta^2)^{-3/2} \tag{2.10}$$

exists in $[0,\infty)$ and

$$-\mathrm{Re}\int_a^b (c\,u')'\,\overline{u}\,|u|^{-1}\,\mathbb{1}_{[u\neq 0]} = L + \int_a^b c\,\big(\mathrm{Im}\,(u'\,\overline{u})\big)^2\,|u|^{-3}\,\mathbb{1}_{[u\neq 0]} - c\,\mathrm{Re}\,\big(u'\,\overline{u}\,|u|^{-1}\big)\big|_a^b,$$

where we use the convention that $(u'\,\overline{u}\,|u|^{-1})(x) = 0$ for all $x \in [a,b]$ such that $u(x) = 0$.

Proof. Let $\delta \in \mathbb{R}\setminus\{0\}$. Taking the real part both sides of (2.8) gives

$$\begin{aligned}
-\mathrm{Re}\int_a^b (c\,u')'\,\overline{u}\,(|u|^2+\delta^2)^{-1/2} &= \int_a^b c\,|u'|^2\,(|u|^2+\delta^2)^{-1/2} \\
&\quad - \int_a^b c\,\big(\mathrm{Re}\,(u'\,\overline{u})\big)^2\,(|u|^2+\delta^2)^{-3/2} \\
&\quad - c\,\mathrm{Re}\,\big(u'\,\overline{u}\,(|u|^2+\delta^2)^{-1/2}\big)\big|_a^b.
\end{aligned}$$

Note that

$$\int_a^b c\,\big(\mathrm{Re}\,(u'\,\overline{u})\big)^2\,(|u|^2+\delta^2)^{-3/2} = \int_a^b c\,|u'\,\overline{u}|^2\,(|u|^2+\delta^2)^{-3/2} - \int_a^b c\,\big(\mathrm{Im}\,(u'\,\overline{u})\big)^2\,(|u|^2+\delta^2)^{-3/2}.$$

Therefore

$$\begin{aligned}
-\mathrm{Re}\int_a^b (c\,u')'\,\overline{u}\,(|u|^2+\delta^2)^{-1/2} &= \int_a^b c\,|u'|^2\,\delta^2\,(|u|^2+\delta^2)^{-3/2} \\
&\quad + \int_a^b c\,\big(\mathrm{Im}\,(u'\,\overline{u})\big)^2\,(|u|^2+\delta^2)^{-3/2} \\
&\quad - c\,\mathrm{Re}\,\big(u'\,\overline{u}\,(|u|^2+\delta^2)^{-1/2}\big)\big|_a^b.
\end{aligned}$$

Clearly

$$\lim_{\delta\to 0} c\,\mathrm{Re}\,\big(u'\,\overline{u}\,(|u|^2+\delta^2)^{-1/2}\big)\big|_a^b = c\,\mathrm{Re}\,\big(u'\,\overline{u}\,|u|^{-1}\big)\big|_a^b,$$

where we use the convention that $(u'\,\overline{u}\,|u|^{-1})(x) = 0$ for all $x \in [a,b]$ such that $u(x) = 0$. It follows from the Lebesgue dominated convergence theorem that

$$\lim_{\delta\to 0} \mathrm{Re} \int_a^b (c\,u')'\,\overline{u}\,(|u|^2+\delta^2)^{-1/2} = \mathrm{Re} \int_a^b (c\,u')'\,\overline{u}\,|u|^{-1}\,\mathbb{1}_{[u\neq 0]}.$$

By the monotone convergence theorem we also have

$$\lim_{\delta\to 0} \int_a^b c\,\big(\mathrm{Im}\,(u'\,\overline{u})\big)^2\,(|u|^2+\delta^2)^{-3/2} = \int_a^b c\,\big(\mathrm{Im}\,(u'\,\overline{u})\big)^2\,|u|^{-3}\,\mathbb{1}_{[u\neq 0]}.$$

Hence by Corollary 2.32 the limit

$$L = \lim_{\delta\to 0} \int_a^b c\,|u'|^2\,\delta^2\,(|u|^2+\delta^2)^{-3/2}$$

exists in $[0,\infty)$ and

$$\mathrm{Re} \int_a^b (c\,u')'\,\overline{u}\,|u|^{-1}\,\mathbb{1}_{[u\neq 0]} = L + \int_a^b c\,\big(\mathrm{Im}\,(u'\,\overline{u})\big)^2\,|u|^{-3}\,\mathbb{1}_{[u\neq 0]} - c\,\mathrm{Re}\,\big(u'\,\overline{u}\,|u|^{-1}\big)\big|_a^b$$

as claimed. □

It is possible that the limit L defined as in (2.10) is strictly positive for some $u \in W^{2,1}(a,b)$, as shown by the following example.

Example 2.34. Let $a = 0$, $b = 1$ and $c = \mathbb{1}_{\mathbb{R}}$. Let $u(x) = x$ for all $x \in (0,1)$. Let L be defined as in (2.10). Then

$$L = \lim_{\delta\to 0} \int_0^1 \frac{\delta^2}{(x^2+\delta^2)^{3/2}}\,dx = \lim_{\delta\to 0} \int_0^{1/|\delta|} \frac{1}{(x^2+1)^{3/2}}\,dx = \int_0^\infty \frac{1}{(x^2+1)^{3/2}}\,dx = 1.$$

Chapter 3

Degenerate elliptic operators in one dimension

3.1 Introduction

The subject to study in this chapter is degenerate elliptic second-order differential operators with bounded complex-valued coefficients in one dimension. We will give a core characterisation for these operators. The results are extensions of those in [CMP98, Theorem 3.5] and [DE15, Theorem 1.5].

The following assumptions will be made throughout the chapter without being mentioned explicitly. If further assumptions are imposed, they will be articulated in the statements in which they are required.

Let $\theta \in [0, \frac{\pi}{2})$. Define

$$\Sigma_\theta = \{r\, e^{i\psi} : r \geq 0 \text{ and } |\psi| \leq \theta\}.$$

Let $c \in W^{1,\infty}(\mathbb{R}, \mathbb{C})$ be such that $c(x) \in \Sigma_\theta$ for all $x \in \mathbb{R}$. Let $m \in L_\infty(\mathbb{R}, \mathbb{C})$ and $M_m > 0$. Suppose

$$|m| \leq M_m \sqrt{\operatorname{Re} c}. \tag{3.1}$$

Let $w \in L_\infty(\mathbb{R}, \mathbb{C})$.

Consider the form $\mathfrak{a}_0$ defined by

$$\mathfrak{a}_0(u, v) = \int_{\mathbb{R}} \Big(c\, u'\, \overline{v'} + m\, u'\, \overline{v} + w\, u\, \overline{v} \Big) \tag{3.2}$$

on the domain

$$D(\mathfrak{a}_0) = C_c^\infty(\mathbb{R}).$$

We will show in Lemma 3.7 in Section 3.2 that $\mathfrak{a}_0$ is closable. Let $\mathfrak{a}$ be the closure of the form $\mathfrak{a}_0$. It follows from the first representation theorem [Kat80, Theorem VI.2.1] that there exists an quasi-m-sectorial operator A associated with the form $\mathfrak{a}$. Formally we can write

$$A = -\frac{d}{dx}\Big(c\,\frac{d}{dx}\Big) + m\,\frac{d}{dx} + w\, I.$$

Let S be the C_0-semigroup generated by $-A$. If A is strongly elliptic, that is, if $\inf \operatorname{Re} c > 0$, then S extends consistently to a C_0-semigroup on $L_p(\mathbb{R})$ for all $p \in [1, \infty)$ by [AMT98, Theorem 2.21]. We prove in Section 3.3 the following extension.

Proposition 3.1. *Let $p \in (1, \infty)$ and $m \in W^{1,\infty}(\mathbb{R}, \mathbb{C})$. Suppose*

(I) $\left|1 - \frac{2}{p}\right| < \cos\theta$ *or*

(II) $\left|1 - \frac{2}{p}\right| \leq \cos\theta$ *and* $m = 0$.

Then S extends consistently to a quasi-contraction C_0-semigroup on $L_p(\mathbb{R})$.

The case $p = 1$ follows directly from [AE12, Corollary 3.13(iii)].

Proposition 3.2. *Suppose $c, m \in W^{1,\infty}(\mathbb{R}, \mathbb{R})$. Then S extends consistently to a quasi-contraction C_0-semigroup on $L_1(\mathbb{R})$.*

Let $p \in [1, \infty)$. Suppose S extends consistently to a C_0-semigroup $S^{(p)}$ on $L_p(\mathbb{R})$. Let $-A_p$ be the generator of $S^{(p)}$. Clearly $C_c^\infty(\mathbb{R}) \subset D(A_p)$. Our aim is to characterise when $C_c^\infty(\mathbb{R})$ is a core for A_p. For this we need to introduce more notation.

Define
$$\mathcal{P} = [\operatorname{Re} c > 0] \quad \text{and} \quad \mathcal{N} = [\operatorname{Re} c = 0].$$
Let $\{I_k \colon k \in K\}$ be the set of connected components of $\mathcal{P}$, with $I_k \neq I_{k'}$ for all $k, k' \in K$ with $k \neq k'$. Write $I_k = (a_k, b_k)$ for all $k \in K$, with $a_k, b_k \in [-\infty, \infty]$. Let
$$E_l = \{a_k \colon k \in K\} \cap \mathbb{R}, \quad E_r = \{b_k \colon k \in K\} \cap \mathbb{R} \quad \text{and} \quad E = E_l \cup E_r$$
be the set of all finite left endpoints, finite right endpoints and finite endpoints respectively. (The intersection with $\mathbb{R}$ is needed to deal with unbounded I_k and to obtain that $E \subset \mathbb{R}$.) For all $k \in K$ define
$$m_k = \begin{cases} \frac{a_k + b_k}{2} & \text{if } a_k \in \mathbb{R} \text{ and } b_k \in \mathbb{R}, \\ a_k + 1 & \text{if } a_k \in \mathbb{R} \text{ and } b_k = \infty, \\ b_k - 1 & \text{if } a_k = -\infty \text{ and } b_k \in \mathbb{R}, \\ 0 & \text{if } a_k = -\infty \text{ and } b_k = \infty. \end{cases}$$
Next define the function $Z \colon \mathbb{R} \longrightarrow \mathbb{R}$ by
$$Z(x) = \begin{cases} \int_x^{m_k} \frac{1}{\operatorname{Re} c} & \text{if } x \in I_k \text{ and } k \in K, \\ \infty & \text{if } x \in \mathcal{N}. \end{cases}$$

The main theorem of this chapter is as follows.

Theorem 3.3. *Let $p \in (1, \infty)$ and $m \in W^{1,\infty}(\mathbb{R}, \mathbb{C})$. Suppose*

(I) $\left|1 - \frac{2}{p}\right| < \cos\theta$ *or*

(II) $\left|1 - \frac{2}{p}\right| \leq \cos\theta$ *and* $m = 0$.

Suppose that for all $x_0 \in E$ there exists a $\delta_0 > 0$ such that
$$\int_{(x_0 - \delta_0, x_0 + \delta_0) \cap \mathcal{P}} \left|\frac{m}{c}\right| < \infty.$$
Then the space $C_c^\infty(\mathbb{R})$ is a core for A_p if and only if $Z|_{(x-\delta, x+\delta)} \notin L_q(x - \delta, x + \delta)$ for all $x \in E$ and $\delta > 0$, where q is the dual exponent of p.

This theorem has the following obvious corollary.

Corollary 3.4. *Let $p \in (1, \infty)$ and $m \in W^{1,\infty}(\mathbb{R}, \mathbb{C})$. Suppose $c \in W^{1,\infty}(\mathbb{R}, \mathbb{C})$ and c' is Hölder continuous of order $1 - \frac{1}{p}$. Moreover, assume that*

(I) $\theta \in [0, \arccos \left|1 - \frac{2}{p}\right|)$ *or*

(II) $m = 0$ *and* $\theta = \arccos \left|1 - \frac{2}{p}\right|$.

Suppose that for all $x_0 \in E$ there exists a $\delta_0 > 0$ such that

$$\int_{(x_0-\delta_0, x_0+\delta_0)\cap\mathcal{P}} \left|\frac{m}{c}\right| < \infty.$$

Then the space $C_c^\infty(\mathbb{R})$ is a core for A_p.

If $p = 1$ then the condition on $\frac{m}{c}$ is superfluous in Theorem 3.3.

Theorem 3.5. *Suppose $c, m \in W^{1,\infty}(\mathbb{R}, \mathbb{R})$. Then the space $C_c^\infty(\mathbb{R})$ is a core for A_1.*

The following is a brief summary of the subsequent sections. In Section 3.2 we give detailed description of the form $\mathfrak{a}$ and its associated operator A. In Section 3.3 we prove Proposition 3.1. We will determine the operator A_p in Section 3.4. The main theorems, Theorems 3.3 and 3.5, are proved in Sections 3.5 and 3.6 respectively. Finally we consider three intriguing examples to illustrate Theorem 3.3 in Section 3.7.

3.2 The form and the operator on L_2

Let $\mathfrak{a}_0$ be as in Section 3.1. We will now show that $\mathfrak{a}_0$ is indeed closable.

Lemma 3.6. *There exist an $\omega > 0$ and a $\theta' \in [0, \frac{\pi}{2})$ such that $\mathfrak{a}_0(u) + \omega\|u\|_2^2 \in \Sigma_{\theta'}$ for all $u \in D(\mathfrak{a}_0)$.*

Proof. Let $u \in D(\mathfrak{a}_0)$. Then

$$\begin{aligned}
\operatorname{Re} \mathfrak{a}_0(u) &\geq \int (\operatorname{Re} c)\, |u'|^2 - \int |m|\, |u'|\, |u| - \int |w|\, |u|^2 \\
&\geq \int (\operatorname{Re} c)\, |u'|^2 - \varepsilon M_m \int (\operatorname{Re} c)\, |u'|^2 - \Big(\frac{1}{4\varepsilon} + \|w\|_\infty\Big)\, \|u\|_2^2 \\
&= (1 - \varepsilon M_m) \int (\operatorname{Re} c)\, |u'|^2 - \Big(\frac{1}{4\varepsilon} + \|w\|_\infty\Big) \|u\|_2^2
\end{aligned}$$

for all $\varepsilon > 0$. Choosing $\varepsilon = \frac{1}{2M_m}$ in the above inequality gives

$$\int (\operatorname{Re} c)\, |u'|^2 \leq 2 \operatorname{Re} \mathfrak{a}_0(u) + (M_m + 2\, \|w\|_\infty)\, \|u\|_2^2. \tag{3.3}$$

It follows that

$$|\operatorname{Im} \mathfrak{a}_0(u)| \leq \tan\theta \int (\operatorname{Re} c)\, |u'|^2 + \int |m|\, |u'|\, |u| + \int |w|\, |u|^2$$

$$
\begin{aligned}
&\le (\tan\theta + M_m) \int (\operatorname{Re} c)\, |u'|^2 + \Big(\frac{1}{4} + \|w\|_\infty\Big) \|u\|_2^2 \\
&\le (\tan\theta + M_m) \Big(2 \operatorname{Re} \mathfrak{a}_0(u) + (M_m + 2\|w\|_\infty)\, \|u\|_2^2\Big) + \Big(\frac{1}{4} + \|w\|_\infty\Big) \|u\|_2^2 \\
&= 2(\tan\theta + M_m) \operatorname{Re} \mathfrak{a}_0(u) + \Big((\tan\theta + M_m)\,(M_m + 2\|w\|_\infty) + \frac{1}{4} + \|w\|_\infty\Big) \|u\|_2^2.
\end{aligned}
$$

That is, $\mathfrak{a}_0(u) + \omega\|u\|_2^2 \in \Sigma_{\theta'}$ where

$$
\omega = \frac{1}{2(\tan\theta + M_m)} \Big((\tan\theta + M_m)\,(M_m + 2\|w\|_\infty) + \frac{1}{4} + \|w\|_\infty\Big)
$$

and θ' is such that

$$
\tan(\theta') = 2(\tan\theta + M_m).
$$

The lemma now follows. □

Proposition 3.7. *The form $\mathfrak{a}_0$ is closable.*

Proof. Define the operator A_0 in $L_2(\mathbb{R})$ by

$$
A_0 u = -(c\,u')' + m\,u' + w\,u
$$

on the domain $D(A_0) = C_c^\infty(\mathbb{R})$. Then $\mathfrak{a}_0(u,v) = (A_0 u, v)$ and $(A_0 u, u) + \omega\,\|u\|_2^2 \in \Sigma_{\theta'}$ for all $u, v \in C_c^\infty(\mathbb{R})$, where ω and θ' are as in Lemma 3.6. It now follows from [Kat80, Theorem VI.1.27] that $\mathfrak{a}_0$ is closable. □

Our next aim is to derive a comprehensive description of the closure of $\mathfrak{a}_0$. If $\Omega \subset \mathbb{R}$ is open and $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, then we denote by $u' \in L_{1,\mathrm{loc}}(\Omega)$ the derivative. If $v \in L_{1,\mathrm{loc}}(\mathbb{R})$ then we say that $v \in C(\overline{\Omega})$ if $v|_\Omega$ has a continuous representative which extends continuously to $\overline{\Omega}$. For all $u \in L_{1,\mathrm{loc}}(\mathbb{R})$ with $u|_{\mathcal{P}} \in W^{1,1}(\mathcal{P})$ define $Du \colon \mathbb{R} \longrightarrow \mathbb{C}$ by

$$
(Du)(x) = \begin{cases} u'(x) & \text{if } x \in \mathcal{P}, \\ 0 & \text{if } x \in \mathcal{N}. \end{cases}
$$

In order to avoid clutter we write

$$
L_2(\mathbb{R}) \cap W^{1,2}_{\mathrm{loc}}(\mathcal{P}) = \{u \in L_2(\mathbb{R}) \colon u|_{\mathcal{P}} \in W^{1,2}_{\mathrm{loc}}(\mathcal{P})\}.
$$

Define the form $\mathfrak{a}$ by

$$
D(\mathfrak{a}) = \{u \in L_2(\mathbb{R}) \cap W^{1,2}_{\mathrm{loc}}(\mathcal{P}) \colon \sqrt{\operatorname{Re} c}\, Du \in L_2(\mathbb{R})\}
$$

and

$$
\mathfrak{a}(u,v) = \int_{\mathbb{R}} \Big(c\,(Du)\,\overline{Dv} + m\,(Du)\,\overline{v} + w\,u\,\overline{v}\Big)
$$

for all $u, v \in D(\mathfrak{a})$. Note that $|c\,Du| \le \frac{\sqrt{\|c\|_\infty}}{\cos\theta} \sqrt{\operatorname{Re} c}\,|Du|$ and $|m\,Du| \le M_m \sqrt{\operatorname{Re} c}\,|Du|$ for all $u \in D(\mathfrak{a})$. Hence $c\,Du$ and $m\,Du$ belong to $L_2(\mathbb{R})$ if $u \in D(\mathfrak{a})$. The domain $D(\mathfrak{a})$ is naturally normed with

$$
u \longmapsto \Big(\operatorname{Re} \mathfrak{a}(u) + (1+\omega)\,\|u\|_2^2\Big)^{1/2},
$$

where ω is as in Lemma 3.6. The following proposition shows that we can replace this norm by a simpler one.

Proposition 3.8. *The norm* $u \longmapsto \Big(\operatorname{Re}\mathfrak{a}(u) + (1+\omega)\,\|u\|_2^2\Big)^{1/2}$ *on* $D(\mathfrak{a})$ *is equivalent to*

$$u \longmapsto \Big(\int_{\mathbb{R}} (\operatorname{Re} c)\,|Du|^2 + \|u\|_2^2\Big)^{1/2},$$

where ω *is as in Lemma 3.6.*

Proof. Let $u \in D(\mathfrak{a})$. Then

$$\begin{aligned}
\operatorname{Re}\mathfrak{a}(u) + (1+\omega)\,\|u\|_2^2 &\le \int (\operatorname{Re} c)\,|u'|^2 + \int |m|\,|u'|\,|u| + (1+\omega+\|w\|_\infty)\,\|u\|_2^2 \\
&\le (1+M_m)\int (\operatorname{Re} c)\,|u'|^2 + \Big(\frac{5}{4}+\omega+\|w\|_\infty\Big)\,\|u\|_2^2 \\
&\le M'\Big(\int (\operatorname{Re} c)\,|u'|^2 + \|u\|_2^2\Big),
\end{aligned}$$

where $M' = (1+M_m) \vee (\frac{5}{4}+\omega+\|w\|_\infty)$. Also by (3.3) we have

$$\begin{aligned}
\int (\operatorname{Re} c)\,|u'|^2 + \|u\|_2^2 &\le 2\operatorname{Re}\mathfrak{a}(u) + (1+M_m+2\,\|w\|_\infty)\,\|u\|_2^2 \\
&\le 2\,(1+M_m+2\,\|w\|_\infty)\,\Big(\operatorname{Re}\mathfrak{a}(u) + (1+\omega)\,\|u\|_2^2\Big).
\end{aligned}$$

This completes the proof. □

Due to Proposition 3.8 we define the norm $\|\cdot\|_{D(\mathfrak{a})}$ on $D(\mathfrak{a})$ by

$$\|u\|_{D(\mathfrak{a})}^2 = \int_{\mathbb{R}} (\operatorname{Re} c)\,|Du|^2 + \|u\|_2^2.$$

Note that if $u \in D(\mathfrak{a})$, then $u\,\mathbb{1}_{\mathcal{P}}, u\,\mathbb{1}_{\mathcal{N}} \in D(\mathfrak{a})$ and

$$\|u\|_{D(\mathfrak{a})}^2 = \|u\,\mathbb{1}_{\mathcal{P}}\|_{D(\mathfrak{a})}^2 + \|u\,\mathbb{1}_{\mathcal{N}}\|_{D(\mathfrak{a})}^2. \tag{3.4}$$

Moreover, $\|u\,\mathbb{1}_{\mathcal{N}}\|_{D(\mathfrak{a})} = \|u\,\mathbb{1}_{\mathcal{N}}\|_{L_2(\mathbb{R})}$.

Lemma 3.9. *The form* $\mathfrak{a}$ *is the closure of* $\mathfrak{a}_0$*. Moreover,* $C_c^\infty(\mathcal{P}) + L_2(\mathcal{N})$ *is dense in* $D(\mathfrak{a})$*.*

Proof. We first show that $\mathfrak{a}$ is closed. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $D(\mathfrak{a})$. Then $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L_2(\mathbb{R})$, so $u = \lim u_n$ exists in $L_2(\mathbb{R})$. Similarly $v = \lim \sqrt{\operatorname{Re} c}\,Du_n$ exists in $L_2(\mathbb{R})$. Let $\tau \in C_c^\infty(\mathcal{P})$. Then

$$(u,\tau')_{L_2(\mathcal{P})} = \lim (u_n,\tau')_{L_2(\mathcal{P})} = -\lim (u_n',\tau)_{L_2(\mathcal{P})} = -\Big(\frac{1}{\sqrt{\operatorname{Re} c}}\,v,\tau\Big)_{L_2(\mathcal{P})}.$$

So $u \in W^{1,2}_{\mathrm{loc}}(\mathcal{P})$ and $u' = \frac{1}{\sqrt{\operatorname{Re} c}}\,v|_{\mathcal{P}}$ in $L_{2,\mathrm{loc}}(\mathcal{P})$. Then $\sqrt{\operatorname{Re} c}\,Du = v \in L_2(\mathbb{R})$. It is now easy to verify that $\lim u_n = u$ in $D(\mathfrak{a})$. So $\mathfrak{a}$ is closed.

Secondly we show that for all $u \in D(\mathfrak{a})$ with $u = u\,\mathbb{1}_{\mathcal{P}}$ and for all $\varepsilon > 0$ there exists a $v \in C_c^\infty(\mathcal{P})$ such that $\|u - v\|_{D(\mathfrak{a})} < \varepsilon$. Since $u\,\mathbb{1}_{I_k} \in D(\mathfrak{a})$ for all $u \in D(\mathfrak{a})$ and $k \in K$ and $\|u\|_{D(\mathfrak{a})}^2 = \sum_{k\in K} \|u\,\mathbb{1}_{I_k}\|^2$ for all $u \in D(\mathfrak{a})$ with $u = u\,\mathbb{1}_{\mathcal{P}}$, it suffices to show that for all $u \in D(\mathfrak{a})$, $k \in K$ and $\varepsilon > 0$ with $u = u \mathbb{1}_{I_k}$, there exists a $v \in C_c^\infty(I_k)$ such that $\|u - v\|_{D(\mathfrak{a})} < \varepsilon$. If I_k is bounded and c takes real-valued, this is the content of [CMP98, Lemma 2.6]. Now suppose that $I_k = (a_k, b_k)$ is unbounded with $a_k \in \mathbb{R}$ and $b_k = \infty$. (The proof for other cases like bounded I_k is similar.) Without loss of generality we may assume that u is real-valued. Since $\lim(-n) \vee u \wedge n = u$ in $D(\mathfrak{a})$ by [GT83, Lemma 7.6], we may assumed that u is bounded. For all $n \in \mathbb{N}$ with $n \geq a_k + 3$ define $\chi_n \colon \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\chi_n(x) = \begin{cases} 0 & \text{if } x \leq a_k + \frac{1}{n}, \\ n(x - a_k - \frac{1}{n}) & \text{if } a_k + \frac{1}{n} < x \leq a_k + \frac{2}{n}, \\ 1 & \text{if } a_k + \frac{2}{n} < x \leq n, \\ -\frac{1}{n}(x - 2n) & \text{if } n < x \leq 2n, \\ 0 & \text{if } x > 2n. \end{cases}$$

Then $\chi_n \in W^{1,2}(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, $0 \leq \chi_n \leq 1$ and

$$\int_{\mathbb{R}} (\operatorname{Re} c)\, |\chi_n'|^2 \leq \int_{(a_k+\frac{1}{n}, a_k+\frac{2}{n})} \|c'\|_\infty\, (x - a_k)\, n^2 + \int_{(n,2n)} \|c\|_\infty n^{-2} \leq 2\, \|c'\|_\infty + \|c\|_\infty.$$

So $\chi_n\, u \in D(\mathfrak{a})$ and

$$\begin{aligned} \|\chi_n\, u\|_{D(\mathfrak{a})}^2 &\leq \int_{\mathbb{R}} |u|^2 + 2 \int_{\mathbb{R}} (\operatorname{Re} c)\, |\chi_n'|^2\, |u|^2 + 2 \int_{\mathbb{R}} (\operatorname{Re} c)\, |\chi_n|^2\, |Du|^2 \\ &\leq 2\, \|u\|_{D(\mathfrak{a})}^2 + (4\, \|c'\|_\infty + 2\, \|c\|_\infty)\, \|u\|_\infty^2 \end{aligned}$$

for all $n \in \mathbb{N}$ with $n \geq a_k + 3$. Passing to a subsequence if necessary, the sequence $(\chi_n\, u)_{n\in\mathbb{N}}$ is weakly convergent in $D(\mathfrak{a})$. But $\lim \chi_n\, u = u$ in $L_2(\mathbb{R})$. So $\lim \chi_n\, u = u$ in $D(\mathfrak{a})$. Let $V = \{v \in D(\mathfrak{a}) \colon \operatorname{supp} v \text{ is compact and } \operatorname{supp} v \subset I_k\}$. We proved that u is in the weak closure of V in $D(\mathfrak{a})$. Since V is convex, the weak closure equals the norm closure and u is therefore in the norm closure of V in $D(\mathfrak{a})$. By regularising elements of V we see that u is an element of the norm closure of $C_c^\infty(I_k)$ in $D(\mathfrak{a})$.

Thirdly, the density of $C_c^\infty(\mathcal{P}) + L_2(\mathcal{N})$ in $D(\mathfrak{a})$ now follows from (3.4) and the previous step.

Finally, let $u \in D(\mathfrak{a})$ and $\varepsilon > 0$. Since $C_c^\infty(\mathbb{R})$ is dense in $L_2(\mathbb{R})$, there exists a $v_1 \in C_c^\infty(\mathbb{R})$ such that $\|u - v_1\|_{L_2(\mathbb{R})} < \varepsilon$. Therefore $\|(u - v_1)\, \mathbb{1}_{\mathcal{N}}\|_{D(\mathfrak{a})} = \|(u - v_1)\, \mathbb{1}_{\mathcal{N}}\|_{L_2(\mathcal{N})} \leq \|u - v_1\|_{L_2(\mathbb{R})} < \varepsilon$. By the above there exists a $v_2 \in C_c^\infty(\mathcal{P})$ such that $\|(u - v_1)\, \mathbb{1}_{\mathcal{P}} - v_2\|_{D(\mathfrak{a})} < \varepsilon$. Using (3.4) it follows that

$$\|u - v_1 - v_2\|_{D(\mathfrak{a})}^2 = \|(u - v_1)\, \mathbb{1}_{\mathcal{P}} - v_2\|_{D(\mathfrak{a})}^2 + \|(u - v_1)\, \mathbb{1}_{\mathcal{N}}\|_{D(\mathfrak{a})}^2 \leq 2\varepsilon^2.$$

So $C_c^\infty(\mathbb{R})$ is dense in $D(\mathfrak{a})$ and $\mathfrak{a}$ is the closure of $\mathfrak{a}_0$. □

Let A be the operator associated with $\mathfrak{a}$. Then the detailed descriptions of A are possible, as shown in the following lemma.

Lemma 3.10. *Let $u \in L_2(\mathbb{R})$. Then the following are equivalent.*

(i) $u \in D(A)$.

(ii) $u \in L_2(\mathbb{R}) \cap W^{1,2}_{\mathrm{loc}}(\mathcal{P})$, $\sqrt{\operatorname{Re} c}\, Du \in L_2(\mathbb{R})$ *and* $c\, Du \in W^{1,2}(\mathbb{R})$.

(iii) $u \in L_2(\mathbb{R}) \cap W^{1,2}_{\mathrm{loc}}(\mathcal{P})$, $\sqrt{\operatorname{Re} c}\, Du \in L_2(\mathbb{R})$ *and* $(c\, Du)|_{\mathcal{P}} \in W^{1,2}(\mathcal{P})$.

(iv) $u \in L_2(\mathbb{R}) \cap W^{1,2}_{\mathrm{loc}}(\mathcal{P})$ *and* $(c\, Du)|_{\mathcal{P}} \in W^{1,2}_0(\mathcal{P})$.

(v) $u \in L_2(\mathbb{R}) \cap W^{1,2}_{\mathrm{loc}}(\mathcal{P})$, $(c\, Du)|_{\mathcal{P}} \in W^{1,2}_{\mathrm{loc}}(\mathcal{P})$, $D(c\, Du) \in L_2(\mathbb{R})$, $\lim_{x\downarrow a}(c\, Du)(x) = 0$ *for all* $a \in E_l$ *and* $\lim_{x\uparrow b}(c\, Du)(x) = 0$ *for all* $b \in E_r$.

(vi) $u \in L_2(\mathbb{R}) \cap W^{1,2}_{\mathrm{loc}}(\mathcal{P})$, $(c\, Du)|_{\mathcal{P}} \in W^{1,2}_{\mathrm{loc}}(\mathcal{P})$, $D(c\, Du) - m\, Du \in L_2(\mathbb{R})$, $\lim_{x\downarrow a}(c\, Du)(x) = 0$ *for all* $a \in E_l$ *and* $\lim_{x\uparrow b}(c\, Du)(x) = 0$ *for all* $b \in E_r$.

Proof. (i $\Longrightarrow$ ii) Let $u \in D(A)$. Write $f = Au \in L_2(\mathbb{R})$. Then $u \in D(\mathfrak{a})$, which implies $u \in L_2(\mathbb{R}) \cap W^{1,2}_{\mathrm{loc}}(\mathcal{P})$ and $\sqrt{\operatorname{Re} c}\, Du \in L_2(\mathbb{R})$. It follows that $c\, Du, m\, Du \in L_2(\mathbb{R})$. Clearly $w\, u \in L_2(\mathbb{R})$ since $w \in L_\infty(\mathbb{R})$ and $u \in L_2(\mathbb{R})$. Let $v \in C_c^\infty(\mathbb{R})$. Then

$$\int_{\mathbb{R}} (c\, Du)\, \overline{v'} + m\,(Du)\,\overline{v} + w\, u\, \overline{v} = \mathfrak{a}(u,v) = \int_{\mathbb{R}} f\, \overline{v}$$

or equivalently

$$\int_{\mathbb{R}} (c\, Du)\, \overline{v'} = \int_{\mathbb{R}} (f - m\, Du - w\, u)\, \overline{v}.$$

Hence $c\, Du \in W^{1,2}(\mathbb{R})$.

(ii $\Longrightarrow$ iii) Trivial.

(iii $\Longrightarrow$ i) By hypothesis $D(c\, Du) \in L_2(\mathbb{R})$ and $w\, u \in L_2(\mathbb{R})$. Since $u \in D(\mathfrak{a})$, we have $m\, Du \in L_2(\mathbb{R})$. Let $f = -D(c\, Du) + m\, Du + w\, u$. Then $f \in L_2(\mathbb{R})$. Moreover,

$$\mathfrak{a}(u,v) = \int_{\mathcal{P}} c\,(Du)\, \overline{v'} + m\,(Du)\,\overline{v} + w\, u\, \overline{v} = \int_{\mathcal{P}} f\, \overline{v} = \int_{\mathbb{R}} f\, \overline{v}$$

for all $v \in C_c^\infty(\mathcal{P})$. If $v \in L_2(\mathcal{N})$ then $\mathfrak{a}(u,v) = \int_{\mathbb{R}} w\, u\, \mathbb{1}_{\mathcal{N}}\, \overline{v} = \int_{\mathbb{R}} f\, \overline{v}$. So $\mathfrak{a}(u,v) = \int_{\mathbb{R}} f\, \overline{v}$ for all $v \in C_c^\infty(\mathcal{P}) + L_2(\mathcal{N})$. Since $C_c^\infty(\mathcal{P}) + L_2(\mathcal{N})$ is a core for $\mathfrak{a}$ by Lemma 3.9, we have $u \in D(A)$.

(iii $\Longrightarrow$ iv) Let $a \in E_l$, $\delta > 0$ and suppose that $(a, a+\delta) \subset \mathcal{P}$. Then $(c\, Du)|_{(a,a+\delta)} \in W^{1,2}(a, a+\delta) \subset C[a, a+\delta]$. So $L = \lim_{x\downarrow a}(c\, Du)(x)$ exists. Moreover, since $\sqrt{\operatorname{Re} c}\, Du \in L_2(\mathcal{P})$ one has

$$\int_{(a,a+\delta)} \frac{|c\, Du|^2}{|c|} \le \frac{1}{\cos\theta} \int_{\mathcal{P}} |\sqrt{\operatorname{Re} c}\, Du|^2 < \infty.$$

So $L = 0$. By a symmetry argument one deduces that $(c\, Du)|_{\mathcal{P}} \in W^{1,2}_0(\mathcal{P})$.

(iv $\Longrightarrow$ v) Trivial.

(v $\Longrightarrow$ iii) Let $a \in E_l$, $s \in (a, \infty)$ and suppose that $(a, s] \subset \mathcal{P}$. By assumption and the Poincaré inequality $(c\, Du)|_{\mathcal{P}} \in W^{1,2}(a,s) \subset C^{1/2}[a,s]$. Since $\lim_{x\downarrow a}(c\, Du)(x) = 0$, there exists an $M > 0$ such that $|(c\, u')(x)| \le M\sqrt{x-a}$ for all $x \in (a,s)$. Let $\varepsilon \in (0, s-a)$. Then $u \in W^{1,2}(a+\varepsilon, s)$. Moreover, $(c\, u')|_{(a+\varepsilon,s)} \in W^{1,2}(a+\varepsilon, s)$. Therefore

$$[c\, u'\, \overline{u}]_{a+\varepsilon}^{s} = \int_{a+\varepsilon}^{s} (c\, u'\, \overline{u})' = \int_{a+\varepsilon}^{s} \overline{u}\,(c\, u')' + \int_{a+\varepsilon}^{s} c\, |u'|^2. \tag{3.5}$$

Note that $(c\,Du)'|_{\mathcal{P}} \in L_2(\mathcal{P})$ and $u \in L_2(\mathbb{R})$. Hence $\Big(\overline{u}\,(c\,u')'\Big)|_{(a,s)} \in L_1(a,s)$ and $\lim_{\varepsilon\downarrow 0}\int_{a+\varepsilon}^s \overline{u}\,(c\,u')' \in \mathbb{C}$ exists. It follows from (3.5) that

$$(c\,u'\,\overline{u})(a+\varepsilon) = (c\,u'\,\overline{u})(s) - \int_{a+\varepsilon}^s \overline{u}\,(c\,u')' - \int_{a+\varepsilon}^s c\,|u'|^2. \tag{3.6}$$

Taking the real parts in (3.6) gives

$$\mathrm{Re}\ (c\,u'\,\overline{u})(a+\varepsilon) = \mathrm{Re}\ (c\,u'\,\overline{u})(s) - \mathrm{Re}\int_{a+\varepsilon}^s \overline{u}\,(c\,u')' - \int_{a+\varepsilon}^s (\mathrm{Re}\,c)\,|u'|^2. \tag{3.7}$$

The limit when $\varepsilon \downarrow 0$ of the right hand side of (3.7) exists in $[-\infty,\infty)$. Now we let $L = \lim_{\varepsilon\downarrow 0}\mathrm{Re}\ (c\,u'\,\overline{u})(a+\varepsilon) \in [-\infty,\infty)$. Suppose $L \neq 0$. Then $|\mathrm{Re}\ (c\,u'\,\overline{u})(x)| \geq \delta$ for all small $x-a$, where $\delta = \frac{1}{2}\,(1\wedge|L|)$. Hence $|(c\,u'\,\overline{u})(x)| \geq \delta$ for all small $x-a$ and

$$|u(x)| = \frac{|(c\,u'\,\overline{u})(x)|}{|(c\,u')(x)|} \geq \frac{\delta}{M\,\sqrt{x-a}}.$$

So $u|_{(a,s)} \notin L_2(a,s)$. This is a contradiction. Hence

$$L = \lim_{x\downarrow a}\mathrm{Re}\ (c\,u'\,\overline{u})(x) = 0. \tag{3.8}$$

Similarly, $\lim_{x\uparrow b}\mathrm{Re}\,(c\,u'\,\overline{u})(x) = 0$ for all $b \in E_r$.

Let $k \in K$ be such that $a = a_k$. If I_k is bounded, then by taking limits in (3.7) one deduces that

$$0 = [\mathrm{Re}\ (c\,u'\,\overline{u})]_{a_k}^{b_k} = \mathrm{Re}\int_{a_k}^{b_k} \overline{u}\,(c\,u')' + \int_{a_k}^{b_k} (\mathrm{Re}\,c)\,|u'|^2.$$

Therefore

$$\int_{I_k} (\mathrm{Re}\,c)\,|u'|^2 = -\mathrm{Re}\int_{I_k} \overline{u}\,(c\,u')'. \tag{3.9}$$

Alternatively, if I_k is unbounded, then as in (3.7) we deduce that $\lim_{x\to\infty}\mathrm{Re}\ (c\,u'\,\overline{u})(x) \in (-\infty,\infty]$ exists. Suppose that $\lim_{x\to\infty}\mathrm{Re}\,(c\,u'\,\overline{u})(x) \neq 0$. Then there are $\delta > 0$ and $N \in (a,\infty)$ such that $|c\,u'\,\overline{u})(x)| \geq \delta$ for all $x \in (N,\infty)$. On the other hand

$$|(c\,u')(x)| = \left|\int_a^x (c\,u')'(t)\,dt\right| \leq \sqrt{x-a}\,\|D(c\,Du)\|_2$$

for all $x \in (a,\infty)$. If $x \in (N,\infty)$ then

$$|u(x)| = \left|\frac{(c\,u'\,\overline{u})(x)}{(c\,u')(x)}\right| \geq \frac{\delta}{\|D(c\,Du)\|_2}\,\frac{1}{\sqrt{x-a}}.$$

This implies $u \notin L_2(\mathbb{R})$, which is a contradiction. Therefore $\lim_{x\to\infty}\mathrm{Re}\,(c\,u'\,\overline{u})(x) = 0$ and (3.9) is valid also for unbounded interval I_k.

Summing over all $k \in K$ and using (3.9) give

$$\int_{\mathcal{P}} (\mathrm{Re}\,c)\,|u'|^2 = -\mathrm{Re}\int_{\mathcal{P}} \overline{u}\,(c\,u')' < \infty.$$

So $\sqrt{\operatorname{Re} c}\, Du \in L_2(\mathbb{R})$. Then also $c\, Du \in L_2(\mathbb{R})$. Since $D(c\, Du) \in L_2(\mathbb{R})$, it follows that $(c\, Du)|_{\mathcal{P}} \in W^{1,2}(\mathcal{P})$.

(v $\Longrightarrow$ vi) First note that $u \in D(\mathfrak{a})$ since (i) and (v) are equivalent. Hence $m\, Du \in L_2(\mathbb{R})$ by (3.1) and the equivalence of (i) and (ii). Therefore $D(c\, Du) - m\, Du \in L_2(\mathbb{R})$. The rest is trivial.

(vi $\Longrightarrow$ v) The aim is to show that $\sqrt{\operatorname{Re} c}\, Du \in L_2(\mathbb{R})$, which will then immediately imply that $m\, Du \in L_2(\mathbb{R})$. This together with the fact that $D(c\, Du) - m\, Du \in L_2(\mathbb{R})$ will give $D(c\, Du) \in L_2(\mathbb{R})$ as required.

The following proof applies for both bounded and unbounded connected components (if there are any) of $\mathcal{P}$. Therefore we will present the proof only for an unbounded case.

Now suppose there exists an $s \in \mathbb{R}$ such that $[s, \infty) \subset \mathcal{P}$. Let $f = D(c\, Du) - m\, Du \in L_2(\mathbb{R})$ by hypothesis. Let $x \in [s, \infty)$. Then

$$[c\, u'\, \overline{u}]_s^x = \int_s^x \overline{u}\, (c\, u')' + \int_s^x c\, |u'|^2 = \int_s^x f\, \overline{u} + \int_s^x c\, |u'|^2 + \int_s^x m\, u'\, \overline{u},$$

which implies that

$$\operatorname{Re}\, (c\, u'\, \overline{u})(x) = \operatorname{Re}\, (c\, u'\, \overline{u})(s) + \operatorname{Re} \int_s^x f\, \overline{u} + \int_s^x (\operatorname{Re} c)\, |u'|^2 + \operatorname{Re} \int_s^x m\, u'\, \overline{u}. \tag{3.10}$$

But

$$|m\, u'\, \overline{u}| \le M_m \sqrt{\operatorname{Re} c}\, |u'|\, |u| \le \frac{1}{2} (\operatorname{Re} c)\, |u'|^2 + \frac{M_m^2}{2}\, |u|^2. \tag{3.11}$$

Therefore

$$\operatorname{Re}\, (c\, u'\, \overline{u})(x) \ge \operatorname{Re}\, (c\, u'\, \overline{u})(s) + \operatorname{Re} \int_s^x f\, \overline{u} + \frac{1}{2} \int_s^x (\operatorname{Re} c)\, |u'|^2 - \frac{M_m^2}{2} \int_s^x |u|^2.$$

As a consequence we have

$$\int_s^x (\operatorname{Re} c)\, |u'|^2 - M_0 \le 2 \operatorname{Re}\, (c\, u'\, \overline{u})(x), \tag{3.12}$$

where

$$M_0 = 2\, |\operatorname{Re}\, (c\, u'\, \overline{u})(s)| + 2 \int_s^\infty |f\, \overline{u}| + M_m^2 \int_s^\infty |u|^2 \in \mathbb{R}$$

since $f, u \in L_2(\mathbb{R})$.

We will show that $\sqrt{\operatorname{Re} c}\, u' \in L_2(s, \infty)$. Now suppose the opposite, that is, suppose that $\int_s^\infty (\operatorname{Re} c)\, |u'|^2 = \infty$. We first notice that

$$\begin{aligned}
|(c\, u')(x)| &= \left| (c\, u')(s) + \int_s^x (f + m\, u') \right| \\
&\le |(c\, u')(s)| + \left(\|f\|_2 + \left(\int_s^x |m\, u'|^2 \right)^{1/2} \right) \sqrt{x - s} \\
&= |(c\, u')(s)| + \left(\|f\|_2 + M_m \left(\int_s^x (\operatorname{Re} c)\, |u'|^2 \right)^{1/2} \right) \sqrt{x - s}.
\end{aligned} \tag{3.13}$$

Since $\int_s^\infty (\mathrm{Re}\, c)\, |u'|^2 = \infty$, there exists an $N_1 > s$ such that

$$|(c\, u')(s)| \leq \left(\|f\|_2 + M_m \left(\int_s^{N_1} (\mathrm{Re}\, c)\, |u'|^2 \right)^{1/2} \right) \sqrt{N_1 - s}.$$

Let $x \in (N_1, \infty)$. Then (3.13) gives

$$|(c\, u')(x)| \leq 2 \left(\|f\|_2 + M_m \left(\int_s^{x} (\mathrm{Re}\, c)\, |u'|^2 \right)^{1/2} \right) \sqrt{x - s}. \tag{3.14}$$

Now (3.12) and (3.14) together yield

$$\int_s^x (\mathrm{Re}\, c)\, |u'|^2 - M_0 \leq 4\, |u(x)| \left(\|f\|_2 + M_m \left(\int_s^{x} (\mathrm{Re}\, c)\, |u'|^2 \right)^{1/2} \right) \sqrt{x - s}. \tag{3.15}$$

Let $s' \in [s, \infty)$ be such that $\int_s^{s'} (\mathrm{Re}\, c)\, |u'|^2 \geq M_0$. Then (3.15) gives

$$\int_{s'}^x (\mathrm{Re}\, c)\, |u'|^2 \leq 4\, |u(x)| \left(M_1 + M_m \left(\int_{s'}^{x} (\mathrm{Re}\, c)\, |u'|^2 \right)^{1/2} \right) \sqrt{x - s} \tag{3.16}$$

for all $x > N_1 \vee s'$, where $M_1 = \|f\|_2 + M_m \left(\int_s^{s'} (\mathrm{Re}\, c)\, |u'|^2 \right)^{1/2}$. Let $N_2 > s'$ be such that $\int_{s'}^{N_2} (\mathrm{Re}\, c)\, |u'|^2 \geq \left(\frac{M_1}{M_m} \right)^2$. If $x \geq N_1 \vee N_2$ then (3.16) gives

$$\int_{s'}^x (\mathrm{Re}\, c)\, |u'|^2 \leq 8\, M_m\, |u(x)| \left(\int_{s'}^{x} (\mathrm{Re}\, c)\, |u'|^2 \right)^{1/2} \sqrt{x - s}$$

or equivalently

$$|u(x)| \geq \frac{\left(\int_{s'}^x (\mathrm{Re}\, c)\, |u'|^2 \right)^{1/2}}{8\, M_m \sqrt{x - s}}.$$

Since $\int_{s'}^\infty (\mathrm{Re}\, c)\, |u'|^2 = \infty$, there exists an $N_3 > s'$ such that $\int_{s'}^{N_3} (\mathrm{Re}\, c)\, |u'|^2 \geq 64\, M_m^2$. It follows that

$$|u(x)| \geq \frac{1}{\sqrt{x - s}}$$

for all $x \geq N_1 \vee N_2 \vee N_3$. This implies $u \notin L_2(\mathbb{R})$, which is a contradiction. Hence $\int_s^\infty (\mathrm{Re}\, c)\, |u'|^2 < \infty$.

Thus so far we have proved that $\sqrt{\mathrm{Re}\, c}\, u' \in L_2(I_k)$ for all $k \in K$. Next we will show that $\sqrt{\mathrm{Re}\, c}\, Du \in L_2(\mathbb{R})$. For that let $k \in K$ and suppose $I_k = (a_k, b_k)$ is bounded. Then $m\, u'\, \overline{u} \in L_1(I_k)$ by (3.11). Let $s \in I_k$. Then (3.10) implies that $L = \lim_{x \uparrow b_k} \mathrm{Re}\, (c\, u'\, \overline{u})(x)$ exists in $\mathbb{R}$. On the other hand $\sqrt{\mathrm{Re}\, c}\, u' \in L_2(I_k)$ implies that $c\, Du \in L_2(I_k)$ and $m\, Du \in L_2(I_k)$. But $D(c\, Du) - m\, Du \in L_2(I_k)$. Consequently $(c\, Du)|_{I_k} \in W^{1,2}(I_k) \subset C^{1/2}(\overline{I_k})$. Since $\lim_{x \uparrow b_k} (c\, Du)(x) = 0$, there exists an $M > 0$ such that $(c\, Du)(x) \leq M \sqrt{b_k - x}$. Now suppose $L \neq 0$. Then for all $x \in I_k$ with $b_k - x$ small enough, we have $|\mathrm{Re}\, (c\, u'\, \overline{u})(x)| \geq \delta$, where $\delta = \frac{1}{2}\, (1 \wedge |L|)$. Hence $|(c\, u'\, \overline{u})(x)| \geq \delta$ for all small $b_k - x$ and

$$|u(x)| \geq \frac{\delta}{M \sqrt{b_k - x}}.$$

Thus $u \notin L_2(I_k)$, which is a contradiction. Therefore we must have $L = 0$. Analogously $\lim_{x \downarrow a_k} \operatorname{Re}(c\, u'\, \overline{u})(x) = 0$. Taking limits to the endpoints of I_k on both sides of (3.10) gives

$$\begin{aligned}\int_{I_k} (\operatorname{Re} c)\, |u'|^2 &= -\operatorname{Re} \int_{I_k} f\, \overline{u} + \operatorname{Re} \int_{I_k} m\, u'\, \overline{u} \\ &\le -\operatorname{Re} \int_{I_k} f\, \overline{u} + \frac{1}{2} \int_{I_k} (\operatorname{Re} c)\, |u'|^2 + \frac{M_m^2}{2} \int_{I_k} |u|^2\end{aligned}$$

or equivalently

$$\int_{I_k} (\operatorname{Re} c)\, |u'|^2 \le -2 \operatorname{Re} \int_{I_k} f\, \overline{u} + M_m^2 \int_{I_k} |u|^2.$$

This is for all $k \in K$ such that I_k is bounded. Let

$$\mathcal{P}_b = \{I_k \subset \mathcal{P} \colon k \in K \text{ and } I_k \text{ is bounded}\}.$$

Then

$$\int_{\mathcal{P}_b} (\operatorname{Re} c)\, |u'|^2 \le -2 \operatorname{Re} \int_{\mathcal{P}_b} f\, \overline{u} + M_m^2 \int_{\mathcal{P}_b} |u|^2 < \infty.$$

Since $\mathcal{P}$ has at most two unbounded connected components, it follows that $\sqrt{\operatorname{Re} c}\, Du \in L_2(\mathbb{R})$.

This completes the proof of the lemma. □

3.3 L_p extension

We will now assume in addition to (3.1) that $m \in W^{1,\infty}(\mathbb{R}, \mathbb{C})$ for the rest of this chapter.

Let S be the C_0-semigroup generated by $-A$. We will show in this section that it is possible to extend S to a quasi-contraction C_0-semigroup on $L_p(\mathbb{R})$ for certain $p \in [1, \infty)$. This is the content of Propositions 3.1 and 3.2. Since Proposition 3.2 was already proved, it remains to prove Proposition 3.1.

Proof of Proposition 3.1. We proceed via two steps.

Step 1: Suppose A is strongly elliptic, i.e. $\operatorname{ess\,inf} \operatorname{Re} c > 0$. Using duality arguments we may assume $p \ge 2$. If $m \ne 0$, we choose $\alpha \in (0,1)$ such that $1 - \frac{2}{p} \le \alpha \cos\theta$, set $\varepsilon = \frac{1}{4}(1-\alpha)\, p$ and $\beta = \frac{M_m^2}{4\,\varepsilon} + \|w\|_\infty$. If $m = 0$, we set $\beta = \|w\|_\infty$. Note that in either case β is independent with $\operatorname{ess\,inf} \operatorname{Re} c$. By [AT98, Theorem 18] the semigroup S extends consistently to a C_0-semigroup $S^{(p)}$ on $L_p(\mathbb{R})$. We will show that $\|S_t\|_{p \to p} \le e^{\beta\, t}$ for all $t > 0$. Let $-A_p$ be the generator of $S^{(p)}$ and $\mathcal{D} = D(A_p) \cap D(A) \cap L_\infty(\mathbb{R})$. Let $u \in \mathcal{D}$ and set $v = |u|^{p-2}\, u$. Then $v \in L_q(\mathbb{R}) \cap L_2(\mathbb{R})$, where q is the dual exponent of p. Since $\operatorname{ess\,inf} c > 0$, we have $D(\overline{\mathfrak{a}_0}) = W^{1,2}(\mathbb{R})$. It follows that $u \in D(A) \subset D(\overline{\mathfrak{a}_0}) = W^{1,2}(\mathbb{R})$. Note that

$$v' = \frac{p}{2}\, |u|^{p-2}\, u' + \frac{p-2}{2}\, |u|^{p-4}\, u^2\, \overline{u'}$$

on the set $\{x \in \mathbb{R} \colon u(x) \ne 0\}$. This together with [GT83, Lemma 7.7] give $v \in W^{1,2}(\mathbb{R})$. We will show that $\operatorname{Re} \int (Au)\, \overline{v} \ge -\beta\, \|u\|_p^p$, where here and in the rest of Step 1 the integral is over the set $\{x \in \mathbb{R} \colon u(x) \ne 0\}$. We have

$$\operatorname{Re} \int (A_p u)\, \overline{v} = \operatorname{Re} \int (Au)\, \overline{v} = \operatorname{Re} \overline{\mathfrak{a}_0}(u, v) = \operatorname{Re} \int (c\, u'\, \overline{v'} + m\, u'\, \overline{v} + w\, u\, \overline{v}). \tag{3.17}$$

But
$$c\,u'\,\overline{v'} = \frac{p}{2}\,c\,|u|^{p-2}\,|u'|^2 + \frac{p-2}{2}\,c\,|u|^{p-4}\,\overline{u}^2\,(u')^2.$$
Since $|c| \leq \frac{1}{\cos\theta}\,\mathrm{Re}\,c$ we have
$$\begin{aligned}\mathrm{Re}\,(c\,u'\,\overline{v'}) &\geq \frac{p}{2}\,(\mathrm{Re}\,c)\,|u|^{p-2}\,|u'|^2 - \frac{p-2}{2}\,\frac{\mathrm{Re}\,c}{\cos\theta}\,|u|^{p-2}\,|u'|^2 \\ &= (\mathrm{Re}\,c)\left(\frac{p}{2} - \frac{p-2}{2}\cdot\frac{1}{\cos\theta}\right)|u|^{p-2}\,|u'|^2. \end{aligned} \tag{3.18}$$

Next we use $w \in L_\infty(\mathbb{R})$ to obtain
$$|w\,u\,\overline{v}| \leq \|w\|_\infty\,|u|^p. \tag{3.19}$$

If $m = 0$ then $1 - \frac{2}{p} \leq \cos\theta$. It follows from (3.17), (3.18) and (3.19) that
$$\mathrm{Re}\int (A_p u)\,\overline{v} \geq -\|w\|_\infty\,\|u\|_p^p = -\beta\,\|u\|_p^p.$$

If $m \neq 0$, then we use $|m| \leq M_m\,\sqrt{\mathrm{Re}\,c}$ to obtain
$$|m\,u'\,\overline{v}| \leq \varepsilon\,(\mathrm{Re}\,c)\,|u|^{p-2}\,|u'|^2 + \frac{M_m^2}{4\varepsilon}|u|^p. \tag{3.20}$$

It follows from (3.17), (3.18), (3.19) and (3.20) that
$$\begin{aligned}\mathrm{Re}\int (A_p u)\,\overline{v} &\geq \int (\mathrm{Re}\,c)\left(\frac{p}{2} - \frac{p-2}{2}\cdot\frac{1}{\cos\theta} - \varepsilon\right)|u|^{p-2}\,|u'|^2 - \left(\frac{M_m^2}{4\varepsilon} + \|w\|_\infty\right)|u|^p \\ &\geq -\beta\,\|u\|_p^p.\end{aligned}$$

Therefore the restriction $(\beta\,I + A_p)|_{\mathcal{D}}$ is accretive. Since $\mathcal{D}$ is a core for A_p by Theorem 2.28, the operator $\beta\,I + A_p$ is also accretive by [LP61, Lemma 3.4]. By the Lumer-Phillips theorem we have that $S^{(p)}$ is a quasi-contraction semigroup and $\|S_t\|_{p\to p} \leq e^{\beta\,t}$ for all $t > 0$.

Step 2: Suppose A is degenerate elliptic. Using duality arguments we may assume $p \in (1, 2)$. Let $n \in \mathbb{N}$ and set $c_n = c + \frac{1}{n}$. Let $A_{[n]} = -\frac{d}{dx}\,c_n\,\frac{d}{dx}$ be the operator associated with the form (3.2) with c replaced by c_n. Moreover, let $S^{[n]}$ be the semigroup generated by $-A_{[n]}$. Then by Step 1 there exists a $\beta > 0$ such that $S^{[n]}$ extends consistently to a C_0-semigroup $S^{(n,p)}$ on $L_p(\mathbb{R})$ and $\|S_t^{(n,p)}\|_{p\to p} \leq e^{\beta\,t}$ for all $t > 0$ and $n \in \mathbb{N}$. Let $t > 0$ and $u \in L_{2,c}(\mathbb{R})$. By [AE12, Corollary 3.9] we have that $\lim_{n\to\infty}\|S_t^{[n]}u - S_t u\|_2 = 0$. Passing to a subsequence if necessary we may assume $\lim_{n\to\infty} S_t^{[n]}u = S_t u$ almost everywhere on $\mathbb{R}$. It follows from Fatou's lemma that
$$\|S_t u\|_p = \|\liminf S_t^{[n]}u\|_p \leq \liminf \|S_t^{[n]}u\|_p \leq e^{\beta t}\,\|u\|_p < \infty.$$

But $L_{2,c}(\mathbb{R})$ is dense in $L_2(\mathbb{R}) \cap L_p(\mathbb{R})$. Therefore $\|S_t u\|_p \leq e^{\beta t}\,\|u\|_p$ for all $u \in L_2(\mathbb{R}) \cap L_p(\mathbb{R})$. Hence $S_t|_{L_2(\mathbb{R})\cap L_p(\mathbb{R})}$ extends consistently to a quasi-contraction operator $S_t^{(p)}$ on $L_p(\mathbb{R})$. Then $S^{(p)}$ is a C_0-semigroup on $L_p(\mathbb{R})$ by [Voi92, Proposition 1]. The proof is now complete. □

3.4 The generator on L_p

Propositions 3.1 and 3.2 provide conditions such that the semigroup S extends consistently to a C_0-semigroup $S^{(p)}$ on $L_p(\mathbb{R})$. Let $-A_p$ be the generator of $S^{(p)}$. In this section we give a detailed description for the domain of A_p.

In addition to (3.1) and $m \in W^{1,\infty}(\mathbb{R})$, we will assume for the rest of the chapter that m also satisfies the following condition: For all $x_0 \in E$ there exists a $\delta > 0$ such that

$$\int_{(x_0-\delta,x_0+\delta)\cap\mathcal{P}} \left|\frac{m}{c}\right| < \infty. \tag{3.21}$$

An easy example is $m \in W^{1,\infty}(\mathbb{R})$ with $|m| \le M \operatorname{Re} c$ for some $M \ge 0$, which also supersedes (3.1) as c is bounded. A more interesting example is provided in Example 3.26.

Condition (3.21) implies in particular that

$$\int_{a_k}^{m_k} \left|\frac{m}{c}\right| < \infty \tag{3.22}$$

for all $k \in K$ with $a_k \in \mathbb{R}$ and

$$\int_{m_k}^{b_k} \left|\frac{m}{c}\right| < \infty \tag{3.23}$$

for all $k \in K$ with $b_k \in \mathbb{R}$. Define $H \colon \mathcal{P} \longrightarrow \mathbb{C}$ by

$$H(x) = e^{\int_x^{m_k} \frac{m}{c}} \tag{3.24}$$

if $x \in I_k$ and $k \in K$. If $k \in K$ is such that $a_k \in \mathbb{R}$, then we obtain from (3.22) that

$$e^{-\int_{a_k}^{m_k} |\frac{m}{c}|} \le |H(x)| \le e^{\int_{a_k}^{m_k} |\frac{m}{c}|}$$

for all $x \in (a_k, m_k]$, the limit $\lim_{x\downarrow a_k} H(x)$ exists in $\mathbb{C}$ and is non-zero. Similarly if $k \in K$ is such that $b_k \in \mathbb{R}$, then we obtain from (3.23) that

$$e^{-\int_{m_k}^{b_k} |\frac{m}{c}|} \le |H(x)| \le e^{\int_{m_k}^{b_k} |\frac{m}{c}|}$$

for all $x \in [m_k, b_k)$, the limit $\lim_{x\uparrow b_k} H(x)$ exists in $\mathbb{C}$ and is also non-zero. It is useful to keep these facts in mind as we will use them frequently later on.

For all $p \in [1,\infty)$ define the operator B_p in $L_p(\mathbb{R})$ by

$$B_p u = -D(c\,Du) + m\,Du + w\,u$$

on the domain

$$\begin{aligned} D(B_p) = \{u \in L_p(\mathbb{R}) \cap W^{1,p}_{\mathrm{loc}}(\mathcal{P}) \colon\ & c\,Du \in W^{1,p}_{\mathrm{loc}}(\mathcal{P}),\ D(c\,Du) - m\,Du \in L_p(\mathbb{R}), \\ & \lim\nolimits_{x\downarrow a}(c\,Du)(x) = 0 \text{ for all } a \in E_l \text{ and} \\ & \lim\nolimits_{x\uparrow b}(c\,Du)(x) = 0 \text{ for all } b \in E_r\}. \end{aligned}$$

Clearly $W^{2,p}(\mathbb{R}) \subset D(B_p)$ for all $p \in [1,\infty)$. We will show that $B_p = A_p$ for all $p \in [1,\infty)$ under suitable conditions.

Lemma 3.11. *The operator B_p is closed for all $p \in [1, \infty)$.*

Proof. Let $(u_n)_{n \in \mathbb{N}}$ be a sequence in $D(B_p)$. Let $u, v \in L_p(\mathbb{R})$. Suppose $\lim u_n = u$ in $L_p(\mathbb{R})$ and $\lim B_p u_n = v$ in $L_p(\mathbb{R})$. Write

$$v_n = B_p u_n \tag{3.25}$$

for all $n \in \mathbb{N}$. By elliptic regularity $\lim u_n(x) = u(x)$ for all $x \in \mathcal{P}$. First suppose that $E \neq \emptyset$. Let $a \in E_l$ and $k \in K$ be such that $a = a_k$. It follows from (3.25) that

$$H(x)\,(c\,u_n')(x) = \int_{a_k}^{x} (-v_n + w\,u_n)\,H$$

for all $x \in I_k$ and $n \in \mathbb{N}$, where H is defined by (3.24). Therefore

$$\lim_{n\to\infty} H(x)\,(c\,u_n')(x) = \int_{a_k}^{x} (-v + w\,u)\,H$$

uniformly for all $x \in I_k$. Let $J \Subset \mathcal{P}$ be an open interval. Then $\operatorname{Re} c$ is strictly positive on J. It follows that u is differentiable on J and $H(x)\,(c\,u')(x) = \int_{a_k}^{x} (-v + w\,u)\,H$ for all $x \in J$. Hence $\lim_{x \downarrow a_k} H(x)\,(c\,u')(x) = 0$. But $\lim_{x \downarrow a_k} |H(x)|$ exists in $\mathbb{R}$ and is non-zero. Therefore we have $\lim_{x \downarrow a_k} (c\,u')(x) = 0$. Moreover, $(H\,c\,u')'(x) = \big((-v + w\,u)\,H\big)(x)$ for almost every $x \in I_k$. Equivalently $-(c\,u')' + m\,u' + w\,u = v$ almost everywhere on I_k. Similarly $\lim_{x \uparrow b} (c\,u')(x) = 0$ for all $b \in E_r$. We proved that $-D(c\,Du) + m\,Du + w\,u = v\,\mathbb{1}_{\mathcal{P}}$. Clearly $v_n|_{\mathcal{N}} = w\,u_n|_{\mathcal{N}}$ for all $n \in \mathbb{N}$. So $v|_{\mathcal{N}} = (w\,u)|_{\mathcal{N}}$ and $-D(c\,Du) + m\,Du + w\,u = v$. Hence $u \in D(B_p)$ and $v = B_p u$. Thus B_p is closed. If $E = \emptyset$, then the proof is similar with small modifications. □

We first concentrate on the case $p \in (1, \infty)$.

Lemma 3.12. *Let $p \in (1, \infty)$. Suppose*

(I) $\left|1 - \frac{2}{p}\right| < \cos\theta$ *or*

(II) $\left|1 - \frac{2}{p}\right| \le \cos\theta$ *and $m = 0$.*

Then B_p is quasi-accretive.

Proof. We will show that there exists a $\beta > 0$ such that $\operatorname{Re} \int (B_p u)\,|u|^{p-2}\,\overline{u} \ge -\beta\,\|u\|_p^p$ for all $u \in D(B_p)$, where here and in the rest of the proof the integral is over the set $\{x \in \mathbb{R} \colon u(x) \neq 0\}$. Let $u \in D(B_p)$. Let $a \in E_l$, $s \in (a, \infty)$ and suppose that $(a, s] \subset \mathcal{P}$. Let $\varepsilon \in (0, s - a)$. Note that $\operatorname{Re} c$ is strictly positive on $(a + \varepsilon, s)$. Therefore elliptic regularity gives $u|_{(a+\varepsilon,s)} \in W^{2,p}(a + \varepsilon, s)$. It follows from [MS08, Theorem 3.1] that

$$\begin{aligned}\operatorname{Re} \int_{a+\varepsilon}^{s} (B_p u)\,|u|^{p-2}\,\overline{u} = \int_{a+\varepsilon}^{s} &\operatorname{Re} \left(\frac{p}{2}\,c\,|u|^{p-2}\,|u'|^2 + \frac{p-2}{2}\,c\,|u|^{p-4}\,\overline{u}^2\,(u')^2 \right) \\ &+ \int_{a+\varepsilon}^{s} \operatorname{Re}\big(m\,u'\,|u|^{p-2}\,\overline{u} + w\,|u|^p\big) \\ &- \operatorname{Re}\,(c\,u'\,|u|^{p-2}\,\overline{u})\Big|_{a+\varepsilon}^{s}.\end{aligned} \tag{3.26}$$

Clearly $\lim_{\varepsilon\downarrow 0}\int_{a+\varepsilon}^s (B_p u)\,|u|^{p-2}\,\overline{u} = \int_a^s (B_p u)\,|u|^{p-2}\,\overline{u} \in \mathbb{C}$.

We next consider the integrands on the right hand side of (3.26). If $m \neq 0$, then we choose $\alpha \in (0,1)$ such that $\left|1-\frac{2}{p}\right| \le \alpha\,\cos\theta$, set $\varepsilon' = \frac{1}{4}\,(1-\alpha)\,p$ and $\beta = \frac{M_m^2}{4\,\varepsilon'} + \|w\|_\infty$. If $m = 0$ then we set $\beta = \|w\|_\infty$. Then it follows as in the proof of Proposition 3.1 Step 1 that

$$\operatorname{Re}\left(\frac{p}{2}\,c\,|u|^{p-2}\,|u'|^2 + \frac{p-2}{2}\,c\,|u|^{p-4}\,\overline{u}^2\,(u')^2 + m\,u'\,|u|^{p-2}\,\overline{u} + w\,|u|^p\right) \ge -\beta\,|u|^p.$$

Therefore

$$\lim_{\varepsilon\downarrow 0}\int_{a+\varepsilon}^s \operatorname{Re}\left(\frac{p}{2}\,c\,|u|^{p-2}\,|u'|^2 + \frac{p-2}{2}\,c\,|u|^{p-4}\,\overline{u}^2\,(u')^2 + m\,u'\,|u|^{p-2}\,\overline{u} + (w+\beta)\,|u|^p\right)$$
$$= \int_a^s \operatorname{Re}\left(\frac{p}{2}\,c\,|u|^{p-2}\,|u'|^2 + \frac{p-2}{2}\,c\,|u|^{p-4}\,\overline{u}^2\,(u')^2 + m\,u'\,|u|^{p-2}\,\overline{u} + (w+\beta)\,|u|^p\right) \in [0,\infty].$$

By a similar argument used in (3.8) we obtain from (3.26) that $\lim_{x\downarrow a}\operatorname{Re}(c\,u'\,|u|^{p-2}\,\overline{u})(x) = 0$. Similarly $\lim_{x\uparrow b}\operatorname{Re}(c\,u'\,|u|^{p-2}\,\overline{u})(x) = 0$ if $b \in E_r$. Arguing as at the end of the proof of Lemma 3.10 gives $\lim_{x\to\infty}\operatorname{Re}(c\,u'\,|u|^{p-2}\,\overline{u})(x) = 0$ if $(a,\infty) \subset \mathcal{P}$. Taking limits in (3.26) we have

$$\operatorname{Re}\int_{I_k} (B_p u)\,|u|^{p-2}\,\overline{u} = \int_{I_k} \operatorname{Re}\left(\frac{p}{2}\,c\,|u|^{p-2}\,|u'|^2 + \frac{p-2}{2}\,c\,|u|^{p-4}\,\overline{u}^2\,(u')^2\right)$$
$$+ \int_{I_k}\left(m\,u'\,|u|^{p-2}\,\overline{u} + w\,|u|^p\right) \ge -\beta\int_{I_k}|u|^p$$

for all $k \in K$. Summing over all k gives the lemma. □

Proposition 3.13. *Let $p \in (1,\infty)$. Suppose*

(I) $\left|1-\frac{2}{p}\right| < \cos\theta$ *or*

(II) $\left|1-\frac{2}{p}\right| \le \cos\theta$ *and $m = 0$.*

Let $S^{(p)}$ be the C_0-semigroup on $L_p(\mathbb{R})$ which is consistent with S. Then $-B_p$ is the generator of $S^{(p)}$.

Proof. Recall that $-A_p$ is the generator of $S^{(p)}$. We first show that $D(A_p) \subset D(B_p)$. Let $u \in D(A_p) \cap D(A)$. Then $u \in L_2(\mathbb{R}) \cap L_p(\mathbb{R})$ and

$$-D(c\,Du) + m\,Du + w\,u = Au = A_p u \in L_p(\mathbb{R}).$$

Let $J \Subset \mathcal{P}$ be an open interval. Since $\operatorname{Re} c$ is strictly positive on J, it follows from elliptic regularity that $u|_J \in W^{2,p}(\mathbb{R})$ and $c\,Du \in W^{1,p}_{\rm loc}(\mathcal{P})$. Since $u \in D(A)$, we know that $(c\,Du)|_{\mathcal{P}} \in W^{1,2}_0(\mathcal{P})$. Hence $\lim_{x\downarrow a}(c\,Du)(x) = 0$ for all $a \in E_l$ and $\lim_{x\uparrow b}(c\,Du)(x) = 0$ for all $b \in E_r$. Therefore $D(A_p) \cap D(A) \subset D(B_p)$ and $A_p|_{D(A_p)\cap D(A)} \subset B_p$. Since B_p is closed and $D(A_p) \cap D(A)$ is a core for A_p, we obtain that $A_p \subset B_p$.

Finally since A_p is the generator of a quasi-contraction C_0-semigroup, A_p is m-quasi-accretive by the Lumer-Phillips theorem. But B_p is quasi-accretive by Lemma 3.12. Therefore $A_p = B_p$ as required. □

The next two results for $p = 1$ are of independent interests, which are not essential to the core characterisation for A_1 in Section 3.6.

Lemma 3.14. *Suppose $c \in W^{1,\infty}(\mathbb{R}, \mathbb{R})$ and $m = 0$. Then B_1 is quasi-accretive.*

Proof. We may assume without loss of generality that $w = 0$. We will show that there exists a $\beta > 0$ such that $\operatorname{Re} \int (B_1 u) \, \overline{u} \, |u|^{-1} \, \mathbb{1}_{[u \neq 0]} \geq -\beta \, \|u\|_1$ for all $u \in D(B_1)$. Let $u \in D(B_1)$. Let $a \in E_l$, $s \in (a, \infty)$ and suppose that $(a, s] \subset \mathcal{P}$. Let $\varepsilon \in (0, s - a)$. Note that $\operatorname{Re} c$ is strictly positive on $(a + \varepsilon, s)$. Therefore elliptic regularity gives $u|_{(a+\varepsilon,s)} \in W^{2,1}(a + \varepsilon, s)$. It follows from Proposition 2.33 that

$$\operatorname{Re} \int_{a+\varepsilon}^{s} (B_1 u) \, \overline{u} \, |u|^{-1} \, \mathbb{1}_{[u \neq 0]} = L(a + \varepsilon, s) + \int_{a+\varepsilon}^{s} c \, \big(\operatorname{Im} (u' \, \overline{u})\big)^2 \, |u|^{-3} \, \mathbb{1}_{[u \neq 0]} \\ - c \operatorname{Re} \big(u' \, \overline{u} \, |u|^{-1}\big)\big|_{a+\varepsilon}^{s}, \tag{3.27}$$

where

$$L(a + \varepsilon, s) = \lim_{\delta \to 0} \int_{a+\varepsilon}^{s} c \, |u'|^2 \, \delta^2 \, (|u|^2 + \delta^2)^{-3/2} \in [0, \infty)$$

and we use the convention that $(u' \, \overline{u} \, |u|^{-1})(x) = 0$ for all $x \in \mathbb{R}$ such that $u(x) = 0$. Clearly $\lim_{\varepsilon \downarrow 0} \int_{a+\varepsilon}^{s} (B_1 u) \, \overline{u} \, |u|^{-1} \, \mathbb{1}_{[u \neq 0]} = \int_{a}^{s} (B_1 u) \, \overline{u} \, |u|^{-1} \, \mathbb{1}_{[u \neq 0]} \in \mathbb{C}$. Furthermore

$$\lim_{\varepsilon \downarrow 0} \int_{a+\varepsilon}^{s} c \, \big(\operatorname{Im} (u' \, \overline{u})\big)^2 \, |u|^{-3} \, \mathbb{1}_{[u \neq 0]} = \int_{a}^{s} c \, \big(\operatorname{Im} (u' \, \overline{u})\big)^2 \, |u|^{-3} \, \mathbb{1}_{[u \neq 0]} \in [0, \infty].$$

Note that $0 \leq \big|\big(c \operatorname{Re} (u' \, \overline{u} \, |u|^{-1})\big)(x)\big| \leq |(c \, u')(x)|$ for all $x \in (a, s)$. It follows that $\lim_{x \downarrow a} \big(c \operatorname{Re} (u' \, \overline{u} \, |u|^{-1})\big)(x) = 0$ since $\lim_{x \downarrow a} (c \, u')(x) = 0$ by hypothesis. Hence we deduce from (3.27) that $\lim_{\varepsilon \downarrow 0} L(a + \varepsilon, s)$ exists in $[0, \infty]$.

Similarly if $b \in E_r$ then $\lim_{x \uparrow b} \big(c \operatorname{Re} (u' \, \overline{u} \, |u|^{-1})\big)(x) = 0$ and $\lim_{\varepsilon \downarrow 0} L(s, b - \varepsilon)$ exists in $[0, \infty]$.

Hence

$$\operatorname{Re} \int_{I_k} (B_1 u) \, \overline{u} \, |u|^{-1} \, \mathbb{1}_{[u \neq 0]} = \lim_{\varepsilon \downarrow 0} \big(L(a_k + \varepsilon, s) + L(s, b_k - \varepsilon)\big) + \int_{I_k} c \, \big(\operatorname{Im} (u' \, \overline{u})\big)^2 \, |u|^{-3} \, \mathbb{1}_{[u \neq 0]} \geq 0$$

for all $k \in K$ with $I_k = (a_k, b_k)$ bounded and $s \in I_k$.

Now let $a \in E_l$ and suppose $(a, \infty) \subset \mathcal{P}$. Without loss of generality we may assume that $a = -1$. Let $s \in (0, \infty)$. Let $\tau \in C_c^\infty(\mathbb{R}, \mathbb{R})$ be such that $\operatorname{supp} \tau \subset (a, \infty)$ and $\tau|_{[0,s+1]} = \mathbb{1}|_{[0,s+1]}$. For all $n \in \mathbb{N}$ define $\tau_n \in C_c^\infty(\mathbb{R})$ by

$$\tau_n(x) = \begin{cases} \tau(x) & \text{if } x \in (-\infty, s), \\ \tau(n^{-1} \, x) & \text{if } x \in [s, \infty). \end{cases}$$

Then $\operatorname{supp} \tau_n \subset (a, \infty)$. Let $u_n = u \, \tau_n$ for all $n \in \mathbb{N}$. Since $u|_{(a,\infty)} \in W^{2,1}_{\mathrm{loc}}(a, \infty)$, we deduce that $u_n \in W^{2,1}(\mathbb{R}) \subset D(B_1)$ for all $n \in \mathbb{N}$. In particular $u_n|_{[s,\infty)} \in W^{2,1}(s, \infty)$ for all $n \in \mathbb{N}$. Moreover, $u_n(n(s + 1)) = 0$ for all $n \in \mathbb{N}$. It follows from Proposition 2.33 that we have

$$\operatorname{Re} \int_{s}^{\infty} (B_1 u_n) \, \overline{u_n} \, |u_n|^{-1} \, \mathbb{1}_{[u_n \neq 0]} = P_n + \big(c \operatorname{Re} (u_n' \, \overline{u_n} \, |u_n|^{-1})\big)(s), \tag{3.28}$$

for all $n \in \mathbb{N}$, where

$$P_n = L(s, n(s+1)) + \int_s^\infty c\,\big(\mathrm{Im}\,(u_n' \,\overline{u_n})\big)^2\, |u_n|^{-3}\, \mathbb{1}_{[u \neq 0]}$$

and

$$L(s, n(s+1)) = \lim_{\delta \to 0} \int_s^{n(s+1)} c\,|u_n'|^2\, \delta^2\, (|u_n|^2 + \delta^2)^{-3/2} \in [0, \infty)$$

and we use the convention that $(u_n' \,\overline{u_n}\, |u_n|^{-1})(x) = 0$ for all $x \in \mathbb{R}$ such that $u_n(x) = 0$. Since $\lim_{x \downarrow a} (c\,u')(x) = 0$, we have

$$|(c\,u')(x)| = \Big| \int_a^x (c\,u')' \Big| \leq \int_a^x |(c\,u')'| \leq \int_a^\infty |(c\,u')'| < \infty$$

for all $x \in (a, \infty)$, as $(c\,u')' \in L_1(a, \infty)$ by hypothesis. Therefore

$$\begin{aligned}
\int_s^\infty |(c\,u_n')' - (c\,u')'| &= \int_s^\infty |(c\,u')'\,(\tau_n - 1) + 2\,c\,u'\,\tau_n' + c'\,u\,\tau_n' + c\,u\,\tau_n''| \\
&\leq \int_{ns}^\infty |(c\,u')'| + \frac{2}{n}\, \|\tau'\|_\infty \int_a^\infty |(c\,u')'| + \frac{1}{n}\, \|\tau'\|_\infty\, \|c'\|_\infty \int_a^\infty |u| \\
&\quad + \frac{1}{n^2}\, \|\tau''\|_\infty\, \|c\|_\infty \int_s^\infty |u|
\end{aligned}$$

for all $n \in \mathbb{N}$. Hence $\lim_{n \to \infty} (c\,u_n')'|_{(s,\infty)} = (c\,u')'|_{(s,\infty)}$ in $L_1(s, \infty)$. Consequently

$$\begin{aligned}
\lim_{n \to \infty} \int_s^\infty (B_1 u_n)\, \overline{u_n}\, |u_n|^{-1}\, \mathbb{1}_{[u_n \neq 0]} &= \lim_{n \to \infty} \int_s^\infty (B_1 u_n)\, \overline{u}\, |u|^{-1}\, \mathbb{1}_{[u \neq 0]}\, \mathbb{1}_{[\tau_n \neq 0]} \\
&= \int_s^\infty (B_1 u)\, \overline{u}\, |u|^{-1}\, \mathbb{1}_{[u \neq 0]}.
\end{aligned}$$

Also

$$|(c\,u_n')(s) - (c\,u')(s)| = |(c\,u)(s)\, \tau_n'(s)| \leq \frac{1}{n}\, \|\tau'\|_\infty\, |(c\,u)(s)|.$$

This implies

$$\lim_{n \to \infty} \big(c\,u_n' \,\overline{u_n}\, |u_n|^{-1}\big)(s) = \lim_{n \to \infty} \big(c\,u_n' \,\overline{u}\, |u|^{-1}\big)(s) = \big(c\,u' \,\overline{u}\, |u|^{-1}\big)(s).$$

It now follows from (3.28) that $P = \lim_{n \to \infty} P_n$ exists in $[0, \infty)$. Hence

$$\mathrm{Re} \int_s^\infty (B_1 u)\, \overline{u}\, |u|^{-1}\, \mathbb{1}_{[u \neq 0]} = P + \big(c\, \mathrm{Re}\,(u' \,\overline{u}\, |u|^{-1})\big)(s). \tag{3.29}$$

By similar arguments as used before in the case of bounded intervals of $\mathcal{P}$, we obtain that $\lim_{\varepsilon \downarrow 0} L(a + \varepsilon, s) \in [0, \infty]$ and

$$\begin{aligned}
\mathrm{Re} \int_a^s (B_1 u)\, \overline{u}\, |u|^{-1}\, \mathbb{1}_{[u \neq 0]} &= \lim_{\varepsilon \downarrow 0} L(a + \varepsilon, s) + \int_a^s c\,\big(\mathrm{Im}\,(u' \,\overline{u})\big)^2\, |u|^{-3}\, \mathbb{1}_{[u \neq 0]} \\
&\quad - \big(c\, \mathrm{Re}\,(u' \,\overline{u}\, |u|^{-1})\big)(s).
\end{aligned} \tag{3.30}$$

Adding (3.29) and (3.30) together gives

$$\mathrm{Re} \int_a^\infty (B_1 u)\,\overline{u}\,|u|^{-1}\,\mathbb{1}_{[u\neq 0]} = P + \lim_{\varepsilon\downarrow 0} L(a+\varepsilon, s) + \int_a^s c\left(\mathrm{Im}\,(u'\,\overline{u})\right)^2 |u|^{-3}\,\mathbb{1}_{[u\neq 0]} \geq 0.$$

Similar result holds for the interval $(-\infty, b)$ if $(-\infty, b) \subset \mathcal{P}$, where $b \in E_r$. Thus we have proved that

$$\mathrm{Re} \int_{I_k} (B_1 u)\,\overline{u}\,|u|^{-1}\,\mathbb{1}_{[u\neq 0]} \geq 0$$

for all intervals $I_k \subset \mathcal{P}$. Summing over all k gives the lemma. □

Proposition 3.15. *Suppose $c \in W^{1,\infty}(\mathbb{R}, \mathbb{R})$ and $m = 0$. Let $S^{(1)}$ be the C_0-semigroup on $L_1(\mathbb{R})$ which is consistent with S. Then $-B_1$ is the generator of $S^{(1)}$.*

Proof. The proof is the same as that of Proposition 3.13. □

We finish this section with some regularity properties of the operator A_p. For that we need some more definitions. Define the function $W \colon \mathcal{P} \longrightarrow \mathbb{C}$ by

$$W(x) = \int_x^{m_k} \frac{1}{H\,c}$$

if $x \in I_k$ and $k \in K$. Recall that we defined the function $Z \colon \mathbb{R} \longrightarrow \mathbb{R}$ by

$$Z(x) = \begin{cases} \int_x^{m_k} \frac{1}{\mathrm{Re}\,c} & \text{if } x \in I_k \text{ and } k \in K, \\ \infty & \text{if } x \in \mathcal{N}. \end{cases}$$

For convenience, in what follows we will write $D(A_p) \subset C[a,b]$ to mean that $u|_{(a,b)} \in C[a,b]$ for all $u \in D(A_p)$, where $p \in (1,\infty)$ and $a, b \in \mathbb{R}$ with $a < b$.

Lemma 3.16. *Let $p \in (1,\infty)$. Suppose*

(I) $\left|1 - \frac{2}{p}\right| < \cos\theta$ *or*

(II) $\left|1 - \frac{2}{p}\right| \leq \cos\theta$ *and* $m = 0$.

Let $-A_p$ be the generator of the C_0-semigroup $S^{(p)}$ on $L_p(\mathbb{R})$ which is consistent with S. Let q be the dual exponent of p and let $k \in K$.

(a) *If $a_k \in \mathbb{R}$ and $Z|_{(a_k, m_k)} \in L_q(a_k, m_k)$, then $D(A_p) \subset C[a_k, m_k]$.*

(b) *If $b_k \in \mathbb{R}$ and $Z|_{(m_k, b_k)} \in L_q(m_k, b_k)$, then $D(A_p) \subset C[m_k, b_k]$.*

(c) *If I_k is bounded and $Z|_{I_k} \in L_q(I_k)$, then*

$$u(b_k) - u(a_k) = \int_{I_k} W \cdot (-A_p u + w\,u) \cdot H$$

and

$$|u(b_k) - u(x)| \leq 3\, e^{\int_{I_k} \left|\frac{m}{c}\right|} (1 + \|w\|_\infty)\, \|Z|_{I_k}\|_{L_q(I_k)}\, \|u\,\mathbb{1}_{I_k}\|_{D(A_p)}$$

for all $u \in D(A_p)$ and $x \in I_k$.

Proof. The proof is inspired by the proof of [CMP98, Theorem 3.3].

(a) Let $u \in D(A_p) = D(B_p)$. Clearly u is continuous on $\mathcal{P}$ since $u \in W^{1,p}_{\mathrm{loc}}(\mathcal{P})$. Let $f = -A_p u + w\, u$. Then $f = (c\, u')' - m\, u'$ on (a_k, b_k). Then $f \in L_p(a_k, b_k)$. Moreover, $\lim_{x \downarrow a_k} (c\, u')(x) = 0$. So if $t \in (a_k, m_k]$ then

$$H(t)\,(c\,u')(t) = \int_{a_k}^t f\,H.$$

Therefore if $x \in (a_k, m_k]$ then

$$\begin{aligned} u(m_k) - u(x) &= \int_x^{m_k} \frac{1}{H(t)\,c(t)} \int_{a_k}^t f(s)\,H(s)\,ds\,dt \\ &= W(x) \int_{a_k}^x f\,H + \int_x^{m_k} W\,f\,H, \end{aligned} \tag{3.31}$$

where we used integration by parts. But $|\int_{a_k}^x f\,H| \le e^{\int_{a_k}^{m_k} |\frac{m}{c}|}\, \|f\|_p\, (x - a_k)^{1/q}$ for all $x \in (a_k, m_k]$. Also $\lim_{x \downarrow a_k} (x - a_k)^{1/q}\, W(x) = 0$ since $Z|_{(a_k, m_k]}$ is decreasing, $Z|_{(a_k, m_k)} \in L_q(a_k, m_k)$ and $|W(x)| \le e^{\int_{a_k}^{m_k} |\frac{m}{c}|}\, |Z(x)|$ for all $x \in (a_k, m_k]$. So $\lim_{x \downarrow a_k} W(x) \int_{a_k}^x f\,H = 0$. Obviously $\lim_{x \downarrow a_k} \int_x^{m_k} W\,f\,H$ exists. So $\lim_{x \downarrow a_k} u(x)$ exists and u has a continuous representative on $[a_k, m_k]$. Note that

$$u(m_k) - u(a_k) = \int_{a_k}^{m_k} W\,f\,H. \tag{3.32}$$

This proves Statement (a).

(b) The proof is similar to that of (a) and we also have

$$u(b_k) - u(m_k) = \int_{m_k}^{b_k} W\,f\,H. \tag{3.33}$$

(c) Now let $k \in K$ and suppose I_k is bounded. The first part of (c) follows by adding (3.32) and (3.33). Also (3.31) and (3.33) together give

$$u(b_k) - u(x) = W(x) \int_{a_k}^x f\,H + \int_x^{b_k} W\,f\,H$$

for all $x \in (a_k, m_k]$. Since I_k is bounded, we have the estimate

$$e^{-\int_{a_k}^{m_k} |\frac{m}{c}|} \le |H(x)| \le e^{\int_{a_k}^{m_k} |\frac{m}{c}|}$$

for all $x \in I_k$. With this in mind we obtain

$$|u(b_k) - u(x)| \le e^{\int_{I_k} |\frac{m}{c}|} \Big(|Z(x)|\,(x - a_k)^{1/q} + \|Z|_{I_k}\|_{L_q(I_k)} \Big)\, \|f\|_{L_p(I_k)}$$

for all $x \in (a_k, m_k]$. Since Z is positive and decreasing on (a_k, m_k), we deduce that

$$\|Z|_{I_k}\|_{L_q(I_k)}^q \ge \int_{(a_k+x)/2}^x |Z(t)|^q\,dt \ge \int_{(a_k+x)/2}^x |Z(x)|^q\,dt = \frac{x - a_k}{2}\,|Z(x)|^q.$$

Hence $Z(x)\,(x-a_k)^{1/q} \le \sqrt[q]{2}\,\|Z|_{I_k}\|_{L_q(I_k)}$. Also

$$\|f\|_{L_p(I_k)} = \|A_p u - w\,u\|_{L_p(I_k)} \le (1+\|w\|_\infty)\,\|u\,\mathbb{1}_{I_k}\|_{D(A_p)}.$$

Therefore

$$|u(b_k)-u(x)| \le (1+\sqrt[q]{2})\,e^{\int_{I_k}\left|\frac{m}{c}\right|}\,(1+\|w\|_\infty)\,\|Z|_{I_k}\|_{L_q(I_k)}\,\|u\,\mathbb{1}_{I_k}\|_{D(A_p)}.$$

The proof is similar if $x \in [m_k, b_k)$. □

3.5 Core characterisation for $p \in (1,\infty)$

Throughout this section we fix $p \in (1,\infty)$ such that

(I) $\left|1-\frac{2}{p}\right| < \cos\theta$ or

(II) $\left|1-\frac{2}{p}\right| \le \cos\theta$ and $m = 0$.

Let $-A_p$ be the generator of the C_0-semigroup $S^{(p)}$ on $L_p(\mathbb{R})$ which is consistent with S. Let q be the dual exponent of p. We will devote this section to proving Theorem 3.3 which gives a characterisation for when the space of test functions $C_c^\infty(\mathbb{R})$ is a core for A_p. We first consider the 'only if' part of the theorem. There are two cases to consider: whether or not the point in E is a cluster point of E. These are the content of the following two lemmas.

Lemma 3.17. *Let $x_0 \in E_l \cap E_r$. Suppose there exists a $\delta > 0$ such that $Z|_{(x_0-\delta,x_0+\delta)} \in L_q(x_0-\delta, x_0+\delta)$. Then $C_c^\infty(\mathbb{R})$ is not a core for A_p.*

Proof. Since $x_0 \in E_l \cap E_r$, there are $k, l \in K$ such that $x_0 = b_k = a_l$. Then $Z|_{(m_k,b_k)} \in L_q(m_k, b_k)$ and $Z|_{(a_l,m_l)} \in L_q(a_l, m_l)$. By Lemma 3.16 and the closed graph theorem, there exists an $M > 0$ such that $D(A_p) \subset C[m_k, b_k]$, $D(A_p) \subset C[a_l, m_l]$ and

$$\|u\|_{C[m_k,b_k]} \le M\,\|u\|_{D(A_p)} \quad \text{and} \quad \|u\|_{C[a_l,m_l]} \le M\,\|u\|_{D(A_p)}$$

for all $u \in D(A_p)$. There exists a $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi|_{[m_k,m_l]} = (2M+1)\,\mathbb{1}$. Then $u = \chi\,\mathbb{1}_{[x_0,\infty)} \in D(A_p)$. Suppose there exists a $v \in C_c^\infty(\mathbb{R})$ such that $\|u-v\|_{D(A_p)} \le 1$. Then $|v(x)| = |u(x)-v(x)| \le M\,\|u-v\|_{D(A_p)} \le M$ for all $x \in (m_k, b_k)$. So $|v(x_0)| \le M$. Similarly $|v(x)-(2M+1)| = |u(x)-v(x)| \le M$ for all $x \in (a_l, m_l)$. Hence $|v(x_0)| \ge M+1$. This is a contradiction. □

Lemma 3.18. *Let $x_0 \in E$. Suppose there exists a $\delta_0 > 0$ such that $Z|_{(x_0-\delta_0,x_0+\delta_0)} \in L_q(x_0-\delta_0, x_0+\delta_0)$. Furthermore suppose that x_0 is a cluster point of E. Then $C_c^\infty(\mathbb{R})$ is not a core for A_p.*

Proof. Without loss of generality we may assume that $[x_0-\delta_0, x_0) \subset \mathcal{P}$, $x_0+\delta_0 \in E_r$ and $\int_{(x_0-\delta_0,x_0+\delta_0)\cap\mathcal{P}} \left|\frac{m}{c}\right| < \infty$. Note that

$$\infty\,|(x_0, x_0+\delta_0) \cap \mathcal{N}| \le \int_{x_0}^{x_0+\delta_0} |Z|^q < \infty.$$

Therefore we must have $|(x_0, x_0 + \delta_0) \cap \mathcal{N}| = 0$.

It follows from Lemma 3.16(b) that $D(A_p) \subset C[x_0 - \delta_0, x_0]$ and there exists an $M_1 > 0$ such that $\|u\|_{C[x_0-\delta_0,x_0]} \leq M_1 \|u\|_{D(A_p)}$ for all $u \in D(A_p)$. Let

$$M_2 = e^{\int_{x_0}^{x_0+\delta_0} |\frac{m}{c}|} (1 + \|w\|_\infty) \, \|Z|_{(x_0,x_0+\delta_0)}\|_{L_q(x_0,x_0+\delta_0)}.$$

There exists a $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi|_{(x_0-\delta_0,x_0+\delta_0)} = 3\,\mathbb{1}$. Set $u = \chi\,\mathbb{1}_{(-\infty,x_0)}$. Then $u \in D(A_p)$. Choose $\varepsilon = \min(M_1^{-1}, (4M_2)^{-1}, \sqrt[p]{\delta_0})$. Suppose there exists a test function $v \in C_c^\infty(\mathbb{R})$ such that $\|u - v\|_{D(A_p)} < \varepsilon$. Then $|u(x) - v(x)| \leq M_1 \|u - v\|_{D(A_p)} \leq 1$ for all $x \in (x_0 - \delta_0, x_0)$, so $|v(x_0)| \geq 2$.

Let $\delta \in (0, \delta_0]$ and suppose that $x_0 + \delta \in E_r$. We use Lemma 3.16(c) to deduce that

$$\begin{aligned} v(x_0 + \delta) - v(x_0) &= \int_{(x_0,x_0+\delta)} v' = \int_{(x_0,x_0+\delta)\cap\mathcal{P}} v' = \sum_{\substack{k\in K \\ I_k\subset(x_0,x_0+\delta)}} \int_{I_k} v' \\ &= \sum_{\substack{k\in K \\ I_k\subset(x_0,x_0+\delta)}} (v(b_k) - v(a_k)) = \sum_{\substack{k\in K \\ I_k\subset(x_0,x_0+\delta)}} \int_{I_k} W \cdot (-A_p v + w\,v) \cdot H. \end{aligned}$$

Nevertheless we have

$$\Big| \int_{I_k} W \cdot (-A_p v + w\,v) \cdot H \Big| \leq e^{\int_{I_k} |\frac{m}{c}|} \int_{I_k} |Z|\,|A_p v - w\,v| \leq e^{\int_{x_0}^{x_0+\delta_0} |\frac{m}{c}|} \int_{I_k} |Z|\,|A_p v - w\,v|$$

for all $k \in K$ such that $I_k \subset (x_0, x_0 + \delta_0)$. Hence

$$\begin{aligned} |v(x_0 + \delta) - v(x_0)| &\leq \sum_{\substack{k\in K \\ I_k\subset(x_0,x_0+\delta)}} \Big| \int_{I_k} W \cdot (-A_p v + w\,v) \cdot H \Big| \\ &\leq \sum_{\substack{k\in K \\ I_k\subset(x_0,x_0+\delta)}} e^{\int_{x_0}^{x_0+\delta_0} |\frac{m}{c}|} \int_{I_k} |Z|\,|A_p v - w\,v| \\ &= e^{\int_{x_0}^{x_0+\delta_0} |\frac{m}{c}|} \int_{(x_0,x_0+\delta_0)} |Z|\,|A_p v - w\,v| \leq M_2 \, \|v\,\mathbb{1}_{(x_0,x_0+\delta_0)}\|_{D(A_p)}. \end{aligned}$$

So $|v(b_k) - v(x_0)| \leq M_2 \|v\,\mathbb{1}_{(x_0,x_0+\delta_0)}\|_{D(A_p)}$ for all $k \in K$ such that $I_k \subset (x_0, x_0 + \delta_0)$. If $k \in K$ and $I_k \subset (x_0, x_0 + \delta_0)$, then it follows from Lemma 3.16(c) that $|v(b_k) - v(x)| \leq 3\,M_2 \|v\,\mathbb{1}_{(x_0,x_0+\delta_0)}\|_{D(A_p)}$ for all $x \in I_k$. Therefore $|v(x_0) - v(x)| \leq 4\,M_2 \|v\,\mathbb{1}_{(x_0,x_0+\delta_0)}\|_{D(A_p)}$ for all $x \in (x_0, x_0 + \delta_0) \cap \mathcal{P}$. But $(x_0, x_0 + \delta_0) \cap \mathcal{P}$ is dense in $(x_0, x_0 + \delta_0)$ because $|(x_0, x_0+\delta_0) \cap \mathcal{N}| = 0$. So $|v(x_0) - v(x)| \leq 4\,M_2 \|v\,\mathbb{1}_{(x_0,x_0+\delta_0)}\|_{D(A_p)}$ for all $x \in (x_0, x_0 + \delta_0)$. Next

$$\|v\,\mathbb{1}_{(x_0,x_0+\delta_0)}\|_{D(A_p)} = \|(u - v)\,\mathbb{1}_{(x_0,x_0+\delta_0)}\|_{D(A_p)} \leq \|u - v\|_{D(A_p)} \leq (4\,M_2)^{-1}.$$

Since $|v(x_0)| \geq 2$, this implies that $|v(x)| \geq 1$ for all $x \in (x_0, x_0 + \delta_0)$. In particular $\int_{(x_0,x_0+\delta_0)} |v|^p \geq \delta_0$.

On the other hand we have

$$\int_{(x_0,x_0+\delta_0)} |v|^p = \int_{(x_0,x_0+\delta_0)} |u - v|^p \leq \|u - v\|_{D(A_p)}^p < \varepsilon^p \leq \delta_0.$$

This is a contradiction. □

We next turn to the 'if' part of Theorem 3.3. Since $-A_p$ is the generator of a quasi-contraction C_0-semigroup, there exists a $\beta > 0$ such that $(\beta I + A_p)(D(A_p)) = L_p(\mathbb{R})$. Without loss of generality we will assume from now on that $\beta = 1$ and that $-A_p$ is the generator of a contraction C_0-semigroup. It follows that the space $C_c^\infty(\mathbb{R})$ is a core for A_p if and only if $(I + A_p)(C_c^\infty(\mathbb{R}))$ is dense in $L_p(\mathbb{R})$. In other words, the space $C_c^\infty(\mathbb{R})$ is a core for A_p if and only if for all $g \in L_q(\mathbb{R})$ with $\int_{\mathbb{R}} \overline{g}\,(I + A_p)u = 0$ for all $u \in C_c^\infty(\mathbb{R})$ it follows that $g = 0$.

We start with a lemma.

Lemma 3.19. *Let $g \in L_q(\mathbb{R})$ and suppose that $\int_{\mathbb{R}} \overline{g}\,(I + A_p)u = 0$ for all $u \in C_c^\infty(\mathbb{R})$. Then we have the following.*

(a) *$g \in W^{2,q}_{\rm loc}(\mathcal{P})$ and*

$$\begin{aligned} -g\,\mathbb{1}_{\mathcal{P}} &= -D(\overline{c}\,Dg) - D(\overline{m}\,g) + \overline{w}\,g \\ &= -D(\overline{c}\,Dg) - \overline{m}\,Dg + (\overline{w} - \overline{m'})\,g. \end{aligned}$$

(b) *Let $k \in K$ and $s \in I_k$. If $a_k \in \mathbb{R}$ then $(\overline{c}\,Dg + \overline{m}\,g)|_{(a_k,s)} \in W^{1,q}(a_k, s)$. If $b_k \in \mathbb{R}$ then $(\overline{c}\,Dg + \overline{m}\,g)|_{(s,b_k)} \in W^{1,q}(s, b_k)$.*

(c) *If $k \in K$, then $(\overline{c}\,Dg)|_{I_k}$ extends to a continuous function on $\overline{I_k}$.*

For all $x_0 \in E_l$ and $y_0 \in E_r$ define

$$L_{x_0} = \lim_{x \downarrow x_0} (\overline{c}\,Dg)(x) \quad \textit{and} \quad R_{y_0} = \lim_{y \uparrow y_0} (\overline{c}\,Dg)(y).$$

Moreover, set $L_{-\infty} = 0$ and $R_\infty = 0$. Then the following hold.

(d) *Let $u \in W^{2,p}_c(\mathbb{R})$. Then with the convention that $u(-\infty) = u(\infty) = 0$ the series $\sum_{k\in K} \Big(R_{b_k}\,u(b_k) - L_{a_k}\,u(a_k)\Big)$ is absolutely convergent and*

$$\sum_{k\in K} \Big(R_{b_k}\,u(b_k) - L_{a_k}\,u(a_k)\Big) = -\int_{\mathcal{N}} (1 + \overline{w})\,g\,u.$$

(e) *Let $x_0 \in E_r$ and $x_1 \in E_l$ with $x_0 < x_1$. Then*

$$|R_{x_0} - L_{x_1}| \le 2\,(1 + \|w\|_\infty)\,\|g\|_q\,(x_1 - x_0)^{1/p}.$$

(f) *Let $k \in K$ and suppose that I_k is bounded. Then*

$$|L_{a_k} - (\overline{c}\,Dg)(x)| \le |(m\,g)(x)| + (1 + \|w\|_\infty)\,\|g\|_{L_q(I_k)}\,(x - a_k)^{1/p}$$

and

$$|R_{b_k} - (\overline{c}\,Dg)(x)| \le |(m\,g)(x)| + (1 + \|w\|_\infty)\,\|g\|_{L_q(I_k)}\,(b_k - x)^{1/p}$$

for all $x \in I_k$.

(g) *Let $x_0 \in E_r$ and suppose that x_0 is a cluster point of E. Then*

$$R_{x_0} = \lim_{\substack{x \downarrow x_0 \\ x \in E_l}} L_x = \lim_{\substack{x \downarrow x_0 \\ x \in E_r}} R_x.$$

(h) *Let $x_0 \in E_r$ and suppose that x_0 is a cluster point of E. Then there exists a $\delta > 0$ such that $(\overline{c}\, Dg)|_{(x_0, x_0+\delta)} \in L_q(x_0, x_0+\delta)$.*

(i) *Let $x_0 \in E$. Then $\lim_{x \to x_0} (\overline{m}\, g)(x) = 0$.*

Proof. (a) Since $\int_{\mathbb{R}} \overline{g}\,(I + A_p)u = 0$ for all $u \in C_c^\infty(\mathcal{P})$ and $\operatorname{Re} c$ is strictly positive on compact subsets of $\mathcal{P}$, it follows from elliptic regularity that $g \in W^{2,q}_{\mathrm{loc}}(\mathcal{P})$. Also

$$-\int_{\mathcal{P}} g\, \overline{u} = \int_{\mathbb{R}} g\, \overline{A_p u} = \int_{\mathcal{P}} g'\, \overline{c\, u'} + g\, \overline{m\, u'} + g\, \overline{w\, u}$$

for all $u \in C_c^\infty(\mathcal{P})$. Therefore

$$\begin{aligned} -g\, \mathbb{1}_{\mathcal{P}} &= -D(\overline{c}\, Dg) - D(\overline{m}\, g) + \overline{w}\, g \\ &= -D(\overline{c}\, Dg) - \overline{m}\, Dg + (\overline{w} - \overline{m'})\, g, \end{aligned}$$

where we used that $m \in W^{1,\infty}(\mathbb{R})$.

(b) Let $k \in K$ and $s \in I_k$. Suppose $a_k \in \mathbb{R}$. Let $x \in (a_k, s)$. Let $\varepsilon \in (0, s - a_k)$. Then

$$(\overline{c}\, Dg + \overline{m}\, g)\Big|_{a_k+\varepsilon}^{x} = \int_{a_k+\varepsilon}^{x} (\overline{c}\, Dg + \overline{m}\, g)'. \tag{3.34}$$

Statement (a) gives $(\overline{c}\, Dg + \overline{m}\, g)'|_{(a_k,s)} \in L_q(a_k, s) \subset L_1(a_k, s)$. Therefore $\lim_{\varepsilon \downarrow 0} \int_{a_k+\varepsilon}^{x} (\overline{c}\, Dg + \overline{m}\, g)'$ exists in $\mathbb{C}$. It follows that $L = \lim_{\varepsilon \downarrow 0} (\overline{c}\, Dg + \overline{m}\, g)(a_k + \varepsilon)$ exists in $\mathbb{C}$. Taking limits when $\varepsilon \downarrow 0$ on both sides of (3.34) gives

$$(\overline{c}\, Dg + \overline{m}\, g)(x) = L + \int_{a_k}^{x} (\overline{c}\, Dg + \overline{m}\, g)'.$$

Therefore

$$|(\overline{c}\, Dg + \overline{m}\, g)(x)| \le |L| + \|(\overline{c}\, Dg + \overline{m}\, g)'\|_{L_q(a_k,s)}\, (s - a_k)^{1/p}.$$

Hence $\overline{c}\, Dg + \overline{m}\, g$ is bounded on (a_k, s), which implies $(\overline{c}\, Dg + \overline{m}\, g)|_{(a_k,s)} \in L_q(a_k, s)$. Recall that $(\overline{c}\, Dg + \overline{m}\, g)'|_{(a_k,s)} \in L_q(a_k, s)$ by Statement (a). These two together justify the claim.

Similarly we derive that $(\overline{c}\, Dg + \overline{m}\, g)|_{(s,b_k)} \in W^{1,q}(s, b_k)$ if $b_k \in \mathbb{R}$.

(c) Let $k \in K$. Then $(\overline{c}\, Dg)|_{I_k} \in W^{1,q}_{\mathrm{loc}}(I_k) \subset C(I_k)$. If $x \in I_k$ then

$$[\overline{H^{-1}}\, \overline{c}\, Dg]_x^{m_k} = \int_x^{m_k} \Big(D(\overline{c}\, Dg) + \overline{m}\, Dg \Big)\, \overline{H^{-1}},$$

where H is defined by (3.24). Since $H|_{I_k}$ extends to a continuous function $\widetilde{H}$ on $\overline{I_k}$ and $\widetilde{H}(x) \neq 0$ for all $x \in \overline{I_k}$, we have $(\overline{c}\, Dg)|_{I_k}$ extends to a continuous function on $\overline{I_k}$.

(d) Since $\int_{\mathbb{R}} \overline{g}\,(I + A_p)u = 0$ for all $u \in C_c^\infty(\mathbb{R})$, it follows by density and continuity that $\int_{\mathbb{R}} \overline{g}\,(I + A_p)u = 0$ for all $u \in W^{2,p}_c(\mathbb{R})$.

Let $k \in K$ and $a', b' \in \mathbb{R}$ be such that $a_k < a' < b' < b_k$. Then

$$- \int_{a'}^{b'} g \cdot \overline{A_p v} = \int_{a'}^{b'} g \, \overline{(c \, v')' - m \, v' - w \, u} = [\overline{c} \, g \, \overline{v'}]_{a'}^{b'} - \int_{a'}^{b'} \Big(g' \, \overline{c \, v'} + g \, \overline{m \, v'} + g \, \overline{w \, v} \Big) \quad (3.35)$$

for all $v \in W_c^{2,p}(\mathbb{R})$. Suppose $a_k \in \mathbb{R}$. Since there exists a $v \in C_c^\infty(\mathbb{R})$ such that $v'|_{(a_k-1, a_k+1)} = \mathbb{1}$, it follows that $L = \lim_{a' \downarrow a_k} (\overline{c} \, g)(a')$ exists. If $L \neq 0$ then

$$\infty = \int_{a_k}^{m_k} \frac{|\overline{c} \, g|^q}{|c|^q} \leq \int_{\mathbb{R}} |g|^q < \infty,$$

which is a contradiction. Hence $\lim_{a' \downarrow a_k} (\overline{c} \, g)(a') = 0$. Similarly $\lim_{b' \uparrow b_k} (\overline{c} \, g)(b') = 0$ if $b_k \in \mathbb{R}$. By taking the limits to the endpoints of the interval I_k on both sides of (3.35) and noticing that v has compact support, we have

$$- \int_{I_k} g \cdot \overline{A_p v} = - \int_{I_k} \Big(\overline{c} \, g' \, \overline{v'} + \overline{m} \, g \, \overline{v'} + \overline{w} \, g \, \overline{v} \Big) \quad (3.36)$$

for all $v \in W_c^{2,p}(\mathbb{R})$. Now let $u \in W_c^{2,p}(\mathbb{R})$ and replace v with $\overline{u}$ in (3.36) to obtain

$$- \int_{I_k} g \cdot \overline{A_p \overline{u}} = - \int_{I_k} \Big(\overline{c} \, g' \, u' + \overline{m} \, g \, u' + \overline{w} \, g \, u \Big).$$

Then

$$\begin{aligned} \int_{\mathbb{R}} g \, u &= - \int_{\mathbb{R}} g \cdot \overline{A_p \overline{u}} = - \sum_{k \in K} \int_{I_k} g \cdot \overline{A_p \overline{u}} - \int_{\mathcal{N}} g \cdot \overline{A_p \overline{u}} \\ &= - \sum_{k \in K} \int_{I_k} \Big(\overline{c} \, g' \, u' + \overline{m} \, g \, u' + \overline{w} \, g \, u \Big) - \int_{\mathcal{N}} \overline{w} \, g \, u. \end{aligned} \quad (3.37)$$

Note that the series in (3.37) is absolutely convergent.

On the other hand

$$\int_{\mathbb{R}} g \, u = \int_{\mathcal{N}} g \, u + \int_{\mathcal{P}} g \, u = \int_{\mathcal{N}} g \, u + \sum_{k \in K} \int_{I_k} g \, u \quad (3.38)$$

where the series is again absolutely convergent. Let $k \in K$ and $a', b' \in \mathbb{R}$ be such that $a_k < a' < b' < b_k$. Then

$$\begin{aligned} \int_{a'}^{b'} g \, u &= \int_{a'}^{b'} \Big((\overline{c} \, g')' + (\overline{m} \, g)' - \overline{w} \, g \Big) \, u \\ &= [\overline{c} \, g' \, u + \overline{m} \, g \, u]_{a'}^{b'} - \int_{a'}^{b'} \Big(\overline{c} \, g' \, u' + \overline{m} \, g \, u' + \overline{w} \, g \, u \Big). \end{aligned} \quad (3.39)$$

Let $s \in I_k$. Suppose $a_k \in \mathbb{R}$. It follows from Statement (b) that $(\overline{c} \, g' + \overline{m} \, g)|_{(a_k, s)} \in W^{1,q}(a_k, s) \subset C^{1/p}[a_k, s]$. Thus $\lim_{a' \downarrow a_k} (\overline{c} \, g' + \overline{m} \, g)(a')$ exists in $\mathbb{C}$. But $\lim_{a' \downarrow a_k} (\overline{c} \, g')(a')$ exists in $\mathbb{C}$ by Statement (c). Hence $L' = \lim_{a' \downarrow a_k} (\overline{m} \, g)(a')$ exists in $\mathbb{C}$. Our next aim is to show that $L' = 0$. Indeed, suppose that $L' \neq 0$. We consider three cases.

Case 1: Suppose there exists a $\delta > 0$ such that $(a_k, a_k + \delta) \subset (a_k, s)$ and $m(x) = 0$ for all

$x \in (a_k, a_k + \delta)$. Then clearly $L' = 0$, which gives a contradiction.
Case 2: Suppose there exists a $\delta > 0$ such that $(a_k, a_k + \delta) \subset (a_k, s)$ and $m(x) \neq 0$ for all $x \in (a_k, a_k + \delta)$. Then

$$\infty = \int_{a_k}^{a_k+\delta} \frac{|\overline{m}\, g|^q}{|m|^q} \leq \int_{\mathbb{R}} |g|^q < \infty,$$

which is a contradiction.
Case 3: Suppose for all $\delta > 0$ there exist $z_1, z_2 \in (a_k, a_k + \delta)$ such that $m(z_1) = 0$ and $m(z_2) \neq 0$. Let $a' \in (a_k, s)$. Note that m is continuous on $\mathbb{R}$. Therefore there exist a $z \in (a_k, a')$ and a $\delta_0 > 0$ such that $[z, z + \delta_0] \subset (a_k, a')$, $m(z) = 0$ and $m(x) \neq 0$ for all $x \in (z, z + \delta_0]$. It follows as in Case 2 that $(\overline{m}\, g)(z) = 0$. Furthermore we have

$$\begin{aligned}
|(\overline{m}\, g)(a')| &= \Big| \int_z^{a'} (\overline{m}\, g)' \Big| = \Big| \int_z^{a'} \big(\overline{m'}\, g + \overline{m}\, g'\big) \Big| \\
&\leq \|m'\|_\infty \, \|g\|_{L_q(\mathbb{R})} \, (a' - z)^{1/p} + \|\overline{c}\, Dg\|_{L_\infty(a_k, s)} \int_z^{a'} \Big|\frac{m}{c}\Big| \\
&\leq \|m'\|_\infty \, \|g\|_{L_q(\mathbb{R})} \, (a' - a_k)^{1/p} + \|\overline{c}\, Dg\|_{L_\infty(a_k, s)} \int_{a_k}^{a'} \Big|\frac{m}{c}\Big|.
\end{aligned}$$

It follows from (3.21) that

$$\lim_{a' \downarrow a_k} (\overline{m}\, g)(a') = 0. \tag{3.40}$$

Similarly

$$\lim_{b' \uparrow b_k} (\overline{m}\, g)(b') = 0 \tag{3.41}$$

if $b_k \in \mathbb{R}$.

Now taking the limits to the endpoints of the interval I_k on both sides of (3.39), using Statement (b) and again noticing that u has compact support, we have

$$\int_{I_k} g\, u = R_{b_k}\, u(b_k) - L_{a_k}\, u(a_k) - \int_{I_k} \Big((\overline{c}\, g' + \overline{m}\, g)\, u' + \overline{w}\, g\, u \Big).$$

Substituting this into (3.38) yields

$$\int_{\mathbb{R}} g\, u = \int_{\mathcal{N}} g\, u + \sum_{k \in K} \Big(R_{b_k}\, u(b_k) - L_{a_k}\, u(a_k) \Big) - \sum_{k \in K} \int_{I_k} \Big(\overline{c}\, g'\, u' + \overline{m}\, g\, u' + \overline{w}\, g\, u \Big). \tag{3.42}$$

Statement (d) now follows by comparing (3.37) and (3.42).

(e) Let $k \in K$ and suppose that I_k is bounded. Let $a', b' \in \mathbb{R}$ be such that $a_k < a' < b' < b_k$. Then

$$\begin{aligned}
(\overline{c}\, g')(b') - (\overline{c}\, g')(a') &= \int_{a'}^{b'} (\overline{c}\, g')' = \int_{a'}^{b'} g - (\overline{m}\, g)' + \overline{w}\, g \\
&= -[\overline{m}\, g]_{a'}^{b'} + \int_{a'}^{b'} (1 + \overline{w})\, g.
\end{aligned} \tag{3.43}$$

By taking limits together with (3.40) and (3.41) we obtain

$$R_{b_k} - L_{a_k} = \int_{I_k} (1 + \overline{w})\, g. \tag{3.44}$$

Now let $k_0, k_1 \in K$ be such that $x_0 = b_{k_0}$ and $x_1 = a_{k_1}$. Let $n \in \mathbb{N}$. Then there exists a $u \in C_c^\infty(\mathbb{R})$ such that $0 \le u \le \mathbb{1}$, $\operatorname{supp} u \subset \Big((x_0 - \frac{1}{n}) \vee a_{k_0}, (x_1 + \frac{1}{n}) \wedge b_{k_1}\Big)$ and $u(x) = 1$ for all $x \in [x_0, x_1]$. Define $h \colon \mathbb{R} \longrightarrow \mathbb{R}$ by

$$h(x) = \begin{cases} 1 & \text{if there exists a } k \in K \text{ such that } x \in I_k \subset \operatorname{supp} u, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\operatorname{supp} h \subset [x_0, x_1]$. Moreover, it follows from Statement (d) and (3.44) that

$$-\int_{\mathcal{N}} (1 + \overline{w})\, g\, u = R_{x_0} - L_{x_1} + \int_{\mathbb{R}} (1 + \overline{w})\, g\, h.$$

This implies

$$\begin{aligned} |R_{x_0} - L_{x_1}| &\le (1 + \|w\|_\infty) \Big(\int_{\mathbb{R}} |g\, u| + \int_{\mathbb{R}} |g\, h| \Big) \\ &\le (1 + \|w\|_\infty)\, \|g\|_{L_q(\mathbb{R})}\, (\|u\|_{L_p(\mathbb{R})} + \|h\|_{L_p(\mathbb{R})}) \\ &\le 2\,(1 + \|w\|_\infty)\, \|g\|_{L_q(\mathbb{R})} \Big(x_1 - x_0 + \frac{2}{n} \Big)^{1/p}. \end{aligned}$$

This is for all $n \in \mathbb{N}$. Therefore Statement (e) follows.

(f) For the first inequality let $b' = x$ in (3.43) and take the limit $a' \downarrow a_k$ to obtain

$$(\overline{c}\, g')(x) - L_{a_k} = -(\overline{m}\, g)(x) + \int_{a_k}^{x} (1 + \overline{w})\, g. \tag{3.45}$$

Thus

$$|L_{a_k} - (\overline{c}\, g')(x)| \le |(m\, g)(x)| + (1 + \|w\|_\infty)\, \|g\|_{L_q(I_k)}\, (x - a_k)^{1/p}.$$

The second inequality is proved similarly.

(g) This follows from Statement (e).

(h) By Statement (g) there exists a $\delta > 0$ such that $|R_{x_0} - L_x| \le 1$ for all $x \in E_l \cap (x_0, x_0 + \delta)$. Without loss of generality we may assume that $x_0 + \delta \in E_r$. Now let $x \in \mathcal{P} \cap (x_0, x_0 + \delta)$. There exists a $k \in K$ such that $x \in I_k$. Then $I_k \subset (x_0, x_0 + \delta)$ and therefore $a_k \in E_l \cap (x_0, x_0 + \delta)$. So $|R_{x_0} - L_{a_k}| \le 1$. Next, by Statement (f) we estimate

$$|L_{a_k} - (\overline{c}\, Dg)(x)| \le |(m\, g)(x)| + (1 + \|w\|_\infty)\, \|g\|_{L_q(\mathbb{R})}\, \sqrt[p]{\delta}.$$

It follows that

$$|(\overline{c}\, Dg)(x)| \le |(m\, g)(x)| + (1 + \|w\|_\infty)\, \|g\|_{L_q(\mathbb{R})}\, \sqrt[p]{\delta} + |R_{x_0}| + 1.$$

Obviously if $x \in \mathcal{N} \cap (x_0, x_0 + \delta)$, then $(\overline{c}\, Dg)(x) = 0$. Since $|m\, g| \le \|m\|_\infty\, |g| \in L_q(\mathbb{R})$, we have $\overline{c}\, Dg \in L_q(x_0, x_0 + \delta)$ as claimed.

(i) By (3.40) and (3.41) we have that $\lim_{x \downarrow a_k} (\overline{m}\, g)(x) = 0$ for all $k \in K$ with $a_k \in \mathbb{R}$ and $\lim_{x \uparrow b_k} (\overline{m}\, g)(x) = 0$ for all $k \in K$ with $b_k \in \mathbb{R}$. Therefore we may assume without loss of generality that $x_0 \in E_r$ and we will show that $\lim_{x \downarrow x_0} (\overline{m}\, g)(x) = 0$.

If x_0 is not a cluster point of E, then there exists a $\delta > 0$ such that $(x_0, x_0 + \delta) \subset \mathcal{N}$. Since $(\overline{m}\, g)(x) = 0$ for all $x \in \mathcal{N}$, we clearly have $\lim_{x \downarrow x_0} (\overline{m}\, g)(x) = 0$.

Now suppose x_0 is a cluster point of E. By (3.21) there exists a $\delta > 0$ such that $\int_{(x_0, x_0+\delta)} \left|\frac{m}{c}\right| < \infty$. If $x \in \mathcal{N}$ then again $(\overline{m}\, g)(x) = 0$. Thus we only need to consider the case $x \in (x_0, x_0 + \delta) \cap \mathcal{P}$. Then there exists a $k \in K$ such that $x \in I_k \cap (x_0, x_0 + \delta)$. It follows from (3.45) that

$$\begin{aligned}
|(\overline{c}\, Dg + \overline{m}\, g)(x)| &\le |L_{a_k}| + (1 + \|w\|_\infty)\, \|g\|_{L_q(\mathbb{R})}\, (x - a_k)^{1/p} \\
&\le |R_{x_0}| + 3\,(1 + \|w\|_\infty)\, \|g\|_{L_q(\mathbb{R})}\, \sqrt[p]{\delta},
\end{aligned}$$

where we used Statement (e) in the second step. Let $M_0 = |R_{x_0}| + 3\,(1 + \|w\|_\infty)\, \|g\|_{L_q(\mathbb{R})}\, \sqrt[p]{\delta}$. Then M_0 is independent with k and we have

$$\begin{aligned}
|(\overline{m}\, g)(x)| &= \left| \int_{a_k}^{x} (\overline{m}\, g)' \right| = \left| \int_{a_k}^{x} (\overline{m'}\, g + \overline{m}\, g') \right| \\
&\le \int_{a_k}^{x} \left| \left(m' - \frac{m^2}{c} \right) g \right| + \int_{a_k}^{x} \left| \frac{m}{c} \right| |\overline{c}\, g' + \overline{m}\, g| \\
&\le (\|m'\|_\infty + M_m)\, \|g\|_{L_q(\mathbb{R})}\, (x - a_k)^{1/p} + M_0 \int_{a_k}^{x} \left| \frac{m}{c} \right| \\
&\le (\|m'\|_\infty + M_m)\, \|g\|_{L_q(\mathbb{R})}\, (x - x_0)^{1/p} + M_0 \int_{(x_0, x) \cap \mathcal{P}} \left| \frac{m}{c} \right|.
\end{aligned}$$

It now follows from (3.21) that $\lim_{x \downarrow x_0} (\overline{m}\, g)(x) = 0$ as required. □

Our next aim is to prove that if $\int_{(x-\delta, x+\delta)} |Z|^q = \infty$ for all $x \in E$ and $\delta > 0$, and if g is as in Lemma 3.19, then $L_x = 0$ for all $x \in E_l$ and $R_x = 0$ for all $x \in E_r$. Again we have to split the proof into several cases, which depend on whether a point in E is a cluster point of E or not.

Lemma 3.20. *Adopt the assumptions and notation as in Lemma 3.19. Then the following are valid.*

(a) *If $x_0 \in E_l$ and $Z|_{(x_0, x_0+\delta)} \notin L_q(x_0, x_0 + \delta)$ for all $\delta > 0$, then $L_{x_0} = 0$.*

(b) *If $x_0 \in E_r$ and $Z|_{(x_0-\delta, x_0)} \notin L_q(x_0 - \delta, x_0)$ for all $\delta > 0$, then $R_{x_0} = 0$.*

Proof. The proof is inspired by the proof of [CMP98, Proposition 3.5]. We only prove Statement (a). Suppose that $L_{x_0} \neq 0$. Replacing g by $\lambda\, g$ for some $\lambda \in \mathbb{C}$ if necessary, we may assume without loss of generality that $L_{x_0} = 2 + \tan\theta$. There exists a $\delta > 0$ such that $(x_0, x_0 + \delta] \subset \mathcal{P}$ and $\left|(\overline{c}\, g')(x) - (2 + \tan\theta)\right| < 1$ for all $x \in (x_0, x_0 + \delta)$. Set $s = x_0 + \delta$. Let $x \in (x_0, s)$. Write $y = (\overline{c}\, g')(x) - (2 + \tan\theta)$. Then $|y| < 1$ and

$$\begin{aligned}
\operatorname{Re} g'(x) &= \operatorname{Re}\left(\frac{2+\tan\theta+y}{\overline{c(x)}}\right) \\
&= \frac{\operatorname{Re} c(x)\cdot\operatorname{Re}(2+\tan\theta+y)-\operatorname{Im} c(x)\cdot\operatorname{Im}(2+\tan\theta+y)}{|c(x)|^2} \\
&\geq \frac{(1+\tan\theta)\operatorname{Re} c(x)-\tan\theta\operatorname{Re} c(x)}{|c(x)|^2} \\
&= \frac{\operatorname{Re} c(x)}{|c(x)|^2} \geq \frac{\cos^2\theta}{\operatorname{Re} c(x)}.
\end{aligned}$$

It follows that

$$\operatorname{Re} g(x) = \operatorname{Re} g(s) - \int_x^s \operatorname{Re} g'(t)\,dt \leq \operatorname{Re} g(s) - \int_x^s \frac{\cos^2\theta}{\operatorname{Re} c(t)}\,dt = \alpha - (\cos^2\theta)\,Z(x)$$

for all $x \in (x_0, x_0+\delta)$, where $\alpha = \operatorname{Re} g(s) + \cos^2\theta \int_s^{m_k} \frac{1}{\operatorname{Re} c}$ and $k \in K$ is such that $x_0 = a_k$. Hence $g \notin L_q(\mathbb{R})$. This is a contradiction. $\square$

Lemma 3.21. *Adopt the assumptions and notation as in Lemma 3.19. Let $x_0 \in E$ and suppose that $\int_{(x_0-\delta,x_0+\delta)} |Z|^q = \infty$ for all $\delta > 0$. Suppose that x_0 is not a cluster point of E. Then $L_{x_0} = 0$ if $x_0 \in E_l$ and $R_{x_0} = 0$ if $x_0 \in E_r$.*

Proof. Without loss of generality we may assume that $x_0 \in E_r$. If $\int_{(x_0-\delta,x_0)} |Z|^q = \infty$ for all $\delta > 0$, then it follows from Lemma 3.20(b) that $R_{x_0} = 0$. Suppose there exists a $\delta_0 > 0$ such that $\int_{(x_0-\delta_0,x_0)} |Z|^q < \infty$. Then $\int_{(x_0,x_0+\delta)} |Z|^q = \infty$ for all $\delta \in (0,\delta_0)$. Since x_0 is not a cluster point of E, there exists a $\delta \in (0,\delta_0)$ such that either $(x_0, x_0+\delta] \subset \mathcal{P}$ or $(x_0, x_0+\delta] \subset \mathcal{N}$. These give rise to two cases.

Case 1: Suppose there exists a $\delta \in (0,\delta_0)$ such that $(x_0, x_0+\delta] \subset \mathcal{P}$.
Then $L_{x_0} = 0$ by Lemma 3.20(a). Without loss of generality we may assume that $(x_0-\delta, x_0) \subset \mathcal{P}$. There exists a $u \in C_c^\infty(\mathbb{R})$ such that $u(x_0) = 1$ and $\operatorname{supp} u \subset (x_0-\delta, x_0+\delta)$. Then it follows from Lemma 3.19(d) that $R_{x_0} - L_{x_0} = 0$. Hence $R_{x_0} = 0$.

Case 2: Suppose there exists a $\delta \in (0,\delta_0)$ such that $(x_0, x_0+\delta] \subset \mathcal{N}$.
Without loss of generality we may assume that $(x_0-\delta, x_0) \subset \mathcal{P}$. Let $n \in \mathbb{N}$. There exists a $u \in C_c^\infty(\mathbb{R})$ such that $u(x_0) = 1$, $0 \leq u \leq \mathbb{1}$ and $\operatorname{supp} u \subset (x_0-\delta, x_0+\frac{1}{n} \wedge \delta)$. Then Lemma 3.19(d) implies that $|R_{x_0}| = |\int_{\mathcal{N}} (1+\overline{w})\,g\,u| \leq (1+\|w\|_\infty)\,\|g\|_{L_q(\mathbb{R})}\,n^{-1/p}$. Hence $R_{x_0} = 0$. $\square$

We next consider the situation when $x_0 \in E$ is a cluster point of E. We divide the proof into two lemmas which deal with two cases whether or not $|(x_0-\delta, x_0+\delta) \cap \mathcal{N}| > 0$ for all $\delta > 0$.

Lemma 3.22. *Adopt the assumptions and notation as in Lemma 3.19. Let $x_0 \in E$ and suppose that $\int_{(x_0-\delta,x_0+\delta)} |Z|^q = \infty$ for all $\delta > 0$. Suppose that x_0 is a cluster point of E. Suppose further that there exists a $\delta_0 > 0$ such that $|(x_0-\delta_0, x_0+\delta_0) \cap \mathcal{N}| = 0$. Then $L_{x_0} = 0$ if $x_0 \in E_l$ and $R_{x_0} = 0$ if $x_0 \in E_r$.*

Proof. Without loss of generality we may assume that $x_0 \in E_r$. If $\int_{(x_0-\delta,x_0)} |Z|^q = \infty$ for all $\delta > 0$, then $R_{x_0} = 0$ by Lemma 3.20(b). Hence we may assume that the interval $(x_0 - \delta_0, x_0) \subset \mathcal{P}$ and $\int_{(x_0-\delta_0,x_0)} |Z|^q < \infty$. Suppose that $R_{x_0} \neq 0$. Replacing g by $\lambda\, g$ for some $\lambda \in \mathbb{C}$ if necessary, we may assume without loss of generality that $R_{x_0} = 2 + \tan\theta$.

By Statements (f), (g) and (i) in Lemma 3.19, there exists a $\delta \in (0, \delta_0)$ such that $|(\overline{c}\, g')(x) - R_{x_0}| < 1$ for all $x \in (x_0, x_0 + \delta) \cap \mathcal{P}$. Using the same argument as in Lemma 3.20, we also have $\operatorname{Re} g'(x) \geq \frac{\cos^2\theta}{\operatorname{Re} c(x)}$ for all $x \in (x_0, x_0 + \delta) \cap \mathcal{P}$. Without loss of generality we may assume that $x_0 + \delta \in E_r$.

Let $k \in K$. Suppose that $I_k \subset (x_0, x_0 + \delta)$. Then $\operatorname{Re} g|_{I_k}$ is increasing. If $\operatorname{Re} g(m_k) \geq 0$ then

$$\begin{aligned} |I_k|\, |\operatorname{Re} g(m_k)|^q &= 2 \int_{m_k}^{b_k} |\operatorname{Re} g(m_k)|^q \, dx \\ &\leq 2 \int_{m_k}^{b_k} |\operatorname{Re} g(x)|^q \, dx \leq 2 \int_{I_k} |\operatorname{Re} g|^q. \end{aligned}$$

Similarly if $\operatorname{Re} g(m_k) \leq 0$ then

$$|I_k|\, |\operatorname{Re} g(m_k)|^q \leq 2 \int_{I_k} |\operatorname{Re} g|^q.$$

Next, for all $x \in (a_k, m_k)$ it follows that

$$\operatorname{Re} g(x) - \operatorname{Re} g(m_k) = -\int_x^{m_k} \operatorname{Re} g'(t)\, dt \leq -\int_x^{m_k} \frac{\cos^2\theta}{\operatorname{Re} c(t)}\, dt = -(\cos^2\theta)\, Z(x).$$

Alternatively if $x \in (m_k, b_k)$ then

$$\operatorname{Re} g(x) - \operatorname{Re} g(m_k) = \int_{m_k}^{x} \operatorname{Re} g'(t)\, dt \geq \int_{m_k}^{x} \frac{\cos^2\theta}{\operatorname{Re} c(t)}\, dt = -(\cos^2\theta)\, Z(x).$$

Therefore

$$(\cos^{2q}\theta)\, |Z(x)|^q \leq |\operatorname{Re} g(x) - \operatorname{Re} g(m_k)|^q \leq 2^q \big(|\operatorname{Re} g(x)|^q + |\operatorname{Re} g(m_k)|^q \big)$$

for all $x \in I_k$. Hence

$$(\cos^{2q}\theta) \int_{I_k} |Z|^q \leq 2^q\, |I_k|\, |\operatorname{Re} g(m_k)|^q + 2^q \int_{I_k} |\operatorname{Re} g|^q \leq 3 \cdot 2^q \int_{I_k} |\operatorname{Re} g|^q.$$

This is for all $k \in K$ with $I_k \subset (x_0, x_0 + \delta)$.

Finally, since $|(x_0, x_0 + \delta) \cap \mathcal{N}| = 0$, we have

$$\begin{aligned} (\cos^{2q}\theta) \int_{(x_0,x_0+\delta)} |Z|^q &= (\cos^{2q}\theta) \int_{(x_0,x_0+\delta)\cap\mathcal{P}} |Z|^q = (\cos^{2q}\theta) \sum_{\substack{k\in K \\ I_k \subset (x_0,x_0+\delta)}} \int_{I_k} |Z|^q \\ &\leq \sum_{\substack{k\in K \\ I_k \subset (x_0,x_0+\delta)}} 3 \cdot 2^q \int_{I_k} |\operatorname{Re} g|^q \leq 3 \cdot 2^q\, \|g\|_{L_q(\mathbb{R})}^q < \infty. \end{aligned}$$

But by assumption $\int_{(x_0,x_0+\delta)} |Z|^q = \infty$. This is a contradiction. Thus $R_{x_0} = 0$. ∎

Lemma 3.23. *Adopt the assumptions and notation as in Lemma 3.19. Let $x_0 \in E$ and suppose that $\int_{(x_0-\delta,x_0+\delta)} |Z|^q = \infty$ for all $\delta > 0$. Suppose that x_0 is a cluster point of E. Suppose further that $|(x_0-\delta,x_0+\delta)\cap\mathcal{N}| > 0$ for all $\delta > 0$. Then $L_{x_0} = 0$ if $x_0 \in E_l$ and $R_{x_0} = 0$ if $x_0 \in E_r$.*

Proof. Without loss of generality we may assume that $0 = x_0 \in E_l$. Let $\delta_0 > 0$ be such that $(0, 2\,\delta_0) \subset \mathcal{P}$. By Lemma 3.20(a) it suffices to consider the case when $\int_{(0,\delta_0)} |Z|^q < \infty$. Then $\int_{(-\delta,0)} |Z|^q = \infty$ for all $\delta > 0$. By Lemma 3.19(h) we may assume that $(\overline{c}\,Dg)|_{(-\delta_0,0)} \in L_q(-\delta_0, 0)$.

Define $\tau\colon \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\tau(x) = \begin{cases} 0 & \text{if } x \in (-\infty, 0], \\ 2x^2 & \text{if } x \in (0, \frac{1}{2}], \\ 1 - 2(1-x)^2 & \text{if } x \in (\frac{1}{2}, 1], \\ 1 & \text{if } x \in (1, \infty). \end{cases}$$

Then τ is differentiable and $\int_{\mathbb{R}} |\tau'|^p = \frac{2^p}{p+1}$. Moreover, $\tau \in W^{2,p}_{\mathrm{loc}}(\mathbb{R})$. Finally let $\chi \in C_c^\infty(\mathbb{R})$ be such that $\chi|_{(-\delta_0,\delta_0)} = \mathbb{1}$ and $\operatorname{supp}\chi \subset (-\infty, 2\,\delta_0)$.

Let $\delta \in (0, \delta_0)$ and suppose that $-\delta \in E_l$. Let $K_1 = \{k \in K \colon I_k \subset (-\delta, 0)\}$. Then K_1 is infinite and countable. Without loss of generality we may assume that $K_1 = \mathbb{N}$ and $a_1 = -\delta$. Let $n \in \mathbb{N}$. Then $(-\delta,0)\setminus(\overline{I_1}\cup\ldots\cup\overline{I_n})$ has at most n connected components. Let $N \in \{1, \ldots, n\}$ be the number of components and denote these components by $J_1, \ldots, J_N$. Let $\lambda_i = |J_i| > 0$ for all $i \in \{1, \ldots, N\}$ and set $\lambda = \sum_{i=1}^N \lambda_i$. Then

$$\lambda = \delta - \sum_{k=1}^N |I_k| \geq \delta - |(-\delta,0)\cap\mathcal{P}| = |(-\delta,0)\cap\mathcal{N}| > 0$$

by assumption. Define $v_n\colon \mathbb{R} \longrightarrow \mathbb{R}$ by

$$v_n(x) = \sum_{i=1}^N \frac{\lambda_i}{\lambda}\, \tau\Big(\frac{x - \inf J_i}{\lambda_i}\Big).$$

Then $v_n \in W^{2,p}_{\mathrm{loc}}(\mathbb{R})$, $\operatorname{supp} v_n \subset [-\delta, \infty)$, $v_n(0) = 1$, $0 \leq v_n \leq \mathbb{1}$ and

$$\int_{\mathbb{R}} |v_n'|^p = \sum_{i=1}^N \Big(\frac{\lambda_i}{\lambda}\Big)^p \cdot \frac{2^p}{(p+1)\,\lambda_i^{p-1}} = \frac{2^p}{(p+1)\,\lambda^{p-1}} \leq \frac{2^p}{(p+1)\,|(-\delta,0)\cup\mathcal{N}|^{p-1}}. \tag{3.46}$$

Define $u_n\colon \mathbb{R} \longrightarrow \mathbb{R}$ by

$$u_n(x) = \begin{cases} v_n(x) & \text{if } x < 0, \\ \chi(x) & \text{if } x \geq 0. \end{cases}$$

Then $u_n \in W^{2,p}_c(\mathbb{R})$. Therefore by Lemma 3.19(d) we have that

$$-L_0 + \sum_{k\in K_1} \Big(R_{b_k}\, u_n(b_k) - L_{a_k}\, u_n(a_k)\Big) = -\int_{\mathcal{N}} (1+\overline{w})\, g\, u_n. \tag{3.47}$$

Note that

$$\Big|\int_{\mathcal{N}} (1+\overline{w})\, g\, u_n\Big| = \Big|\int_{(-\delta,0)\cap\mathcal{N}} (1+\overline{w})\, g\, u_n\Big| \leq (1 + \|w\|_\infty)\, \|g\|_{L_q(\mathbb{R})}\, \sqrt[p]{\delta}.$$

Recall that $K_1 = \mathbb{N}$. If $k \in \{1,\ldots,n\}$, then $u_n(b_k) = u_n(a_k)$ by construction. So

$$\Big|\sum_{k=1}^{n} \Big(R_{b_k}\, u_n(b_k) - L_{a_k}\, u_n(a_k)\Big)\Big| \le \sum_{k=1}^{n} u_n(a_k)\, |R_{b_k} - L_{a_k}| \le \sum_{k=1}^{n} \Big| \int_{I_k} (1+\overline{w})\, g\Big|$$
$$\le \int_{I_1\cup\ldots\cup I_n} |(1+\overline{w})\, g| \le (1+\|w\|_\infty)\, \|g\|_{L_q(\mathbb{R})} \sqrt[p]{\delta},$$

where we used (3.44) to obtain the second inequality.

Let $k \in \mathbb{N}$ be such that $k > n$. Let $a, b \in \mathbb{R}$ and suppose that $a_k < a < b < b_k$. Then

$$(\overline{c}\, g'\, u_n)(b) - (\overline{c}\, g'\, u_n)(a) = \int_a^b (\overline{c}\, g'\, u_n)' = \int_a^b \Big((\overline{c}\, g')'\, u_n + \overline{c}\, g'\, u_n'\Big)$$
$$= \int_a^b \Big(\Big((1+\overline{w})\, g - (\overline{m}\, g)'\Big)\, u_n + \overline{c}\, g'\, u_n'\Big)$$
$$= -[\overline{m}\, g\, u_n]_a^b + \int_a^b \Big((1+\overline{w})\, g\, u_n + (\overline{c}\, g' + \overline{m}\, g)\, u_n'\Big),$$

where we used Lemma 3.19(a) in the third step. Taking limits and using Statement (i) in Lemma 3.19 give

$$R_{b_k}\, u_n(b_k) - L_{a_k}\, u_n(a_k) = \int_{I_k} \Big((1+\overline{w})\, g\, u_n + (\overline{c}\, g' + \overline{m}\, g)\, u_n'\Big).$$

It follows that

$$\Big|\sum_{k=n+1}^{\infty} \Big(R_{b_k}\, u_n(b_k) - L_{a_k}\, u_n(a_k)\Big)\Big| = \Big|\sum_{k=n+1}^{\infty} \int_{I_k} \Big((1+\overline{w})\, g\, u_n + (\overline{c}\, g' + \overline{m}\, g)\, u_n'\Big)\Big|$$
$$= \Big|\int_{\bigcup_{k\ge n+1} I_k} \Big((1+\overline{w})\, g\, u_n + (\overline{c}\, g' + \overline{m}\, g)\, u_n'\Big)\Big|$$
$$\le \int_{(-\delta,0)} |(1+\overline{w})\, g\, u_n| + \int_{\bigcup_{k=n+1}^{\infty} I_k} |(\overline{c}\, g' + \overline{m}\, g)\, u_n'|.$$

Obviously

$$\int_{(-\delta,0)} |(1+\overline{w})\, g\, u_n| \le (1+\|w\|_\infty)\, \|g\|_{L_q(\mathbb{R})} \sqrt[p]{\delta}.$$

For the second term we estimate

$$\int_{\bigcup_{k=n+1}^{\infty} I_k} |(\overline{c}\, g' + \overline{m}\, g)\, u_n'| \le \|v_n'\|_{L_p(\mathbb{R})} \cdot \Big(\int_{\bigcup_{k=n+1}^{\infty} I_k} |\overline{c}\, g' + \overline{m}\, g|^q\Big)^{1/q}$$
$$\le \Big(\frac{2^p}{(p+1)\, |(-\delta,0)\cup\mathcal{N}|^{p-1}}\Big)^{1/p} \Big(\int_{\bigcup_{k=n+1}^{\infty} I_k} |\overline{c}\, g' + \overline{m}\, g|^q\Big)^{1/q} \tag{3.48}$$

by (3.46). Note that $(\overline{c}\, Dg)|_{(-\delta,0)} \in L_q(-\delta,0)$ by Lemma 3.19(b), $|\overline{m}\, g| \leq \|m\|_\infty \, |g| \in L_q(\mathbb{R})$ and $\bigcup_{k=n+1}^\infty I_k \downarrow \emptyset$ if $n \longrightarrow \infty$. Therefore the right-hand side of (3.48) tends to 0 by Lebesgue dominated convergence theorem. It now follows from (3.47) that

$$L_0 \leq 3\,(1+\|w\|_\infty)\,\|g\|_{L_q(\mathbb{R})}\,\sqrt[p]{\delta}.$$

This is for all $\delta > 0$ with $-\delta \in E_l$. Hence $L_0 = 0$. □

Proof of Theorem 3.3. ($\Longrightarrow$) This follows from Lemmas 3.17 and 3.18.

($\Longleftarrow$) Adopt the assumptions and notations as in Lemma 3.19. Then it follows from Lemmas 3.20–3.23 that $L_x = 0$ for all $x \in E_l$ and $R_x = 0$ for all $x \in E_r$. By Lemma 3.19(a) we have that $g \in W^{1,q}_{\rm loc}(\mathcal{P})$, $(\overline{c}\, Dg)|_{\mathcal{P}} \in W^{1,q}_{\rm loc}(\mathcal{P})$ and $D(\overline{c}\, Dg) + D(\overline{m}\, g) \in L_q(\mathbb{R})$. Thus $g \in D(A_p^*)$ by Proposition 3.13. Moreover, $(I + A_p)^*(g) = 0$. Since $(I + A_p)^*$ is invertible, we have $g = 0$. Hence $C_c^\infty(\mathbb{R})$ is a core for A_p.

The proof of Theorem 3.3 is complete. □

3.6 Core characterisation for $p = 1$

If c and m are real-valued, then we show in this section that $C_c^\infty(\mathbb{R})$ is a core for the generator $-A_1$ of the C_0-semigroup $S^{(1)}$ on $L_1(\mathbb{R})$ which is consistent with S.

Proof of Theorem 3.5. Since $c \in W^{1,\infty}(\mathbb{R})$, it follows that $Z|_{(a_k,m_k)} \notin L_\infty(a_k,m_k)$ for all $k \in K$ with $a_k \in \mathbb{R}$ and $Z|_{(m_k,b_k)} \notin L_\infty(m_k,b_k)$ for all $k \in K$ with $b_k \in \mathbb{R}$. Without loss of generality we may assume that $w = 0$. Let $\omega > 0$ be as in Lemma 3.6. Let $\beta > \omega + 2\,M_m^2$ be such that $\beta\, I + A_1$ is accretive (cf. Theorem 3.2). Let $g \in L_\infty(\mathbb{R})$. Suppose that $\int_{\mathbb{R}} \overline{g}\,(\beta\, I + A_1)u = 0$ for all $u \in C_c^\infty(\mathbb{R})$. Without loss of generality we may assume that g is real-valued. It follows from elliptic regularity that $g \in W^{2,r}_{\rm loc}(\mathcal{P})$ for all $r \in (1,\infty)$ and $D(c\, Dg) + m\, Dg = \beta\, g\, \mathbb{1}_{\mathcal{P}}$.

Let $k \in K$. Using similar arguments as in Lemma 3.19(c) we have that $(c\, g')|_{I_k}$ extends to a continuous function on $\overline{I_k}$. Suppose $a_k \in \mathbb{R}$. Let $L = \lim_{x \downarrow a_k} (c\, g')(x) \in \mathbb{C}$. Suppose $L \neq 0$. Without loss of generality we may assume that $L = 2$. We argue as in the proof of Lemma 3.20 to obtain that there exist $s \in I_k$ and $\alpha \in \mathbb{R}$ such that

$$({\rm Re}\, g)(x) \leq ({\rm Re}\, g)(s) - Z(x) + \alpha$$

for all $x \in (a_k, s)$. This implies that $g \notin L_\infty(\mathbb{R})$, which is a contradiction. Hence $\lim_{x \downarrow x_0} (c\, g')(x) = 0$ for all $x_0 \in E_l$. Analogously $\lim_{x \uparrow x_0} (c\, g')(x) = 0$ for all $x_0 \in E_r$.

Let $k \in K$ and suppose that I_k is bounded. Then $g\, \mathbb{1}_{I_k} \in D(A)$ by Lemma 3.10(vi $\Longrightarrow$ i). Therefore

$$0 \leq \mathfrak{a}(g\, \mathbb{1}_{I_k}) + \omega \int_{I_k} g^2 = \big(A(g\, \mathbb{1}_{I_k}), g\, \mathbb{1}_{I_k}\big) + \omega \int_{I_k} g^2 = -(\beta - \omega) \int_{I_k} g^2 \leq 0.$$

It follows that $g\, \mathbb{1}_{I_k} = 0$.

Next let $k \in K$ and suppose that I_k is unbounded. We may assume without loss of generality that I_k is bounded from below and $I_k = (0,\infty)$. Let $\tau \in C_c^\infty(\mathbb{R},\mathbb{R})$ be such that $\operatorname{supp} \tau \subset (-1,2)$ and $\tau|_{[0,1]} = \mathbb{1}$. For all $n \in \mathbb{N}$ define $\tau_n \in C_c^\infty(\mathbb{R})$ by $\tau_n(x) = \tau(n^{-1}\, x)$ for all $x \in \mathbb{R}$. Let $n \in \mathbb{N}$. Then $g\,\tau_n\,\mathbb{1}_{(0,\infty)} \in D(A)$ by Lemma 3.10(vi $\Longrightarrow$ i). Moreover, we have

$$-\omega \int_0^\infty g^2\,\tau_n^2 \le \mathfrak{a}(g\,\tau_n\,\mathbb{1}_{(0,\infty)}) = \int_0^\infty c\,|(g\,\tau_n)'|^2 + m\,(g\,\tau_n)'\,g\,\tau_n. \tag{3.49}$$

Note that

$$\int_0^\infty c\,|(g\,\tau_n)'|^2 \le 2\int_0^\infty c\,(g')^2\,\tau_n^2 + 2\int_0^\infty c\,g^2\,(\tau_n')^2$$

and

$$\begin{aligned}\int_0^\infty m\,(g\,\tau_n)'\,g\,\tau_n &= \int_0^\infty m\,\big(g'\,g\,\tau_n^2 + g^2\,\tau_n'\,\tau_n\big)\\ &\le \int_0^\infty M_m\,\sqrt{c}\,\big(|g'|\,|g|\,\tau_n^2 + g^2\,|\tau_n'|\,\tau_n\big)\\ &\le \int_0^\infty c\,(g')^2\,\tau_n^2 + \int_0^\infty c\,g^2\,(\tau_n')^2 + \frac{M_m^2}{2}\int_0^\infty g^2\,\tau_n^2,\end{aligned}$$

where we used (3.1) in the second step. It follows from (3.49) that

$$-\omega \int_0^\infty g^2\,\tau_n^2 \le 3\int_0^\infty c\,(g')^2\,\tau_n^2 + 3\int_0^\infty c\,g^2\,(\tau_n')^2 + \frac{M_m^2}{2}\int_0^\infty g^2\,\tau_n^2$$

or equivalently

$$-\big(\omega + \frac{M_m^2}{2}\big)\int_0^\infty g^2\,\tau_n^2 \le 3\int_0^\infty c\,(g')^2\,\tau_n^2 + 3\int_0^\infty c\,g^2\,(\tau_n')^2. \tag{3.50}$$

We note that $(c\,g')'(x) + (m\,g')(x) = \beta\,g(x)$ for a.e. $x \in (0,\infty)$ and τ_n has a compact support. Let $\varepsilon > 0$. Then

$$\begin{aligned}\int_\varepsilon^\infty c\,(g')^2\,\tau_n^2 &= g\,c\,g'\,\tau_n^2|_\varepsilon^\infty - \int_\varepsilon^\infty g\,(c\,g'\,\tau_n^2)'\\ &= -(g\,c\,g'\,\tau_n^2)(\varepsilon) - \int_\varepsilon^\infty g\,\big((c\,g')'\,\tau_n^2 + 2\,c\,g'\,\tau_n\,\tau_n'\big)\\ &= -(g\,c\,g'\,\tau_n^2)(\varepsilon) - \beta\int_\varepsilon^\infty g^2\,\tau_n^2 + \int_\varepsilon^\infty m\,g'\,g\,\tau_n^2 - 2\int_\varepsilon^\infty c\,g'\,g\,\tau_n\,\tau_n'\\ &\le -(g\,c\,g'\,\tau_n^2)(\varepsilon) - \beta\int_\varepsilon^\infty g^2\,\tau_n^2 + \int_\varepsilon^\infty M_m\,\sqrt{c}\,|g'|\,|g|\,\tau_n^2 - 2\int_\varepsilon^\infty c\,g'\,g\,\tau_n\,\tau_n'\\ &\le -(g\,c\,g'\,\tau_n^2)(\varepsilon) - \beta\int_\varepsilon^\infty g^2\,\tau_n^2 + \frac{1}{4}\int_\varepsilon^\infty c\,(g')^2\,\tau_n^2 + M_m^2\int_\varepsilon^\infty g^2\,\tau_n^2\\ &\qquad + \frac{1}{2}\int_\varepsilon^\infty c\,(g')^2\,\tau_n^2 + 2\int_\varepsilon^\infty c\,g^2\,(\tau_n')^2\\ &= -(g\,c\,g'\,\tau_n^2)(\varepsilon) + (-\beta + M_m^2)\int_\varepsilon^\infty g^2\,\tau_n^2 + 2\int_\varepsilon^\infty c\,g^2\,(\tau_n')^2\\ &\qquad + \frac{3}{4}\int_\varepsilon^\infty c\,(g')^2\,\tau_n^2.\end{aligned}$$

By rearrangement it follows that

$$\int_\varepsilon^\infty c\,(g')^2\,\tau_n^2 \le -4\,(g\,c\,g'\,\tau_n^2)(\varepsilon) + 4\,(-\beta + M_m^2)\int_\varepsilon^\infty g^2\,\tau_n^2 + 8\int_\varepsilon^\infty c\,g^2\,(\tau_n')^2.$$

Since g is bounded and $\lim_{x\downarrow 0}(c\,g')(x) = 0$, taking the limits both sides when $\varepsilon \downarrow 0$ gives

$$\int_0^\infty c\,(g')^2\,\tau_n^2 \le 4\,(-\beta + M_m^2)\int_0^\infty g^2\,\tau_n^2 + 8\int_0^\infty c\,g^2\,(\tau_n')^2.$$

This together with (3.50) give

$$-\big(\omega + \frac{M_m^2}{2}\big)\int_0^\infty g^2\,\tau_n^2 \le 12\,(-\beta + M_m^2)\int_0^\infty g^2\,\tau_n^2 + 27\int_0^\infty c\,g^2\,(\tau_n')^2$$

or equivalently

$$\gamma\int_0^\infty g^2\,\tau_n^2 \le 27\int_0^\infty g^2\,(\tau_n')^2 \le 54\,\|c\|_\infty\,\|g\|_\infty^2\,\|\tau'\|_\infty^2\,n^{-1},$$

where

$$\gamma = 12\beta - \omega - \frac{25}{2}M_m^2 > 0.$$

Using Fatou's lemma we deduce that $\int_0^\infty g^2 = 0$, which implies $g|_{(0,\infty)} = 0$. Hence $g|_{\mathcal{P}} = 0$. Then $\int_{\mathbb{R}} g\,\mathbb{1}_{\mathcal{N}}\,u = 0$ for all $u \in C_c^\infty(\mathbb{R})$ and $g = 0$. Thus $C_c^\infty(\mathbb{R})$ is a core for A_1. □

3.7 Examples

In this section we provide two examples to illustrate the main theorem of this chapter and one example to illustrate condition (3.21).

Example 3.24. Let $b \in (0,\infty)$ and $\kappa \in (1,\infty)$. Define $c\colon \mathbb{R} \longrightarrow [0,\infty)$ by $c(x) = \Big(d(x, b\mathbb{Z})\Big)^\kappa$. Then $c \in W^{1,\infty}(\mathbb{R})$. Set $s = b/2$. Then $Z(x) = (\kappa-1)^{-1}\Big(x^{-(\kappa-1)} - s^{-(\kappa-1)}\Big)$ for all $x \in (0,s)$. So if $x \in (0,s/2)$ then

$$\frac{1 - 2^{-(\kappa-1)}}{\kappa - 1}\,x^{-(\kappa-1)} \le Z(x) \le \frac{1}{\kappa-1}\,x^{-(\kappa-1)}$$

and if $x \in [s/2, s)$ then

$$0 \le Z(x) \le \frac{1}{\kappa-1}\,x^{-(\kappa-1)}.$$

Let $p \in (1,\infty)$ and let q be the dual exponent. Then $Z \in L_q(0,s)$ if and only if $q(\kappa-1) < 1$ and if $q(\kappa-1) < 1$ then

$$\frac{2\,(1 - 2^{-(\kappa-1)})^q}{(\kappa-1)^q\,4^{1-(\kappa-1)q}\,(1 - (\kappa-1)q)}\,b^{1-(\kappa-1)q}$$
$$\le \int_0^b Z^q \le \frac{2}{(\kappa-1)^q\,(1-(\kappa-1)q)}\,b^{1-(\kappa-1)q}. \tag{3.51}$$

It follows from Theorem 3.3 that $C_c^\infty(\mathbb{R})$ is a core for A_p if and only if $\kappa \ge 1 + \frac{1}{q}$.

We next 'extend' this example to a Cantor-like set instead of the grid $b\mathbb{Z}$.

Example 3.25. Fix $\lambda \in [0,1)$. Let $K_\infty = \bigcap_{n=0}^\infty K_n \subset [0,1]$ be the generalized Cantor set with $K_0 = [0,1]$ and for any $n \in \mathbb{N}_0$ construct K_{n+1} by removing the central open interval of length $(1-\lambda)\, 3^{-(n+1)}$ from each of the 2^n intervals of K_n. So if $\lambda = 0$, then K_∞ is the usual Cantor set. It is easy to verify that $|K_n| = \lambda + (1-\lambda)\,(\frac{2}{3})^n$ for all $n \in \mathbb{N}_0$. Hence $|K_\infty| = \lambda$.

Fix $\kappa \in (1,\infty)$ and define $c\colon \mathbb{R} \longrightarrow [0,\infty)$ by

$$c(x) = \Big(\frac{d(x,K_\infty)}{1+d(x,K_\infty)}\Big)^\kappa .$$

(The denominator is to make c bounded.) Let $p \in (1,\infty)$ and let q be the dual exponent of p. Then it follows from Example 3.24 that $\int_{(x-\delta,x+\delta)} Z^q = \infty$ for all $x \in E$ and $\delta > 0$ if $q(\kappa-1) \geq 1$.

Next suppose that $q(\kappa - 1) < 1$. Then (3.51) gives

$$\int_{\mathcal{P}\cap[0,1]} Z^q \leq \sum_{n=0}^\infty 2^n \, \frac{2}{(\kappa-1)^q\,(1-(\kappa-1)q)} \, \Big((1-\lambda)3^{-(n+1)}\Big)^{1-(\kappa-1)q} .$$

Using also the lower bound in (3.51) it follows that $Z|_{\mathcal{P}\cap[0,1]} \in L_q(\mathcal{P}\cap[0,1])$ if and only if $2 < 3^{1-(\kappa-1)q}$.

Now $\mathcal{N} = K_\infty$ is negligible if and only if $\lambda = 0$. Moreover, $Z(x) = \infty$ for all $x \in K_\infty$ by definition. Hence if $\lambda = 0$ and $2 < 3^{1-(\kappa-1)q}$, then $\int_{(x-\delta,x+\delta)} Z^q < \infty$ for all $x \in E$ and $\delta > 0$. Alternatively, if $\lambda > 0$ or $2 \geq 3^{1-(\kappa-1)q}$, then by symmetry and self-similarity one deduces that $\int_{(x-\delta,x+\delta)} Z^q = \infty$ for all $x \in E$ and $\delta > 0$. Therefore Theorem 3.3 gives that $C_c^\infty(\mathbb{R})$ is a core for A_p if and only if $\lambda > 0$ or $2 \geq 3^{1-(\kappa-1)q}$.

Example 3.26. Adopt the assumptions and notation as in Example 3.25. Recall that we denote by $\{I_k : k \in K\}$ the set of distinct connected components of $\mathcal{P}$. Since K is countable, we can set $K = \mathbb{N}$. Let $\alpha \in (0,1) \cap \big(0, \frac{\kappa}{2} \wedge (\kappa-1)\big]$. Define $m\colon \mathbb{R} \longrightarrow [0,\infty)$ by

$$m(x) = \begin{cases} \dfrac{1}{k^2}\Big(\dfrac{d(x,K_\infty)}{1+d(x,K_\infty)}\Big)^{\kappa-\alpha} & \text{if } x \in I_k \text{ and } k \in K, \\ 0 & \text{if } x \in \mathcal{N}. \end{cases}$$

Then $m \in W^{1,\infty}(\mathbb{R})$. Moreover, m satisfies (3.1) and (3.21).

Chapter 4

Higher dimensions

4.1 Introduction

In this chapter we investigate degenerate elliptic second-order differential operators with bounded complex-valued coefficients in higher dimensions. We will provide many sufficient conditions for when $C_c^\infty(\mathbb{R}^d)$ is a core for these operators. The results are generalisations of those in [WD83, Theorem I] and [Ouh05, Theorem 5.2].

Let $d \in \mathbb{N}$ and $\theta \in [0, \frac{\pi}{2})$. Let $c_{kl} \in W^{2,\infty}(\mathbb{R}^d, \mathbb{C})$ for all $k, l \in \{1, \ldots, d\}$. Define $C = (c_{kl})_{1 \le k,l \le d}$ and $\Sigma_\theta = \{r\, e^{i\psi} : r \ge 0 \text{ and } |\psi| \le \theta\}$. Assume that

$$(C(x)\,\xi, \xi) \in \Sigma_\theta \tag{4.1}$$

for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$. Later on we will usually refer to (4.1) as *C takes values in the sector Σ_θ.*

Define the form

$$\mathfrak{a}_0(u, v) = \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} c_{kl}\,(\partial_k u)\, \partial_l \overline{v}$$

on the domain $D(\mathfrak{a}_0) = C_c^\infty(\mathbb{R}^d)$. Then it follows from (4.1) that

$$\mathfrak{a}_0(u, u) = \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} c_{kl}\,(\partial_k u)\, \partial_l \overline{u} = \int_{\mathbb{R}^d} (C\,\nabla u, \nabla u) \in \Sigma_\theta$$

for all $u \in C_c^\infty(\mathbb{R}^d)$. Using [Kat80, Theorem VI.1.27] we deduce as in the proof of Proposition 3.7 that $\mathfrak{a}_0$ is closable.

Let A be the operator associated with the closure of the form $\mathfrak{a}_0$. Then $W^{2,2}(\mathbb{R}^d) \subset D(A)$ and

$$Au = -\sum_{k,l=1}^{d} \partial_l(c_{kl}\,\partial_k u)$$

for all $u \in W^{2,2}(\mathbb{R}^d)$. Furthermore, by [Kat80, Theorem VI.2.1], the operator A is an m-sectorial operator. Let S be the C_0-semigroup generated by $-A$. If A is strongly elliptic, that is, if there exists a $\mu > 0$ such that

$$\mathrm{Re}\,(C(x)\,\xi, \xi) \ge \mu\,\|\xi\|^2$$

for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$, then S extends consistently to a C_0-semigroup on $L_p(\mathbb{R}^d)$ for all $p \in [1, \infty)$ by [Aus96, Theorem 4.8]. In the general case where the coefficient matrix merely satisfies

$$(C(x)\,\xi, \xi) \in \Sigma_\theta$$

for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$, then we prove in Section 4.3 that an extension is possible for certain $p \in (1, \infty)$. Before presenting the precise statement, we need to introduce the following notation. We write

$$C = R + i\,B,$$

where R and B are $d \times d$ matrix-valued functions with real-valued entries. Let B_a be the anti-symmetric part of B, that is, $B_a = \frac{1}{2}(B - B^T)$. The result about semigroup extension is as follows.

Proposition 4.1. *Let $p \in (1, \infty)$. Suppose $|1 - \frac{2}{p}| \le \cos\theta$ and $B_a = 0$. Then S extends consistently to a contraction C_0-semigroup $S^{(p)}$ on $L_p(\mathbb{R}^d)$.*

Let $p \in (1, \infty)$ be such that $\left|1 - \frac{2}{p}\right| \le \cos\theta$. Using Proposition 4.1 we can now extend S consistently to a C_0-semigroup $S^{(p)}$ on $L_p(\mathbb{R}^d)$. Let $-A_p$ be the generator of $S^{(p)}$. Clearly $C_c^\infty(\mathbb{R}^d) \subset D(A_p)$. We wish to show that $C_c^\infty(\mathbb{R}^d)$ is a core for A_p under certain conditions on the coefficients. The first main result of this chapter is as follows.

Theorem 4.2. *Let $p \in (1, \infty)$ be such that $|1 - \frac{2}{p}| < \cos\theta$. Suppose $B_a = 0$. Then the space $C_c^\infty(\mathbb{R}^d)$ is a core for A_p.*

Next let R_s and B_s be the symmetric parts of the matrices R and B respectively, that is, $R_s = \frac{1}{2}\,(R + R^T)$ and $B_s = \frac{1}{2}(B + B^T)$. Since A is naturally defined in $L_2(\mathbb{R}^d)$ via the closure of the form $\mathfrak{a}_0$, the condition $B_a = 0$ can be dropped in the proof of the core properties for A. In this case we prove the following sufficient conditions for $C_c^\infty(\mathbb{R}^d)$ to be a core for A.

Theorem 4.3. *Suppose one of the following holds.*

(i) *The matrix B_s has constant entries.*

(ii) *There exist $\theta_1, \theta_2 \in [0, \frac{\pi}{2})$, $\phi \in W^{2,\infty}(\mathbb{R}^d)$ and a $d \times d$ matrix $\widetilde{C}$ with entries in $W^{2,\infty}(\mathbb{R}^d)$ such that $\theta = \theta_1 + \theta_2$, $\phi(x) \in \Sigma_{\theta_1}$ for all $x \in \mathbb{R}^d$, $\widetilde{C}$ takes values in Σ_{θ_2} and $C = \phi\,\widetilde{C}$. Write $\widetilde{C} = \widetilde{R} + i\,\widetilde{B}$, where $\widetilde{R}$ and $\widetilde{B}$ are $d \times d$ matrix-valued functions with real-valued entries. Set $\widetilde{R}_s = \frac{1}{2}\,(\widetilde{R} + \widetilde{R}^T)$. Also define $\operatorname{Re}\widetilde{C} = \frac{1}{2}\left(\widetilde{C} + (\widetilde{C})^*\right)$. Suppose further that there exists an $h > 0$ such that*

$$\operatorname{tr}\,(U\,(\operatorname{Re}\widetilde{C})\,\overline{U}) \ge h\,\operatorname{tr}\,(U\,\widetilde{R}_s\,\overline{U})$$

for all $u \in C_c^\infty(\mathbb{R}^d)$, where $U = (\partial_l \partial_k u)_{1 \le k,l \le d}$.

(iii) *There exists an $M > 0$ such that $\|(\partial_l B_a)\,U\|_{HS}^2 \le M\,\operatorname{tr}\,(U\,R_s\,\overline{U})$ for all $l \in \{1, \ldots, d\}$ and $u \in C_c^\infty(\mathbb{R}^d)$, where $U = (\partial_l \partial_k u)_{1 \le k,l \le d}$.*

Then $C_c^\infty(\mathbb{R}^d)$ is a core for A.

If functions in $D(A)$ are known to possess certain smoothness properties, then $C_c^\infty(\mathbb{R}^d)$ is always a core for A regardless of B_a.

Theorem 4.4. *Suppose $D(A) \subset W^{1,2}(\mathbb{R}^d)$. Then $C_c^\infty(\mathbb{R}^d)$ is a core for A.*

An overview of the contents of the subsequent sections is as follows. In Section 2 we examine the matrix of coefficients C closely. We will prove various results concerning the correspondences between R_s, B_s and B_a. In Section 3 we prove the extension of the semigroup S to L_p-spaces. We will analyse the operator A_p in detail and then prove that $C_c^\infty(\mathbb{R}^d)$ is a core for A_p in Sections 4 and 5. In Section 6 we deal specifically with the operator A in $L_2(\mathbb{R}^d)$ and we present the proofs of Theorems 4.3 and 4.4. In Section 7 we provide some interesting examples.

4.2 The coefficient matrix C

Define

$$\operatorname{Re} C = \frac{C + C^*}{2} \quad \text{and} \quad \operatorname{Im} C = \frac{C - C^*}{2i},$$

where C^* is the conjugate transpose of C. Then $(\operatorname{Re} C)(x)$ and $(\operatorname{Im} C)(x)$ are self-adjoint for all $x \in \mathbb{R}^d$ and

$$C = \operatorname{Re} C + i \operatorname{Im} C. \tag{4.2}$$

We will also consider the coefficient matrix C in the form

$$C = R + i\, B, \tag{4.3}$$

where R and B are real matrices. Write $R = R_s + R_a$, where $R_s = \frac{R+R^T}{2}$ is the symmetric part of R and $R_a = \frac{R-R^T}{2}$ is the anti-symmetric part of R. Similarly $B = B_s + B_a$, where $B_s = \frac{B+B^T}{2}$ and $B_a = \frac{B-B^T}{2}$. A comparison between (4.2) and (4.3) gives

$$\operatorname{Re} C = R_s + i\, B_a \quad \text{and} \quad \operatorname{Im} C = B_s - i\, R_a.$$

In this section we will list various relations among R_s, R_a, B_s and B_a which will be used in subsequent sections.

Lemma 4.5. *We have*

$$|(B_s\, \xi, \eta)| \le \frac{1}{2} \tan\theta \Big((R_s\, \xi, \xi) + (R_s\, \eta, \eta) \Big)$$

for all $\xi, \eta \in \mathbb{R}^d$.

Proof. Since C takes values in Σ_θ, we have

$$\Big| \big((\operatorname{Im} C(x))\, \xi, \xi \big) \Big| \le \tan\theta \, \big((\operatorname{Re} C(x))\, \xi, \xi \big)$$

for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$. It follows that

$$|(B_s\, \xi, \xi)| \le \tan\theta \, (R_s\, \xi, \xi)$$

for all $\xi \in \mathbb{R}^d$. We next use polarisation to obtain

$$|(B_s\, \xi, \eta)| \le \tan\theta \, (R_s\, \xi, \xi)^{1/2} \, (R_s\, \eta, \eta)^{1/2} \le \frac{1}{2} \tan\theta \Big((R_s\, \xi, \xi) + (R_s\, \eta, \eta) \Big)$$

for all $\xi, \eta \in \mathbb{R}^d$ as required. □

Lemma 4.6. *Let $j \in \{1, \ldots, d\}$. Let $f \in W^{2,\infty}(\mathbb{R}^d)$ be such that $f(x) \geq 0$ for all $x \in \mathbb{R}^d$. Then*

$$|\partial_j f|^2 \leq 2\, \|\partial_j^2 f\|_\infty\, f.$$

Proof. Let $j \in \{1, \ldots, d\}$, $x \in \mathbb{R}^d$ and $h \in \mathbb{R}$. For each $n \in \mathbb{N}$ let $f_n = J_n * f$, where J_n denotes the usual mollifier with respect to a suitable function in $C_c^\infty(\mathbb{R}^d)$. Then $f_n \geq 0$ and $f_n \in C^\infty(\mathbb{R}^d)$ for all $n \in \mathbb{N}$. Using the Taylor expansion we have

$$0 \leq f_n(x) + h\,(\partial_j f_n)(x) + \frac{h^2}{2}\, \|\partial_j^2 f_n\|_\infty$$

for all $n \in \mathbb{N}$. Letting $n \longrightarrow \infty$ we obtain

$$0 \leq f(x) + h\,(\partial_j f)(x) + \frac{h^2}{2}\, \|\partial_j^2 f\|_\infty.$$

This is true for all $h \in \mathbb{R}$. Hence $|\partial_j f(x)|^2 \leq 2\, \|\partial_j^2 f\|_\infty\, f(x)$ as required. □

Lemma 4.7. *Let $j \in \{1, \ldots, d\}$. Let $f \in W^{2,\infty}(\mathbb{R}^d)$ be such that $f(x) \in \Sigma_\theta$ for all $x \in \mathbb{R}^d$. Then*

$$|\partial_j f|^2 \leq 4\,(1 + \tan\theta)^2 \sup_{1 \leq j \leq d} \|\partial_j^2 f\|_\infty\, \mathrm{Re}\, f.$$

Proof. Since $f(x) \in \Sigma_\theta$ for all $x \in \mathbb{R}^d$, we have $\mathrm{Re}\, f \geq 0$. Therefore by Lemma 4.6 we have

$$|\partial_j(\mathrm{Re}\, f)|^2 \leq 2\, \|\partial_j^2(\mathrm{Re}\, f)\|_\infty\, \mathrm{Re}\, f \leq 2 \sup_{1 \leq j \leq d} \|\partial_j^2 f\|_\infty\, \mathrm{Re}\, f.$$

Also $|\mathrm{Im}\, f| \leq (\tan\theta)\, \mathrm{Re}\, f$. That is, $(\tan\theta)\, \mathrm{Re}\, f \pm \mathrm{Im}\, f \geq 0$. Applying Lemma 4.6 again we obtain

$$\begin{aligned} |\partial_j((\tan\theta)\, \mathrm{Re}\, f + \mathrm{Im}\, f)|^2 &\leq 2\, \|\partial_j^2((\tan\theta)\, \mathrm{Re}\, f + \mathrm{Im}\, f)\|_\infty\, ((\tan\theta)\, \mathrm{Re}\, f + \mathrm{Im}\, f) \\ &\leq 2\,(1 + \tan\theta) \sup_{1 \leq j \leq d} \|\partial_j^2 f\|_\infty\, ((\tan\theta)\, \mathrm{Re}\, f + \mathrm{Im}\, f) \end{aligned}$$

and

$$\begin{aligned} |\partial_j((\tan\theta)\, \mathrm{Re}\, f - \mathrm{Im}\, f)|^2 &\leq 2\, \|\partial_j^2((\tan\theta)\, \mathrm{Re}\, f - \mathrm{Im}\, f)\|_\infty\, ((\tan\theta)\, \mathrm{Re}\, f - \mathrm{Im}\, f) \\ &\leq 2\,(1 + \tan\theta) \sup_{1 \leq j \leq d} \|\partial_j^2 f\|_\infty\, ((\tan\theta)\, \mathrm{Re}\, f - \mathrm{Im}\, f). \end{aligned}$$

Adding the two inequalities gives

$$(\tan\theta)^2\, |\partial_j(\mathrm{Re}\, f)|^2 + |\partial_j(\mathrm{Im}\, f)|^2 \leq 2\,(1 + \tan\theta)^2 \sup_{1 \leq j \leq d} \|\partial_j^2 f\|_\infty\, \mathrm{Re}\, f.$$

Hence

$$|\partial_j f|^2 = |\partial_j(\mathrm{Re}\, f)|^2 + |\partial_j(\mathrm{Im}\, f)|^2 \leq 4\,(1 + \tan\theta)^2 \sup_{1 \leq j \leq d} \|\partial_j^2 f\|_\infty\, \mathrm{Re}\, f$$

as required. □

Lemma 4.8. *Let $j \in \{1, \ldots, d\}$. Let $\xi, \eta \in \mathbb{C}^d$. Then the following are valid.*

(a) $|((\partial_j C)\,\xi,\eta)|^2 \le M\Big(((\mathrm{Re}\,C)\,\xi,\xi)+((\mathrm{Re}\,C)\,\eta,\eta)\Big)$, *where*

$$M = 8\,(1+\tan\theta)^2\,(\|\xi\|^2+\|\eta\|^2)\sup_{1\le l\le d}\|\partial_l^2 C\|_\infty.$$

(b) $|((\partial_j \mathrm{Im}\,C)\,\xi,\eta)|^2 \le M\Big(((\mathrm{Re}\,C)\,\xi,\xi)+((\mathrm{Re}\,C)\,\eta,\eta)\Big)$, *where*

$$M = 8\,(1+\tan\theta)^2\,(\|\xi\|^2+\|\eta\|^2)\sup_{1\le l\le d}\|\partial_l^2 C\|_\infty.$$

Proof. We will prove Statement (a). The proof for Statement (b) is similar.

Since C takes values in Σ_θ, we have

$$|(C\,\xi,\xi)| \le (1+\tan\theta)\,((\mathrm{Re}\,C)\,\xi,\xi).$$

Polarisation gives

$$\begin{aligned}|(C\,\xi,\eta)| &\le 2\,(1+\tan\theta)\,((\mathrm{Re}\,C)\,\xi,\xi)^{1/2}\,((\mathrm{Re}\,C)\,\eta,\eta)^{1/2}\\ &\le (1+\tan\theta)\Big(((\mathrm{Re}\,C)\,\xi,\xi)+((\mathrm{Re}\,C)\,\eta,\eta)\Big).\end{aligned}$$

Let

$$X = (1+\tan\theta)\Big(((\mathrm{Re}\,C)\,\xi,\xi)+((\mathrm{Re}\,C)\,\eta,\eta)\Big)$$

and

$$Y = (C\,\xi,\eta) = Y_1 + i\,Y_2,$$

where Y_1 and Y_2 are real-valued functions. Since $X - Y_1 \ge 0$, it follows from Lemma 4.6 that

$$|\partial_j(X-Y_1)|^2 \le 2\,\|\partial_j^2(X-Y_1)\|_\infty\,(X-Y_1) \le 2\,(\|\partial_j^2 X\|_\infty + \|\partial_j^2 Y\|_\infty)\,(X-Y_1).$$

Arguing similarly for $X+Y_1 \ge 0$ we yield

$$|\partial_j(X+Y_1)|^2 \le 2\,(\|\partial_j^2 X\|_\infty + \|\partial_j^2 Y\|_\infty)\,(X+Y_1).$$

By adding the two inequalities we obtain

$$|\partial_j X|^2 + |\partial_j Y_1|^2 \le 2\,(\|\partial_j^2 X\|_\infty + \|\partial_j^2 Y\|_\infty)\,X.$$

Analogously

$$|\partial_j X|^2 + |\partial_j Y_2|^2 \le 2\,(\|\partial_j^2 X\|_\infty + \|\partial_j^2 Y\|_\infty)\,X.$$

Hence

$$\begin{aligned}|((\partial_j C)\,\xi,\eta)|^2 = |\partial_j Y_1|^2 + |\partial_j Y_2|^2 &\le 4\,(\|\partial_j^2 X\|_\infty + \|\partial_j^2 Y\|_\infty)\,X\\ &\le M\Big(((\mathrm{Re}\,C)\,\xi,\xi)+((\mathrm{Re}\,C)\,\eta,\eta)\Big),\end{aligned}$$

where

$$M = 8\,(1+\tan\theta)^2\,(\|\xi\|^2+\|\eta\|^2)\sup_{1\le l\le d}\|\partial_l^2 C\|_\infty.$$

The proof is complete. □

Proposition 4.9 (Complex version of Oleinik's inequality)**.** *Let $j \in \{1, \ldots, d\}$. Let U be a complex $d \times d$ matrix. Then the following are valid.*

(a) $|\mathrm{tr}\,((\partial_j C)\, U)|^2 \le M \Big(\mathrm{tr}\,(U^*\,(\mathrm{Re}\, C)\, U) + \mathrm{tr}\,(U\,(\mathrm{Re}\, C)\, U^*)\Big)$, *where*

$$M = 16\, d\, (1 + \tan\theta)^2 \sup_{1 \le l \le d} \|\partial_l^2 C\|_\infty.$$

(b) $|\mathrm{tr}\,((\partial_j \mathrm{Im}\, C)\, U)|^2 \le M \Big(\mathrm{tr}\,(U^*\,(\mathrm{Re}\, C)\, U) + \mathrm{tr}\,(U\,(\mathrm{Re}\, C)\, U^*)\Big)$, *where*

$$M = 16\, d\, (1 + \tan\theta)^2 \sup_{1 \le l \le d} \|\partial_l^2 C\|_\infty.$$

Proof. We will prove Statement (a). The proof for Statement (b) is similar.

Let $j \in \{1, \ldots, d\}$ and

$$M = 16\, d\, (1 + \tan\theta)^2 \sup_{1 \le l \le d} \|\partial_l^2 C\|_\infty.$$

Let V be a unitary matrix such that $U = V\,|U|$, where $|U| = \sqrt{U^*\,U}$. Since $|U|$ is positive and Hermitian, there exists a unitary matrix W such that $|U| = W\,D\,W^*$, where D is a positive diagonal matrix. It follows that

$$\begin{aligned}
|\mathrm{tr}\,((\partial_j C)\, U)|^2 &= |\mathrm{tr}\,((\partial_j C)\, V\,|U|)|^2 = |\mathrm{tr}\,(W^*\,(\partial_j C)\, V\, W\, W^*\,|U|\, W)|^2 \\
&= |\mathrm{tr}\,(W^*\,(\partial_j C)\, V\, W\, D)|^2 = \Big|\sum_{k=1}^{d} (W^*\,(\partial_j C)\, V\, W)_{kk}\, D_{kk}\Big|^2 \\
&\le d \sum_{k=1}^{d} |(W^*\,(\partial_j C)\, V\, W)_{kk}|^2\, |D_{kk}|^2 \\
&\le M \sum_{k=1}^{d} \Big((W^*\,(\mathrm{Re}\, C)\, W)_{kk} + (W^*\, V^*\,(\mathrm{Re}\, C)\, V\, W)_{kk}\Big)\, |D_{kk}|^2 \\
&\le M \sum_{k=1}^{d} \Big(D_{kk}\,(W^*\,(\mathrm{Re}\, C)\, W)_{kk}\, D_{kk} + D_{kk}\,(W^*\, V^*\,(\mathrm{Re}\, C)\, V\, W)_{kk}\, D_{kk}\Big) \\
&\le M \Big(\mathrm{tr}\,(|U|\,(\mathrm{Re}\, C)\,|U|) + \mathrm{tr}\,(|U|\, V^*\,(\mathrm{Re}\, C)\, V\,|U|)\Big) \\
&= M \Big(\mathrm{tr}\,(U\,(\mathrm{Re}\, C)\, U^*) + \mathrm{tr}\,(U^*\,(\mathrm{Re}\, C)\, U)\Big),
\end{aligned}$$

where we used Lemma 4.8 in the second inequality. □

Corollary 4.10. *Let $j \in \{1, \ldots, d\}$. Suppose U is a complex $d \times d$ matrix with $U^T = U$. Then the following are valid.*

(a) $|\mathrm{tr}\,((\partial_j C)\, U)|^2 \le M\, \mathrm{tr}\,(U\, R_s\, \overline{U})$, *where*

$$M = 32\, d\, (1 + \tan\theta)^2 \sup_{1 \le l \le d} \|\partial_l^2 C\|_\infty.$$

(b) $|\mathrm{tr}\,((\partial_j \mathrm{Im}\, C)\, U)|^2 \le M\, \mathrm{tr}\,(U\, R_s\, \overline{U})$, *where*

$$M = 32\, d\, (1 + \tan\theta)^2 \sup_{1 \le l \le d} \|\partial_l^2 C\|_\infty.$$

Proof. Since $U^T = U$ we have

$$\begin{aligned} \mathrm{tr}\,(U^* \,(\mathrm{Re}\, C)\, U) + \mathrm{tr}\,(U\, (\mathrm{Re}\, C)\, U^*) &= \mathrm{tr}\,(\overline{U}\, (\mathrm{Re}\, C)\, U) + \mathrm{tr}\,(U\, (\mathrm{Re}\, C)\, \overline{U}) \\ &= \mathrm{tr}\,(\overline{U}\, (\mathrm{Re}\, C)\, U) + \mathrm{tr}\,(\overline{U}\, (\mathrm{Re}\, C)^T\, U) \\ &= 2\, \mathrm{tr}\,(U\, R_s\, \overline{U}). \end{aligned}$$

Next we use Proposition 4.9 to derive the result. □

Lemma 4.11. *Let U be a complex $d \times d$ matrix. Then*

$$((\mathrm{Re}\, C)\, U\, \xi, U\, \xi) \le \mathrm{tr}\,(U^*\, (\mathrm{Re}\, C)\, U)\, \|\xi\|^2$$

for all $\xi \in \mathbb{C}^d$.

Proof. By hypothesis $\mathrm{Re}\, C \ge 0$. Therefore $((\mathrm{Re}\, C)\, U\, \xi, U\, \xi) \ge 0$ for all $\xi \in \mathbb{C}^d$. It follows that $U^*\, (\mathrm{Re}\, C)\, U \ge 0$. Hence $U^*\, (\mathrm{Re}\, C)\, U \le \mathrm{tr}\,(U^*\, (\mathrm{Re}\, C)\, U)\, I$, where I denotes the identity matrix. This justifies the claim. □

Lemma 4.12. *We have*

$$|(B_a\, \xi, \xi)| \le (R_s\, \xi, \xi)$$

for all $\xi \in \mathbb{C}^d$.

Proof. Write $\xi = \xi_1 + i\, \xi_2$, where $\xi_1, \xi_2 \in \mathbb{R}^d$. Then $(R_s\, \xi, \xi) = (R_s\, \xi_1, \xi_1) + (R_s\, \xi_2, \xi_2)$ and $(B_a\, \xi, \xi) = -2i\, (B_a\, \xi_1, \xi_2)$. Since C takes values in Σ_θ, we have $((\mathrm{Re}\, C)\, \xi, \xi) \ge 0$ for all $\xi \in \mathbb{C}^d$. Equivalently

$$-2\, (B_a\, \xi_1, \xi_2) \le (R_s\, \xi_1, \xi_1) + (R_s\, \xi_2, \xi_2).$$

Replacing ξ by $\overline{\xi}$ and repeating the same process as above we also obtain

$$2\, (B_a\, \xi_1, \xi_2) \le (R_s\, \xi_1, \xi_1) + (R_s\, \xi_2, \xi_2).$$

The result now follows. □

Lemma 4.13. *Let $l \in \{1, \ldots, d\}$ and $\xi \in \mathbb{C}^d$. Then*

$$|((\partial_l B_a)\, \xi, \xi)|^2 \le M\, (R_s\, \xi, \xi),$$

where $M = 2\, \|\xi\|^2 \sup_{1 \le l \le d} \|\partial_l^2 C\|_\infty$.

Proof. Let $l \in \{1, \ldots, d\}$ and $\xi \in \mathbb{C}^d$. By Lemma 4.12 we deduce that $R_s \pm i\, B_a \ge 0$. Now we use Lemma 4.6 to derive

$$\begin{aligned} |(\partial_l (R_s + i\, B_a)\, \xi, \xi)|^2 &\le 2\, \|(\partial_l^2 (R_s + i\, B_a)\, \xi, \xi)\|_\infty\, ((R_s + i\, B_a)\, \xi, \xi) \\ &\le 2\, \|\xi\|^2 \sup_{1 \le l \le d} \|\partial_l^2 C\|_\infty\, ((R_s + i\, B_a)\, \xi, \xi) \end{aligned}$$

and

$$|(\partial_l (R_s - i\,B_a)\,\xi,\xi)|^2 \le 2\,\|(\partial_l^2 (R_s - i\,B_a)\,\xi,\xi)\|_\infty\,((R_s - i\,B_a)\,\xi,\xi)$$
$$\le 2\,\|\xi\|^2 \sup_{1\le l\le d} \|\partial_l^2 C\|_\infty\,((R_s - i\,B_a)\,\xi,\xi).$$

Adding the two inequalities together gives

$$|((\partial_l R_s)\,\xi,\xi)|^2 + |((\partial_l B_a)\,\xi,\xi)|^2 \le 2\,\|\xi\|^2 \sup_{1\le l\le d} \|\partial_l^2 C\|_\infty\,(R_s\,\xi,\xi),$$

which clearly implies the result. □

Lemma 4.14. *Let Q be a complex $d\times d$ matrix. Suppose there exists an $M>0$ such that $|(Q\,\xi,\xi)| \le M\,(R_s\,\xi,\xi)$ for all $\xi\in\mathbb{C}^d$. Then $\|Q\,\xi\|^2 \le 4\,M^2\,\|R_s\|_\infty\,(R_s\,\xi,\xi)$ for all $\xi\in\mathbb{C}^d$.*

Proof. Since $|(Q\,\xi,\xi)| \le M\,(R_s\,\xi,\xi)$ for all $\xi\in\mathbb{C}^d$, polarisation gives

$$|(Q\,\xi,\eta)| \le 2\,M\,(R_s\,\xi,\xi)^{1/2}\,(R_s\,\eta,\eta)^{1/2} \le 2\,M\,\|R_s\|_\infty^{1/2}\,\|\eta\|\,(R_s\,\xi,\xi)^{1/2}$$

for all $\xi,\eta\in\mathbb{C}^d$. It follows that

$$\|Q\,\xi\| \le 2\,M\,\|R_s\|_\infty^{1/2}\,(R_s\,\xi,\xi)^{1/2}$$

for all $\xi\in\mathbb{C}^d$, which justifies the claim. □

Lemma 4.15. *We have*

$$\|C\,\xi\|^2 \le 16\,(1+\tan\theta)^2\,\|R_s\|_\infty\,(R_s\,\xi,\xi)$$

for all $\xi\in\mathbb{C}^d$.

Proof. Let $\xi\in\mathbb{C}^d$. Since C takes values in Σ_θ, we have

$$|(C\,\xi,\xi)| \le ((\operatorname{Re} C)\,\xi,\xi) + |((\operatorname{Im} C)\,\xi,\xi)| \le (1+\tan\theta)\,((\operatorname{Re} C)\,\xi,\xi).$$

However $((\operatorname{Re} C)\,\xi,\xi) \le 2\,(R_s\,\xi,\xi)$ by Lemma 4.12. It follows that

$$|(C\,\xi,\xi)| \le 2\,(1+\tan\theta)\,(R_s\,\xi,\xi).$$

Using Lemma 4.14 we obtain

$$\|C\,\xi\|^2 \le 16\,(1+\tan\theta)^2\,\|R_s\|_\infty\,(R_s\,\xi,\xi)$$

as required. □

Recall that the Hilbert-Schmidt norm for a matrix $V\in M_{d\times d}(\mathbb{C})$ is defined by

$$\|V\|_{HS} = (\operatorname{tr}(V^*\,V))^{1/2} = \left(\sum_{j=1}^d \|V\,e_j\|^2\right)^{1/2}.$$

Lemma 4.16. *Let U a complex $d\times d$ matrix with $U^T = U$. Then*

$$\|C\,U\|_{HS}^2 \le 16\,(1+\tan\theta)^2\,\|R_s\|_\infty\,\operatorname{tr}(U\,R_s\,\overline{U}).$$

Proof. We note that

$$\|C\,U\|_{HS}^2 = \sum_{j=1}^d \|C\,U e_j\|_2^2 \le 16\,(1+\tan\theta)^2\,\|R_s\|_\infty \sum_{j=1}^d (R_s\,U e_j, U e_j)$$
$$= 16\,(1+\tan\theta)^2\,\|R_s\|_\infty\,\operatorname{tr}(U\,R_s\,\overline{U}),$$

where we used Lemma 4.15 in the second step. □

4.3 L_p extension

Let S be the contraction C_0-semigroup generated by $-A$. In this section we will extend S to a contraction C_0-semigroup on $L_p(\mathbb{R}^d)$ for all $p \in (1,\infty)$ with $|1-\frac{2}{p}| \le \cos\theta$, under the condition that $B_a = 0$.

Proof of Proposition 4.1. We proceed via two steps.
Step 1: Suppose that A is strongly elliptic.
Then S extends consistently to a C_0-semigroup $S^{(p)}$ on $L_p(\mathbb{R}^d)$ by [AMT98, Theorem 2.21]. Using duality arguments we can assume without loss of generality that $p \ge 2$. Let $-A_p$ be the generator of $S^{(p)}$. Let $u \in \mathcal{D}$, where $\mathcal{D} = D(A) \cap D(A_p) \cap L_\infty(\mathbb{R}^d)$. Since A is strongly elliptic, the form $\mathfrak{a}_0$ is closable and $D(\overline{\mathfrak{a}_0}) = W^{1,2}(\mathbb{R}^d)$. By construction $D(A) \subset D(\overline{\mathfrak{a}_0})$. Therefore $u \in W^{1,2}(\mathbb{R}^d)$. Set $v = |u|^{p-2}\, u$. Then $v \in L_q(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$, where q is the dual exponent of p. By [GT83, Lemma 7.7] we have

$$\partial_l v = \frac{p}{2}\, |u|^{p-2}\, \partial_l u + \frac{p-2}{2}\, |u|^{p-4}\, u^2\, \partial_l \overline{u}$$

for all $l \in \{1,\ldots,d\}$. It follows that $v \in W^{1,2}(\mathbb{R}^d)$. Our aim is to prove the inequality $\mathrm{Re} \int (A_p u)\, \overline{v} \ge 0$, where here and in the rest of this paragraph the integral is over the set $\{x \in \mathbb{R}^d : u(x) \ne 0\}$. Indeed we have

$$\begin{aligned}
\int (A_p u)\, \overline{v} = \int (Au)\, \overline{v} = \overline{\mathfrak{a}_0}(u,v) &= \sum_{k,l=1}^d \int c_{kl}\, (\partial_k u)\, \partial_l \overline{v} \\
&= \sum_{k,l=1}^d \int c_{kl}\, (\partial_k u) \Big(\frac{p}{2}\, |u|^{p-2}\, \partial_l \overline{u} + \frac{p-2}{2}\, |u|^{p-4}\, \overline{u}^2\, \partial_l u \Big) \\
&= \frac{1}{2} \int |u|^{p-4} \sum_{k,l=1}^d \Big(p\, c_{kl}\, |u|^2\, (\partial_k u)\, \partial_l \overline{u} + (p-2)\, c_{kl}\, \overline{u}^2\, (\partial_k u)\, \partial_l u \Big) \\
&= \frac{1}{2} \int |u|^{p-4} \Big(p\, (C\, u\, \nabla \overline{u}, u\, \nabla \overline{u}) + (p-2)\, (C\, \overline{u}\, \nabla u, u\, \nabla \overline{u}) \Big).
\end{aligned}$$

Write $u\, \nabla \overline{u} = \xi + i\, \eta$, where $\xi, \eta \in \mathbb{R}^d$. Then

$$\mathrm{Re}\, (C\, u\, \nabla \overline{u}, u\, \nabla \overline{u}) = (R_s\, \xi, \xi) + (R_s\, \eta, \eta) + 2\, (B_a\, \xi, \eta) = (R_s\, \xi, \xi) + (R_s\, \eta, \eta)$$

as $B_a = 0$ by hypothesis and

$$\mathrm{Re}\, (C\, \overline{u}\, \nabla u, u\, \nabla \overline{u}) = (R_s\, \xi, \xi) - (R_s\, \eta, \eta) + 2\, (B_s\, \xi, \eta).$$

Therefore

$$\begin{aligned}
\mathrm{Re} \int (A_p u)\, \overline{v} &= \int |u|^{p-4} \Big((p-1)\, (R_s\, \xi, \xi) + (R_s\, \eta, \eta) + (p-2)\, (B_s\, \xi, \eta) \Big) \\
&= \int |u|^{p-4} \Big((R_s\, \xi', \xi') + (R_s\, \eta, \eta) + \frac{p-2}{\sqrt{p-1}}\, (B_s\, \xi', \eta) \Big),
\end{aligned}$$

where $\xi' = \sqrt{p-1}\,\xi$. If $\theta = 0$ then it follows from Lemma 4.5 that $(B_s\,\xi', \eta) = 0$. Consequently

$$\operatorname{Re} \int (A_p u)\,\overline{v} = \int |u|^{p-4} \Big((R_s\,\xi', \xi') + (R_s\,\eta, \eta) \Big) \geq 0.$$

If $\theta \neq 0$ then

$$\operatorname{Re} \int (A_p u)\,\overline{v} \geq \int |u|^{p-4} \Big((R_s\,\xi', \xi') + (R_s\,\eta, \eta) - 2\,\cot\theta\,|(B_s\,\xi', \eta)| \Big) \geq 0,$$

where we again used Lemma 4.5 and the fact that $|1 - \frac{2}{p}| \leq \cos\theta$ is equivalent to $|p - 2|\,\tan\theta \leq 2\sqrt{p-1}$. In either case the restriction $A_p|_{\mathcal{D}}$ is accretive. Since $\mathcal{D}$ is a core for A_p, we also have that A_p is accretive by [LP61, Lemma 3.4]. By the Lumer-Phillips theorem, $S^{(p)}$ is a contraction semigroup.

Step 2: Suppose that A is degenerate elliptic.
Let $n \in \mathbb{N}$. Let $A_{[n]} = A - \frac{1}{n}\,\Delta$, where $\Delta = \partial_1^2 + \ldots + \partial_d^2$. Then $A_{[n]}$ is strongly elliptic. Let $S^{[n]}$ be the contraction C_0-semigroup generated by $A_{[n]}$. Then $S^{[n]}$ extends consistently to a contraction C_0-semigroup $S^{(n,p)}$ on $L_p(\mathbb{R}^d)$ by Step 1. Using duality arguments we can assume without loss of generality that $p \in (1,2)$.

Let $t > 0$ and $u \in L_{2,c}(\mathbb{R}^d)$. By [AE12, Corollary 3.9] we have $\lim_{n\to\infty} S_t^{[n]} u = S_t u$ in $L_2(\mathbb{R}^d)$. Also by [AE12, Lemma 4.5] we obtain $\lim_{n\to\infty} S_t^{[n]} u = S_t u$ in $L_1(\mathbb{R}^d)$. Interpolation then gives $\lim_{n\to\infty} S_t^{[n]} u = S_t u$ in $L_p(\mathbb{R}^d)$. It follows that $\|S_t u\|_p \leq \|u\|_p$ as $S^{(n,p)}$ is contractive on $L_p(\mathbb{R}^d)$. But $L_{2,c}(\mathbb{R}^d)$ is dense in $L_2(\mathbb{R}^d) \cap L_p(\mathbb{R}^d)$. Therefore $\|S_t u\|_p \leq \|u\|_p$ for all $u \in L_2(\mathbb{R}^d) \cap L_p(\mathbb{R}^d)$. That is, $S_t|_{L_2(\mathbb{R}^d)\cap L_p(\mathbb{R}^d)}$ extends continuously to a contraction operator $S_t^{(p)}$ on $L_p(\mathbb{R}^d)$. We now use [Voi92, Proposition 1] to conclude that $S^{(p)}$ is a C_0-semigroup on $L_p(\mathbb{R}^d)$. □

4.4 The operator B_p

Let $p \in (1,\infty)$. Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Define

$$H_q u = -\sum_{k,l=1}^{d} \partial_k(\overline{c_{kl}}\,\partial_l u) \tag{4.4}$$

on the domain

$$D(H_q) = C_c^\infty(\mathbb{R}^d).$$

Next define $B_p = (H_q)^*$, which is the dual of H_q. Then B_p is closed by [Kat80, Subsection III.5.5]. Also note that $W^{2,p}(\mathbb{R}^d) \subset D(B_p)$ and

$$B_p u = -\sum_{k,l=1}^{d} \partial_l(c_{kl}\,\partial_k u)$$

for all $u \in W^{2,p}(\mathbb{R}^d)$.

We will prove at the end of this section that $C_c^\infty(\mathbb{R}^d)$ is a core for B_p if $|1 - \frac{2}{p}| < \cos\theta$ and $B_a = 0$. In the next section we will prove that $A_p = B_p$ under the same assumptions. The proofs require a lot of preparation.

Proposition 4.17. *Suppose* $|1-\frac{2}{p}| \le \cos\theta$ *and* $B_a = 0$. *Then*

$$\mathrm{Re}\,(B_p u, |u|^{p-2}\, u) \ge 0$$

for all $u \in W^{2,p}(\mathbb{R}^d)$.

Proof. Let $u \in W^{2,p}(\mathbb{R}^d)$. It follows from Theorem 2.30 that

$$\begin{aligned}(B_p u, |u|^{p-2}\, u) = &\int_{[u\neq 0]} |u|^{p-2}\,(C\,\nabla\overline{u}, \nabla\overline{u}) \\ &+ (p-2)\int_{[u\neq 0]} |u|^{p-4}\,\big(C\,\mathrm{Re}\,(u\,\nabla\overline{u}), \mathrm{Re}\,(u\,\nabla\overline{u})\big) \\ &- i\,(p-2)\int_{[u\neq 0]} |u|^{p-4}\,\big(C\,\mathrm{Re}\,(u\,\nabla\overline{u}), \mathrm{Im}\,(u\,\nabla\overline{u})\big). \end{aligned} \tag{4.5}$$

Write $u\,\nabla\overline{u} = \xi + i\,\eta$, where $\xi, \eta \in \mathbb{R}^d$. Then

$$\begin{aligned}|u|^2\,(C\,\nabla\overline{u}, \nabla\overline{u}) &= (C\,u\,\nabla\overline{u}, u\,\nabla\overline{u}) = \big(C(\xi + i\,\eta), \xi + i\,\eta\big) \\ &= (R\,\xi, \xi) + (R\,\eta, \eta) + (B\,\xi, \eta) - (B\,\eta, \xi) \\ &\quad - i\,\big((R\,\eta, \xi) - (R\,\xi, \eta) + (B\,\xi, \xi) + (B\,\eta, \eta)\big).\end{aligned}$$

Therefore

$$\begin{aligned}\mathrm{Re}\,\big(|u|^2\,(C\,\nabla\overline{u}, \nabla\overline{u})\big) &= (R\,\xi, \xi) + (R\,\eta, \eta) + (B\,\xi, \eta) - (B\,\eta, \xi) \\ &= (R_s\,\xi, \xi) + (R_s\,\eta, \eta) + 2\,(B_a\,\xi, \eta) \\ &= (R_s\,\xi, \xi) + (R_s\,\eta, \eta)\end{aligned}$$

since $B_a = 0$. We also have

$$\mathrm{Re}\,\big(C\,\mathrm{Re}\,(u\,\nabla\overline{u}), \mathrm{Re}\,(u\,\nabla\overline{u})\big) = \mathrm{Re}\,(C\,\xi, \xi) = (R\,\xi, \xi) = (R_s\,\xi, \xi).$$

Similarly

$$\mathrm{Re}\,\Big(i\,\big(C\,\mathrm{Re}\,(u\,\nabla\overline{u}), \mathrm{Im}\,(u\,\nabla\overline{u})\big)\Big) = \mathrm{Re}\,\big(i\,(C\,\xi, \eta)\big) = -(B\,\xi, \eta) = -(B_s\,\xi, \eta)$$

since $B_a = 0$. Hence taking the real parts on both sides of (4.5) yields

$$\mathrm{Re}\,(B_p u, |u|^{p-2}\, u) = \int_{[u\neq 0]} |u|^{p-4}\,\Big((p-1)\,(R_s\,\xi, \xi) + (R_s\,\eta, \eta) + (p-2)\,(B_s\,\xi, \eta)\Big)$$

since $B_a = 0$. Now we argue as in Step 1 of the proof of Proposition 4.1 to derive the claim. □

Let $J \in C_c^\infty(\mathbb{R}^d, \mathbb{R})$ be such that $J \ge 0$, $\mathrm{supp}\, J \subset B_1(0)$ and $\int_{\mathbb{R}^d} J = 1$. For each $n \in \mathbb{N}$ and $x \in \mathbb{R}^d$ define $J_n(x) = n^d\, J(n\,x)$. For all $n \in \mathbb{N}$ define the bounded operator $T_n^{(1)} : W^{1,p}(\mathbb{R}^d) \longrightarrow L_p(\mathbb{R}^d)$ by

$$T_n^{(1)} u = -\sum_{k,l=1}^{d} \int_{\mathbb{R}^d} J_n(y)\,\Big((I - L_y)\,(\partial_l c_{kl})\Big)\,L_y(\partial_k u)\,dy$$

and the bounded operator $T_n^{(2)} : W^{1,p}(\mathbb{R}^d) \longrightarrow L_p(\mathbb{R}^d)$ by

$$T_n^{(2)} u = - \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} \left(\frac{\partial}{\partial y_l} \Big(J_n(y)\,(I - L_y)\,c_{kl} \Big) \right) L_y(\partial_k u)\,dy,$$

where $(L_y u)(x) = u(x-y)$ for all $x, y \in \mathbb{R}^d$. Also define for all $n \in \mathbb{N}$ the operator $T_n : W^{1,p}(\mathbb{R}^d) \longrightarrow L_p(\mathbb{R}^d)$ by

$$T_n = T_n^{(1)} + T_n^{(2)}. \tag{4.6}$$

Lemma 4.18. *The sequence $\{T_n^{(1)}\}_{n\in\mathbb{N}}$ is bounded. Furthermore $\lim_{n\to\infty} \|T_n^{(1)} u\|_p = 0$ for all $u \in W^{1,p}(\mathbb{R}^d)$.*

Proof. Let $n \in \mathbb{N}$ and $u \in W^{1,p}(\mathbb{R}^d)$. For all $k, l \in \{1, \ldots, d\}$ we have $c_{kl} \in W^{2,\infty}(\mathbb{R}^d)$, which implies

$$|(\partial_l c_{kl})(x) - (\partial_l c_{kl})(x-y)| \le \|c_{kl}\|_{W^{2,\infty}}\,|y| \tag{4.7}$$

for all $x, y \in \mathbb{R}^d$. It follows that

$$\begin{aligned}
\|T_n^{(1)} u\|_p &\le \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} |J_n(y)| \left\| \Big((I - L_y)\,(\partial_l c_{kl}) \Big)\, L_y(\partial_k u) \right\|_p dy \\
&\le \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} |J_n(y)|\, \|(I - L_y)\,(\partial_l c_{kl})\|_\infty\, \|L_y(\partial_k u)\|_p\,dy \\
&\le \Big(\sum_{k,l=1}^{d} \|c_{kl}\|_{W^{2,\infty}} \Big) \|u\|_{W^{1,p}} \int_{\mathbb{R}^d} |J_n(y)|\,|y|\,dy \\
&= \Big(\sum_{k,l=1}^{d} \|c_{kl}\|_{W^{2,\infty}} \Big) \|u\|_{W^{1,p}}\, \frac{1}{n} \int_{\mathbb{R}^d} |J(y)|\,|y|\,dy,
\end{aligned}$$

where we used $J_n(y) = n^d\, J(n\,y)$ in the last step. Note that

$$\lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{R}^d} |J(y)|\,|y|\,dy = 0.$$

Hence $\lim_{n\to\infty} \|T_n^{(1)} u\|_p = 0$. Moreover, $\{T_n^{(1)}\}_{n\in\mathbb{N}}$ is bounded. □

Lemma 4.19. *The sequence $\{T_n^{(2)}\}_{n\in\mathbb{N}}$ is bounded. Furthermore $\lim_{n\to\infty} \|T_n^{(2)} u\|_p = 0$ for all $u \in W^{1,p}(\mathbb{R}^d) \cap L_{p,c}(\mathbb{R}^d)$.*

Proof. Let $n \in \mathbb{N}$ and $u \in W^{1,p}(\mathbb{R}^d)$. Expanding $T_n^{(2)}$ gives

$$T_n^{(2)} u = - \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} \Big(J_n(y)\, L_y(\partial_l c_{kl}) + (\partial_l J_n)(y)\,(I - L_y)\,c_{kl} \Big)\, L_y(\partial_k u)\,dy,$$

where we used $L_y(\partial_l c_{kl}) = -\frac{\partial}{\partial y_l}(L_y c_{kl})$ for all $k, l \in \{1, \ldots, d\}$. Therefore

$$\begin{aligned}\|T_n^{(2)}u\|_p &\le \sum_{k,l=1}^d \left(\|(\partial_l c_{kl})\,(\partial_k u)\|_p + \int_{\mathbb{R}^d} |(\partial_l J_n)(y)|\,\|((I-L_y)\,c_{kl})\,L_y(\partial_k u)\|_p\,dy \right)\\ &\le \sum_{k,l=1}^d \left(\|\partial_l c_{kl}\|_\infty\,\|(\partial_k u)\|_p + \int_{\mathbb{R}^d} |(\partial_l J_n)(y)|\,\|(I-L_y)\,c_{kl}\|_\infty\,\|L_y(\partial_k u)\|_p\,dy \right)\\ &\le M\,\|u\|_{W^{1,p}}, \end{aligned} \tag{4.8}$$

where

$$M = \sum_{k,l=1}^d \left(\|c_{kl}\|_{W^{2,\infty}} \left(1 + \int_{\mathbb{R}^d} |(\partial_l J)(y)|\,|y|\,dy \right) \right) \tag{4.9}$$

and we used (4.7) in the last step. Therefore $\{T_n^{(2)}\}_{n\in\mathbb{N}}$ is bounded.

To prove to the latter statement of the lemma, we consider two cases.

Case 1: Suppose $u \in C_c^\infty(\mathbb{R}^d)$.

Since J_n has a compact support, we have

$$\sum_{k,l=1}^d \int_{\mathbb{R}^d} \left(\frac{\partial}{\partial y_l} \Big(J_n(y)\,(I-L_y)\,c_{kl} \Big) \right) (\partial_k u)\,dy = 0.$$

Consequently

$$\begin{aligned} T_n^{(2)}u &= \sum_{k,l=1}^d \int_{\mathbb{R}^d} \left(\frac{\partial}{\partial y_l} \Big(J_n(y)\,(I-L_y)\,c_{kl} \Big) \right) (I-L_y)\,(\partial_k u)\,dy\\ &= \sum_{k,l=1}^d \int_{\mathbb{R}^d} \Big(J_n(y)\,L_y(\partial_l c_{kl}) + (\partial_l J_n)(y)\,(I-L_y)\,c_{kl} \Big)\,(I-L_y)\,(\partial_k u)\,dy. \end{aligned}$$

It follows that

$$\|T_n^{(2)}u\|_p \le \sum_{k,l=1}^d \int_{\mathbb{R}^d} \Big(J_n(y)\,\|L_y(\partial_l c_{kl})\|_\infty + |(\partial_l J_n)(y)|\,\|(I-L_y)\,c_{kl}\|_\infty \Big)\,\|(I-L_y)(\partial_k u)\|_p\,dy.$$

Note that

$$\begin{aligned} \|(I-L_y)(\partial_k u)\|_p &= \left(\int_{\mathbb{R}^d} |(\partial_k u)(x) - (\partial_k u)(x-y)|^p\,dx \right)^{\frac{1}{p}}\\ &\le \left(\int_{\mathbb{R}^d} (\|u\|_{W^{2,\infty}}\,|y|)^p\,\mathbb{1}_{\operatorname{supp}\partial_k u \cup \operatorname{supp} L_y(\partial_k u)}\,dx \right)^{\frac{1}{p}}\\ &\le 2\,|\operatorname{supp}\partial_k u|^{1/p}\,\|u\|_{W^{2,\infty}}\,|y| \le \frac{2}{n}\,|\operatorname{supp} u|^{1/p}\,\|u\|_{W^{2,\infty}} \end{aligned}$$

for all $k \in \{1,\ldots,d\}$ and $y \in \mathbb{R}^d$ such that $|y| < \frac{1}{n}$, where in the last step we used the fact that $\operatorname{supp}\partial_k u \subset \operatorname{supp} u$ for all $k \in \{1,\ldots,d\}$. Therefore

$$\|T_n^{(2)}u\|_p \le \frac{2M}{n}\,|\operatorname{supp} u|^{1/p}\,\|u\|_{W^{2,\infty}}, \tag{4.10}$$

where M is defined by (4.9) and we used the fact that $\int_{\mathbb{R}^d} J_n(y)\,dy = 1$. Hence (4.10) gives $\lim_{n\to\infty} \|T_n^{(2)} u\|_p = 0$.
Case 2: Suppose $u \in W^{1,p}(\mathbb{R}^d) \cap L_{p,c}(\mathbb{R}^d)$.
Let $\varepsilon > 0$. Let $v \in C_c^\infty(\mathbb{R}^d)$ be such that $\|u - v\|_{W^{1,p}} < \frac{\varepsilon}{2M}$. Choose an $n \in \mathbb{N}$ such that $\frac{2M}{n} \, |\mathrm{supp}\, v|^{1/p} \, \|v\|_{W^{2,\infty}} < \frac{\varepsilon}{2}$. Then it follows from (4.8) and (4.10) that

$$\|T_n^{(2)} u\|_p \le \|T_n^{(2)}(u-v)\|_p + \|T_n^{(2)} v\|_p \le M \, \|u - v\|_{W^{1,p}} + \frac{2M}{n} \, |\mathrm{supp}\, v|^{1/p} \, \|v\|_{W^{2,\infty}} < \varepsilon.$$

The proof is complete. □

Lemma 4.20. *The sequence $\{T_n\}_{n\in\mathbb{N}}$ is bounded. Furthermore $\lim_{n\to\infty} \|T_n u\|_p = 0$ for all $u \in W^{1,p}(\mathbb{R}^d) \cap L_{p,c}(\mathbb{R}^d)$.*

Proof. This is a consequence of Lemmas 4.18 and 4.19. □

We have the following approximation proposition (cf. [Fri44] and [Kat72] for a special case of the proposition when the coefficient c_{kl} are real-valued for all $k, l \in \{1, \ldots, d\}$).

Proposition 4.21. *Let $u \in D(B_p) \cap W^{1,p}(\mathbb{R}^d) \cap L_{p,c}(\mathbb{R}^d)$. Then $\lim_{n\to\infty} B_p(J_n * u) = B_p u$ in $L_p(\mathbb{R}^d)$.*

Proof. Let $u \in D(B_p) \cap W^{1,p}(\mathbb{R}^d) \cap L_{p,c}(\mathbb{R}^d)$. It is well-known that $\lim_{n\to\infty} J_n * (B_p u) = B_p u$ in $L_p(\mathbb{R}^d)$. Therefore it suffices to show that

$$\lim_{n\to\infty} \|B_p(J_n * u) - J_n * (B_p u)\|_p = 0.$$

In what follows note that $L_y(\partial_l u) = -\frac{\partial}{\partial_l}(L_y u)$ and $\partial_l (J_n * u) = (\partial_l J_n) * u$ for all $l \in \{1, \ldots, d\}$. We first calculate $J_n * (B_p u)$. Let $x \in \mathbb{R}^d$. Define $\phi(y) = J_n(x - y)$ for all $y \in \mathbb{R}^d$. Then $\phi \in C_c^\infty(\mathbb{R}^d)$. By the definition of B_p we have

$$\begin{aligned}
(J_n * (B_p u))(x) &= \int_{\mathbb{R}^d} J_n(x-y)\,(B_p u)(y)\,dy = (B_p u, \phi) = (u, H_q \phi) \\
&= -\sum_{k,l=1}^d \int_{\mathbb{R}^d} \Big(\frac{\partial}{\partial y_k} \big(c_{kl}(y) \, \frac{\partial}{\partial y_l} J_n(x-y) \big) \Big) \, u(y)\,dy \\
&= \sum_{k,l=1}^d \int_{\mathbb{R}^d} \Big(c_{kl}(y) \, \frac{\partial}{\partial y_l} J_n(x-y) \Big) \, (\partial_k u)(y)\,dy \\
&= -\sum_{k,l=1}^d \int_{\mathbb{R}^d} (\partial_l J_n)(x-y) \, (c_{kl} \, \partial_k u)(y)\,dy \\
&= -\sum_{k,l=1}^d \int_{\mathbb{R}^d} (\partial_l J_n)(y) \, (c_{kl} \, \partial_k u)(x-y)\,dy
\end{aligned}$$

for all $n \in \mathbb{N}$. Hence

$$J_n * (B_p u) = -\sum_{k,l=1}^d \int_{\mathbb{R}^d} (\partial_l J_n)(y) \, L_y(c_{kl} \, \partial_k u)\,dy$$

for all $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. We have

$$
\begin{aligned}
&B_p(J_n * u) - J_n * (B_p u) \\
&= -\sum_{k,l=1}^{d} \Big(\partial_l \Big(c_{kl} \int_{\mathbb{R}^d} J_n(y)\, L_y(\partial_k u)\, dy \Big) - \int_{\mathbb{R}^d} (\partial_l J_n)(y)\, L_y(c_{kl}\, \partial_k u)\, dy \Big) \\
&= -\sum_{k,l=1}^{d} \Big((\partial_l c_{kl}) \int_{\mathbb{R}^d} J_n(y)\, L_y(\partial_k u)\, dy + c_{kl}\, \partial_l \Big(\int_{\mathbb{R}^d} J_n(y)\, L_y(\partial_k u)\, dy \Big) \\
&\qquad - \int_{\mathbb{R}^d} (\partial_l J_n)(y)\, L_y(c_{kl}\, \partial_k u)\, dy \\
&= -\sum_{k,l=1}^{d} \Big((\partial_l c_{kl}) \int_{\mathbb{R}^d} J_n(y)\, L_y(\partial_k u)\, dy + c_{kl} \int_{\mathbb{R}^d} (\partial_l J_n)(y)\, L_y(\partial_k u)\, dy \\
&\qquad - \int_{\mathbb{R}^d} (\partial_l J_n)(y)\, L_y(c_{kl}\, \partial_k u)\, dy \Big).
\end{aligned}
$$

On the other hand expanding $T_n^{(1)}$ and $T_n^{(2)}$ gives

$$
T_n^{(1)} u = -\sum_{k,l=1}^{d} \int_{\mathbb{R}^d} \Big(J_n(y)\, (\partial_l c_{kl})\, L_y(\partial_k u) - J_n(y)\, L_y\big((\partial_l c_{kl})\, \partial_k u\big) \Big)\, dy
$$

and

$$
\begin{aligned}
T_n^{(2)} u &= -\sum_{k,l=1}^{d} \int_{\mathbb{R}^d} \Big(J_n(y)\, L_y(\partial_l c_{kl}) + (\partial_l J_n)(y)\, (I - L_y)\, c_{kl} \Big)\, L_y(\partial_k u)\, dy \\
&= -\sum_{k,l=1}^{d} \int_{\mathbb{R}^d} \Big(J_n(y)\, L_y\big((\partial_l c_{kl})\, \partial_k u\big) + (\partial_l J_n)(y)\, c_{kl}\, L_y(\partial_k u) \\
&\qquad - (\partial_l J_n)(y)\, L_y(c_{kl}\, \partial_k u) \Big)\, dy.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
T_n u = T_n^{(1)} u + T_n^{(2)} u = -\sum_{k,l=1}^{d} \Big(&(\partial_l c_{kl}) \int_{\mathbb{R}^d} J_n(y)\, L_y(\partial_k u)\, dy + c_{kl} \int_{\mathbb{R}^d} (\partial_l J_n)(y)\, L_y(\partial_k u)\, dy \\
&- \int_{\mathbb{R}^d} (\partial_l J_n)(y)\, L_y(c_{kl}\, \partial_k u)\, dy \Big).
\end{aligned}
$$

Hence

$$
B_p(J_n * u) - J_n * (B_p u) = T_n u. \tag{4.11}
$$

The claim now follows from Lemma 4.20. □

Let $\tau \in C_c^\infty(\mathbb{R}^d)$ be such that $0 \le \tau \le \mathbb{1}$, $\tau|_{B_1(0)} = 1$ and $\operatorname{supp} \tau \subset B_2(0)$. Define $\tau_n(x) = \tau(n^{-1}x)$ for all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$.

Lemma 4.22. *Let $u \in D(B_p) \cap W^{1,p}(\mathbb{R}^d)$. Then $\tau_n u \in D(B_p)$ for all $n \in \mathbb{N}$ and we have $\lim_{n\to\infty} \tau_n u = u$ in $D(B_p)$. If u satisfies further that $u \in W^{2,p}(\mathbb{R}^d)$ and $\nabla(B_p u) \in (L_p(\mathbb{R}^d))^d$, then $\nabla(B_p(\tau_n u)) \in (L_p(\mathbb{R}^d))^d$ and $\lim_{n\to\infty} \nabla(B_p(\tau_n u)) = \nabla(B_p u)$ in $(L_p(\mathbb{R}^d))^d$.*

Proof. Let $n \in \mathbb{N}$ and $\phi \in C_c^\infty(\mathbb{R}^d)$. Then

$$(\tau_n u, H_q \phi) = (v, \phi),$$

where

$$v = \tau_n (B_p u) + (B_p \tau_n) u - \sum_{k,l=1}^d c_{kl} (\partial_k u) (\partial_l \tau_n) - \sum_{k,l=1}^d c_{kl} (\partial_l u) (\partial_k \tau_n). \tag{4.12}$$

It follows that

$$\|v\|_p \le M_1 \|u\|_{W^{1,p}} + \|B_p u\|_p < \infty,$$

where $M_1 = 3 \sup\{\|c_{kl}\, \tau\|_{W^{2,\infty}} : 1 \le k,l \le d\}$. Therefore $\tau_n u \in D(B_p)$ and $B_p(\tau_n u) = v$.

Next we consider the expression for v in (4.12). For the first term we have $\|\tau_n (B_p u)\|_p \le \|B_p u\|$ for all $n \in \mathbb{N}$ and $\{\tau_n (B_p u)\}_{n\in\mathbb{N}}$ converges to $B_p u$ pointwise. As a consequence $\lim_{n\to\infty} \tau_n (B_p u) = B_p u$ in $L_p(\mathbb{R}^d)$ by the Lebesgue dominated convergence theorem. For the second term we notice that

$$(\partial_k \tau_n)(x) = \frac{1}{n} (\partial_k \tau)(n^{-1} x) \quad \text{and} \quad (\partial_l \partial_k \tau_n)(x) = \frac{1}{n^2} (\partial_l \partial_k \tau)(n^{-1} x) \tag{4.13}$$

for all $x \in \mathbb{R}^d$, $n \in \mathbb{N}$ and $k,l \in \{1,\dots,d\}$. Since $c_{kl} \in W^{2,\infty}(\mathbb{R}^d)$ for all $k,l \in \{1,\dots,d\}$, we obtain

$$\|(B_p \tau_n)\, u\|_p = \Big\| \Big(\sum_{k,l=1}^d (\partial_l c_{kl}) (\partial_k \tau_n) + c_{kl} (\partial_l \partial_k \tau_n) \Big) u \Big\|_p \le \frac{2d^2}{n} \|c_{kl}\|_{W^{2,\infty}} \|\tau\|_{W^{2,\infty}} \|u\|_p \tag{4.14}$$

for all $n \in \mathbb{N}$. It follows that $\lim_{n\to\infty} \|(B_p \tau_n)\, u\|_p = 0$. Similarly the last two terms also converge to 0 in $L_p(\mathbb{R}^d)$. Clearly $\lim_{n\to\infty} \tau_n u = u$ in $L_p(\mathbb{R}^d)$. Hence $\lim_{n\to\infty} \tau_n u = u$ in $D(B_p)$.

To prove the second statement let $j \in \{1,\dots,d\}$ and $n \in \mathbb{N}$. Using (4.12) we have

$$\begin{aligned}
\partial_j(B_p(\tau_n u)) &= \tau_n\, \partial_j(B_p u) + (\partial_j \tau_n)(B_p u) + (B_p \tau_n)(\partial_j u) + (\partial_j(B_p \tau_n))\, u \\
&\quad - \sum_{k,l=1}^d (\partial_j c_{kl}) (\partial_k u) (\partial_l \tau_n) + c_{kl} (\partial_j \partial_k u) (\partial_l \tau_n) + c_{kl} (\partial_k u) (\partial_j \partial_l \tau_n) \\
&\quad - \sum_{k,l=1}^d (\partial_j c_{kl}) (\partial_l u) (\partial_k \tau_n) + c_{kl} (\partial_j \partial_l u) (\partial_k \tau_n) + c_{kl} (\partial_l u) (\partial_j \partial_k \tau_n).
\end{aligned} \tag{4.15}$$

It follows that

$$\|\partial_j(B_p(\tau_n u))\|_p \le M_2 \|u\|_{W^{2,p}} + (1 \wedge \|\tau\|_{W^{1,\infty}}) \|B_p u\|_{W^{1,p}} < \infty,$$

where $M_2 = 8 \sup\{\|c_{kl}\|_{W^{2,\infty}} \|\tau\|_{W^{3,\infty}} : 1 \le k,l \le d\}$. Therefore $\partial_j(B_p(\tau_n u)) \in L_p(\mathbb{R}^d)$. Furthermore notice that

$$(\partial_j\partial_l\partial_k\tau_n)(x) = \frac{1}{n^3}(\partial_j\partial_l\partial_k\tau)(n^{-1}x) \tag{4.16}$$

for all $x \in \mathbb{R}^d$ and $k,l \in \{1,\dots,d\}$. Using (4.13), (4.16) and repeating the arguments used in (4.14) we see that all terms in the expression for $\partial_j(B_p(\tau_n u))$ in (4.15) converge to 0 in $L_p(\mathbb{R}^d)$ as n tends to infinity except for the first one, whereas the first term converges to $\partial_j(B_p u)$ in $L_p(\mathbb{R}^d)$ as n tends to infinity. Hence $\lim_{n\to\infty} \partial_j(B_p(\tau_n u)) = \partial_j(B_p u)$ in $L_p(\mathbb{R}^d)$. This completes the proof. □

Proposition 4.23. *The space $C_c^\infty(\mathbb{R}^d)$ is dense in $(D(B_p) \cap W^{1,p}(\mathbb{R}^d), \|\cdot\|_{D(B_p)})$.*

Proof. Let $u \in D(B_p) \cap W^{1,p}(\mathbb{R}^d)$ and $\varepsilon > 0$. For all $n \in \mathbb{N}$ set $u_n = \tau_n u \in D(B_p) \cap W^{1,p}(\mathbb{R}^d) \cap L_{p,c}(\mathbb{R}^d)$. By Lemma 4.22 we can choose an $n \in \mathbb{N}$ such that $\|u - u_n\|_{D(B_p)} < \frac{\varepsilon}{2}$. Next for all $m \in \mathbb{N}$ set $v_m = J_m * (\tau_n u) \in C_c^\infty(\mathbb{R}^d)$. We now use Lemma 4.21 to choose an $m \in \mathbb{N}$ such that $\|u_n - v_m\|_{D(B_p)} < \frac{\varepsilon}{2}$. Then

$$\|u - v_m\|_{D(B_p)} \le \|u - u_n\|_{D(B_p)} + \|u_n - v_m\|_{D(B_p)} < \varepsilon.$$

This verifies the claim. □

Proposition 4.24. *Suppose $|1 - \frac{2}{p}| < \cos\theta$ and $B_a = 0$. Then there exists an $M > 0$ such that*

$$\mathrm{Re}\,(\nabla(B_p u), |\nabla u|^{p-2}\nabla u) \ge -M\,\|\nabla u\|_p^p$$

for all $u \in W^{2,p}(\mathbb{R}^d)$ such that $\nabla(B_p u) \in (L_p(\mathbb{R}^d))^d$.

Proof. The condition $|1 - \frac{2}{p}| < \cos\theta$ is equivalent to $|p-2|\,\tan\theta < 2\sqrt{p-1}$. Let $\varepsilon_0 \in (0, 1 \wedge (p-1))$ be such that

$$|p-2|\,\tan\theta \le 2\sqrt{(1-\varepsilon)\,(p-1-\varepsilon)}$$

for all $\varepsilon \in (0,\varepsilon_0)$. Let $\varepsilon \in (0,\varepsilon_0)$ be such that

$$\varepsilon < \frac{\varepsilon_0}{32\,d\,(1+\tan\theta)^2\,\sup_{1\le l\le d}\|\partial_l^2 C\|_\infty}. \tag{4.17}$$

Let $u \in W^{2,p}(\mathbb{R}^d)$. By Lemma 4.22 we can assume without loss of generality that u has a compact support. For the rest of the proof, all integrations are over the set $\{x \in \mathbb{R}^d : |(\nabla u)(x)| \ne 0\}$. We have

$$\begin{aligned}(\nabla(B_p u), |\nabla u|^{p-2}\nabla u) &= -\sum_{k,l,j=1}^d \int \Big(\partial_j\partial_l(c_{kl}\,\partial_k u)\Big)\,|\nabla u|^{p-2}\,\partial_j\overline{u} \\ &= -\sum_{k,l,j=1}^d \int \Big(\partial_l\big((\partial_j c_{kl})\,(\partial_k u) + c_{kl}\,(\partial_j\partial_k u)\big)\Big)\,|\nabla u|^{p-2}\,\partial_j\overline{u}\end{aligned}$$

$$\begin{aligned} &= -\sum_{k,l,j=1}^{d} \int \Big(\partial_l \big((\partial_j c_{kl})\,(\partial_k u)\big)\Big)\, |\nabla u|^{p-2}\, \partial_j \overline{u} \\ &\quad + \sum_{k,l,j=1}^{d} \int c_{kl}\,(\partial_j \partial_k u)\, \partial_l \big(|\nabla u|^{p-2}\, \partial_j \overline{u}\big) \\ &= (\mathrm{I}) + (\mathrm{II}). \end{aligned}$$

We first consider the real part of (I). We have

$$\begin{aligned} -\mathrm{Re} \sum_{k,l,j=1}^{d} \int \Big(\partial_l \big((\partial_j c_{kl})\,(\partial_k u)\big)\Big)\, |\nabla u|^{p-2}\, \partial_j \overline{u} &= -\mathrm{Re} \sum_{k,l,j=1}^{d} \int (\partial_l \partial_j c_{kl})\,(\partial_k u)\,(\partial_j \overline{u})\, |\nabla u|^{p-2} \\ &\quad - \mathrm{Re} \sum_{k,l,j=1}^{d} \int (\partial_j c_{kl})\,(\partial_l \partial_k u)\,(\partial_j \overline{u})\, |\nabla u|^{p-2} \\ &= (\mathrm{Ia}) + (\mathrm{Ib}). \end{aligned}$$

For (Ia) we have

$$(\mathrm{Ia}) \geq -\frac{1}{2} \sum_{k,l,j=1}^{d} \|c_{kl}\|_{W^{2,\infty}} \int (|\partial_k u|^2 + |\partial_j u|^2)\, |\nabla u|^{p-2} \geq -M_1\, \|\nabla u\|_p^p,$$

where $M_1 = d^2 \sup\{\|c_{kl}\|_{W^{2,\infty}} : 1 \leq k,l \leq d\}$. Let $U = (\partial_l \partial_k u)_{1 \leq k,l \leq d}$. For (Ib) we estimate

$$\begin{aligned} (\mathrm{Ib}) &= -\mathrm{Re} \sum_{j=1}^{d} \int \mathrm{tr}\,((\partial_j C)\, U)\,(\partial_j \overline{u})\, |\nabla u|^{p-2} \\ &\geq -\sum_{j=1}^{d} \int \Big(\varepsilon\, |\mathrm{tr}\,((\partial_j C)\, U)|^2\, |\nabla u|^{p-2} + \frac{1}{4\varepsilon}\, |\partial_j \overline{u}|^2\, |\nabla u|^{p-2}\Big) \\ &\geq -\varepsilon' \int \mathrm{tr}\,(U\, R_s\, \overline{U})\, |\nabla u|^{p-2} - M_2\, \|\nabla u\|_p^p, \end{aligned}$$

where we used Corollary 4.10(a) in the last step with $\varepsilon' = 32\,\varepsilon\, d\,(1+\tan\theta)^2 \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty$ and $M_2 = \frac{1}{4\varepsilon}$. Note that $\varepsilon' \in (0, \varepsilon_0)$ by (4.17).

Next we consider the real part of (II). Note that

$$\begin{aligned} \mathrm{Re} \sum_{k,l,j=1}^{d} \int c_{kl}\,(\partial_j \partial_k u)\, \partial_l \big(|\nabla u|^{p-2}\, \partial_j \overline{u}\big) &= \mathrm{Re} \sum_{k,l,j=1}^{d} \int c_{kl}\,(\partial_j \partial_k u)\,(\partial_l \partial_j \overline{u})\, |\nabla u|^{p-2} \\ &\quad + \mathrm{Re} \sum_{k,l,j=1}^{d} \int c_{kl}\,(\partial_j \partial_k u)\,(\partial_j \overline{u})\, \partial_l (|\nabla u|^{p-2}) \\ &= (\mathrm{IIa}) + (\mathrm{IIb}). \end{aligned}$$

For (IIa) we have

$$\text{(IIa)} = \int \operatorname{tr}\,(\overline{U}\,(\operatorname{Re} C)\,U)\,|\nabla u|^{p-2} = \int \operatorname{tr}\,(U\,R_s\,\overline{U})\,|\nabla u|^{p-2}$$

as $B_a = 0$. For (IIb) we have the following estimate

$$\begin{aligned}
\text{(IIb)} &= \operatorname{Re} \sum_{k,l,i,j=1}^{d} \frac{p-2}{2} \int c_{kl}\,(\partial_j \partial_k u)\,(\partial_j \overline{u}) \Big((\partial_l \partial_i u)\,(\partial_i \overline{u}) + (\partial_l \partial_i \overline{u})\,(\partial_i u) \Big)\,|\nabla u|^{p-4} \\
&= \frac{p-2}{2} \int \operatorname{Re} \Big((C\,U\,\nabla\overline{u}, \overline{U\,\nabla\overline{u}}) + (C\,U\,\nabla\overline{u}, U\,\nabla\overline{u}) \Big)\,|\nabla u|^{p-4} \\
&= (p-2) \int \Big((R_s\,\xi, \xi) - (B_s\,\xi, \eta) \Big)\,|\nabla u|^{p-4},
\end{aligned}$$

where $\xi, \eta \in \mathbb{R}^d$ and $U\,\nabla\overline{u} = \xi + i\,\eta$.

In total we obtain

$$\begin{aligned}
\operatorname{Re}\,(\nabla(B_p u), |\nabla u|^{p-2}\,\nabla u) \geq -(M_1 + M_2)\,\|\nabla u\|_p^p + (1-\varepsilon') \int \operatorname{tr}\,(U\,R_s\,\overline{U})\,|\nabla u|^{p-2} \\
+ (p-2) \int \Big((R_s\,\xi, \xi) - (B_s\,\xi, \eta) \Big)\,|\nabla u|^{p-4} \\
= -(M_1 + M_2)\,\|\nabla u\|_p^p + P,
\end{aligned}$$

where

$$P = (1-\varepsilon') \int \operatorname{tr}\,(U\,R_s\,\overline{U})\,|\nabla u|^{p-2} + (p-2) \int \Big((R_s\,\xi, \xi) - (B_s\,\xi, \eta) \Big)\,|\nabla u|^{p-4}.$$

Next we will show that $P \geq 0$. Since $B_a = 0$, it follows from Lemma 4.11 that

$$\begin{aligned}
(R_s\,\xi, \xi) + (R_s\,\eta, \eta) = ((\operatorname{Re} C)\,U\,\nabla\overline{u}, U\,\nabla\overline{u}) &\leq \operatorname{tr}\,(U^*\,(\operatorname{Re} C)\,U)\,|\nabla u|^2 \\
&= \operatorname{tr}\,(\overline{U}\,R_s\,U)\,|\nabla u|^2 = \operatorname{tr}\,(U\,R_s\,\overline{U})\,|\nabla u|^2.
\end{aligned}$$

Therefore

$$\begin{aligned}
P &\geq \int \Big((p-1-\varepsilon')\,(R_s\,\xi, \xi) + (1-\varepsilon')\,(R_s\,\eta, \eta) - (p-2)\,(B_s\,\xi, \eta) \Big)\,|\nabla u|^{p-4} \\
&= \int \Big((R_s\,\xi', \xi') + (R_s\,\eta', \eta') - \frac{p-2}{\sqrt{(1-\varepsilon')\,(p-1-\varepsilon')}}\,(B_s\,\xi', \eta') \Big)\,|\nabla u|^{p-4}, \qquad (4.18)
\end{aligned}$$

where $\xi' = \sqrt{p-1-\varepsilon'}\,\xi$ and $\eta' = \sqrt{1-\varepsilon'}\,\eta$. If $\theta = 0$ then it follows from Lemma 4.5 that $(B_s\,\xi', \eta') = 0$. Therefore (4.18) gives

$$P \geq \int \Big((R_s\,\xi', \xi') + (R_s\,\eta', \eta') \Big)\,|\nabla u|^{p-4} \geq 0.$$

If $\theta \neq 0$ then (4.18) can be estimated by

$$P \geq \int \Big((R_s\,\xi', \xi') + (R_s\,\eta', \eta') - 2\,\cot\theta\,|(B_s\,\xi', \eta')| \Big)\,|\nabla u|^{p-4} \geq 0,$$

where we again used Lemma 4.5. Either way we always have

$$\operatorname{Re}\left(\nabla(B_p u), |\nabla u|^{p-2}\,\nabla u\right) \geq -(M_1 + M_2)\,\|\nabla u\|_p^p$$

as claimed. ☐

Proposition 4.25. *Suppose $|1 - \frac{2}{p}| < \cos\theta$ and $B_a = 0$. Then B_p is m-accretive. Furthermore $C_c^\infty(\mathbb{R}^d)$ is a core for B_p.*

Proof. We will proceed in three steps.
Step 1: We will show that $\overline{B_p|_{D(B_p)\cap W^{1,p}(\mathbb{R}^d)}}$ is m-accretive.
It follows from Propositions 4.17 and 4.23 that $B_p|_{D(B_p)\cap W^{1,p}(\mathbb{R}^d)}$ is accretive. Hence $\overline{B_p|_{D(B_p)\cap W^{1,p}(\mathbb{R}^d)}}$ is also accretive.

Next we will show that there exists a $\lambda > 0$ such that $(\lambda + B_p)(D(B_p) \cap W^{1,p}(\mathbb{R}^d))$ is dense in $L_p(\mathbb{R}^d)$. In fact we will show that there exists a $\lambda > 0$ such that $W^{1,p}(\mathbb{R}^d) \subset (\lambda + B_p)(D(B_p) \cap W^{1,p}(\mathbb{R}^d))$. Since $-\Delta$ satisfies the same conditions as those of B_p, Proposition 4.24 also applies to $-\Delta$. In particular there exists an $M' > 0$ such that

$$\operatorname{Re}\Big(\nabla(\Delta u),\, |\nabla u|^{p-2}\,\nabla u\Big) \geq -M'\,\|\nabla u\|_p^p$$

for all $u \in W^{3,p}(\mathbb{R}^d)$.

For all $n \in \mathbb{N}$ define the operator $B_{p,n}$ by

$$B_{p,n} u = B_p u - \frac{1}{n}\,\Delta u$$

on the domain

$$D(B_{p,n}) = W^{2,p}(\mathbb{R}^d),$$

where $\Delta = \partial_1^2 + \ldots + \partial_d^2$. Note that for each $n \in \mathbb{N}$ the operator $B_{p,n}$ is strongly elliptic, which implies that $B_{p,n}$ is closed.

Let M be as in Proposition 4.24 and $\lambda = M + M' + 1$. Let $f \in W^{1,p}(\mathbb{R}^d)$. Let $n \in \mathbb{N}$. Then there exists a $u_n \in W^{2,p}(\mathbb{R}^d)$ such that $(\lambda + B_{p,n})u_n = f$. Elliptic regularity gives $u_n \in W^{3,p}(\mathbb{R}^d)$. It follows that $\nabla(B_{p,n}u_n) = \nabla(f - \lambda\, u_n) \in (L_p(\mathbb{R}^d))^d$ and $\nabla(B_p u_n) = \nabla(B_{p,n}u_n) + \frac{1}{n}\,\nabla(\Delta u_n) \in (L_p(\mathbb{R}^d))^d$. By Proposition 4.17 we have

$$(f, |u_n|^{p-2}\,u_n) = \lambda\,\|u_n\|_p^p + (B_{p,n}u_n, |u_n|^{p-2}\,u_n) \geq \lambda\,\|u_n\|_p^p \geq \|u_n\|_p^p.$$

However

$$(f, |u_n|^{p-2}\,u_n) \leq \|f\|_p\,\||u_n|^{p-2}\,u_n\|_q = \|f\|_p\,\|u_n\|_p^{p/q}$$

by Hölder's inequality. Therefore $\|u_n\|_p^p \leq \|f\|_p\,\|u_n\|_p^{p/q}$, or equivalently $\|u_n\|_p \leq \|f\|_p$. Also it follows from Proposition 4.24 that

$$\begin{aligned}
(\nabla f, |\nabla u_n|^{p-2}\,\nabla u_n) &= \lambda\,\|\nabla u_n\|_p^p + \operatorname{Re}\Big(\nabla(B_{p,n}u_n), |\nabla u_n|^{p-2}\,\nabla u_n\Big) \\
&= \lambda\,\|\nabla u_n\|_p^p + \operatorname{Re}\Big(\nabla(B_p u_n), |\nabla u_n|^{p-2}\,\nabla u_n\Big) \\
&\quad - \frac{1}{n}\operatorname{Re}\Big(\nabla(\Delta u_n),\, |\nabla u_n|^{p-2}\,\nabla u_n\Big) \\
&\geq (\lambda - M - M')\,\|\nabla u_n\|_p^p = \|\nabla u_n\|_p^p.
\end{aligned}$$

Again the Hölder's inequality gives $\|\nabla u_n\|_p \leq \|\nabla f\|_p$. Hence $\|u_n\|_{W^{1,p}} \leq \|f\|_{W^{1,p}}$. In particular $\{u_k\}_{k\in\mathbb{N}}$ is bounded in $W^{1,p}(\mathbb{R}^d)$. Passing to a subsequence if necessary we may assume that $\{u_k\}_{k\in\mathbb{N}}$ converges weakly to a $u \in W^{1,p}(\mathbb{R}^d)$. Note that $B_{p,n}u_n = f - \lambda\, u_n$. Therefore $\{B_{p,n}u_n\}_{k\in\mathbb{N}}$ is bounded in $L_p(\mathbb{R}^d)$. Passing to a subsequence if necessary we again assume that $\{B_{p,k}u_k\}_{k\in\mathbb{N}}$ converges weakly to a $v \in L_p(\mathbb{R}^d)$. Then $v = f - \lambda\, u$. We will show that $B_p u = v$. Indeed let $\phi \in C_c^\infty(\mathbb{R}^d)$. Then $\lim_{n\to\infty} B_{p,n}^*\phi = B_p^*\phi$ strongly in $L_q(\mathbb{R}^d)$ and

$$(v, \phi) = \lim_{n\to\infty} (B_{p,n}u_n, \phi) = \lim_{n\to\infty} (u_n, B_{p,n}^*\phi) = (u, B_p^*\phi).$$

Therefore $u \in D(B_p)$ and $B_p u = v$. Hence $(\lambda + B_p)u = f$.

Step 2: We will show that $\overline{B_p|_{D(B_p)\cap W^{1,p}(\mathbb{R}^d)}} = B_p$, which implies B_p is m-accretive.

Clearly $\overline{D(B_p)\cap W^{1,p}(\mathbb{R}^d)}^{\|\cdot\|_{D(B_p)}} \subset D(B_p)$. For the reverse inclusion let $u \in D(B_p)$ and λ be defined as in Step 1. Since $(\lambda + B_p)u \in L_p(\mathbb{R}^d)$ and $\overline{B_p|_{D(B_p)\cap W^{1,p}(\mathbb{R}^d)}}$ is m-accretive, there exists a $v \in \overline{D(B_p)\cap W^{1,p}(\mathbb{R}^d)}^{\|\cdot\|_{D(B_p)}}$ such that $(\lambda + B_p)v = (\lambda + B_p)u$. Equivalently

$$(u - v, (\lambda + H_q)\phi) = 0 \tag{4.19}$$

for all $\phi \in C_c^\infty(\mathbb{R}^d)$.

Define $G_q = (B_p|_{C_c^\infty(\mathbb{R}^d)})^*$. Then $H_q \subset G_q$. Note that $|1 - \frac{2}{p}| < \cos\theta$ is equivalent to $|1 - \frac{2}{q}| < \cos\theta$. Furthermore C^* satisfies the same condition as those of C. Therefore all previous results apply to G_q. In particular, Proposition 4.23 gives $C_c^\infty(\mathbb{R}^d)$ is dense in $(D(G_q)\cap W^{1,q}(\mathbb{R}^d), \|\cdot\|_{D(G_q)})$ and Step 1 gives $\overline{G_q|_{D(G_q)\cap W^{1,q}(\mathbb{R}^d)}}$ is m-accretive.

Now it follows from (4.19) that

$$(u - v, (\lambda + G_q)\phi) = 0$$

for all $\phi \in \overline{(D(G_q)\cap W^{1,q}(\mathbb{R}^d), \|\cdot\|_{D(G_q)})}$. Since $\overline{G_q|_{D(G_q)\cap W^{1,q}(\mathbb{R}^d)}}$ is m-accretive, we must have $u = v$.

Step 3: We will show that $C_c^\infty(\mathbb{R}^d)$ is a core for B_p.

This follows immediately from Proposition 4.23 and Step 2. □

4.5 The core property for A_p

Let $p \in (1, \infty)$ be such that $|1 - \frac{2}{p}| < \cos\theta$. Suppose $B_a = 0$. In Section 4.3, we proved that the contraction C_0-semigroup S generated by A extends consistently to a contraction C_0-semigroup $S^{(p)}$ on $L_p(\mathbb{R}^d)$. Let $-A_p$ be the generator of $S^{(p)}$. In this section we will show that the operator A_p and B_p are in fact the same. Consequently the space of test functions $C_c^\infty(\mathbb{R}^d)$ is a core for A_p. This is the content of Theorem 4.2, which is also the main theorem of the chapter.

Proposition 4.26. *Let $p \in (1, \infty)$ be such that $|1 - \frac{2}{p}| < \cos\theta$. Suppose $B_a = 0$. Then $A_p = B_p$.*

Proof. Let $u \in D(A) \cap D(A_p)$. Then

$$(A_p u, \phi) = (Au, \phi) = \mathfrak{a}(u, \phi) = (u, H_q\phi)$$

for all $\phi \in C_c^\infty(\mathbb{R}^d)$. It follows that $u \in D(B_p)$ and $B_p u = A_p u$. In particular this implies that $D(A) \cap D(A_p) \subset D(B_p)$. However $D(A) \cap D(A_p)$ is a core for A_p and B_p is closed. Hence $D(A_p) \subset D(B_p)$. On the other hand note that $-A_p$ generates a contraction C_0-semigroup. Therefore A_p is m-accretive. By Proposition 4.25 the operator B_p is also m-accretive. Hence $A_p = B_p$ as required. □

The main result of this chapter now follows immediately from the above proposition.

Proof of Theorem 4.2. By Proposition 4.25 the space $C_c^\infty(\mathbb{R}^d)$ is a core for B_p. Since $A_p = B_p$ by Proposition 4.26, it follows that $C_c^\infty(\mathbb{R}^d)$ is also a core for A_p. □

4.6 More sufficient conditions in L_2

This section is motivated by the fact that B_2 is accretive on $W^{2,2}(\mathbb{R}^d)$ without the requirement that $B_a = 0$ (cf. Proposition 4.17). In fact more is true.

Proposition 4.27. *We have*

$$\operatorname{Re}(B_2 u, u) \geq 0$$

for all $u \in W^{1,2}(\mathbb{R}^d) \cap D(B_2)$.

Proof. Let $u \in W^{1,2}(\mathbb{R}^d) \cap D(B_2)$. Then

$$\begin{aligned}\operatorname{Re}(B_2 u, u) &= -\operatorname{Re} \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} (\partial_l (c_{kl}\, \partial_k u))\, \overline{u} = \operatorname{Re} \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} c_{kl}\, (\partial_k u)\, \partial_l \overline{u} \\ &= \operatorname{Re} \int_{\mathbb{R}^d} (C\, \nabla u, \nabla u) = \int_{\mathbb{R}^d} ((\operatorname{Re} C)\, \nabla u, \nabla u) \geq 0\end{aligned}$$

as claimed. □

Define the operator $Z = \overline{B_2|_{C_c^\infty(\mathbb{R}^d)}}$. Then Z is closed. Furthermore we have the following.

Proposition 4.28. *The operator Z is accretive and $Z = \overline{B_2|_{W^{1,2}(\mathbb{R}^d) \cap D(B_2)}}$.*

Proof. It suffices to show $Z = \overline{B_2|_{W^{1,2}(\mathbb{R}^d) \cap D(B_2)}}$. This follows immediately from Proposition 4.23. □

From now on we drop the condition that $B_a \neq 0$. In this section we will provide many sufficient conditions for the space of test functions $C_c^\infty(\mathbb{R}^d)$ to be a core for the operator A. Define the operator L in $L_2(\mathbb{R}^d)$ as follows.

$$Lu = -\sum_{k,l=1}^{d} \partial_k \big(\overline{(B_a)_{kl}}\, \partial_l u\big) \tag{4.20}$$

on the domain

$$D(L) = C_c^\infty(\mathbb{R}^d).$$

Next define the operator associated with B_a as $(B_a)^{\mathrm{op}} = L^*$, which is the dual of L. In what follows we denote $(\partial_k B_a)_{kl} = \partial_k\big((B_a)_{kl}\big)$ for all $k, l \in \{1, \ldots, d\}$. Although $(B_a)^{\mathrm{op}}$ appears to be a differential operator of second order, it is in fact a first-order differential operator. Indeed for all $u \in D((B_a)^{\mathrm{op}})$ and $\phi \in C_c^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned}
\big((B_a)^{\mathrm{op}} u, \phi\big) = (u, L\phi) &= -\sum_{k,l=1}^{d} \int_{\mathbb{R}^d} u\, \partial_k\big((B_a)_{kl}\, \partial_l \phi\big) \\
&= -\sum_{k,l=1}^{d} \int_{\mathbb{R}^d} u \Big((\partial_k B_a)_{kl}\, \partial_l \phi + (B_a)_{kl}\, \partial_k \partial_l \phi\Big) \\
&= -\sum_{k,l=1}^{d} \int_{\mathbb{R}^d} u\, (\partial_k B_a)_{kl}\, \partial_l \phi, \qquad (4.21)
\end{aligned}$$

where the last step follows from the anti-symmetry of B_a. Note that $(B_a)_{kl} \in W^{2,\infty}(\mathbb{R}^d)$ for all $k, l \in \{1, \ldots, d\}$. Therefore it follows from (4.21) that $W^{1,2}(\mathbb{R}^d) \subset D((B_a)^{\mathrm{op}})$ and

$$\begin{aligned}
\big((B_a)^{\mathrm{op}} u, \phi\big) &= \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} \partial_l\big((\partial_k B_a)_{kl}\, u\big)\, \phi = \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} \big((\partial_l \partial_k B_a)_{kl}\, u + (\partial_k B_a)_{kl}\, (\partial_l u)\big)\, \phi \\
&= \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} (\partial_k B_a)_{kl}\, (\partial_l u)\, \phi = -\sum_{k,l=1}^{d} \int_{\mathbb{R}^d} (\partial_l B_a)_{kl}\, (\partial_k u)\, \phi
\end{aligned}$$

for all $u \in W^{1,2}(\mathbb{R}^d)$ and $\phi \in C_c^\infty(\mathbb{R}^d)$ since B_a is anti-symmetric. Hence

$$(B_a)^{\mathrm{op}} u = \sum_{k,l=1}^{d} \partial_l\big((\partial_k B_a)_{kl}\, u\big) = -\sum_{k,l=1}^{d} (\partial_l B_a)_{kl}\, \partial_k u$$

for all $u \in W^{1,2}(\mathbb{R}^d)$.

Lemma 4.29. *For all $\varepsilon > 0$ there exists an $M > 0$ such that*

$$\big|\big((B_a)^{\mathrm{op}} u, -\Delta u\big)\big| \le \varepsilon \int_{\mathbb{R}^d} \|(\partial_l B_a)\, U\|_{HS}^2 + M\, \|\nabla u\|_2^2$$

for all $u \in C_c^\infty(\mathbb{R}^d)$, where $U = (\partial_l \partial_k u)_{1 \le k,l \le d}$.

Proof. Let $u \in C_c^\infty(\mathbb{R}^d)$ and write $U = (\partial_l \partial_k u)_{1 \le k,l \le d}$. Then

$$\begin{aligned}
\big|\big((B_a)^{\mathrm{op}} u, -\Delta u\big)\big| &= \Big| \sum_{k,l,j=1}^{d} \int_{\mathbb{R}^d} (\partial_l B_a)_{kl}\, (\partial_k u)\, \partial_j^2 \overline{u} \Big| \\
&= \Big| \sum_{k,l,j=1}^{d} \int_{\mathbb{R}^d} \Big((\partial_j \partial_l B_a)_{kl}\, \partial_k u + (\partial_l B_a)_{kl}\, \partial_k \partial_j u\Big)\, \partial_j \overline{u} \Big| \\
&\le \Big| \sum_{k,l,j=1}^{d} \int_{\mathbb{R}^d} (\partial_j \partial_l B_a)_{kl}\, (\partial_k u)\, \partial_j \overline{u} \Big| + \Big| \sum_{k,l,j=1}^{d} \int_{\mathbb{R}^d} (\partial_l B_a)_{kl}\, (\partial_k \partial_j u)\, \partial_j \overline{u} \Big| \\
&= \qquad\qquad (\mathrm{I}) \qquad\qquad + \qquad\qquad (\mathrm{II}).
\end{aligned}$$

For (I) we have

$$\Big|\sum_{k,l,j=1}^{d}\int_{\mathbb{R}^d}(\partial_j\partial_l B_a)_{kl}\,(\partial_k u)\,\partial_j\overline{u}\Big| \le d^2 \sup_{1\le k,l\le d}\|(B_a)_{kl}\|_{W^{2,\infty}}\,\|\nabla u\|_2^2.$$

We estimate the term (II) by

$$\begin{aligned}\Big|\sum_{k,l,j=1}^{d}\int_{\mathbb{R}^d}(\partial_l B_a)_{kl}\,(\partial_k\partial_j u)\,\partial_j\overline{u}\Big| &= \Big|\sum_{l,j=1}^{d}\int_{\mathbb{R}^d}\big((\partial_l B_a)\,U\big)_{lj}\,\partial_j\overline{u}\Big|\\ &\le \varepsilon\sum_{l,j=1}^{d}\int_{\mathbb{R}^d}\big|\big((\partial_l B_a)\,U\big)_{lj}\big|^2 + \frac{d}{4\varepsilon}\,\|\nabla u\|_2^2\\ &= \varepsilon\int_{\mathbb{R}^d}\|(\partial_l B_a)\,U\|_{HS}^2 + \frac{d}{4\varepsilon}\,\|\nabla u\|_2^2.\end{aligned}$$

Hence

$$\big|\big((B_a)^{\mathrm{op}}u,-\Delta u\big)\big| \le \varepsilon\int_{\mathbb{R}^d}\|(\partial_l B_a)\,U\|_{HS}^2 + M\,\|\nabla u\|_2^2,$$

where

$$M = d^2\sup_{1\le k,l\le d}\|(B_a)_{kl}\|_{W^{2,\infty}} + \frac{d}{4\varepsilon}$$

as required. □

Lemma 4.30. *For all $\varepsilon>0$ there exists an $M>0$ such that*

$$\Big|\int_{\mathbb{R}^d}\mathrm{tr}\,(U\,B_a\,\overline{U}) \le \varepsilon\int_{\mathbb{R}^d}\|(\partial_l B_a)\,U\|_{HS}^2 + M\,\|\nabla u\|_2^2$$

for all $u\in C_c^\infty(\mathbb{R}^d)$, where $U=(\partial_l\partial_k u)_{1\le k,l\le d}$.

Proof. Let $u\in C_c^\infty(\mathbb{R}^d)$ and write $U=(\partial_l\partial_k u)_{1\le k,l\le d}$. Then

$$\begin{aligned}((B_a)^{\mathrm{op}}u,-\Delta u) &= \sum_{k,l,j=1}^{d}\int_{\mathbb{R}^d}\big(\partial_l((B_a)_{kl}\,\partial_k u)\big)\,\partial_j^2\overline{u}\\ &= -\sum_{k,l,j=1}^{d}\int_{\mathbb{R}^d}\Big(\partial_l\big((\partial_j B_a)_{kl}\,\partial_k u + (B_a)_{kl}\,\partial_j\partial_k u\big)\Big)\,\partial_j\overline{u}\\ &= -\sum_{k,l,j=1}^{d}\int_{\mathbb{R}^d}(\partial_l\partial_j B_a)_{kl}\,(\partial_k u)\,\partial_j\overline{u} + (\partial_j B_a)_{kl}\,(\partial_l\partial_k u)\,\partial_j\overline{u}\\ &\quad+\sum_{k,l,j=1}^{d}\int_{\mathbb{R}^d}(B_a)_{kl}\,(\partial_j\partial_k u)\,\partial_l\partial_j\overline{u}\\ &= -\sum_{k,l,j=1}^{d}\int_{\mathbb{R}^d}(\partial_l\partial_j B_a)_{kl}\,(\partial_k u)\,\partial_j\overline{u} + \int_{\mathbb{R}^d}\mathrm{tr}\,(U\,B_a\,\overline{U}),\end{aligned}$$

where in the last step we used $\sum_{k,l=1}^d (\partial_j B_a)_{kl}\,(\partial_l\partial_k u) = 0$ for all $j \in \{1,\dots,d\}$, which follows from the anti-symmetry of B_a.

Let $\varepsilon > 0$ and M be as in Lemma 4.29. Then

$$\Big|\int_{\mathbb{R}^d} \operatorname{tr}\,(U\,B_a\,\overline{U})\Big| \le |((B_a)^{\mathrm{op}}u, -\Delta u)| + \Big|\sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_l\partial_j B_a)_{kl}\,(\partial_k u)\,\partial_j\overline{u}\Big|$$

$$\le \varepsilon \int_{\mathbb{R}^d} \|(\partial_l B_a)\,U\|_{HS}^2 + (M + d^2\,\|B_a\|_{W^{2,\infty}})\,\|\nabla u\|_2^2,$$

where we used Lemma 4.29 in the last step. □

Lemma 4.31. *Let $u \in C_c^\infty(\mathbb{R}^d)$. Then*

$$\operatorname{Re}\big((B_2 - B_2^*)u, -\Delta u\big) = 2\,\operatorname{Im}\sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_l\partial_j \operatorname{Im} C)_{kl}\,(\partial_k u)\,\partial_j\overline{u}$$

$$+ 2\,\operatorname{Im}\sum_{j=1}^d \int_{\mathbb{R}^d} \operatorname{tr}\,((\partial_j \operatorname{Im} C)\,U)\;\partial_j\overline{u}.$$

Proof. Let $u \in C_c^\infty(\mathbb{R}^d)$ and write $U = (\partial_l\partial_k u)_{1\le k,l\le d}$. Then

$$\begin{aligned}
\big((B_2 - B_2^*)u, -\Delta u\big) &= \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} \Big(\partial_l\big((c_{kl} - \overline{c_{lk}})\,\partial_k u\big)\Big)\,\partial_j^2\overline{u} \\
&= 2i \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} \Big(\partial_l\big((\operatorname{Im} C)_{kl}\,\partial_k u\big)\Big)\,\partial_j^2\overline{u} \\
&= -2i \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} \Big(\partial_l\big((\partial_j \operatorname{Im} C)_{kl}\,\partial_k u + (\operatorname{Im} C)_{kl}\,\partial_j\partial_k u\big)\Big)\,\partial_j\overline{u} \\
&= -2i \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} \Big((\partial_l\partial_j \operatorname{Im} C)_{kl}\,(\partial_k u) + (\partial_j \operatorname{Im} C)_{kl}\,(\partial_l\partial_k u)\Big)\,\partial_j\overline{u} \\
&\quad + 2i \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\operatorname{Im} C)_{kl}\,(\partial_j\partial_k u)\,(\partial_l\,\partial_j\overline{u}) \\
&= -2i \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_l\partial_j \operatorname{Im} C)_{kl}\,(\partial_k u)\,\partial_j\overline{u} - 2i \sum_{j=1}^d \int_{\mathbb{R}^d} \operatorname{tr}\,((\partial_j \operatorname{Im} C)\,U)\;\partial_j\overline{u} \\
&\quad + 2i \int_{\mathbb{R}^d} \operatorname{tr}\,(U\,(\operatorname{Im} C)\,\overline{U}).
\end{aligned}$$

Taking the real parts both sides gives the statement since $\operatorname{tr}\,(U\,(\operatorname{Im} C)\,\overline{U}) \in \mathbb{R}$. □

Lemma 4.32. *Let $u \in C_c^\infty(\mathbb{R}^d)$. Then*

$$\begin{aligned}\mathrm{Re}\sum_{j=1}^d \int_{\mathbb{R}^d} \mathrm{tr}\,\big((\partial_j C)\,U\big)\,\partial_j\overline{u} &= \frac{1}{2}\,\mathrm{Re}\sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_j^2 c_{kl})\,(\partial_l u)\,\partial_k\overline{u} - 2\,(\partial_k\partial_j c_{kl})\,(\partial_l u)\,\partial_j\overline{u}\\ &\quad + \mathrm{Im}\sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_l\partial_j \mathrm{Im}\, C)_{kl}\,(\partial_j u)\,\partial_k\overline{u}\\ &\quad + \mathrm{Im}\sum_{j=1}^d \int_{\mathbb{R}^d} \mathrm{tr}\,((\partial_j \mathrm{Im}\, C)\,\overline{U})\;\partial_j u.\end{aligned}$$

Proof. Let $u \in C_c^\infty(\mathbb{R}^d)$ and write $U = (\partial_l\partial_k u)_{1\le k,l\le d}$. Then

$$\begin{aligned}(B_2 u, -\Delta u) &= \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_l(c_{kl}\,\partial_k u)\,\partial_j^2 u = -\sum_{k,l,j=1}^d \int_{\mathbb{R}^d} \Big(\partial_l((\partial_j c_{kl})\,\partial_k u + c_{kl}\,\partial_j\partial_k u)\Big)\,\partial_j\overline{u}\\ &= \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} ((\partial_j c_{kl})\,\partial_k u + c_{kl}\,\partial_j\partial_k u)\,\partial_l\partial_j\overline{u}\\ &= -\sum_{k,l,j=1}^d \int_{\mathbb{R}^d} \Big((\partial_j^2 c_{kl})\,\partial_k u + (\partial_j c_{kl})\,\partial_j\partial_k u + (\partial_j c_{kl})\,\partial_j\partial_k u + c_{kl}\,\partial_j^2\partial_k u\Big)\,\partial_l\overline{u}\\ &= -\sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_j^2 c_{kl})\,(\partial_k u)\,\partial_l\overline{u} + 2\,(\partial_j c_{kl})\,(\partial_j\partial_k u)\,\partial_l\overline{u} - (\partial_j^2 u)\,\partial_k(c_{kl}\,\partial_l\overline{u})\\ &= -\sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_j^2 c_{kl})\,(\partial_k u)\,\partial_l\overline{u} - 2\,(\partial_j u)\Big((\partial_k\partial_j c_{kl})\,\partial_l\overline{u} + (\partial_j c_{kl})\,(\partial_k\partial_l\overline{u})\Big)\\ &\quad + (-\Delta u, B_2^* u).\end{aligned}$$

Hence

$$\begin{aligned}\sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_j c_{kl})\,(\partial_l\partial_k\overline{u})\,\partial_j u &= \frac{1}{2}\sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_j^2 c_{kl})\,(\partial_k u)\,\partial_l\overline{u} - 2\,(\partial_k\partial_j c_{kl})\,(\partial_l\overline{u})\,(\partial_j u)\\ &\quad + \frac{1}{2}\Big((B_2 u, -\Delta u) - (-\Delta u, B_2^* u)\Big).\end{aligned}$$

Replacing u by $\overline{u}$ in the above equation and taking the real parts on both sides gives

$$\begin{aligned}\mathrm{Re}\sum_{j=1}^d \int_{\mathbb{R}^d} \mathrm{tr}\,\big((\partial_j C)\,U\big)\,\partial_j\overline{u} &= \mathrm{Re}\sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_j c_{kl})\,(\partial_l\partial_k u)\,\partial_j\overline{u}\\ &= \frac{1}{2}\,\mathrm{Re}\sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_j^2 c_{kl})\,(\partial_k\overline{u})\,\partial_l u - 2\,(\partial_k\partial_j c_{kl})\,(\partial_l u)\,(\partial_j\overline{u})\\ &\quad + \frac{1}{2}\,\mathrm{Re}\,\big((B_2 - B_2^*)\overline{u}, -\Delta\overline{u}\big).\end{aligned}$$

Using Lemma 4.31 we yield the result. □

Proposition 4.33. *Suppose one of the following holds.*

(i) *The matrix B_s has constant entries.*

(ii) *There exist $\theta_1, \theta_2 \in [0, \frac{\pi}{2})$, $\phi \in W^{2,\infty}(\mathbb{R}^d)$ and a $d \times d$ matrix $\widetilde{C}$ with entries in $W^{2,\infty}(\mathbb{R}^d)$ such that $\theta = \theta_1 + \theta_2$, $\phi(x) \in \Sigma_{\theta_1}$ for all $x \in \mathbb{R}^d$, $\widetilde{C}$ takes values in Σ_{θ_2} and $C = \phi\,\widetilde{C}$. Write $\widetilde{C} = \widetilde{R} + i\,\widetilde{B}$, where $\widetilde{R}$ and $\widetilde{B}$ are $d \times d$ matrix-valued functions with real-valued entries. Set $\widetilde{R}_s = \frac{1}{2}\,(\widetilde{R} + \widetilde{R}^T)$. Also define $\operatorname{Re}\widetilde{C} = \frac{1}{2}\big(\widetilde{C} + (\widetilde{C})^*\big)$. Suppose further that there exists an $h > 0$ such that*

$$\operatorname{tr}\big(U\,(\operatorname{Re}\widetilde{C})\,\overline{U}\big) \geq h\,\operatorname{tr}\big(U\,\widetilde{R}_s\,\overline{U}\big)$$

for all $u \in C_c^\infty(\mathbb{R}^d)$, where $U = (\partial_l\partial_k u)_{1\leq k,l\leq d}$.

(iii) *There exists an $M > 0$ such that $\|(\partial_l B_a)\,U\|_{HS}^2 \leq M\,\operatorname{tr}\big(U\,R_s\,\overline{U}\big)$ for all $l \in \{1,\ldots,d\}$ and $u \in C_c^\infty(\mathbb{R}^d)$, where $U = (\partial_l\partial_k u)_{1\leq k,l\leq d}$.*

Then Z is m-accretive.

Proof. By Proposition 4.28 we have that that $D(-\Delta) = W^{2,2}(\mathbb{R}^d) \subset D(Z)$. We will show that there exists a $\beta \in \mathbb{R}$ such that

$$\operatorname{Re}\,(Zu, -\Delta u) \geq -\beta\,\|\nabla u\|_2^2 \tag{4.22}$$

for all $u \in D(-\Delta) = W^{2,2}(\mathbb{R}^d)$. It then follows from [Ouh05, Theorem 1.50] that Z is m-accretive. Since $C_c^\infty(\mathbb{R}^d)$ is dense in $W^{2,2}(\mathbb{R}^d)$ and is a core for Z, it suffices to show that (4.22) holds for all $u \in C_c^\infty(\mathbb{R}^d)$.

Let $u \in C_c^\infty(\mathbb{R}^d)$ and $U = (\partial_l\partial_k u)_{1\leq k,l\leq d}$. Using integration by parts we obtain

$$\begin{aligned}
(Zu, -\Delta u) &= \sum_{k,l,j=1}^d \int_{\mathbb{R}^d} \big(\partial_l(c_{kl}\,\partial_k u)\big)\,\partial_j^2\overline{u} = -\sum_{k,l,j=1}^d \int_{\mathbb{R}^d} \Big(\partial_l\big((\partial_j c_{kl})\,(\partial_k u) + c_{kl}\,(\partial_j\partial_k u)\big)\Big)\,\partial_j\overline{u} \\
&= -\sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_l\partial_j c_{kl})\,(\partial_k u)\,\partial_j\overline{u} + (\partial_j c_{kl})\,(\partial_l\partial_k u)\,\partial_j\overline{u} - c_{kl}\,(\partial_j\partial_k u)\,\partial_l\partial_j\overline{u} \\
&= -\sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_l\partial_j c_{kl})\,(\partial_k u)\,\partial_j\overline{u} - \sum_{j=1}^d \int_{\mathbb{R}^d} \operatorname{tr}\big((\partial_j C)\,U\big)\,\partial_j\overline{u} + \int_{\mathbb{R}^d} \operatorname{tr}\big(U\,C\,\overline{U}\big).
\end{aligned}$$

Therefore

$$\begin{aligned}
\operatorname{Re}\,(Zu, -\Delta u) = -\operatorname{Re} &\sum_{k,l,j=1}^d \int_{\mathbb{R}^d} (\partial_l\partial_j c_{kl})\,(\partial_k u)\,\partial_j\overline{u} - \operatorname{Re}\sum_{j=1}^d \int_{\mathbb{R}^d} \operatorname{tr}\big((\partial_j C)\,U\big)\,\partial_j\overline{u} \\
&+ \int_{\mathbb{R}^d} \operatorname{tr}\big(U\,(\operatorname{Re} C)\,\overline{U}\big) \\
&= (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}).
\end{aligned}$$

The estimate for (I) is straightforward as

$$(\mathrm{I}) \geq - \sum_{k,l,j=1}^{d} \int_{\mathbb{R}^d} \left|(\partial_l \partial_j c_{kl})\,(\partial_k u)\,\partial_j \overline{u}\right| \geq -M_1 \|\nabla u\|_2^2, \tag{4.23}$$

where $M_1 = d^2 \sup_{1 \leq k,l \leq d} \|c_{kl}\|_{W^{2,\infty}}$. The estimates for (II) and (III) are more involved. We consider three cases according to the three conditions (i), (ii) and (iii) imposed above.
Case 1: Suppose (i) holds.
Since $U = U^T$ and $R_a = -R_a^T$, we have

$$\operatorname{tr}\big((\partial_j R_a)\,U\big) = \operatorname{tr}\big(U^T\,(\partial_j R_a)^T\big) = -\operatorname{tr}\big(U\,(\partial_j R_a)\big) = -\operatorname{tr}\big((\partial_j R_a)\,U\big).$$

Therefore $\operatorname{tr}\big((\partial_j R_a)\,U\big) = 0$. This implies

$$\operatorname{tr}\big((\partial_j \mathrm{Im}\, C)\,U\big) = \operatorname{tr}\big((\partial_j B_s)\,U\big) - i\,\operatorname{tr}\big((\partial_j R_a)\,U\big) = \operatorname{tr}\big((\partial_j B_s)\,U\big) = 0,$$

where the last equality follows from the hypothesis. Using Lemma 4.32 we obtain that

$$(\mathrm{II}) = \sum_{k,l,j=1}^{d} \int_{\mathbb{R}^d} \mathrm{Re}\left(\frac{1}{2}\,(\partial_j^2 c_{kl})\,(\partial_l u)\,\partial_k \overline{u} - (\partial_k \partial_j c_{kl})\,(\partial_l u)\,\partial_j \overline{u}\right) + \mathrm{Im}\,\Big((\partial_l \partial_j \mathrm{Im}\, C)_{kl}\,(\partial_j u)\,\partial_k \overline{u}\Big).$$

Consequently

$$(\mathrm{II}) \geq -M_2\,\|\nabla u\|_2^2,$$

where $M_2 = 3\,d^2 \sup_{1 \leq k,l \leq d} \|c_{kl}\|_{W^{2,\infty}}$. Note that (III) ≥ 0. Hence

$$\mathrm{Re}\,(Zu, -\Delta u) \geq -(M_1 + M_2)\,\|\nabla u\|_2^2.$$

Case 2: Suppose (ii) holds.
We first consider (II). We have

$$\begin{aligned}
(\mathrm{II}) &= -\mathrm{Re} \sum_{j=1}^{d} \int_{\mathbb{R}^d} \operatorname{tr}\big(\partial_j(\phi\, \widetilde{C})\,U\big)\,\partial_j \overline{u} \\
&= -\mathrm{Re} \sum_{j=1}^{d} \int_{\mathbb{R}^d} (\partial_j \phi)\,\operatorname{tr}(\widetilde{C}\,U)\,\partial_j \overline{u} - \mathrm{Re} \sum_{j=1}^{d} \int_{\mathbb{R}^d} \phi\,\operatorname{tr}\big((\partial_j \widetilde{C})\,U\big)\,\partial_j \overline{u} \\
&= \qquad (\mathrm{IIa}) \qquad + \qquad (\mathrm{IIb}).
\end{aligned}$$

Let

$$M_3 = 64\,d\,(1 + \tan\theta_1)^2\,(1 + \tan\theta_2)^2\,\|\widetilde{R}_s\|_\infty \sup_{1 \leq j \leq d} \|\partial_j^2 \phi\|_\infty$$

and

$$M_4 = 32\,d^2\,(1 + \tan\theta_1)\,(1 + \tan\theta_2)^2 \sup_{1 \leq j \leq d} \|\partial_j^2 \widetilde{C}\|_\infty.$$

Let

$$\varepsilon = \frac{(1 - \tan\theta_1\,\tan\theta_2)\,h}{4\,(M_3 \vee M_4 \vee 1)}.$$

Note that $\varepsilon > 0$ as $1 - \tan\theta_1 \tan\theta_2 > 0$. Indeed, if $\tan\theta = 0$ then $\theta = \theta_1 = \theta_2 = 0$, which implies $1 - \tan\theta_1 \tan\theta_2 = 1 > 0$. If $\tan\theta > 0$ then $1 - \tan\theta_1 \tan\theta_2 = \frac{\tan\theta_1 + \tan\theta_2}{\tan\theta} > 0$.

For (IIa) we estimate

$$(\mathrm{IIa}) \geq -\varepsilon \int_{\mathbb{R}^d} \Big(\sum_{j=1}^d |\partial_j \phi|^2 \Big) \, |\mathrm{tr}\,(\widetilde{C}\, U)|^2 - \frac{1}{4\varepsilon} \, \|\nabla u\|_2^2.$$

Note that

$$|\partial_j \phi|^2 \leq 4\,(1 + \tan\theta_1)^2 \sup_{1 \leq j \leq d} \|\partial_j^2 \phi\|_\infty \, \mathrm{Re}\,\phi$$

for all $j \in \{1 \ldots, d\}$ by Lemma 4.7. Moreover,

$$|\mathrm{tr}\,(\widetilde{C}\, U)|^2 \leq d \, \|\widetilde{C}\, U\|_{HS}^2 \leq 16\, d\,(1 + \tan\theta_2)^2 \, \|\widetilde{R}_s\|_\infty \, \mathrm{tr}\,(U\, \widetilde{R}_s\, \overline{U}),$$

where we used Lemma 4.16 in the last step. Consequently

$$\begin{aligned}
(\mathrm{IIa}) &\geq -\varepsilon\, M_3 \int_{\mathbb{R}^d} (\mathrm{Re}\,\phi)\, \mathrm{tr}\,(U\, \widetilde{R}_s\, \overline{U}) - \frac{1}{4\varepsilon}\, \|\nabla u\|_2^2 \\
&\geq -\frac{(1 - \tan\theta_1 \tan\theta_2)\, h}{4} \int_{\mathbb{R}^d} (\mathrm{Re}\,\phi)\, \mathrm{tr}\,(U\, \widetilde{R}_s\, \overline{U}) - \frac{1}{4\varepsilon}\, \|\nabla u\|_2^2. \qquad (4.24)
\end{aligned}$$

For (IIb) we estimate as follows. Since $\phi(x) \in \Sigma_{\theta_1}$ for all $x \in \mathbb{R}^d$, we have

$$|\phi| \leq |\mathrm{Re}\,\phi| + |\mathrm{Im}\,\phi| \leq (1 + \tan\theta_1)\, \mathrm{Re}\,\phi.$$

Therefore

$$\begin{aligned}
(\mathrm{IIb}) &\geq -\sum_{j=1}^d \int_{\mathbb{R}^d} |\phi| \, \big|\mathrm{tr}\,\big((\partial_j \widetilde{C})\, U\big)\big| \, |\partial_j \overline{u}| \\
&\geq -(1 + \tan\theta_1) \sum_{j=1}^d \int_{\mathbb{R}^d} (\mathrm{Re}\,\phi) \Big(\varepsilon\, \big|\mathrm{tr}\,\big((\partial_j \widetilde{C})\, U\big)\big|^2 + \frac{1}{4\varepsilon}\, |\partial_j \overline{u}|^2 \Big) \\
&\geq -\varepsilon\,(1 + \tan\theta_1) \sum_{j=1}^d \int_{\mathbb{R}^d} (\mathrm{Re}\,\phi)\, \big|\mathrm{tr}\,\big((\partial_j \widetilde{C})\, U\big)\big|^2 - \frac{(1 + \tan\theta_1)\, \|\phi\|_\infty}{4\varepsilon}\, \|\nabla u\|_2^2 \\
&\geq -\varepsilon\, M_4 \int_{\mathbb{R}^d} (\mathrm{Re}\,\phi)\, \mathrm{tr}\,(U\, \widetilde{R}_s\, \overline{U}) - \frac{(1 + \tan\theta_1)\, \|\phi\|_\infty}{4\varepsilon}\, \|\nabla u\|_2^2 \\
&\geq -\frac{(1 - \tan\theta_1 \tan\theta_2)\, h}{4} \int_{\mathbb{R}^d} (\mathrm{Re}\,\phi)\, \mathrm{tr}\,(U\, \widetilde{R}_s\, \overline{U}) - \frac{(1 + \tan\theta_1)\, \|\phi\|_\infty}{4\varepsilon}\, \|\nabla u\|_2^2, \quad (4.25)
\end{aligned}$$

where we used Corollary 4.10(a) in the fourth step.

On the other hand, estimating (III) gives

$$(\mathrm{III}) = \int_{\mathbb{R}^d} \mathrm{tr}\,\big(U\,(\mathrm{Re}\,(\phi\, \widetilde{C}))\, \overline{U}\big) = \int_{\mathbb{R}^d} (\mathrm{Re}\,\phi)\, \mathrm{tr}\,(U\,(\mathrm{Re}\,\widetilde{C})\, \overline{U}) - (\mathrm{Im}\,\phi)\, \mathrm{tr}\,(U\,(\mathrm{Im}\,\widetilde{C})\, \overline{U}).$$

Since $\phi(x) \in \Sigma_{\theta_1}$ for all $x \in \mathbb{R}^d$, we have $|\mathrm{Im}\,\phi| \leq (\tan\theta_1)\,\mathrm{Re}\,\phi$. Also as $\widetilde{C}$ takes values in Σ_{θ_2}, we deduce that $|(\mathrm{Im}\,(\widetilde{C}\,Ue_j, Ue_j)| \leq (\tan\theta_2)\,\mathrm{Re}\,(\widetilde{C}\,Ue_j, Ue_j)$, which in turns implies that $|\mathrm{tr}\,(U\,(\mathrm{Im}\,\widetilde{C})\,\overline{U})| \leq (\tan\theta_2)\,\mathrm{tr}\,(U\,(\mathrm{Re}\,\widetilde{C})\,\overline{U})$. Therefore

$$\begin{aligned}
(\mathrm{III}) &\geq \int_{\mathbb{R}^d} (1-\tan\theta_1\,\tan\theta_2)\,(\mathrm{Re}\,\phi)\,\mathrm{tr}\,(U\,(\mathrm{Re}\,\widetilde{C})\,\overline{U}) \\
&\geq \int_{\mathbb{R}^d} (1-\tan\theta_1\,\tan\theta_2)\,h\,(\mathrm{Re}\,\phi)\,\mathrm{tr}\,(U\,\widetilde{R}_s\,\overline{U})
\end{aligned} \tag{4.26}$$

by the hypothesis. Hence by (4.23), (4.24), (4.25) and (4.26) we have

$$\begin{aligned}
\mathrm{Re}\,(Zu, -\Delta u) &\geq \frac{(1-\tan\theta_1\,\tan\theta_2)\,h}{2}\int_{\mathbb{R}^d} (\mathrm{Re}\,\phi)\,\mathrm{tr}\,(U\,\widetilde{R}_s\,\overline{U}) \\
&\quad - \big(M_1 + \frac{1+(1+\tan\theta_1)\,\|\phi\|_\infty}{4\varepsilon}\big)\,\|\nabla u\|_2^2 \\
&\geq -\big(M_1 + \frac{1+(1+\tan\theta_1)\,\|\phi\|_\infty}{4\varepsilon}\big)\,\|\nabla u\|_2^2.
\end{aligned}$$

Case 3: Suppose (iii) holds.
Let $\varepsilon_1 = \frac{1}{2M}$ and M' be corresponding to ε_1 as in Lemma 4.30. Then

$$\begin{aligned}
(\mathrm{III}) &= \int_{\mathbb{R}^d} \mathrm{tr}\,(U\,R_s\,\overline{U}) + i\int_{\mathbb{R}^d} \mathrm{tr}\,(U\,B_a\,\overline{U}) \\
&\geq \int_{\mathbb{R}^d} \mathrm{tr}\,(U\,R_s\,\overline{U}) - \varepsilon_1 \int_{\mathbb{R}^d} \|(\partial_l B_a)\,U\|_{HS}^2 - M'\,\|\nabla u\|_2^2 \\
&\geq \frac{1}{2}\int_{\mathbb{R}^d} \mathrm{tr}\,(U\,R_s\,\overline{U}) - M'\,\|\nabla u\|_2^2
\end{aligned}$$

since $\|(\partial_l B_a)\,U\|_{HS}^2 \leq M\,\mathrm{tr}\,(U\,R_s\,\overline{U})$ by hypothesis.

Let $\varepsilon_2 = \frac{1}{4dM''}$, where M'' is the constant as in Corollary 4.10(a). Then

$$(\mathrm{II}) \geq -\varepsilon_2 \sum_{j=1}^d \int_{\mathbb{R}^d} \big|\mathrm{tr}\,\big((\partial_j C)\,U\big)\big|^2 - \frac{1}{4\varepsilon_2}\,\|\nabla u\|_2^2 \geq -\frac{1}{4}\int_{\mathbb{R}^d} \mathrm{tr}\,(U\,R_s\,\overline{U}) - \frac{1}{4\varepsilon_2}\,\|\nabla u\|_2^2,$$

where we used Corollary 4.10(a) in the last step. Hence

$$\mathrm{Re}\,(Zu, -\Delta u) \geq \frac{1}{4}\int_{\mathbb{R}^d} \mathrm{tr}\,(U\,R_s\,\overline{U}) - (\frac{1}{4\varepsilon_2} + M_1 + M')\,\|\nabla u\|_2^2.$$

The proof is complete. □

We emphasise that it is not known yet whether B_2 is accretive if $B_a \neq 0$.

Theorem 4.34. *Suppose one of the following holds.*

(i) *The matrix B_s has constant entries.*

(ii) *There exist* $\theta_1, \theta_2 \in [0, \frac{\pi}{2})$, $\phi \in W^{2,\infty}(\mathbb{R}^d)$ *and a* $d \times d$ *matrix* $\widetilde{C}$ *with entries in* $W^{2,\infty}(\mathbb{R}^d)$ *such that* $\theta = \theta_1 + \theta_2$, $\phi(x) \in \Sigma_{\theta_1}$ *for all* $x \in \mathbb{R}^d$, $\widetilde{C}$ *takes values in* Σ_{θ_2} *and* $C = \phi\, \widetilde{C}$. *Write* $\widetilde{C} = \widetilde{R} + i\, \widetilde{B}$, *where* $\widetilde{R}$ *and* $\widetilde{B}$ *are* $d \times d$ *matrix-valued functions with real-valued entries. Set* $\widetilde{R}_s = \frac{1}{2}(\widetilde{R} + \widetilde{R}^T)$. *Also define* $\operatorname{Re} \widetilde{C} = \frac{1}{2}\left(\widetilde{C} + (\widetilde{C})^*\right)$. *Suppose further that there exists an* $h > 0$ *such that*

$$\operatorname{tr}(U\,(\operatorname{Re}\widetilde{C})\,\overline{U}) \geq h \operatorname{tr}(U\,\widetilde{R}_s\,\overline{U})$$

for all $u \in C_c^\infty(\mathbb{R}^d)$, *where* $U = (\partial_l \partial_k u)_{1 \leq k,l \leq d}$.

(iii) *There exists an* $M > 0$ *such that* $\|(\partial_l B_a)\, U\|_{HS}^2 \leq M \operatorname{tr}(U\, R_s\, \overline{U})$ *for all* $l \in \{1, \ldots, d\}$ *and* $u \in C_c^\infty(\mathbb{R}^d)$, *where* $U = (\partial_l \partial_k u)_{1 \leq k,l \leq d}$.

Then $A = B_2 = Z$.

Proof. By Proposition 4.33 the operator Z is m-accretive. We will show that $Z = B_2$. Clearly $Z \subset B_2$. For the reverse inclusion let $u \in D(B_2)$. Then $(I + B_2)u \in L_2(\mathbb{R}^d)$. Since Z is m-accretive, there exists a $v \in D(Z)$ such that $(I + Z)v = (I + B_2)u$. But B_2 is an extension of Z. Therefore $(I + B_2)v = (I + B_2)u$. Let $\phi \in C_c^\infty(\mathbb{R}^d)$. Then

$$0 = ((I + B_2)(u - v), \phi) = (u - v, (I + H_2)\phi),$$

where H_2 is defined by (4.4). But H_2 satisfies the same criteria as those of $B_2|_{C_c^\infty(\mathbb{R}^d)}$. Therefore analogous arguments give that $\overline{H_2}$ is also m-accretive. Consequently $u = v$. Hence $Z = B_2$. It follows that B_2 is m-accretive. In particular B_2 is accretive. Note that A is m-accretive and $A \subset B_2$. Therefore we must have $A = B_2 = Z$. □

Theorem 4.3 now follows from Theorem 4.34 as an easy consequence.

Proof of Theorem 4.3. By Theorem 4.34 we have $A = Z$. Since $C_c^\infty(\mathbb{R}^d)$ is a core for Z, it is also a core for A. □

The next proposition provides three easy criteria to verify Condition (iii) in Theorem 4.3.

Proposition 4.35. *Suppose* C *satisfies one of the following.*

(a) *There exists an* $r \in \mathbb{R} \setminus \{0\}$ *such that* $R_s + ir\, \partial_l B_a \geq 0$ *for all* $l \in \{1, \ldots, d\}$.

(b) *The matrices* R_s *and* $\partial_l B_a$ *commute for all* $l \in \{1, \ldots, d\}$.

(c) *There exist a real-valued function* $\phi \in W^{2,\infty}(\mathbb{R}^d)$ *which satisfies* $\phi \geq 0$ *and a* $d \times d$ *matrix* $\widetilde{C}$ *which has constant entries and takes values in* Σ_θ *such that* $C = \phi\, \widetilde{C}$.

Then there exists an $M > 0$ *such that* $\|(\partial_l B_a)\, U\|_{HS}^2 \leq M \operatorname{tr}(U\, R_s\, \overline{U})$ *for all* $l \in \{1, \ldots, d\}$ *and* $u \in C_c^\infty(\mathbb{R}^d)$, *where* $U = (\partial_l \partial_k u)_{1 \leq k,l \leq d}$.

Proof. Let $l \in \{1, \ldots, d\}$. Let $u \in C_c^\infty(\mathbb{R}^d)$ and $U = (\partial_l \partial_k u)_{1 \leq k,l \leq d}$.

We first deal with (a) and (b). Set $P = \sqrt{U\, U^*} \geq 0$. Let V be a unitary matrix such that $P = V\, D_P\, V^*$, where D_P is a positive diagonal matrix. Then

$$\begin{aligned}\|(\partial_l B_a)\, U\|_{HS}^2 &= -\mathrm{tr}\,(U^*\,(\partial_l B_a)^2\, U) = -\mathrm{tr}\,((\partial_l B_a)^2\, P^2) = -\mathrm{tr}\,((\partial_l B_a)^2\, V\, D_P^2\, V^*) \\ &= -\mathrm{tr}\,(V^*\,(\partial_l B_a)^2\, V\, D_P^2) = \sum_{k=1}^d |(V^*\,(\partial_l B_a)^2\, V)_{kk}|\,|(D_P)_{kk}|^2.\end{aligned}$$

We consider two cases.
Case 1: Suppose (a) holds.
Then $|((\partial_l B_a)\,\xi,\xi)| \le \frac{1}{|r|}\,(R_s\,\xi,\xi)$ for all $\xi \in \mathbb{C}^d$. By Lemma 4.14 we have $\|(\partial_l B_a)\,\xi\|^2 \le \frac{4}{r^2}\,\|R_s\|_\infty\,(R_s\,\xi,\xi)$ for all $\xi \in \mathbb{C}^d$. In particular $\|(\partial_l B_a)\,V e_k\|^2 \le \frac{4}{r^2}\,\|R_s\|_\infty\,(V^*\,R_s\,V)_{kk}$ for all $k \in \{1,\ldots,d\}$. It follows that

$$\begin{aligned}\|(\partial_l B_a)\, U\|_{HS}^2 &\le \frac{4}{r^2}\,\|R_s\|_\infty \sum_{k=1}^d (V^*\,R_s\,V)_{kk}\,|(D_P)_{kk}|^2 = \frac{4}{r^2}\,\|R_s\|_\infty\,\mathrm{tr}\,(V^*\,R_s\,V\,D_P^2) \\ &= \frac{4}{r^2}\,\|R_s\|_\infty\,\mathrm{tr}\,(R_s\,P^2) = \frac{4}{r^2}\,\|R_s\|_\infty\,\mathrm{tr}\,(U^*\,R_s\,U) \\ &= \frac{4}{r^2}\,\|R_s\|_\infty\,\mathrm{tr}\,(U\,R_s\,U^*) = \frac{4}{r^2}\,\|R_s\|_\infty\,\mathrm{tr}\,(U\,R_s\,\overline{U}),\end{aligned}$$

where the last equality follows from the fact that $U = U^T$.
Case 2: Suppose (b) holds.
Let W be a unitary matrix such that $\partial_l B_a = W\,D\,W^*$, where D is diagonal. Therefore

$$|D_{kk}|^2 = |(W^*\,(\partial_l B_a)\,W)_{kk}|^2 \le 2 \sup_{1\le l\le d} \|\partial_l^2 C\|_\infty\,(W^*\,R_s\,W)_{kk}$$

for all $k \in \{1,\ldots,d\}$ by Lemma 4.13. Since R_s and $\partial_l B_a$ commute, we may assume without loss of generality that the matrix W also diagonalises R_s. It follows that

$$\begin{aligned}|(V^*\,(\partial_l B_a)^2\,V)_{kk}| &= |(V^*\,W\,D^2\,W^*\,V)_{kk}| = |((W^*\,V)^*\,D^2\,W^*\,V)_{kk}| \\ &= \sum_{j=1}^d \big((W^*\,V)^*\big)_{kj}\,|D_{jj}|^2\,(W^*\,V)_{jk} \\ &\le 2 \sup_{1\le l\le d} \|\partial_l^2 C\|_\infty \sum_{j=1}^d \big((W^*\,V)^*\big)_{kj}\,(W^*\,R_s\,W)_{jj}\,(W^*\,V)_{jk} \\ &= 2 \sup_{1\le l\le d} \|\partial_l^2 C\|_\infty\,(V^*\,R_s\,V)_{kk}\end{aligned}$$

for all $k \in \{1,\ldots,d\}$. Hence

$$\begin{aligned}\|(\partial_l B_a)\, U\|_{HS}^2 &\le 2 \sup_{1\le l\le d} \|\partial_l^2 C\|_\infty \sum_{k=1}^d (V^*\,R_s\,V)_{kk}\,|(D_P)_{kk}|^2 \\ &= 2 \sup_{1\le l\le d} \|\partial_l^2 C\|_\infty\,\mathrm{tr}\,(V^*\,R_s\,V\,D_P^2) = 2 \sup_{1\le l\le d} \|\partial_l^2 C\|_\infty\,\mathrm{tr}\,(R_s\,P^2) \\ &= 2 \sup_{1\le l\le d} \|\partial_l^2 C\|_\infty\,\mathrm{tr}\,(U\,R_s\,\overline{U}).\end{aligned}$$

This completes the proof of the proposition under the assumptions (a) and (b).

Next we turn to (c). Suppose (c) holds. Write $\widetilde{C} = \widetilde{R} + i\,\widetilde{B}$. Set $\widetilde{R}_s = \frac{1}{2}\,(\widetilde{R} + \widetilde{R}^T)$ and $\widetilde{B}_a = \frac{1}{2}\,(\widetilde{B} - \widetilde{B}^T)$. Since ϕ is real-valued, we have $R_s = \phi\,\widetilde{R}_s$ and $B_a = \phi\,\widetilde{B}_a$. Applying Lemma 4.6 to ϕ we obtain $(\partial_l\phi)^2 \leq 2\,\|\phi\|_{W^{2,\infty}}\,\phi$. By Lemmas 4.12 and 4.14 we also have $\|\widetilde{B}_a\,\xi\|^2 \leq 4\,\|\widetilde{R}_s\|_\infty\,(\widetilde{R}_s\,\xi, \xi)$ for all $\xi \in \mathbb{C}^d$. Therefore

$$\|(\partial_l B_a)\,U\|_{HS}^2 = \sum_{j=1}^d \|(\partial_l B_a)\,U\,e_j\|_2^2 = (\partial_l\phi)^2 \sum_{j=1}^d \|\widetilde{B}_a\,U\,e_j\|_2^2$$

$$\leq 8\,\|\phi\|_{W^{2,\infty}}\,\|\widetilde{R}_s\|_\infty\,\phi \sum_{j=1}^d (\widetilde{R}_s\,U\,e_j, U\,e_j) = 8\,\|\phi\|_{W^{2,\infty}}\,\|\widetilde{R}_s\|_\infty\,\mathrm{tr}\,(U\,R_s\,\overline{U}).$$

The proof is complete. □

Our next aim is to show that if $D(A) \subset W^{1,2}(\mathbb{R}^d)$, then $C_c^\infty(\mathbb{R}^d)$ is a core for A.

Lemma 4.36. *Suppose $D(A) \subset W^{1,2}(\mathbb{R}^d)$. Then*

$$\sum_{k,l=1}^d \int_{\mathbb{R}^d} c_{kl}\,\eta\,(\partial_k u)\,\partial_l\overline{\phi} = \Big(\eta\,Au - \sum_{k,l=1}^d c_{kl}\,(\partial_k u)\,\partial_l\eta, \phi\Big)$$

for all $u \in D(A)$ and $\eta, \phi \in C_c^\infty(\mathbb{R}^d)$.

Proof. Let $u \in D(A)$ and $\eta, \phi \in C_c^\infty(\mathbb{R}^d)$. Then

$$(\eta\,Au, \phi) = (Au, \overline{\eta}\,\phi) = \mathfrak{a}(u, \overline{\eta}\,\phi) = \sum_{k,l=1}^d \int_{\mathbb{R}^d} c_{kl}\,(\partial_k u)\,\partial_l(\eta\,\overline{\phi})$$

$$= \sum_{k,l=1}^d \int_{\mathbb{R}^d} c_{kl}\,(\partial_k u)\,(\partial_l\eta)\,\overline{\phi} + \sum_{k,l=1}^d \int_{\mathbb{R}^d} c_{kl}\,(\partial_k u)\,\eta\,\partial_l\overline{\phi}.$$

Next we rearrange the terms to derive the lemma. □

Recall that J_n is the usual mollifier with respect to a suitable function in $C_c^\infty(\mathbb{R}^d)$ for all $n \in \mathbb{N}$.

Proposition 4.37. *Suppose $D(A) \subset W^{1,2}(\mathbb{R}^d)$. Then $C_c^\infty(\mathbb{R}^d)$ is a core for A if and only if $\lim_{n\to\infty} A(J_n * u) = Au$ in $L_2(\mathbb{R}^d)$ for all $u \in D(A)$.*

Proof. ($\Longrightarrow$) It is well-known that $\lim_{n\to\infty} J_n * (Au) = Au$ in $L_2(\mathbb{R}^d)$. Therefore it suffices to show that $\lim_{n\to\infty} \|A(J_n * u) - J_n * (Au)\|_2 = 0$.

By a similar calculation as in (4.11) we yield

$$A(J_n * u) - J_n * Au = T_n u \tag{4.27}$$

for all $n \in \mathbb{N}$ and $u \in C_c^\infty(\mathbb{R}^d)$, where the bounded operator $T_n : W^{1,2}(\mathbb{R}^d) \longrightarrow L_2(\mathbb{R}^d)$ is defined by (4.6). Let $n \in \mathbb{N}$ and $u \in D(A)$. Since $C_c^\infty(\mathbb{R}^d)$ is a core for $D(A)$, there exists a sequence $\{\phi_j\}_{j\in\mathbb{N}}$ in $C_c^\infty(\mathbb{R}^d)$ such that

$$\lim_{j\to\infty} \phi_j = u \tag{4.28}$$

in $D(A)$. By hypothesis $D(A) \subset W^{1,2}(\mathbb{R}^d)$. Therefore the inclusion $D(A) \hookrightarrow W^{1,2}(\mathbb{R}^d)$ is continuous. It follows from (4.28) that $\lim_{j\to\infty} \phi_j = u$ in $W^{1,2}(\mathbb{R}^d)$. Recall that the operator T_n is bounded. As a consequence $\lim_{j\to\infty} T_n\phi_j = T_n u$ in $L_2(\mathbb{R}^d)$. We also derive from (4.28) that $\lim_{j\to\infty} J_n * \phi_j = J_n * u$ in $L_2(\mathbb{R}^d)$ and $\lim_{j\to\infty} J_n * (A\phi_j) = J_n * (Au)$ in $L_2(\mathbb{R}^d)$. Therefore (4.27) gives

$$\lim_{j\to\infty} A(J_n * \phi_j) = \lim_{j\to\infty} \big(T_n\phi_j + J_n * (A\phi_j)\big) = T_n u + J_n * (Au)$$

in $L_2(\mathbb{R}^d)$. Since T_n is bounded, it is also closed. Hence $J_n * u \in D(A)$ and $A(J_n * u) = T_n u + J_n * (Au)$. That is,

$$A(J_n * u) - J_n * Au = T_n u \tag{4.29}$$

also holds for all $n \in \mathbb{N}$ and $u \in D(A)$.

Let $\psi \in W^{2,2}(\mathbb{R}^d)$. Then $\lim_{n\to\infty} J_n * \psi = \psi$ in $W^{2,2}(\mathbb{R}^d)$. Consequently $\lim_{n\to\infty} A(J_n * \psi) = A\psi$ in $L_2(\mathbb{R}^d)$. Also $\lim_{n\to\infty} J_n * (A\psi) = A\psi$ in $L_2(\mathbb{R}^d)$. Therefore it follows from (4.29) that $\lim_{n\to\infty} \|T_n u\|_2 = 0$. This is for all $\psi \in W^{2,2}(\mathbb{R}^d)$. Since $W^{2,2}(\mathbb{R}^d)$ is dense in $W^{1,2}(\mathbb{R}^d)$ and $\{T_n\}_{n\in\mathbb{N}}$ is bounded by Lemma 4.20, we deduce that $\lim_{n\to\infty} \|T_n u\|_2 = 0$ for all $u \in W^{1,2}(\mathbb{R}^d)$. In particular $\lim_{n\to\infty} \|T_n u\|_2 = 0$ for all $u \in D(A)$ as $D(A) \subset W^{1,2}(\mathbb{R}^d)$ by hypothesis.

($\Longleftarrow$) Let $\tau \in C_c^\infty(\mathbb{R}^d)$ be such that $0 \le \tau \le \mathbb{1}$, $\tau|_{B_1(0)} = 1$ and $\operatorname{supp} \tau \subset B_2(0)$. Define $\tau_n(x) = \tau(n^{-1} x)$ for all $x \in \mathbb{R}^d$ and $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. Let $u \in D(A)$ and $\phi \in C_c^\infty(\mathbb{R}^d)$. Then $u \in W^{1,2}(\mathbb{R}^d)$ and hence $\tau_n u \in W^{1,2}(\mathbb{R}^d)$. Moreover

$$\mathfrak{a}(\tau_n u, \phi) = \sum_{k,l=1}^d \int_{\mathbb{R}^d} c_{kl}\, \partial_k(\tau_n u)\, \partial_l\overline{\phi} = \sum_{k,l=1}^d \int_{\mathbb{R}^d} c_{kl}\, ((\partial_k\tau_n)\, u + \tau_n\, \partial_k u)\, \partial_l\overline{\phi} = (f_n, \phi),$$

where

$$f_n = (A\tau_n)\, u + \tau_n\, Au - \sum_{k,l=1}^d c_{kl}\, (\partial_k\tau_n)\, \partial_l u - \sum_{k,l=1}^d c_{kl}\, (\partial_l\tau_n)\, \partial_k u$$

and we used Lemma 4.36 in the last equality. Since $f_n \in L_2(\mathbb{R}^d)$, we have $\tau_n u \in D(A)$ and $A(\tau_n u) = f_n$. Next we will show that $\lim_{n\to\infty} f_n = Au$ in $L_2(\mathbb{R}^d)$. Clearly $\lim_{n\to\infty} \tau_n\, Au = Au$ in $L_2(\mathbb{R}^d)$. Note that

$$\|(A\tau_n)\, u\|_2 = \Big\| - \sum_{k,l=1}^d (\partial_l(c_{kl}\, \partial_k\tau_n))\, u\Big\|_2 = \Big\| \sum_{k,l=1}^d \Big((\partial_l c_{kl})\, \partial_k\tau_n + c_{kl}\, \partial_l\partial_k\tau_n\Big)\, u\Big\|_2$$
$$\le \sum_{k,l=1}^d \|c_{kl}\|_{W^{2,\infty}} \Big(\frac{1}{n}\, \|\partial_k\tau\|_\infty + \frac{1}{n^2}\, \|\partial_l\partial_k\tau\|_\infty\Big)\, \|u\|_2.$$

Similarly

$$\Big\| \sum_{k,l=1}^d c_{kl}\, (\partial_k\tau_n)\, \partial_l u\Big\|_2 \le \frac{1}{n} \sum_{k,l=1}^d \|c_{kl}\|_\infty\, \|\partial_k\tau\|_\infty\, \|\partial_l u\|_2$$

and

$$\Big\| \sum_{k,l=1}^{d} c_{kl}\, (\partial_l \tau_n)\, \partial_k u \Big\|_2 \le \frac{1}{n} \sum_{k,l=1}^{d} \|c_{kl}\|_\infty \, \|\partial_l \tau\|_\infty \, \|\partial_k u\|_2.$$

It follows that these three terms go to 0 in $L_2(\mathbb{R}^d)$ as n tends to infinity. Hence

$$\lim_{n\to\infty} \|A(\tau_n\, u) - Au\|_2 = 0. \tag{4.30}$$

Finally we will show that $C_c^\infty(\mathbb{R}^d)$ is a core for A. Let $u \in D(A)$. The hypothesis gives

$$\lim_{k\to\infty} \|A(J_k * (\tau_n\, u)) - A(\tau_n\, u)\|_2 = 0 \tag{4.31}$$

for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. By (4.30) we can choose an $n \in \mathbb{N}$ such that $\|A(\tau_n\, u) - Au\|_2 < \frac{\varepsilon}{2}$. Next we use (4.31) to choose a $k \in \mathbb{N}$ such that $\|A(J_k * (\tau_n\, u)) - A(\tau_n\, u)\|_2 < \frac{\varepsilon}{2}$. Then

$$\|A(J_k * (\tau_n\, u)) - Au\|_2 \le \|A(J_k * (\tau_n\, u)) - A(\tau_n\, u)\|_2 + \|A(\tau_n\, u) - Au\|_2 < \varepsilon.$$

Note that $J_k * (\tau_n\, u) \in C_c^\infty(\mathbb{R}^d)$. Hence $C_c^\infty(\mathbb{R}^d)$ is indeed a core for A. □

Let $\delta \in (0,1)$. Define

$$C_\delta = (R_s + i\delta\, B_a) + i\,(B_s - i\, R_a).$$

Lemma 4.38. *The matrix C_δ takes values in Σ_ψ, where $\psi \in [0, \frac{\pi}{2})$ is such that $\tan\psi = \frac{1}{\delta}\tan\theta$.*

Proof. Let $\xi \in \mathbb{C}^d$. Then

$$\begin{aligned} |((\operatorname{Im} C_\delta)\,\xi,\xi)| = |((\operatorname{Im} C)\,\xi,\xi)| \le \tan\theta\,((\operatorname{Re} C)\,\xi,\xi) = \frac{1}{\delta}\tan\theta\,((\delta\, R_s + i\delta\, B_a)\,\xi,\xi) \\ \le \frac{1}{\delta}\tan\theta\,((R_s + i\delta\, B_a)\,\xi,\xi) = \frac{1}{\delta}\tan\theta\,((\operatorname{Re} C_\delta)\,\xi,\xi) \end{aligned}$$

since C takes values in Σ_θ and $(R_s\,\xi,\xi) \ge 0$ by Lemma 4.12. The statement now follows. □

Define the form

$$\mathfrak{a}_{0,\delta}(u,v) = \int_{\mathbb{R}^d} (C_\delta\, \nabla u, \nabla u)$$

on the domain $D(\mathfrak{a}_{0,\delta}) = C_c^\infty(\mathbb{R}^d)$. Then by the same analysis as in Section 4.1, the form $\mathfrak{a}_{0,\delta}$ is closable. Let A_δ be the operator associated with the closure of $\mathfrak{a}_{0,\delta}$. Then we also have that $W^{2,2}(\mathbb{R}^d) \subset D(A_\delta)$ and

$$A_\delta u = -\sum_{k,l=1}^{d} \partial_l((C_\delta)_{kl}\, \partial_k u)$$

for all $u \in W^{2,2}(\mathbb{R}^d)$. Define

$$H_\delta = -\sum_{k,l=1}^{d} \partial_k(\overline{(C_\delta)_{kl}}\, \partial_l u)$$

on the domain $D(H_\delta) = C_c^\infty(\mathbb{R}^d)$. Then we have the following.

Proposition 4.39. *The space $C_c^\infty(\mathbb{R}^d)$ is a core for A_δ. Furthermore $A_\delta = (H_\delta)^*$.*

Proof. We note that

$$\operatorname{tr}(U\,(\operatorname{Re} C_\delta)\,\overline{U}) = (1-\delta)\operatorname{tr}(U\,R_s\,\overline{U}) + \delta\operatorname{tr}(U\,(\operatorname{Re} C)\,\overline{U}) \geq (1-\delta)\operatorname{tr}(U\,R_s\,\overline{U})$$

for all $u \in C_c^\infty(\mathbb{R}^d)$, where $U = (\partial_l\partial_k u)_{1\leq k,l\leq d}$. That is, C_δ satisfies Condition (ii) in Theorem 4.3. Hence $C_c^\infty(\mathbb{R}^d)$ is a core for A_δ and $A_\delta = (H_\delta)^*$ by Theorem 4.3. □

Lemma 4.40. *Suppose $D(A) \subset W^{1,2}(\mathbb{R}^d)$. Then $D(A) \subset D(A_\delta) \cap D((B_a)^{\mathrm{op}})$ and*

$$Au = A_\delta u + i(1-\delta)\,(B_a)^{\mathrm{op}} u$$

for all $u \in D(A)$.

Proof. Recall that the operators H_2 and L are defined by (4.4) and (4.20) respectively. First note that $D(A) \subset W^{1,2}(\mathbb{R}^d) \subset D((B_a)^{\mathrm{op}})$. Moreover, the condition $D(A) \subset W^{1,2}(\mathbb{R}^d)$ implies that

$$(u, H_2\phi) = -\int_{\mathbb{R}^d} u\,\partial_k(c_{kl}\,\partial_l\overline{\phi}) = \int_{\mathbb{R}^d} c_{kl}\,(\partial_k u)\,\partial_l\overline{\phi} = \mathfrak{a}(u,\phi) = (Au,\phi)$$

for all $u \in D(A)$ and $\phi \in C_c^\infty(\mathbb{R}^d)$, where we used integration by parts in the second step. Since $Au \in L_2(\mathbb{R}^d)$, we conclude that $u \in D(B_2)$ and

$$B_2 u = Au \tag{4.32}$$

for all $u \in D(A)$. Therefore we also have $D(A) \subset D(B_2)$.

Next let $u \in D(A)$. Then

$$\begin{aligned}(u, H_\delta\phi) &= (u, H_2\phi) - i(1-\delta)\,(u, L\phi) = (B_2 u, \phi) - i(1-\delta)\,\big((B_a)^{\mathrm{op}} u, \phi\big)\\ &= \big(B_2 u - i(1-\delta)\,(B_a)^{\mathrm{op}} u, \phi\big)\end{aligned}$$

for all $\phi \in C_c^\infty(\mathbb{R}^d)$. Note that $B_2 u - i(1-\delta)\,(B_a)^{\mathrm{op}} u \in L_2(\mathbb{R}^d)$. Hence $u \in D(A_\delta)$ and

$$A_\delta u = B_2 u - i(1-\delta)\,(B_a)^{\mathrm{op}} u = Au - i(1-\delta)\,(B_a)^{\mathrm{op}} u,$$

where we used (4.32) in the last step. The lemma now follows. □

Lemma 4.41. *Suppose $D(A) \subset W^{1,2}(\mathbb{R}^d)$. Then there exists a $\delta_0 \in (0,1)$ such that for all $\delta \in [\delta_0, 1)$ there exists an $M > 0$ such that $D(A_\delta) \subset W^{1,2}(\mathbb{R}^d)$ and $\|u\|_{W^{1,2}} \leq M\,\|u\|_{D(A_\delta)}$ for all $u \in D(A_\delta)$.*

Proof. Since $D(A) \subset W^{1,2}(\mathbb{R}^d)$, there exists an $M_1 > 0$ such that $\|u\|_{W^{1,2}} \leq M_1\,\|u\|_{D(A)}$ for all $u \in D(A)$ by the closed graph theorem. Similarly the inclusion $W^{1,2}(\mathbb{R}^d) \subset D((B_a)^{\mathrm{op}})$ implies that there exists an $M_2 > 0$ which satisfies $\|u\|_{D((B_a)^{\mathrm{op}})} \leq M_2\,\|u\|_{W^{1,2}}$ for all $u \in D((B_a)^{\mathrm{op}})$. Let $\delta_0 = (1 - \frac{1}{2M_1M_2}) \vee \frac{1}{2}$ and $\delta \in [\delta_0, 1)$. If $u \in D(A)$ then $u \in D(A_\delta)$ by Lemma 4.40. Therefore

$$
\begin{aligned}
\|u\|_{W^{1,2}} &\leq M_1 \left(\|u\|_2 + \|Au\|_2\right) \leq M_1 \left(\|u\|_2 + \|A_\delta u\|_2 + (1-\delta)\,\|(B_a)^{\mathrm{op}} u\|_2\right) \\
&= M_1\,\|u\|_{D(A_\delta)} + (1-\delta)\,M_1 \|(B_a)^{\mathrm{op}} u\|_2 \leq M_1\,\|u\|_{D(A_\delta)} + (1-\delta)\,M_1\,M_2\,\|u\|_{W^{1,2}}
\end{aligned}
$$

for all $u \in D(A)$. It follows that

$$\|u\|_{W^{1,2}} \leq \frac{M_1}{1-(1-\delta)\,M_1 M_2}\,\|u\|_{D(A_\delta)}$$

for all $u \in D(A)$. In particular

$$\|u\|_{W^{1,2}} \leq \frac{M_1}{1-(1-\delta)\,M_1 M_2}\,\|u\|_{D(A_\delta)} \tag{4.33}$$

for all $u \in C_c^\infty(\mathbb{R}^d)$. Note that $C_c^\infty(\mathbb{R}^d)$ is a core for A_δ by Lemma 4.39 and the space $W^{1,2}(\mathbb{R}^d)$ is complete. Consequently (4.33) implies that $D(A_\delta) \subset W^{1,2}(\mathbb{R}^d)$ and

$$\|u\|_{W^{1,2}} \leq \frac{M_1}{1-(1-\delta)\,M_1 M_2}\,\|u\|_{D(A_\delta)}.$$

for all $u \in D(A_\delta)$ as required. □

Lemma 4.42. *Let $u \in D(A)$. Then $\lim_{n\to\infty} A_\delta(J_n * u) = A_\delta u$ in $L_2(\mathbb{R}^d)$.*

Proof. The proof is the same as that of the 'only if' part of Proposition 4.37. Note that $C_c^\infty(\mathbb{R}^d)$ is a core for A_δ by Lemma 4.39 and $D(A_\delta) \subset W^{1,2}(\mathbb{R}^d)$ by Lemma 4.41. □

We are now in the position to prove Theorem 4.4.

Proof of Theorem 4.4. Let $\delta = \delta_0$, where δ_0 is defined as in Lemma 4.41. By Lemma 4.42 we have $\lim_{n\to\infty} A_\delta(J_n * u) = A_\delta u$ in $L_2(\mathbb{R}^d)$ for all $u \in D(A)$. Furthermore [ERS11, Proposition 2.1] gives that $\lim_{n\to\infty}(B_a)^{\mathrm{op}}(J_n * u) = (B_a)^{\mathrm{op}} u$ in $L_2(\mathbb{R}^d)$ for all $u \in D((B_a)^{\mathrm{op}})$. Hence $\lim_{n\to\infty} A(J_n * u) = Au$ in $L_2(\mathbb{R}^d)$ for all $u \in D(A)$ as $A \subset A_\delta + i(1-\delta)\,(B_a)^{\mathrm{op}}$. Using Proposition 4.37 we can conclude that $C_c^\infty(\mathbb{R}^d)$ is a core for A. □

4.7 Examples

In this section we present several applications of Theorems 4.2, 4.3 and 4.4 in showing the core properties for some specific degenerate elliptic operators in higher dimensions.

Example 4.43. For all $(x, y) \in \mathbb{R}^2$ let $\phi(x, y) = \frac{\pi}{4}\cos(\sin(x + y))$. Let

$$C = \begin{pmatrix} 2\cos\phi + i\,\sin\phi & \sin\phi \\ -\sin\phi & 2\cos\phi + i\,\sin\phi \end{pmatrix}.$$

Then $\big(C(x, y)\,\xi, \xi\big) \in \Sigma_{\frac{\pi}{4}}$ for all $(x, y) \in \mathbb{R}^2$ and $\xi \in \mathbb{C}^2$. Note that $B_a = 0$.

Consider the form $\mathfrak{a}_0$ defined by

$$\mathfrak{a}_0(u, v) = \int_{\mathbb{R}^2} (C\,\nabla u, \nabla v)$$

on the domain $D(\mathfrak{a}_0) = C_c^\infty(\mathbb{R}^2)$. Then $\mathfrak{a}_0$ is closable by [Kat80, Theorem VI.1.27]. Let A be the operator associated with the closure of $\mathfrak{a}_0$ in $L_2(\mathbb{R}^2)$. Since $B_a = 0$, we can extend the contraction C_0-semigroup S generated by $-A$ to a contraction C_0-semigroup $S^{(p)}$ on $L_p(\mathbb{R}^2)$ for all $p \in [4-2\sqrt{2}, 4+2\sqrt{2}]$ by Proposition 4.1. Let $-A_p$ be the generator of $S^{(p)}$ for all $p \in [4-2\sqrt{2}, 4+2\sqrt{2}]$. Then the space $C_c^\infty(\mathbb{R}^2)$ is a core for A_p for all $p \in (4-2\sqrt{2}, 4+2\sqrt{2})$ by Theorem 4.2.

Example 4.44. For all $(x,y) \in \mathbb{R}^2$ let

$$C(x,y) = \begin{pmatrix} \frac{1}{\sqrt{2}}(1+i) & e^{i\,(x+y)} \\ i\,e^{-i\,(x+y)} & \frac{1}{\sqrt{2}}(1+i) \end{pmatrix}.$$

Note that

$$C = (1+i)\,(\operatorname{Re} C), \tag{4.34}$$

where

$$(\operatorname{Re} C)(x,y) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{\cos(x+y)+\sin(x+y)}{2} - i\frac{\cos(x+y)-\sin(x+y)}{2} \\ \frac{\cos(x+y)+\sin(x+y)}{2} + i\frac{\cos(x+y)-\sin(x+y)}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Therefore $\big(C(x,y)\,\xi, \xi\big) \in \Sigma_{\frac{\pi}{4}}$ for all $(x,y) \in \mathbb{R}^2$ and $\xi \in \mathbb{C}^2$.

Consider the form $\mathfrak{a}_0$ defined by

$$\mathfrak{a}_0(u,v) = \int_{\mathbb{R}^2} (C\,\nabla u, \nabla v)$$

on the domain $D(\mathfrak{a}_0) = C_c^\infty(\mathbb{R}^2)$. Then $\mathfrak{a}_0$ is closable by [Kat80, Theorem VI.1.27]. Let A be the operator associated with the closure of $\mathfrak{a}_0$ in $L_2(\mathbb{R}^2)$.

Using (4.34) and the fact that $\operatorname{Re} C$ is self-adjoint, we conclude that the space $C_c^\infty(\mathbb{R}^2)$ is a core for A by Theorem 4.3(i).

Example 4.45. Let $c_{kl} \in \mathbb{C}$ for all $k,l \in \{1,2\}$. Suppose there exists a constant $\mu > 0$ such that

$$\operatorname{Re}\,(C\,\xi, \xi) \ge \mu\,\|\xi\|^2$$

for all $\xi \in \mathbb{C}^2$, where $C = (c_{kl})_{1 \le k,l \le 2}$. Define $A_1 = \partial_x$ and $A_2 = \cos x\,\partial_y + \sin x\,\partial_z$. Consider the form $\mathfrak{a}_0$ defined by

$$\mathfrak{a}_0(u,v) = \sum_{k,l=1}^{2} \int_{\mathbb{R}^3} c_{kl}(A_k u)\,A_l v$$

on the domain $D(\mathfrak{a}_0) = C_c^\infty(\mathbb{R}^3)$. Then $\mathfrak{a}_0$ is closable by [Kat80, Theorem VI.1.27]. Let A be the operator associated with the closure of $\mathfrak{a}_0$ in $L_2(\mathbb{R}^3)$. Then formally

$$A = -\sum_{k,l=1}^{2} c_{kl} A_l\,A_k.$$

We have $D(A) \subset W^{1,2}(\mathbb{R}^3)$. This follows from the regularity of sub-elliptic operators on Lie groups associated to unitary representations. Specifically it follows from [ER98, Theorem 9.2.II] together with [ER94, Lemma 6.1] and [ER94, Theorem 7.2.(VI and V)] applied to the standard representation of the covering group of the Euclidean motion group (cf. [DER03, Example II.5.1]).

Hence $C_c^\infty(\mathbb{R}^3)$ is a core for A by Theorem 4.4.

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