Comprehensive Analysis of Uniform Discrete Truncated Random Variables

by

Dr Stevan Berber

Faculty of Engineering Report

Department of Electrical and Computer Engineering
The University of Auckland

November 2015

THE UNIVERSITY OF AUCKLAND
NEW ZEALAND

Abstract

This Report presents a brief review of uniform continuous density functions, both un-truncated and truncated, which are well described in existing literature. Then the discrete density function is derived and expressed in terms of Dirac's delta functions and related mean and variance are derived and analyzed. The necessity of having truncated discrete density function, from the application point of view in communication systems, for example, is explained and related density and distribution functions are derived. For these functions, the mean and variance are expressed as functions of the length of the defined truncation interval and compared with related moments of the continuous truncated density function. The important advancement is achieved by deriving the truncated discrete density functions and expressing them in terms of Dirac's delta and unit step functions. In this way it became possible to solve integrals which contain these density and distribution functions. Analyses of density functions with zero mean are repeated in the Appendix 1 for the case when the mean has a finite value.

Contents

1. Continuous uniform density function

1.1 Continuous density and distribution functions

The density and distribution function of a uniform continuous random variable τ can be expressed as

$$f_c(\tau) = \begin{Bmatrix} \frac{1}{2T_c} & -T_c \le \tau < T_c \\ 0 & otherwise \end{Bmatrix}, \tag{1}$$

$$F_c(\tau) = \int_{-T_c}^{\tau} \frac{1}{2T_c} dx = \begin{Bmatrix} \frac{1}{2T_c}(\tau + T_c) & -T_c \le \tau < T_c \\ 0 & otherwise \end{Bmatrix} \tag{2}$$

and graphically presented, as shown in Fig. 1, for the mean value equal zero and the variance σ_c^2.

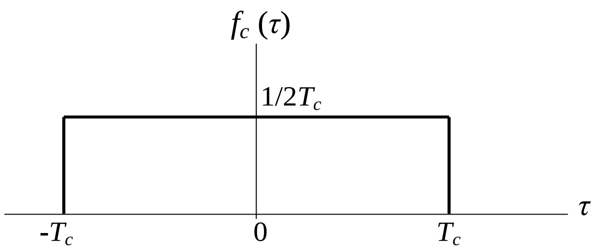

Fig. 1 Continuous uniform density function.

1.2 Moments of the uniform continuous distribution

The mean value is zero and the variance can be obtained as follows

$$\eta_c = E\{\tau\} = \int_{\tau=-\infty}^{\infty} \tau \cdot f_d(\tau) d\tau = \int_{\tau=-T_c}^{T_c} \tau \frac{1}{2T_c} d\tau = \frac{1}{2T_c}\left(\frac{T_c^2}{2} - \frac{T_c^2}{2}\right) = 0 \tag{3}$$

$$E\{\tau^2\} = \int_{\tau=-\infty}^{\infty} \tau^2 \cdot f_d(\tau) d\tau = \int_{\tau=-T_c}^{T_c} \tau^2 \frac{1}{2T_c} d\tau = \frac{1}{2T_c}\left(\frac{T_c^3}{3} + \frac{T_c^3}{3}\right) = \frac{T_c^2}{3} \tag{4}$$

$$\sigma_c^2 = E\{\tau^2\} - \eta_c^2 = \frac{T_c^2}{3} \tag{5}$$

Now, we may express the density function in terms of the mean and variance values. The T_c interval is

$$T_c = \sqrt{3\sigma_c^2} = \sigma_c\sqrt{3} \tag{6}$$

and the density function is

$$f_\tau(\tau) = \begin{cases} \dfrac{1}{2\sigma_c\sqrt{3}} & -T_c \le \tau < T_c \\ 0 & otherwise \end{cases} \tag{7}$$

Like in literature when the functions is defined in an arbitrary interval (a, b)

$$Var(\tau) = E\{\tau^2\} - \eta_c^2 = \frac{T_c^2}{3} = \frac{((b-a)/2)^2}{3} = \frac{(b-a)^2}{12}\,. \tag{8}$$

2. Truncated continuous uniform density function

Suppose the uniform density function is truncated having the values inside the interval ($-T_c + A \le \tau < T_c - A$), as shown in Fig. 2.

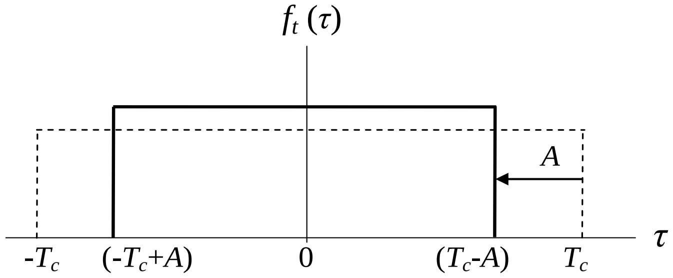

Fig. 2 Continuous truncated uniform density function

According to the definition of the truncated function the truncated uniform density and distribution function (1) can be expressed as

$$f_{ct}(\tau) = \frac{f_c(\tau)}{P(-T_c + A \le \tau < T_c - A)} = \frac{1/2T_c}{[(T_c - A) - (-T_c + A)]/2T_c}$$
$$= \frac{1/2T_c}{(2T_c - 2A)/2T_c} = \begin{Bmatrix} \dfrac{1}{2(T_c - A)} & -T_c + A \le \tau < T_c - A,\ A \ge 0 \\ 0 & otherwise \end{Bmatrix}, \tag{9}$$

$$F_{ct}(\tau) = \int_{-T_c + A}^{\tau} \frac{1}{2(T_c - A)} dx = \begin{Bmatrix} \dfrac{1}{2(T_c - A)}(\tau + T_c - A) & -T_c + A \le \tau < T_c - A \\ 0 & otherwise \end{Bmatrix}.$$

The mean value is zero and the variance, having in mind (9), can be obtained as follows

$$\sigma_{ct}^2 = E\{\tau^2\} = \int_{\tau=-\infty}^{\infty} \tau^2 \cdot f_d(\tau) d\tau = \int_{\tau=-T_c+A}^{T_c-A} \tau^2 \frac{1}{2(T_c - A)} d\tau$$
$$= \frac{1}{2(T_c - A)} \left(\frac{(T_c - A)^3}{3} - \frac{(-T_c + A)^3}{3} \right)$$
$$= \frac{1}{2(T_c - A)} \left(\frac{(T_c - A)(T_c - A)^2}{3} - \frac{(-T_c + A)(-T_c + A)^2}{3} \right) \tag{10}$$
$$= \frac{(T_c - A)^2}{3} = \frac{T_c^{\ 2}}{3}(1 - A/T_c)^2 = \sigma_c^2 (1 - A/T_c)^2$$

Now, we may express the density function in terms of the mean and variance values. We may have

$$\sigma_{ct}^2 = \frac{(T_c - A)^2}{3} \Rightarrow T_c - A = \sqrt{3\sigma_{ct}^2} = \sigma_{ct}\sqrt{3}\,, \tag{11}$$

And the density function is

$$f_{ct}(\tau) = \left\{ \begin{array}{cc} \dfrac{1}{2\sigma_{ct}\sqrt{3}} & -T_c + A \le \tau < T_c - A,\ A \ge 0 \\ 0 & otherwise \end{array} \right\}. \tag{12}$$

The problem of relating and comparing this density and its variance with the densities and variance of discrete density functions will be addressed in the later Sections.

3. Derivations for the discrete density and distribution functions

3.1 Discrete density and distribution functions

Uniform discrete density: If a uniform continuous density function, expressed as

$$f_c(\tau) = \begin{Bmatrix} \dfrac{1}{2T_c} & -T_c \le \tau < T_c \\ 0 & otherwise \end{Bmatrix},$$

is discretised in respect to τ, as shown in Fig. 3 with the interval of discretisation of T_s. the probability value in the first interval around zero, $n = 0$, is

$$P\{-T_s/2 \le \tau < T_s/2\} = \frac{1}{2T_c}T_s\,. \tag{13}$$

For the first positive value $n = 1$ the probability is

$$P\{T_s - T_s/2 \le \tau < T_s - T_s/2\} = \frac{1}{2T_c}T_s\,, \tag{14}$$

and for the first negative value $n = -1$ is

$$P\{-T_s - T_s/2 \le \tau < -T_s + T_s/2\} = \frac{1}{2T_c}T_s\,. \tag{15}$$

For any interval defined by n the probability can be calculated as

$$P\{(2n-1)T_s/2 \le \tau < (2n+1)T_s/2\} = \frac{1}{2T_c}T_s\,. \tag{16}$$

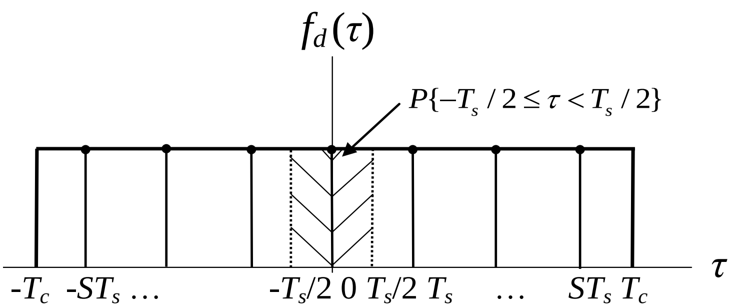

Fig. 3 Discretisation of an uniform density function.

These probabilities can be understood as the weights of Dirac's delta functions that define the discrete density function, which can be expressed as

$$f_d(\tau) = \sum_{n=-S}^{S} \frac{T_s}{2T_c} \cdot \delta(\tau - nT_s). \tag{17}$$

In the case the number of positive and negative discrete intervals is *S*, the whole interval is

$$2T_c = 2ST_s + T_s, \tag{18}$$

And the relations between the values T_c, T_s and *S*, which will be used later, can be found in these forms

$$\frac{2T_c}{T_s} = 2S+1, \; T_c = (2S+1)T_s/2, \; S = \frac{2T_c - T_s}{2T_s} = \frac{T_c}{T_s} - \frac{1}{2}. \tag{19}$$

Now, based on (16) and (19) the probability that the random variable is in the *n*-th interval can be expressed as

$$P\{(2n-1)T_s/2 \le \tau < (2n+1)T_s/2\} = \frac{T_s}{2T_c} = \frac{1}{(2S+1)} \tag{20}$$

Therefore, the discrete density and distribution functions can be expressed as

$$f_d(\tau) = \sum_{n=-S}^{n=S} \frac{1}{2S+1} \cdot \delta(\tau - nT_s),$$

$$F_d(\tau) = \sum_{n=-S}^{\tau/T_s} \frac{1}{2S+1} U(\tau - nT_s) \tag{21}$$

and, for a unit interval $T_s = 1$, it is

$$f_d(\tau) = \sum_{n=-S}^{n=S} \frac{1}{2S+1} \cdot \delta(\tau - n) \tag{22}$$

This function is presented in Fig. 4 for *S* =3.

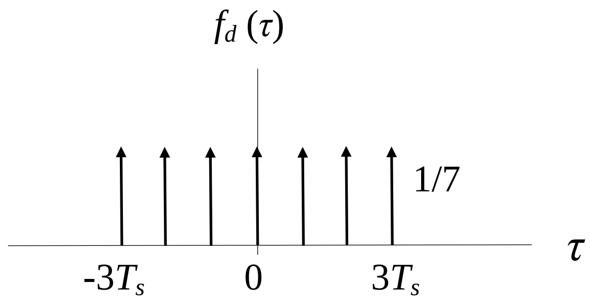

Fig. 4 Discrete uniform density function presented using Dirac's delta functions

Proposition: Function (21) fulfils condition to be a density function. The sum of terms in is one.The distribution function fulfils condition of a dis

Proof:

$$\sum_{n=-S}^{n=S} \frac{1}{2S+1} = \frac{1}{2S+1}(2S+1) = 1$$

3.2 Comments on the discretisation procedure

The discretisation procedure applied in the previous Section was motivated by these reasons:

1. Preservation of the symmetry: The discrete density function is obtained by assigning probability values as the weights of Dirac's delta functions that are placed in the middle of the sampling interval. In this way the discrete density function preserved symmetry in respect to *y*-axis and the mean value remained to be zero as in the case of the corresponding continuous function.

2. Preservation of the value of the sampling interval T_S: Representing the density function in terms of T_c and T_s, and relating them to the discrete sampling interval $(2S+1)T_s$ it is easy to reconstruct the sampling interval and relate it to the real values in practical application. For example, in the case of defining delay in communication systems these sampling intervals will be expressed in appropriate time units.

3. Expression of density functions in closed form: By using Dirac's delta functions, and possibly Kronecker's functions, the obtained density function of a random variable can be easily used to calculate the mean values of the functions which have that variable as an argument. In those cases it is simple to solve the integral that defines the mean value of a function.

Other possible discretisation can be used, as presented in Fig. 3, for example. Two cases can be distinguished:

1. The calculated probability in T_s interval (for example shaded interval in Fig. 5) can be assigned as the weight of the left of the interval resulting in discrete values presented in black colour in Fig. 5. This procedure should be repeated 2S time and a stream of samples can be obtained that starts at $-ST_s$ and finishes at $(S\text{-}1)T_s$. in this case two issues have to be mentioned. Firstly, a mean value will exist for the discrete random variable which is result of discretisation and does not exist in the continuous density function. Secondly, continuous random values inside particular interval will be assigned to the lowest value of the interval which will reduce statistical accuracy in generating random variates in the case of simulation.

2. This case is similar to the previous. In this case the calculated probability in T_s interval (for example shaded interval in Fig. 5) is assigned as the weight of the discrete value on the right of the interval, and the discrete values are shifted to the right (represented by arrows in Fig. 5). This procedure has the same characteristics as the previous one.

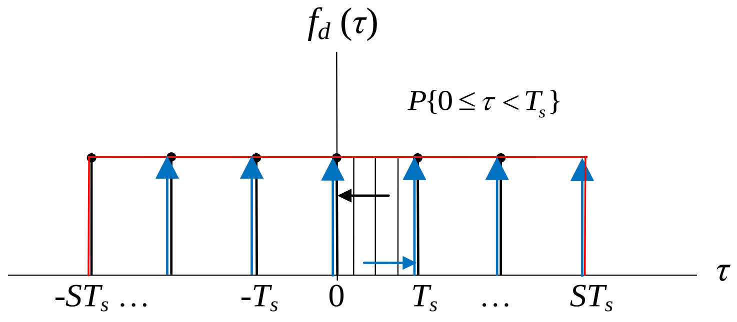

Fig. 5 Possible discretisation of a uniform density function.

3.3 Derivations of the moments for the discrete random variable

Proposition: The mean, mean square and variance are expressed as

$$\eta_d = 0, \tag{23}$$

$$E\{\tau^2\} = T_s^2 \frac{S(S+1)}{3} \tag{24}$$

$$\sigma_d^2 = E\{\tau^2\} - \eta_d^2 = E\{\tau^2\} = T_s^2 \frac{S(S+1)}{3}, \tag{25}$$

and for the unit interval $T_s = 1$, they are

$$E\{\tau^2\} = \frac{S(S+1)}{3} = \sigma_d^2 \tag{26}$$

Proofs: The mean of the discrete density function is

$$\begin{aligned}\eta_d &= \int_{\tau=-\infty}^{\infty} \tau \cdot f_d(\tau) d\tau = \int_{\tau=0}^{\infty} \tau \cdot \sum_{n=-S}^{n=S} \frac{1}{2S+1} \cdot \delta(\tau - nT_s) d\tau = \sum_{n=-S}^{n=S} \frac{1}{2S+1} \cdot \int_{\tau=T_s}^{\infty} \tau \cdot \delta(\tau - nT_s) d\tau \\ &= \frac{1}{2S+1} \sum_{n=-S}^{n=S} nT_s = T_s \frac{1}{2S+1} \left(\sum_{n=-S}^{n=-1} n + \sum_{n=1}^{n=S} n \right) = 0\end{aligned} \tag{27}$$

The mean square value is

$$\begin{aligned}E\{\tau^2\} &= \int_{\tau=-\infty}^{\infty} \tau^2 \cdot f_d(\tau) d\tau = \sum_{n=-S}^{n=S} \frac{1}{2S+1} \cdot \int_{\tau=-\infty}^{\infty} \tau^2 \cdot \delta(\tau - nT_s) d\tau = \frac{T_s^2}{2S+1} \sum_{n=-S}^{n=S} n^2 \\ &= \frac{2T_s^2}{2S+1} \sum_{n=1}^{n=S} n^2 = \frac{2T_s^2}{2S+1} \frac{S(S+1)(2S+1)}{6} = T_s^2 \frac{S(S+1)}{3}\end{aligned} . \tag{28}$$

The variance is

$$\sigma_d^2 = E\{\tau^2\} - \eta_d^2 = E\{\tau^2\} = T_s^2 \frac{S(S+1)}{3}, \tag{29}$$

which, for a unit sampling interval, $T_s = 1$, becomes

$$\sigma_d^2 = \frac{S(S+1)}{3}. \tag{30}$$

The variance can be expressed in terms of the variance of the continuous density. Having in mind that

$$T_c = (2S+1)T_s / 2 \Rightarrow T_s = 2T_c / (2S+1), \tag{31}$$

and the variance expression of the continuous density

$$\sigma_c^2 = E\{\tau^2\} - \eta_c^2 = \frac{T_c^2}{3} \qquad (32)$$

the variance of the discrete density is

$$\sigma_d^2 = T_s^2 \frac{S(S+1)}{3} = \frac{4T_c^2}{(2S+1)^2} \frac{S(S+1)}{3} = \frac{T_c^2}{3} \frac{4S(S+1)}{(2S+1)^2} = \sigma_c^2 \frac{4S(S+1)}{(2S+1)^2} \qquad (33)$$

4. Truncated discrete uniform density and distribution functions

4.1 Derivation of the truncated density and distribution function

In practical application the discrete delays are taking values in a limited interval defined as the truncated interval ($-S + a$, $S - a$), where $a \leq S$ is a positive whole number named truncation factor. Therefore, the function which describes the delay distribution is truncated and has the values in the truncated interval, as shown in Fig. 6.

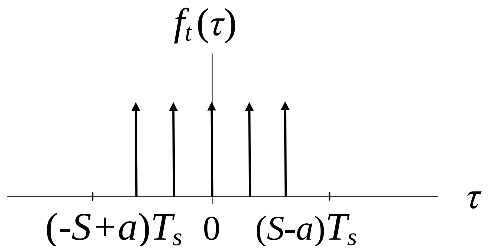

Fig. 6 Discrete truncated uniform density function presented using Dirac's delta functions.

Proposition: The density and distribution functions are given in closed by these expressions form

$$f_t(\tau) = \frac{1}{2(S-a)+1} \sum_{n=-S+a}^{n=S-a} \delta(\tau - nT_s) . \tag{34}$$

$$F_t(\tau) = \frac{1}{2(S-a)+1} \sum_{n=-S+a}^{\tau} U(\tau - nT_s) . \tag{35}$$

Proof: Based on the definition of a truncated density function as a conditional density function, the truncated discrete uniform density function can be expressed as

$$f_t(\tau) = f_d(\tau \mid -S+a \leq \tau \leq S-a) = \frac{f_d(\tau)}{P(-S+a \leq \tau \leq S-a)}$$
$$= \frac{\sum_{n=-S+a}^{S-a} \frac{1}{2S+1} \cdot \delta(\tau - nT_s)}{\sum_{n=-S+a}^{S-a} \frac{1}{2S+1}} = \frac{\sum_{n=-S+a}^{S-a} \frac{1}{2S+1} \cdot \delta(\tau - nT_s)}{P(S)} . \tag{36}$$

The value $P(S)$ can be calculated as

$$P(S)=\sum_{n=-S+a}^{S-a}\frac{1}{2S+1}=\frac{1}{2S+1}(S-a+S-a+1)=\frac{2S-2a+1}{2S+1} \tag{37}$$

By inserting this expression into (41), the density function can be expressed in this closed form, as stated in the proposition, i.e.,

$$f_t(\tau)=\frac{2S+1}{2S-2a+1}\sum_{n=-S+a}^{S-a}\frac{1}{2S+1}\cdot\delta(\tau-nT_s)=\frac{1}{2S-2a+1}\sum_{n=-S+a}^{S-a}\delta(\tau-nT_s)\,, \tag{38}$$

and the related distribution function can be obtained by integration the last expression as

$$F_t(\tau)=\frac{1}{2S-2a+1}\sum_{n=-S+a}^{\tau/T_s}U(\tau-nT_s)\,. \tag{39}$$

4.2 Mean and variance of the truncated discrete uniform density function

Proposition: The mean of the discrete truncated random variable is zero.

Proof: Based on the expression (38) for the discrete density function we may have

$$\eta_t=\int_{-\infty}^{\infty}\tau f_\tau(\tau)d\tau=\frac{1}{2S-2a+1}\sum_{n=-S+a}^{S-a}\int_{-\infty}^{\infty}\tau\cdot\delta(\tau-nT_s)d\tau=\frac{T_s}{2S-2a+1}\sum_{n=-S+a}^{S-a}n=0 \tag{40}$$

Proposition: The mean squared value of the discrete truncated random variable is

$$E\{\tau^2\}=T_s^2\,\frac{(S-a)(S-a+1)}{3}=\sigma_t^2 \tag{41}$$

Proof: Based on the expression (38) for the discrete density function we may have

$$\begin{aligned}E\{\tau^2\}&=\int_{-\infty}^{\infty}\tau^2 f_\tau(\tau)d\tau=\frac{1}{2S-2a+1}\sum_{n=-S+a}^{S-a}\int_{-\infty}^{\infty}\tau^2\delta(\tau-nT_s)d\tau=\frac{T_s^2}{2S-2a+1}\sum_{n=-S+a}^{S-a}n^2\\&=\frac{2T_s^2}{2S-2a+1}\sum_{n=1}^{S-a}n^2=\frac{2T_s^2}{2S-2a+1}\,\frac{(S-a)(S-a+1)(2(S-a)+1)}{6}=T_s^2\,\frac{(S-a)(S-a+1)}{3}\end{aligned}\,. \tag{42}$$

The variance can be calculated form (41) and (42) as

$$\sigma_t^2 = E\{\tau^2\} - \mu_t^2 = T_s^2 \frac{(S-a)(S-a+1)}{3}, \tag{43}$$

which completes our proofs. For a unit sampling interval, $T_s = 1$, we can easily get expression for the variance.

The variance (43) can be expressed in terms of the variance of the continuous density. By inserting the expression for T_s from (19) and then expression (5), we may have the truncated variance as a function of continuous in this form

$$\sigma_t^2 = E\{\tau^2\} - \mu_t^2 = T_s^2 \frac{(S-a)(S-a+1)}{3} = \frac{T_c^2}{3} \frac{4(S-a)(S-a+1)}{(2S+1)^2} = \sigma_c^2 \frac{4(S-a)(S-a+1)}{(2S+1)^2}. \tag{44}$$

5. Comparison of variances

We are interest in statistical characteristics of these distributions and their mutual interrelationship. For that purpose we will compare their variances. As a reference we will use the variance of continuous density. All other variances will be expressed in terms of this variance. The variances of the discrete density and truncated discrete density are already expressed as functions of the variance of continuous density, which is presented in (33) and (). and of the continuous density. Having in mind

$$T_c = (2S+1)T_s / 2 \Rightarrow T_s = 2T_c / (2S+1) \tag{45}$$

and the variance expression of continuous density,

$$\sigma_c^2 = E\{\tau^2\} - \eta_c^2 = \frac{T_c^2}{3}, \tag{46}$$

the variance of the discrete density can be found as

$$\sigma_d^2 = T_s^2 \frac{S(S+1)}{3} = \frac{4T_c^2}{(2S+1)^2} \frac{S(S+1)}{3} = \frac{T_c^2}{3} \frac{4S(S+1)}{(2S+1)^2} = \sigma_c^2 \frac{4S(S+1)}{(2S+1)^2} \tag{47}$$

$$\sigma_t^2 = T_s^2 \frac{(S-a)(S-a+1)}{3} = \frac{4T_c^2}{(2S+1)^2} \frac{(S-a)(S-a+1)}{3}$$
$$= \frac{T_c^2}{3} \frac{4(S-a)(S-a+1)}{(2S+1)^2} = \sigma_c^2 \frac{4(S-a)(S-a+1)}{(2S+1)^2}$$

(48)

The continuous truncated variance is

$$\sigma_{ct}^2 = \sigma_c^2 (1 - A / T_c)^2 . \tag{49}$$

In order to compare this variance with the variance of the truncated discrete density function, both of them need to be calculated the same truncation interval. Therefore, the ratio A/T_c, which would correspond to the truncated value of the discrete function, should be found. The corresponding truncation intervals for continuous and discrete density can be found from the truncating probabilities. If we take them from (36) –(37) and (9) and equate them we can get

$$\frac{2S-2a+1}{2S+1} = \frac{2T_c - 2A}{2T_c}, \tag{50}$$

as illustrated in Fig. 7. From (5) we may have

$$(1 - \frac{A}{T_c}) = 1 - \frac{2a}{2S+1} \tag{51}$$

which can be inserted in (49) to get the variance of truncated continuous density in this form

$$\sigma_{ct}^2 = \sigma_c^2 (1 - \frac{2a}{2S+1})^2 .$$

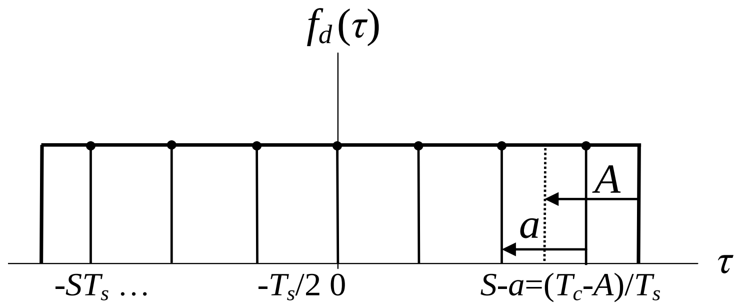

Fig. 7 Relations between truncation intervals of the discrete and continuous truncated uniform density function.

Calculated expressions for the variances as functions of the variance of continuous uniform variable are presented in Table 1 in the form to clearly see relationship between truncated and non-truncated variances. Namelly, the variance of truncated density function should be les tha or equal to the variance of continuous random variable.

Table 1 Variance expressions

Uniform distributions	**Variances**
Continuous	$\sigma_c^2 = \frac{T_c^2}{3}$
continuous truncated	$\sigma_{ct}^2 = \sigma_c^2 \left(\frac{2S-2a+1}{2S+1} \right)^2 = \sigma_c^2 \left(1 - \frac{2a}{2S+1} \right)^2$
Discrete	$\sigma_d^2 = \sigma_c^2 \frac{4S(S+1)}{(2S+1)^2} = \sigma_c^2 \left(1 - \frac{1}{(2S+1)^2} \right) =$
Discrete truncated	$\sigma_t^2 = \sigma_c^2 \frac{4(S-a)(S-a+1)}{(2S+1)^2} = \sigma_c^2 \left(1 - \frac{1-4a(2S-a+1)}{(2S+1)^2} \right)$

The graphs for the variances of the continuous, continuous truncated, discrete and discrete truncated random variables as functions of the variance of the continuous random variable, for the truncation factor *a* and sampling interval 2*S* as parameters, are presented in Fig. 8. The non-truncated continuous and discrete random variables have very similar, nearly the same variance, as we can expect, because they are calculated on the same interval of possible values of random variable. The small difference occurs due to that the variances of continuous density are calculated on the continuum of random variable values while the variance of discrete density are calculated for a finite number of discrete values.

The variances of the truncated density functions are smaller than the variances of the non-truncated functions due to the truncation of the function which is defined by truncation factors *A* and *a*. The higher these factors are the smaller gradients of these curves are and higher the difference is between variances of truncated and non-truncated densities.

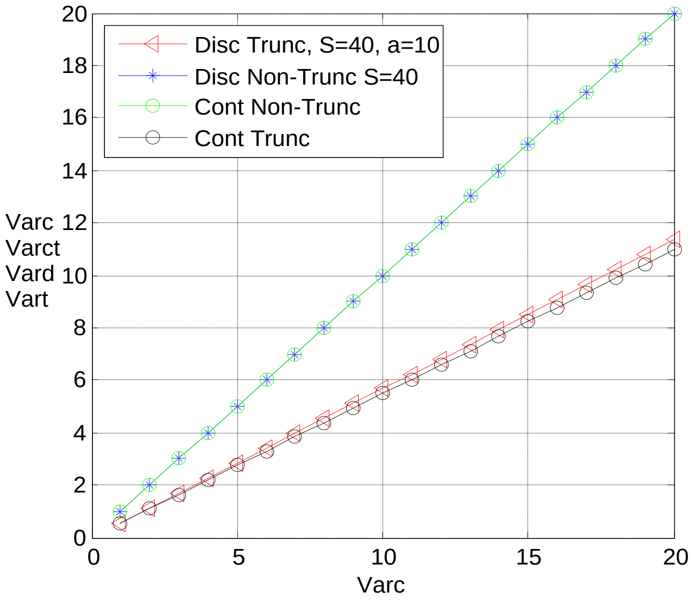

Fig. 8 Continuous, continuous truncated, discrete and discrete truncated variances versus continuous variance.

The variances of the continuous and discrete random variable are nearly the same, because the interval of their existence is nearly the same. In contrast to that, the truncation values are inside a narrower interval and consequently the variance of the truncated density functions is smaller than those of non-truncated.

6. Conclusions

In this Report a uniform discrete truncated density function is derived and investigated. The function is expressed in closed form in terms of Dirac's delta functions. Expressions for the first and second moments are derived. If complete discretisation is necessary, it is possible to express the density and distribution function in terms of Kronecker's delta function and discrete unit step functions.

APENDIX 1 Density and distribution functions for a finite value of the mean

1. Continuous uniform density function

1.1 Continuous density and distribution functions

The density and distribution function of a uniform continuous random variable τ with a finite mean can be expressed as

$$f_c(\tau) = \begin{Bmatrix} \frac{1}{2T_c} & \eta - T_c \le \tau < \eta + T_c \\ 0 & otherwise \end{Bmatrix}, \quad (1)$$

$$F_c(\tau) = \int_{\eta - T_c}^{\tau} \frac{1}{2T_c} dx = \begin{Bmatrix} \frac{1}{2T_c}(\tau - \eta + T_c) & \eta - T_c \le \tau < \eta + T_c \\ 0 & otherwise \end{Bmatrix} \quad (2)$$

and graphically presented, as shown in Fig. 1, for the mean value equal zero and the variance σ_c^2.

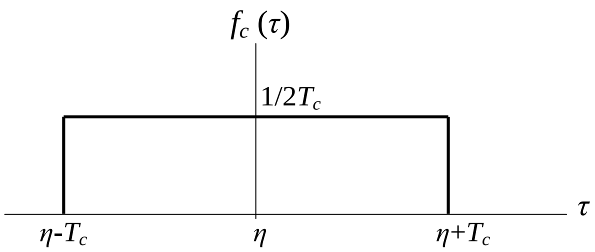

Fig. 1 Continuous uniform density function.

1.2 Moments of the uniform continuous distribution

The mean value is zero and the variance can be obtained as follows

$$\eta_c = E\{\tau\} = \int_{\tau=-\infty}^{\infty} \tau \cdot f_d(\tau) d\tau = \int_{\tau=\eta-T_c}^{\eta+T_c} \tau \frac{1}{2T_c} d\tau = \frac{1}{2T_c}\left(\frac{(\eta+T_c)^2}{2} - \frac{(\eta-T_c)^2}{2} \right) = \frac{1}{2T_c} 2T_c \eta = \eta \tag{3}$$

$$\begin{aligned} E\{\tau^2\} &= \int_{\tau=\eta-T_c}^{\eta+T_c} \tau^2 \frac{1}{2T_c} d\tau = \frac{1}{2T_c}\left(\frac{(\eta+T_c)^3}{3} - \frac{(\eta-T_c)^3}{3} \right) = \frac{1}{2T_c}\left(\frac{(\eta+T_c)^3}{3} - \frac{(\eta-T_c)^3}{3} \right) \\ &= \frac{1}{2T_c}\left(\frac{\eta^3 + 3\eta^2 T_c + 3\eta T_c^2 + T_c^3}{3} - \frac{\eta^3 - 3\eta^2 T_c + 3\eta T_c^2 - T_c^3}{3} \right) = \frac{1}{2T_c} \frac{6\eta^2 T_c + 2T_c^3}{3} \\ &= \frac{1}{2T_c} \frac{2T_c(3\eta^2 + T_c^2)}{3} = \frac{3\eta^2 + T_c^2}{3} = \eta^2 + \frac{T_c^2}{3} \end{aligned} \tag{4}$$

$$\sigma_c^2 = E\{\tau^2\} - \eta_c^2 = \eta^2 + \frac{T_c^2}{3} - \eta_c^2 = \frac{T_c^2}{3} \tag{5}$$

Now, we may express the density function in terms of the mean and variance values. The T_c interval is

$$T_c = \sqrt{3\sigma_c^2} = \sigma_c \sqrt{3} \tag{6}$$

and the density function is

$$f_c(\tau) = \left\{ \begin{matrix} \dfrac{1}{2\sigma_c \sqrt{3}} & \eta - T_c \le \tau < \eta + T_c \\ 0 & otherwise \end{matrix} \right\} \tag{7}$$

2. Truncated continuous uniform density function

Suppose the uniform density function is truncated having the values inside the interval ($\eta - T_c + A \le \tau < \eta + T_c - A$), as shown in Fig. 2.

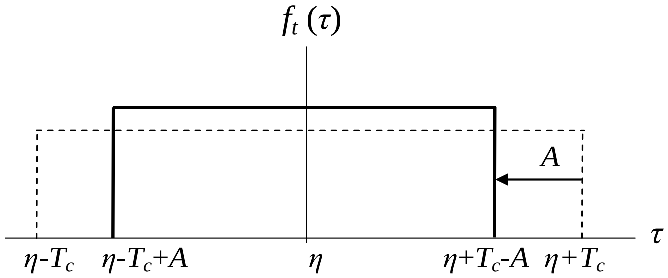

Fig. 2 Continuous truncated uniform density function

According to the definition of the truncated function the truncated uniform density and distribution function (1) can be expressed as

$$f_{ct}(\tau) = \frac{f_c(\tau)}{P(\eta - T_c + A \le \tau < \eta + T_c - A)} = \frac{1/2T_c}{[(\eta + T_c - A) - (\eta - T_c + A)]/2T_c}$$
$$= \frac{1/2T_c}{(2T_c - 2A)/2T_c} = \begin{cases} \dfrac{1}{2(T_c - A)} & \eta - T_c + A \le \tau < \eta + T_c - A,\ A \ge 0 \\ 0 & otherwise \end{cases}, \qquad (8)$$

$$F_{ct}(\tau) = \int_{\eta - T_c + A}^{\tau} \frac{1}{2(T_c - A)} dx = \begin{cases} \dfrac{1}{2(T_c - A)}(\tau - \eta + T_c - A) & \eta - T_c + A \le \tau < \eta + T_c - A \\ 0 & otherwise \end{cases}.$$

The mean value is zero and the variance, having in mind (9), can be obtained as follows

$$E\{\tau^2\} = \int_{\tau=-\infty}^{\infty} \tau^2 \cdot f_d(\tau) d\tau = \int_{\tau=\eta - T_c + A}^{\eta + T_c - A} \tau^2 \frac{1}{2(T_c - A)} d\tau = \frac{1}{2(T_c - A)} \left(\frac{(\eta + (T_c - A))^3}{3} - \frac{(\eta - (T_c - A))^3}{3} \right)$$
$$= \frac{1}{2(T_c - A)} \left(\frac{\eta^3 + 3\eta^2 (T_c - A) + 3\eta (T_c - A)^2 + (T_c - A)^3}{3} - \frac{\eta^3 - 3\eta^2 (T_c - A) + 3\eta (T_c - A)^2 - (T_c - A)^3}{3} \right) \quad (9)$$
$$= \frac{1}{2(T_c - A)} \left(\frac{6\eta^2 (T_c - A) + 2(T_c - A)^3}{3} \right) = \frac{3\eta^2 + (T_c - A)^2}{3} = \eta^2 + \frac{(T_c - A)^2}{3}$$

Now, we may express the density function in terms of the mean and variance values. We may have

$$\sigma_{ct}^2 = \frac{(T_c - A)^2}{3} \Rightarrow T_c - A = \sqrt{3\sigma_{ct}^2} = \sigma_{ct}\sqrt{3}\,, \tag{10}$$

And the density function is

$$f_{ct}(\tau) = \left\{ \begin{array}{ll} \frac{1}{2\sigma_{ct}\sqrt{3}} & \eta - T_c + A \le \tau < \eta + T_c - A,\ A \ge 0 \\ 0 & otherwise \end{array} \right\}. \tag{11}$$

The problem of relating and comparing this density and its variance with the densities and variance of discrete density functions will be addressed in the later Sections.

3. Derivations for the discrete density and distribution functions

3.1 Discrete density and distribution functions

Uniform discrete density: If a uniform continuous density function, expressed as

$$f_c(\tau) = \begin{Bmatrix} \frac{1}{2T_c} & \eta - T_c \le \tau < \eta + T_c \\ 0 & otherwise \end{Bmatrix}, \tag{12}$$

is discretised in respect to τ, as shown in Fig. 3 with the interval of discretisation of T_s. the probability value in the first interval around the mean value, n = 0, is

$$P\{\eta - T_s/2 \le \tau < \eta + T_s/2\} = \frac{1}{2T_c} T_s . \tag{13}$$

For the first positive value n = 1 the probability is

$$P\{\eta + T_s - T_s/2 \le \tau < \eta + T_s + T_s/2\} = \frac{1}{2T_c} T_s , \tag{14}$$

and for the first negative value n = -1 is

$$P\{\eta - T_s - T_s/2 \le \tau < \eta - T_s + T_s/2\} = \frac{1}{2T_c} T_s . \tag{15}$$

For any interval defined by n the probability can be calculated as

$$P\{\eta + (2n-1)T_s/2 \le \tau < \eta + (2n+1)T_s/2\} = \frac{1}{2T_c} T_s . \tag{16}$$

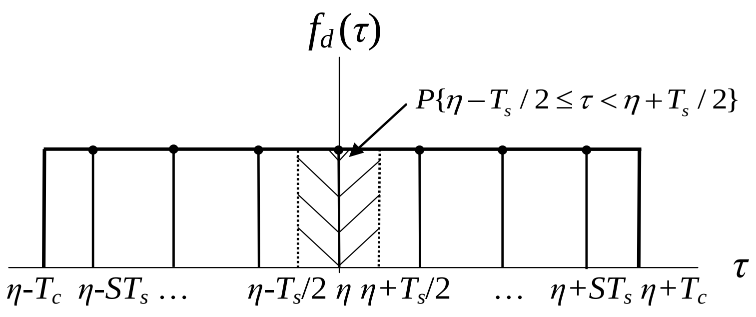

Fig. 3 Discretisation of uniform density function.

These probabilities can be understood as the weights of Dirac's delta functions that define the discrete density function, which can be expressed as

$$f_d(\tau) = \sum_{n=-S}^{S} \frac{T_s}{2T_c} \cdot \delta\big(\tau - (\eta + nT_s)\big) . \tag{17}$$

In the case the number of positive and negative discrete intervals is S, the whole interval is

$$2T_c = 2ST_s + T_s , \tag{18}$$

And the relations between the values T_c, T_s and S, which will be used later, can be found in these forms

$$\frac{2T_c}{T_s} = 2S + 1 , \; T_c = (2S+1)T_s / 2 , \; S = \frac{2T_c - T_s}{2T_s} = \frac{T_c}{T_s} - \frac{1}{2} . \tag{19}$$

Now, based on (16) and (19) the probability that the random variable is in the n-th interval can be expressed as

$$P\{\eta + (2n-1)T_s / 2 \le \tau < \eta + (2n+1)T_s / 2\} = \frac{T_s}{2T_c} = \frac{1}{(2S+1)} \tag{20}$$

Therefore, the discrete density and distribution functions can be expressed as

$$f_d(\tau) = \sum_{n=-S}^{n=S} \frac{1}{2S+1} \cdot \delta(\tau - \eta - nT_s) , \tag{20}$$

$$F_d(\tau) = \sum_{n=-S}^{\tau/T_s} \frac{1}{2S+1} \cdot U(\tau - \eta - nT_s) \tag{21}$$

and, for a unit interval $T_s = 1$, it is

$$f_d(\tau) = \sum_{n=-S}^{n=S} \frac{1}{2S+1} \cdot \delta(\tau - \eta - n) \tag{22}$$

This function is presented in Fig. 4 for S =3.

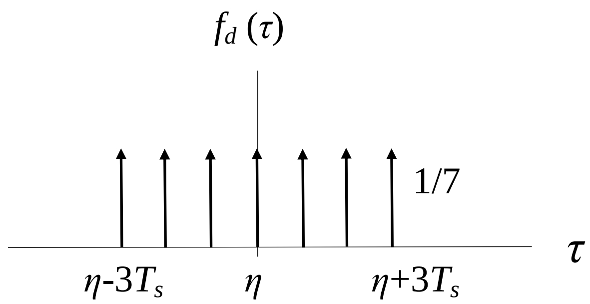

Fig. 4 Discrete uniform density function presented using Dirac's delta functions

3.2 Comments on the discretisation procedure

The discretisation procedure applied in the previous Section was motivated by these reasons:

1. Preservation of the symmetry: The discrete density function is obtained by assigning probability values as the weights of Dirac's delta functions that are placed in the middle of the sampling interval. In this way the discrete density function preserved symmetry in respect to the mean value as in the case of the corresponding continuous function.

2. Preservation of the value of the sampling interval T_s: Representing the density function in terms of T_c and T_s, and relating them to the discrete sampling interval $(2S+1)T_s$ it is easy to reconstruct the sampling interval and relate it to the real values in practical application. For example, in the case of defining delay in communication systems these sampling intervals will be expressed in appropriate time units.

3. Expression of density functions in closed form: By using Dirac's delta functions, and possibly Kronecker's functions, the obtained density function of a random variable can be easily used to calculate the mean values of the functions which have that variable as an argument. In those cases it is simple to solve the integral that defines the mean value of a function.

Other possible discretisation can be used, as presented in Fig. 3, for example. Two cases can be distinguished:

3. The calculated probability in T_s interval (for example shaded interval in Fig. 5) can be assigned as the weight of the left of the interval resulting in discrete values presented in black colour in Fig. 5. This procedure should be repeated 2S times and a stream of samples can be obtained that starts at $-ST_s$ and finishes at $(S\text{-}1)T_s$. In this case two issues have to be mentioned. Firstly, the mean value will be different from the mean of continuous density which is the result of discretisation. Secondly, continuous random values inside particular interval will be assigned to the lowest value of the interval which will reduce statistical accuracy in generating random variates in the case of simulation.
4. This case is similar to the previous. In this case the calculated probability in T_s interval (for example shaded interval in Fig. 5) is assigned as the weight of the discrete value on the right of the interval, and the discrete values are shifted to the right (represented by arrows in Fig. 5). This procedure has the same characteristics as the previous one.

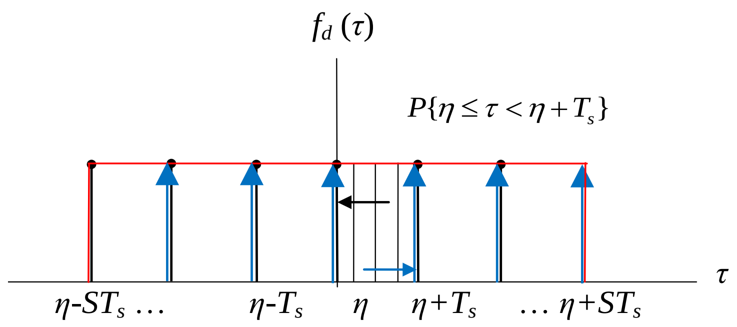

Fig. 5 Possible discretisation of a uniform density function.

3.3 Derivations of the moments for the discrete random variable

Proposition: The mean, mean square and variance are expressed as

$$\eta_d = \eta, \tag{23}$$

$$E\{\tau^2\} = \eta^2 + T_s^2 \frac{S(S+1)}{3} \tag{24}$$

$$\sigma_d^2 = E\{\tau^2\} - \eta_d^2 = E\{\tau^2\} = T_s^2 \frac{S(S+1)}{3}, \tag{25}$$

and for the unit interval $T_s = 1$, they are

$$E\{\tau^2\} = \eta^2 + \frac{S(S+1)}{3} \tag{26}$$

Proofs: The mean of the discrete density function is

$$\begin{aligned}\eta_d &= \int_{\tau=-\infty}^{\infty} \tau \cdot f_d(\tau) d\tau = \sum_{n=-S}^{n=S} \frac{1}{2S+1} \cdot \int_{\tau=-T_s}^{\infty} \tau \cdot \delta(\tau - \eta - nT_s) d\tau \\ &= \frac{1}{2S+1} \sum_{n=-S}^{n=S} (\eta + nT_s) = \eta + T_s \frac{1}{2S+1} \left(\sum_{n=-S}^{n=-1} n + \sum_{n=1}^{n=S} n \right) = \eta\end{aligned} \tag{27}$$

The mean square value is

$$\begin{aligned}E\{\tau^2\} &= \int_{\tau=-\infty}^{\infty} \tau^2 \cdot f_d(\tau) d\tau = \sum_{n=-S}^{n=S} \frac{1}{2S+1} \cdot \int_{\tau=-\infty}^{\infty} \tau^2 \delta(\tau - \eta - nT_s) d\tau = \frac{1}{2S+1} \sum_{n=-S}^{n=S} (\eta + nT_s)^2 \\ &= \eta^2 + \frac{2\eta T_s}{2S+1} \sum_{n=-S}^{n=S} n + \frac{2T_s^2}{2S+1} \sum_{n=1}^{n=S} n^2 = \eta^2 + \frac{2T_s^2}{2S+1} \frac{S(S+1)(2S+1)}{6} = \eta^2 + T_s^2 \frac{S(S+1)}{3}\end{aligned} \tag{28}$$

The variance is

$$\sigma_d^2 = E\{\tau^2\} - \eta_d^2 = E\{\tau^2\} = T_s^2 \frac{S(S+1)}{3}, \tag{29}$$

which, for a unit sampling interval, $T_s = 1$, becomes

$$\sigma_d^2 = \frac{S(S+1)}{3}. \tag{30}$$

The variance can be expressed in terms of the variance of the continuous density. Having in mind that

$$T_c = (2S+1)T_s / 2 \Rightarrow T_s = 2T_c / (2S+1), \tag{31}$$

and the variance expression of the continuous density

$$\sigma_c^2 = E\{\tau^2\} - \eta_c^2 = \frac{T_c^2}{3} \tag{32}$$

the variance of the discrete density is

$$\sigma_d^2 = T_s^2 \frac{S(S+1)}{3} = \frac{4T_c^2}{(2S+1)^2} \frac{S(S+1)}{3} = \frac{T_c^2}{3} \frac{4S(S+1)}{(2S+1)^2} = \sigma_c^2 \frac{4S(S+1)}{(2S+1)^2} \tag{33}$$

4. Truncated discrete uniform density and distribution functions

4.1 Derivation of the truncated density and distribution function

In practical application the discrete delays are taking values in a limited interval defined as the truncated interval (η-S + a, η+S - a), where $a \leq S$ is a positive whole number named truncation factor. Therefore, the function which describes the delay distribution is truncated and has the values in the truncated interval, as shown in Fig. 6.

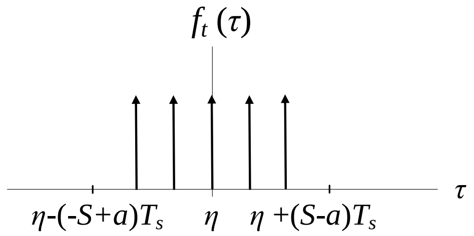

Fig. 6 Discrete truncated uniform density function presented using Dirac's delta functions.

Proposition: The density and distribution functions are given in closed by these expressions form

$$f_t(\tau) = \frac{1}{2(S-a)+1} \sum_{n=-S+a}^{n=S-a} \delta(\tau - \eta - nT_s) . \tag{34}$$

$$F_t(\tau) = \frac{1}{2(S-a)+1} \sum_{n=-S+a}^{\tau/T_s} U(\tau - \eta - nT_s) . \tag{35}$$

Proof: Based on the definition of a truncated density function as a conditional density function, the truncated discrete uniform density function can be expressed as

$$\begin{aligned} f_t(\tau) &= f_d(\tau \mid -S+a \leq \tau \leq S-a) = \frac{f_d(\tau)}{P(\eta - S + a \leq \tau \leq \eta + S - a)} \\ &= \frac{\sum_{n=-S+a}^{n=S-a} \frac{1}{2S+1} \cdot \delta(\tau - \eta - nT_s)}{\sum_{n=-S+a}^{S-a} \frac{1}{2S+1}} = \frac{\sum_{n=-S+a}^{S-a} \frac{1}{2S+1} \cdot \delta(\tau - \eta - nT_s)}{P(S)} . \end{aligned} \tag{36}$$

The value $P(S)$ can be calculated as

$$P(S)=\sum_{n=-S+a}^{S-a}\frac{1}{2S+1}=\sum_{n=-S+a}^{S-a}\frac{1}{2S+1}=\frac{1}{2S+1}(S-a+S-a+1)=\frac{2S-2a+1}{2S+1} \qquad (37)$$

By inserting this expression into (36), the density function can be expressed in this closed form, as stated in the proposition, i.e.,

$$f_t(\tau)=\frac{2S+1}{2S-2a+1}\sum_{n=-S+a}^{S-a}\frac{1}{2S+1}\cdot\delta(\tau-\eta-nT_s)=\frac{1}{2S-2a+1}\sum_{n=-S+a}^{S-a}\delta(\tau-\eta-nT_s)\ , \qquad (38)$$

and the related distribution function can be obtained by integration the last expression as

$$F_t(\tau)=\frac{1}{2S-2a+1}\sum_{n=-S+a}^{\tau/T_s}U(\tau-\eta-nT_s)\ . \qquad (39)$$

as it was stated.

4.2 Mean and variance of the truncated discrete uniform density function

Proposition: The mean of the discrete truncated random variable is η.

Proof: Based on the expression (38) for the discrete density function we may have

$$\begin{aligned}\eta_t &= \int_{-\infty}^{\infty}\tau f_\tau(\tau)d\tau=\frac{1}{2S-2a+1}\sum_{n=-S+a}^{S-a}\int_{-\infty}^{\infty}\tau\cdot\delta(\tau-\eta-nT_s)d\tau \\ &=\frac{1}{2S-2a+1}\sum_{n=-S+a}^{S-a}(\eta+nT_s)=\frac{1}{2S-2a+1}\sum_{n=-S+a}^{S-a}\eta+\frac{1}{2S-2a+1}\sum_{n=-S+a}^{S-a}nT_s=\eta\end{aligned} \qquad (40)$$

Proposition: The mean squared value of the discrete truncated random variable is

$$\eta_t=E\{\tau^2\}=T_s^2\,\frac{(S-a)(S-a+1)}{3}=\sigma_t^2 \qquad (41)$$

Proof: Based on the expression (38) for the discrete density function we may have

$$E\{\tau^2\} = \int_{-\infty}^{\infty} \tau^2 f_\tau(\tau) d\tau = \frac{1}{2S-2a+1} \sum_{n=-S+a}^{S-a} \int_{-\infty}^{\infty} \tau^2 \delta(\tau - \eta - nT_s) d\tau = \frac{1}{2S-2a+1} \sum_{n=-S+a}^{S-a} (\eta + nT_s)^2$$

$$= \frac{1}{2S-2a+1} \sum_{n=-S+a}^{S-a} \eta^2 + \frac{1}{2S-2a+1} \sum_{n=-S+a}^{S-a} 2\eta n T_s + \frac{1}{2S-2a+1} \sum_{n=-S+a}^{S-a} n^2$$

$$= \frac{2S-2a+1}{2S-2a+1} \eta^2 + 0 + \frac{2T_s^2}{2S-2a+1} \sum_{n=1}^{S-a} n^2 \quad . \quad (42)$$

$$= \eta^2 + \frac{2T_s^2}{2S-2a+1} \frac{(S-a)(S-a+1)(2(S-a)+1)}{6} = \eta^2 + T_s^2 \frac{(S-a)(S-a+1)}{3}$$

The variance can be calculated form (41) and (42) as

$$\sigma_t^2 = E\{\tau^2\} - \eta_t^2 = T_s^2 \frac{(S-a)(S-a+1)}{3}, \quad (43)$$

which completes our proofs. For a unit sampling interval, $T_s = 1$, we can easily get expression for the variance.

The variance (43) can be expressed in terms of the variance of the continuous density. By inserting the expression for T_s from (19) and then expression (5), we may have the truncated variance as a function of continuous in this form

$$\sigma_t^2 = E\{\tau^2\} - \mu_t^2 = T_s^2 \frac{(S-a)(S-a+1)}{3} = \frac{T_c^2}{3} \frac{4(S-a)(S-a+1)}{(2S+1)^2} = \sigma_c^2 \frac{4(S-a)(S-a+1)}{(2S+1)^2} . \quad (44)$$

5. Comparison of variances

We are interest in statistical characteristics of these distributions and their mutual interrelationship. For that purpose we will compare their variances. As a reference we will use the variance of continuous density. All other variances will be expressed in terms of this variance. The variances of the discrete density and truncated discrete density are already expressed as functions of the variance of continuous density, which is presented in (33) and (). and of the continuous density. Having in mind

$$T_c = (2S+1)T_s / 2 \Rightarrow T_s = 2T_c / (2S+1) \tag{45}$$

and the variance expression of continuous density,

$$\sigma_c^2 = \frac{T_c^2}{3} \tag{46}$$

the variance of the discrete density can be found as

$$\sigma_d^2 = T_s^2 \frac{S(S+1)}{3} = \frac{4T_c^2}{(2S+1)^2} \frac{S(S+1)}{3} = \frac{T_c^2}{3} \frac{4S(S+1)}{(2S+1)^2} = \sigma_c^2 \frac{4S(S+1)}{(2S+1)^2} \tag{47}$$

$$\begin{aligned} \sigma_t^2 &= T_s^2 \frac{(S-a)(S-a+1)}{3} = \frac{4T_c^2}{(2S+1)^2} \frac{(S-a)(S-a+1)}{3} \\ &= \frac{T_c^2}{3} \frac{4(S-a)(S-a+1)}{(2S+1)^2} = \sigma_c^2 \frac{4(S-a)(S-a+1)}{(2S+1)^2} \end{aligned} \tag{48}$$

The continuous truncated variance is

$$\sigma_{ct}^2 = \sigma_c^2 (1 - A / T_c)^2 . \tag{49}$$

In order to compare this variance with the variance of the truncated discrete density function, both of them need to be calculated the same truncation interval. Therefore, the ratio A/T_c, which would correspond to the truncated value of the discrete function, should be found. The corresponding truncation intervals for continuous and discrete density can be found from the truncating probabilities. If we take them from (36) –(37) and (10) and equate them we can get

$$\frac{2S-2a+1}{2S+1} = (1 - \frac{A}{T_c}) , \tag{50}$$

as illustrated in Fig. 7. From (5) we may have

$$(1 - \frac{A}{T_c}) = 1 - \frac{2a}{2S+1} \tag{51}$$

which can be inserted in (49) to get the variance of truncated continuous density in this form

$$\sigma_{ct}^2 = \sigma_c^2 (1 - \frac{2a}{2S+1})^2 . \tag{52}$$

Calculated expressions for the variances as functions of the variance of continuous uniform variable are presented in Table 1 in the form to clearly see relationship between truncated and non-truncated variances. Namely, the variance of truncated density function should be less than or equal to the variance of continuous random variable.

$$\sigma_{ct}^2 = \frac{(T_c - A)^2}{3} \Rightarrow T_c - A = \sqrt{3\sigma_{ct}^2} = \sigma_{ct}\sqrt{3} \tag{53}$$

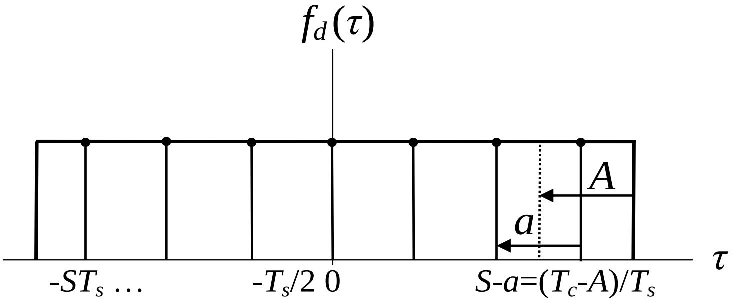

Fig. 7 Relations between truncation intervals of the discrete and continuous truncated uniform density function.

Table 1 Variance expressions

Uniform distributions	**Variances**
Continuous	$\sigma_c^2 = \frac{T_c^2}{3}$
continuous truncated	$\sigma_{ct}^2 = \sigma_c^2 \left(\frac{2S - 2a + 1}{2S+1} \right)^2 = \sigma_c^2 \left(1 - \frac{2a}{2S+1} \right)^2$
Discrete	$\sigma_d^2 = \sigma_c^2 \frac{4S(S+1)}{(2S+1)^2} = \sigma_c^2 \left(1 - \frac{1}{(2S+1)^2} \right) =$
Discrete truncated	$\sigma_t^2 = \sigma_c^2 \frac{4(S-a)(S-a+1)}{(2S+1)^2} = \sigma_c^2 \left(1 - \frac{1-4a(2S-a+1)}{(2S+1)^2} \right)$

The graphs for the variances of the continuous, continuous truncated, discrete and discrete truncated random variables as functions of the variance of the continuous random variable, for the truncation factor a and sampling interval $2S$ as parameters, are presented in Fig. 8. The non-truncated continuous and discrete random variables have very similar, nearly the same variance, as we can expect, because they are calculated on the same interval of possible values of random variable. The small difference occurs due to that the variances of continuous density are calculated on the continuum of random variable values while the variance of discrete density are calculated for a finite number of discrete values.

The variances of the truncated density functions are smaller than the variances of the non-truncated functions due to the truncation of the function which is defined by truncation factors *A* and *a*. The higher these factors are the smaller gradients of these curves are and higher the difference is between variances of truncated and non-truncated densities.

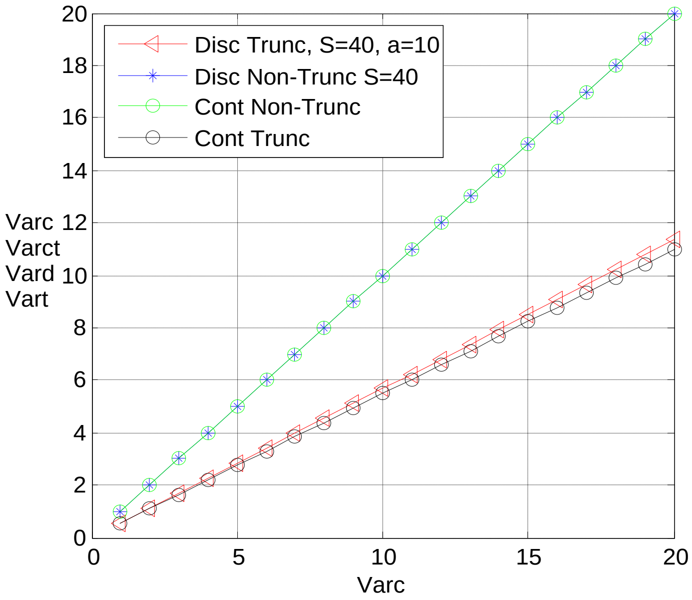

Fig. 8 Continuous, continuous truncated, discrete and discrete truncated variances versus continuous variance.

The variances of the continuous and discrete random variable are nearly the same, because the interval of their existence is nearly the same. In contrast to that, the truncation values are inside a narrower interval and consequently the variance of the truncated density functions is smaller than those of non-truncated.