Comprehensive Analysis of Exponential Discrete Truncated Stochastic Processes

by

Dr Stevan Berber

Faculty of Engineering Report

Department of Electrical and Computer Engineering
The University of Auckland

November 2015

THE UNIVERSITY OF AUCKLAND
NEW ZEALAND

Abstract

There are numerous applications of exponential continuous random variables and stochastic processes. However, the theoretical analysis and application of these processes, in particular when the random variable is both discrete and truncated, is not completed. This Report presents firstly a review of exponential continuous density functions, both un-truncated and truncated. Then the discrete density function is derived and expressed in terms of Dirac's delta functions. For this case, the mean and variance are derived and analyzed. The necessity of having truncated discrete density function, from the application point of view in communication systems, is explained and related density and distribution functions are derived. For these functions, the mean and variance expressions are expressed as functions of the length of the defined truncation interval and compared with related moments of the continuous truncated density function. The important advancement is achieved by deriving the truncated discrete density functions and expressing them in terms of Dirac's delta and unit step functions. How to design a discrete truncated exponential stochastic process is illustrated on several examples.

Contents

1. Continuous exponential density function

Theoretical characteristics and applications of the exponential continuous variables are well known. In spite of that it is still subject of research [1-3]. In particular, the development of discrete time systems, like one presented in [4], requires the presentation of the exponential and similar density functions in discrete form, which is subject of late and present research interests [5 - 13] The density function of a random delay τ, e.g., in a communication channel, can be represented by exponential density and distribution function, as shown in Fig. 1, and expressed as

$$f_\tau(\tau) = \lambda e^{-\lambda\tau}, \tag{1}$$

$$F_\tau(\tau) = 1 - e^{-\lambda\tau} \tag{2}$$

Where parameter λ defines the mean value and variance

$$\eta_\tau = \frac{1}{\lambda},\ E\{\tau^2\} = \frac{2}{\lambda^2} \text{ and } \sigma_\tau^2 = \frac{1}{\lambda^2}.$$

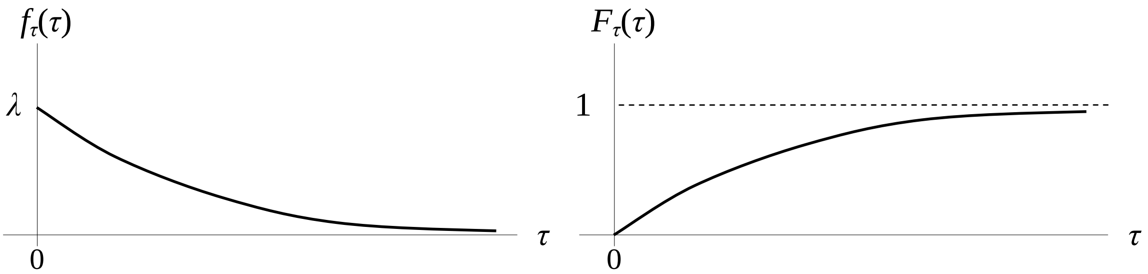

Figure 1 Continuous exponential density function.

In order to specify the interval of argument values these densities can be expressed in terms of unit step functions as

$$f_\tau(\tau) = \lambda e^{-\lambda\tau} U(\tau), \tag{3}$$

$$F_\tau(\tau) = \left(1 - e^{-\lambda\tau}\right) U(\tau). \tag{4}$$

2. Truncated exponential continuous density function

If the expected delay is finite and can have values in a limited interval T, the truncated continuous density function needs to be found.

2.1 Truncated exponential continuous density function

Proposition: The truncated continuous density function of an exponential density function is expressed as

$$f_t(\tau) = \frac{1}{1-e^{-\lambda T}} \lambda e^{-\lambda \tau}, \quad (5)$$

where T is truncation interval of the function.

Proof: Suppose the domain of argument values of the continuous density function defined in (1) is subdivided into infinite number of intervals of the same duration T. If the corresponding values of all intervals are added, the truncated continuous density function $f_t(\tau)$ is obtained, as shown in Fig. 2.

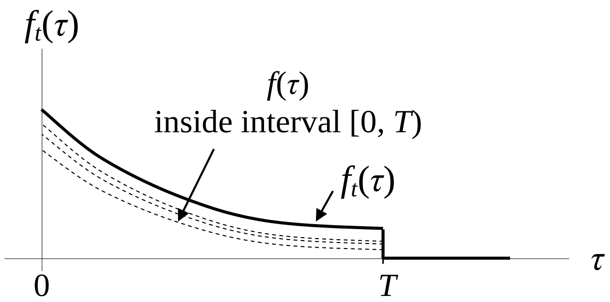

Figure 2 Truncated continuous density function.

If the expression for the density function at one point inside T interval is found, the same expression will be valid for any value in the interval. The value of the truncated function for any τ inside T interval is

$$f_t(\tau) = f_\tau(\tau) + f_\tau(\tau+T) + f_\tau(\tau+2T) + \ldots + f_\tau(\tau+mT) + \ldots$$
$$= \sum_{k=1}^{\infty} f_\tau(\tau+kT) = \sum_{k=1}^{\infty} \lambda e^{-\lambda(\tau+kT)} = \lambda e^{-\lambda\tau} \sum_{k=1}^{\infty} e^{-\lambda kT} = \frac{\lambda}{1-\lambda e^{-\lambda T}} e^{-\lambda\tau} \,. \quad (6)$$

We can prove that the integral of the truncated density function inside T interval is one and the distribution function fulfils its main requirements, which completes our proof.

Proof:

The integral of the truncated density function inside truncation interval T is one, i.e.,

$$\int_0^T f_t(\tau)d\tau = \frac{1}{1-e^{-\lambda T}}\lambda \int_0^T e^{-\lambda\tau}d\tau = \frac{1}{1-e^{-\lambda T}}\lambda \left[\frac{e^{-\lambda\tau}}{-\lambda}\right]_0^T = \frac{1}{1-e^{-\lambda T}}\lambda \left[\frac{e^{-\lambda\tau}}{-\lambda}\right]_0^T = \frac{-e^{-\lambda T}+1}{1-e^{-\lambda T}} = 1 \quad (7)$$

The truncated distribution function $F_t(\tau)$ can be calculated as follows

$$\int_{-\infty}^{\tau} f_t(x)dx = \frac{1}{1-e^{-\lambda T}}\lambda \int_0^{\tau} e^{-\lambda x}dx = \frac{1}{1-e^{-\lambda T}}\lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^{\tau} = \frac{1-e^{-\lambda\tau}}{1-e^{-\lambda T}}$$

$$F_t(\tau) = \begin{cases} 0 & \tau < 0 \\ \frac{1-e^{-\lambda\tau}}{1-e^{-\lambda T}} & 0 < \tau < T \\ 1 & \tau > T \end{cases} \quad (8)$$

In this case, if $\tau < 0$ then $F_t(\tau) = 0$ and if $\tau > T$ then $F_t(\tau) = 1$. Figure 3 present graphs for the truncated continuous density function (pdfct) for $\lambda = 1$ and then for the corresponding truncated density function for three chosen values of T including $T = 3$.

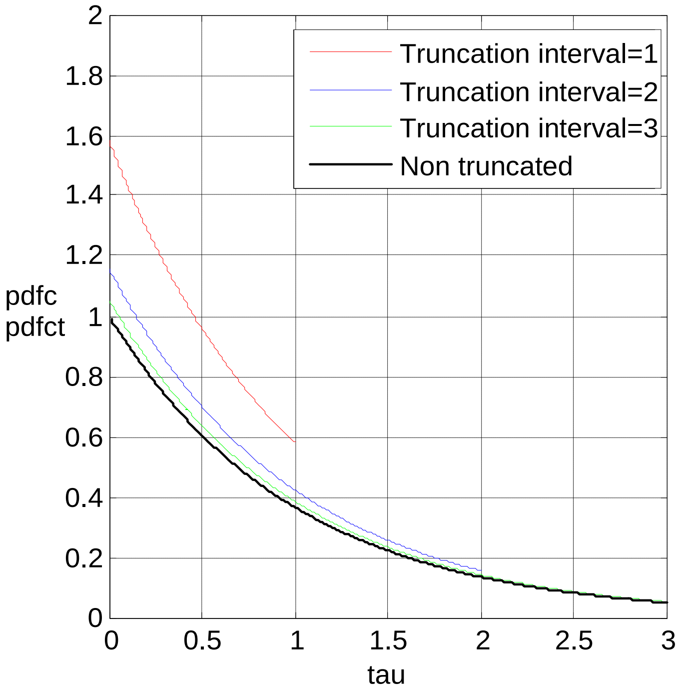

Figure 3 Graphs for the continuous and truncated density function (pdfc and pdfct) for $\lambda = 1$.

2.2 The mean and variance calculations

The mean and variance of this truncated density function can be calculated as follows. The mean is

$$E\{\tau\} = \int_0^T \tau f_t(\tau) d\tau = \int_0^T \tau \frac{\lambda}{1-e^{-\lambda T}} . e^{-\lambda\tau} d\tau \frac{\lambda}{1-e^{-\lambda T}} \int_0^T \tau . e^{-\lambda\tau} d\tau = \frac{\lambda}{1-e^{-\lambda T}} \int_0^T \tau . e^{-\lambda\tau} d\tau$$

$$E\{\tau\} = \frac{\lambda}{1-e^{-k}} \left\{ \frac{-\tau}{\lambda} e^{-\lambda\tau} \right]_0^T + \int_0^T e^{-\lambda\tau}\, d\tau \right\} = \frac{\lambda}{1-e^{-\lambda T}} \left\{ \frac{-\lambda T}{\lambda} e^{-\lambda T} + \left[\frac{1}{\lambda} e^{-\lambda\tau}\right]_T^0 \right\}$$

$$E\{\tau\} = \frac{1}{1-e^{-\lambda T}} \left(-Te^{-\lambda T} + \frac{e^{-\lambda T}}{-\lambda} + \frac{1}{\lambda} \right) = \frac{1-e^{-\lambda T}(\lambda T+1)}{\lambda(1-e^{-\lambda T})} = \frac{1}{\lambda} - \frac{T}{e^{\lambda T}-1} \tag{9}$$

The mean square value and variance can be found using integration by parts which results in

$$E\{\tau^2\} = \frac{2-(\lambda^2 T^2 + 2\lambda T + 2)e^{-\lambda T}}{\lambda^2(1-e^{-\lambda T})} \tag{10}$$

$$VAR(\tau) = \frac{2-(\lambda^2T^2+2\lambda T+2)e^{-\lambda T}}{\lambda^2(1-e^{-\lambda T})} - \left(\frac{1-e^{-\lambda T}(\lambda T+1)}{\lambda(1-e^{-\lambda T})}\right)^2 \tag{11}$$

This moment of the truncated density can be related to the corresponding values for continuous density function. Fig. 4 presents the variance, the mean square value and the mean of the truncated and continuous density function as a function of the mean value $1/\lambda$ for the truncation interval T as a parameter. Random variables with truncated functions have smaller values of the moments that the corresponding random variables with the continuous non-truncated density function. The moments are increasing when the truncation interval increases and tends to the moments of the continuous density function when T tends to infinity, as expected.

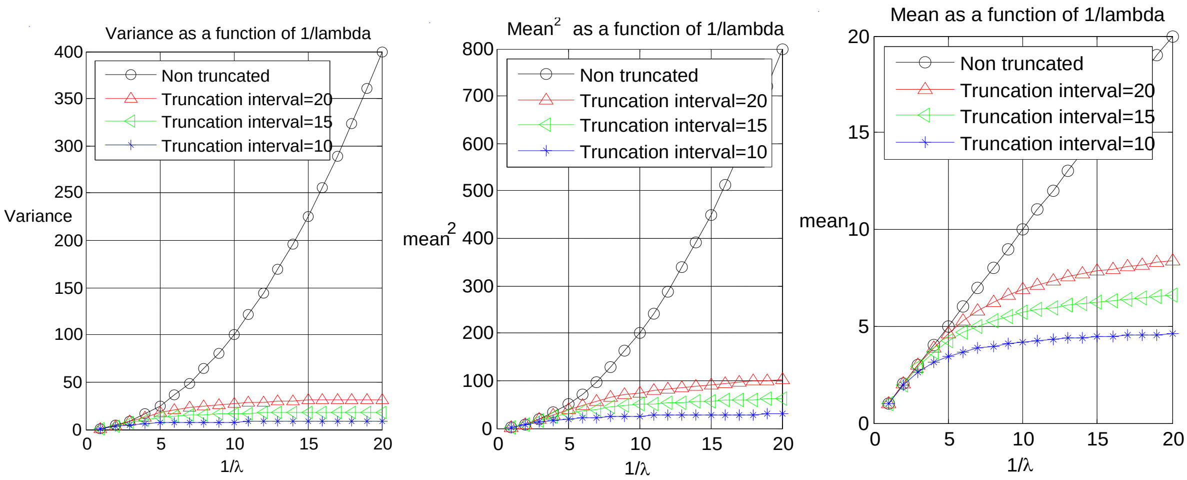

Figure 4 Graphs for the moments of continuous and truncated density function for different values of $1/\lambda$.

3. Discrete exponential density function

3.1 Signal representation in discrete time domain

In theory of discrete time signal processing and its application in telecommunication systems [4], a digital binary signal is represented as a time series of binary symbols (bits), x_i. Each bit is represented by interpolated samples, as shown in Fig. 3 (here S equidistant samples, each in the unit interval T_s = 1). The i-th symbol can be received with different delays τ_{i1}, τ_{i2}, … τ_{iM}, depending on propagation conditions.

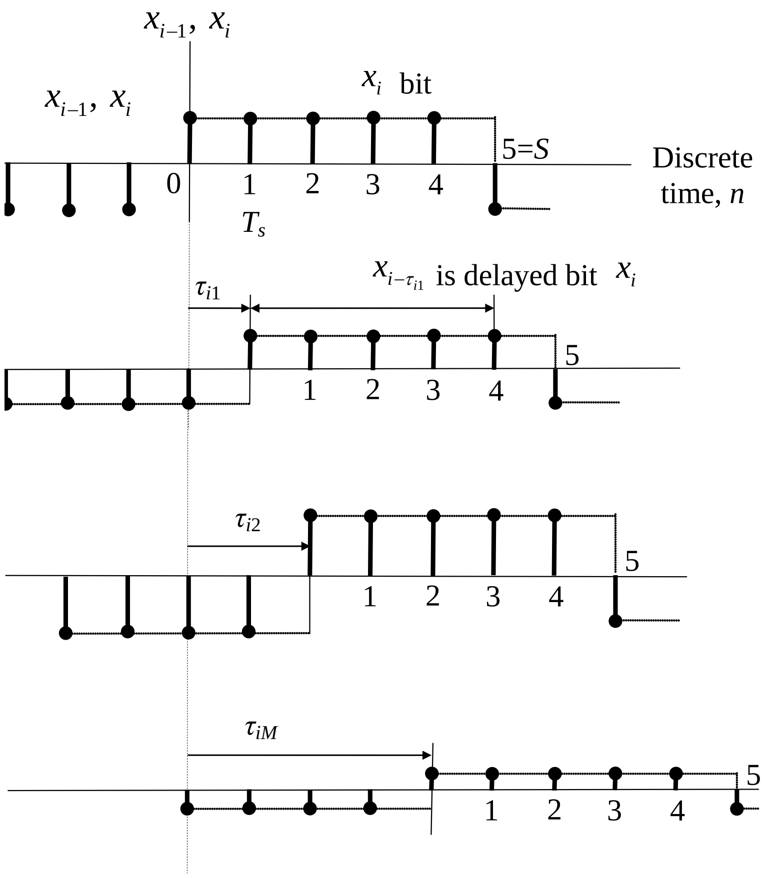

Figure 5 Signals with random delays in discrete time domain.

The delay of any symbol in respect to the reference symbol (for first one with the delay zero) needs to be represented by certain whole number of samples τ_{tj}. These random delays are having exponential distribution [4] defined by its discrete exponential function.

There are two problems in the proper characterisation of this density function of these delays:

1. Firstly, the density function needs to be uniformly discretised with the interval of discretisation T_s. This discretisation should be performed in such a way to obtain discrete density that fulfils conditions of its definition.
2. Secondly, the maximum value of the discrete delay is finite. Therefore, the obtained discrete density function should be truncated to get discrete truncated density function. This function again needs to fulfil conditions of its definition.
3. Thirdly, the obtained discrete and truncated density function should be expressed as function of discrete delay τ_d. This density also needs to be in a closed form in order to easily use in finding the mean value of random function where the argument contains the delay as a variable. This is of particular interest in communication theory and practice where the mean value of the probability of error functions needs to be calculated. For these reason, the discrete density function and its truncated version will be expressed in terms of delta functions (Dirac's and Kronecker's functions).

3.2 Derivations for the discrete density and distribution functions

The procedure of a continuous density discretisation is presented in Fig. 6. The discretisation is performed at uniformly spaced discrete interval T_s. The problem is how to find the expression of this discrete density function and how to find its mean and variance.

In the process of discretisation the probability of each interval needs to be calculated and assigned to the discrete time instants nT_s. There are two possibilities is this case:

1. The probability are assigned to the left side of the interval starting with the discrete $\tau = 0$.
2. The probability are assigned to the right side of the interval starting with the discrete $\tau = 1$.

These two possibilities will have some consequences to the expressions for the mean and variances of the truncated discrete random variable and the care should be taken about this issue. Firstly, we will derive the discrete density and distribution functions in closed forms and related moments.

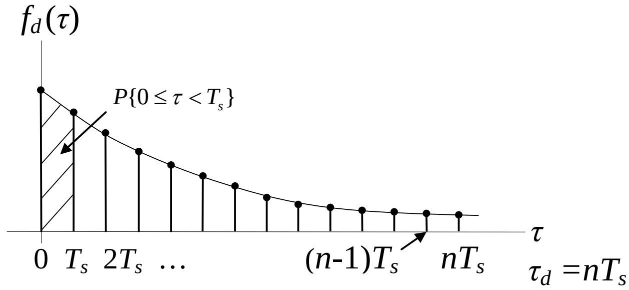

Figure 6 Discretisation of continuous exponential density function.

Proposition: The discrete density and distribution functions, having the values at the uniformly spaced instants T_s, is expressed as

$$f_d(\tau) = (e^{\lambda T_s} - 1)\sum_{n=1}^{\infty} e^{-n\lambda T_s}\delta(\tau - nT_s) \text{, and} \tag{12}$$

$$F_d(\tau) = \sum_{n=1}^{\tau/T_s} (e^{\lambda T_s} - 1)e^{-\lambda n T_s} U(\tau - nT_s) \text{,} \tag{13}$$

where $\delta(.)$ are Dirac's delta functions and τ is continuous variable. The values of the density function are discrete and defined by the positions of delta functions.

Proof: If this density is uniformly discretized in respect to τ, with the interval of discretisation of T_s as shown in Fig. 6, the probability value in each interval can be calculated as

$$P\{0 \le \tau < T_s\} = \int_0^{T_s} f_{\mathrm{T}}(\tau)d\tau = \int_0^{T_s} \lambda e^{-\lambda\tau} d\tau = \frac{\lambda}{-\lambda} e^{-\lambda\tau}\Big|_0^{T_s} = -e^{-\lambda\tau}\Big|_0^{T_s} = 1 - e^{-\lambda T_s} \text{,} \tag{14}$$

$$P\{T_s \le \tau < 2T_s\} = \int_{T_s}^{2T_s} f_{\mathrm{T}}(\tau)d\tau = e^{-\lambda T_s} - e^{-2\lambda T_s} = e^{-\lambda T_s}(1 - e^{-\lambda T_s}) \text{,} \tag{15}$$

$$P\{2T_s \le \tau < 3T_s\} = \int_{2T_s}^{3T_s} f_{\mathrm{T}}(\tau)d\tau = e^{-2\lambda T_s} - e^{-3\lambda T_s} = e^{-2\lambda T_s}(1 - e^{-\lambda T_s}) \text{,} \tag{16}$$

or, in general form, for any interval defined by n, as

$$P\{(n-1)T_s \le \tau < nT_s\} = \int_{(n-1)T_s}^{nT_s} f_{\mathrm{T}}(\tau)d\tau = e^{-(n-1)\lambda T_s} - e^{-n\lambda T_s} = (e^{\lambda T_s} - 1)e^{-n\lambda T_s} \text{.} \tag{17}$$

To express the density in a closed form, these probabilities can be used as weights of Dirac's delta functions representing the discrete exponential density function at time instants $\tau = nT_s$, as presented in Fig. 7, which results in this expression

$$f_d(\tau)=\sum_{n=1}^{\infty}P\{(n-1)T_s\le\tau<nT_s\}\delta(\tau-nT_s)=\sum_{nT_s=T_s}^{\infty}(e^{\lambda T_s}-1)e^{-n\lambda T_s}\delta(\tau-nT_s)$$
$$=(e^{\lambda T_s}-1)\sum_{n=1}^{\infty}e^{-n\lambda T_s}\delta(\tau-nT_s) \quad . \qquad (18)$$

Here the delay is a discrete variable. A hypothetical discrete density function is presented in Fig. 7.

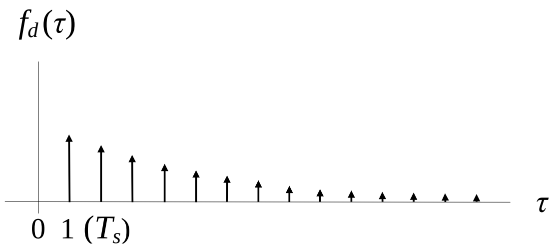

Fig. 7 Discrete exponential density function presented using Dirac's delta functions.

The discrete exponential distribution function can be obtained as follows. Because the Dirac's delta function is used, the discrete density function values are different from zero at the discrete points defined by nT_s on the continuum of τ random values. Thus, the discrete distribution function can be calculated by integrating the density function as follows

$$F_d(\tau)=\int_{x=-\infty}^{\tau}f_d(x)dx=\int_{x=0}^{\tau}\sum_{nT_s=T_s}^{\infty}(e^{\lambda T_s}-1)e^{-\lambda nT_s}\delta(x-nT_s)dx \qquad (19)$$

Because the integral includes all the values from 0 to τ, or the values specified by n =1 to τ/T_s. Also, the integral of the delta function is the unit step function. Therefore, the solution of integral is the discrete distribution function

$$F_d(\tau)=\sum_{nT_s=T_s}^{\tau}(e^{\lambda T_s}-1)e^{-\lambda nT_s}\int_{x=T_s}^{\tau}\delta(x-nT_s)dx=\sum_{n=1}^{\tau/T_s}(e^{\lambda T_s}-1)e^{-\lambda nT_s}U(\tau-nT_s)\,. \qquad (20)$$

Furthermore, for a finite discrete value of τ we may calculate the related value of distribution function as follows

$$
\begin{aligned}
F_d(\tau) &= \sum_{n=1}^{\tau/T_s}(e^{\lambda T_s}-1)e^{-\lambda n T_s}U(\tau-nT_s) \\
&= (e^{\lambda T_s}-1)\left[e^{-\lambda n T_s}U(\tau-T_s)+e^{-\lambda n T_s}U(\tau-2T_s)+...+e^{-\lambda\tau}U(\tau-\tau)\right] \\
&= (e^{\lambda T_s}-1)\left[e^{-\lambda T_s}+e^{-\lambda 2T_s}+...+e^{-\lambda\tau}\right]=(e^{\lambda T_s}-1)\sum_{n=1}^{\tau/T_s}(e^{-\lambda T_s})^n \qquad , \qquad (21) \\
&= (e^{\lambda T_s}-1)\frac{e^{-\lambda T_s}}{1-e^{-\lambda T_s}}(1-e^{-\lambda T_s \tau/T_s})=1-e^{-\lambda\tau}
\end{aligned}
$$

which completed our proofs related to density and distribution functions.

SPECIAL CASE: For a unit interval of $T_s = 1$, we may have density function expressed as

$$f_d(\tau)=(e^{\lambda}-1)\sum_{n=1}^{\infty}e^{-n\lambda}\delta(\tau-n)\,, \qquad (22)$$

and the distribution function as

$$F_d(\tau)=\sum_{n=1}^{\tau}(e^{\lambda}-1)e^{-\lambda n}=1-e^{-\lambda\tau}\,, \qquad (23)$$

Proposition: The derived functions are fulfilling conditions of the density and distribution function definition.

Proof: The integral of the density function is equal to one.

$$
\begin{aligned}
\int_{\tau=-\infty}^{\infty} f_d(\tau)d\tau &= \int_{\tau=0}^{\infty}\sum_{nT_s=T_s}^{\infty}(e^{\lambda T_s}-1)e^{-n\lambda T_s}\delta(\tau-nT_s)d\tau \;=\; \sum_{nT_s=T_s}^{\infty}(e^{\lambda T_s}-1)e^{-\lambda n T_s}\int_{\tau=T_s}^{\infty}\delta(\tau-nT_s)d\tau \qquad (24) \\
&= \sum_{nT_s=T_s}^{\infty}(e^{\lambda T_s}-1)e^{-\lambda n T_s}\cdot 1 \;=(e^{\lambda T_s}-1)\sum_{nT_s=T_s}^{\infty}\left(e^{-\lambda T_s}\right)^n \;=(e^{\lambda T_s}-1)\frac{e^{-\lambda T_s}}{1-e^{-\lambda T_s}}=1
\end{aligned}
$$

Also, it is easy to prove that the distribution function fulfils necessary conditions of its definition: For $\tau = 0$ the function value is 0 and when τ tends to infinity the function tends to 1. The graph of the discrete density function for $\lambda = 1$ is shown in Fig. 8, for two cases mentioned at the beginning of this Chapter. The blue graph represents Dirac's delta functions and their weights for the case when the first function value is assigned to the argument value $n = 1$. The red discrete graph represents the discrete density function which starts at $n = 0$.

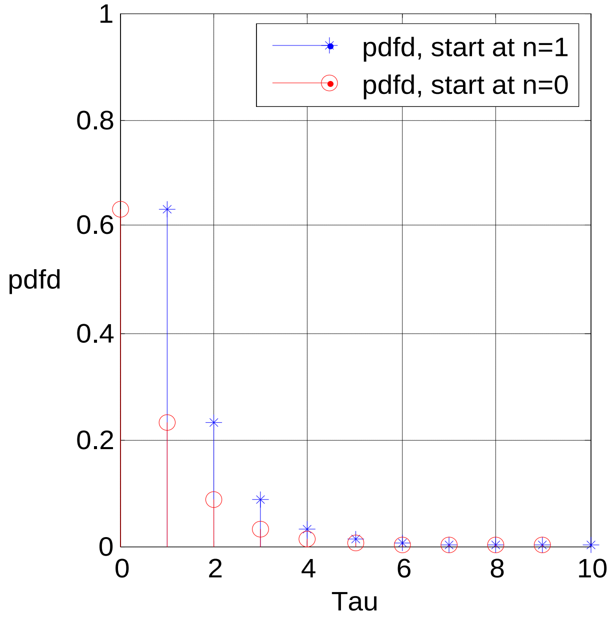

Figure 8 Discrete density function for $\lambda = 1$.

3.3 Derivations of the moments for the discrete random variable

Proposition: The mean, mean square an variance are expressed as

$$\eta_d = \int_{\tau=-\infty}^{\infty} \tau \cdot f_d(\tau) d\tau = \frac{T_s}{1-e^{-\lambda T_s}} = \frac{T_s e^{\lambda T_s}}{e^{\lambda T_s} - 1}, \tag{25}$$

$$E\{\tau^2\} = \int_{\tau=-\infty}^{\infty} \tau^2 \cdot f_d(\tau) d\tau = T_s^2 \frac{(e^{-\lambda T_s} + 1)}{(e^{-\lambda T_s} - 1)^2} \tag{26}$$

$$Var(\tau) = E\{\tau^2\} - \eta_d^2 = \frac{e^{-\lambda T_s}}{(e^{-\lambda T_s} - 1)^2}, \tag{27}$$

and for the unit interval $T_s = 1$, they are

$$\eta_d = \frac{1}{1-e^{-\lambda}} = \frac{e^{\lambda}}{e^{\lambda}-1}, \tag{28}$$

$$E\{\tau^2\} = \frac{e^{-\lambda}+1}{(e^{-\lambda}-1)^2} \tag{29}$$

$$\sigma_\tau^2 = E\{\tau^2\} - E^2\{\tau\} = \frac{e^{\lambda}}{(e^{\lambda}-1)^2}. \tag{30}$$

Proofs: The proofs for the unit discretisation interval $T_s = 1$ will be presented. In this case the mean of the discrete density function is

$$\eta_d = \int_{\tau=-\infty}^{\infty} \tau \cdot f_d(\tau) d\tau = \int_{\tau=0}^{\infty} \tau \cdot \sum_{n=1}^{\infty} (e^{\lambda}-1) e^{-n\lambda} \delta(\tau-n) d\tau \ = \sum_{n=1}^{\infty} (e^{\lambda}-1) e^{-\lambda n} \int_{\tau=T_s}^{\infty} \tau \cdot \delta(\tau-n) d\tau$$
$$= (e^{\lambda}-1) \sum_{n=1}^{\infty} n \cdot e^{-\lambda n} = (e^{\lambda}-1) \cdot \Sigma_1$$

where

$$\Sigma_1 = \sum_{n=1}^{\infty} n \cdot e^{-\lambda n} = e^{-\lambda} + 2e^{-2\lambda} + 3e^{-3\lambda} + \ldots$$
$$e^{\lambda}\Sigma_1 - \Sigma_1 = (1 + 2e^{-\lambda} + 3e^{-2\lambda} + \ldots) - (e^{-\lambda} + 2e^{-2\lambda} + 3e^{-3\lambda} + \ldots) = 1 + e^{-\lambda} + e^{-2\lambda} + e^{-3\lambda} + \ldots$$
$$= \sum_{n=0}^{\infty} (e^{-\lambda})^n = \frac{1}{1-e^{-\lambda}} = \frac{e^{\lambda}}{e^{\lambda}-1},$$
$$\Sigma_1 = \frac{e^{\lambda}}{(e^{\lambda}-1)^2}$$

Therefore, the mean is

$$\eta_d = (e^\lambda - 1)\cdot\Sigma_1 = (e^\lambda - 1)\cdot\frac{e^\lambda}{(e^\lambda - 1)^2} = \frac{e^\lambda}{e^\lambda - 1} = \frac{1}{1 - e^{-\lambda}}\,. \tag{31}$$

The mean square value is

$$E\{\tau^2\} = \int_{\tau=-\infty}^{\infty} \tau^2 \cdot f_d(\tau)d\tau = (e^\lambda - 1)\sum_{n=1}^{\infty} n^2 \cdot e^{-\lambda n} = (e^\lambda - 1)\cdot\Sigma_2 \tag{32}$$

where the sigma two can be calculated as a function of sigma one as

$$\begin{aligned}
\Sigma_2 &= \sum_{n=1}^{\infty} n^2 \cdot e^{-\lambda n} = e^{-\lambda} + 4e^{-2\lambda} + 9e^{-3\lambda} + \ldots \\
e^\lambda\Sigma_2 - \Sigma_2 &= (1 + 4e^{-\lambda} + 9e^{-2\lambda} + \ldots) - (e^{-\lambda} + 4e^{-2\lambda} + 9e^{-3\lambda} + \ldots) = 1 + 3e^{-\lambda} + 5e^{-2\lambda} + \ldots \\
&= e^{-\lambda} + 2e^{-2\lambda} + \ldots + 1 + 2e^{-\lambda} + 3e^{-2\lambda} + \ldots = \Sigma_1 + e^\lambda\Sigma_1 \\
\Sigma_2 &= \frac{\Sigma_1 + e^\lambda\Sigma_1}{e^\lambda - 1}
\end{aligned}$$

And then the mean square value can be calculated as

$$E\{\tau^2\} = (e^\lambda - 1)\cdot\Sigma_2 = (e^\lambda - 1)\cdot\frac{\Sigma_1(1 + e^\lambda)}{e^\lambda - 1} = \frac{e^\lambda}{(e^\lambda - 1)^2}(1 + e^\lambda) = \frac{e^\lambda + e^{2\lambda}}{(e^\lambda - 1)^2} = \frac{e^{-\lambda} + 1}{(e^{-\lambda} - 1)^2}\,.$$

The variance is

$$\sigma_\tau^2 = E\{\tau^2\} - E^2\{\tau\} = \frac{e^{-\lambda} + 1}{(e^{-\lambda} - 1)^2} - \frac{1}{(1 - e^{-\lambda})^2} = \frac{e^{-\lambda}}{(1 - e^{-\lambda})^2}$$, or, in this form

$$\sigma_\tau^2 = E\{\tau^2\} - E^2\{\tau\} = \frac{e^\lambda + e^{2\lambda}}{(1 - e^\lambda)^2} - \frac{e^{2\lambda}}{(e^\lambda - 1)^2} = \frac{e^\lambda}{(e^\lambda - 1)^2}\,. \tag{33}$$

Plots of the three graphs for the mean, mean square value and variance as a function of parameter $1/\lambda$, starting with $1/\lambda = 1$ and finishing with $1/\lambda = 20$, are shown in Fig. 9.

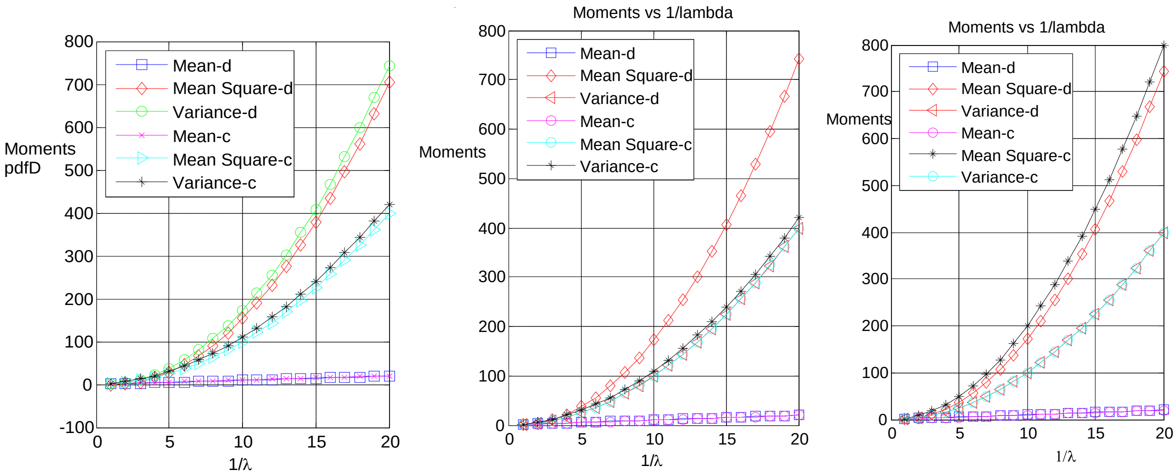

Figure 9 The moments for continuous and discrete density functions.

The mean of discrete and continuous densities are close to each other. To see clearly their difference they are separatelly presented in Fig. 10. There is one graph for the continuous density and two graphs for the discrete density. The upper graph (blue) for the mean of the discrete random variable (Meand) correponds to the density function which, by definition, starts at $n = 1$. The lower graph (black) correponds to the mean of the discrete random variable (Meand) which, by definition, starts at $n = 0$. Horisontal distance between these two graphs is equal to $T_s = 1$, as we expect according to the proof of the following proposition.

The difference in the mean value for the density function defined for different starting pont, $n = 0$ and $n = 1$, is not so important when the discretisation intervals T_s are small. However, if the interval T_s is large the difference in the mean value can be significant and should be taken into account. This problem will be addressed again in our analysis of the truncated discrete variable.

Proposition: The mean value for the discrete random variable starting with $n = 0$ is expressed as

$$\eta_{d0} = \frac{1}{e^{\lambda} - 1}. \tag{34}$$

The difference of this mean and the mean of the random variable starting with $n = 1$, as presented in (29), is equal to one.

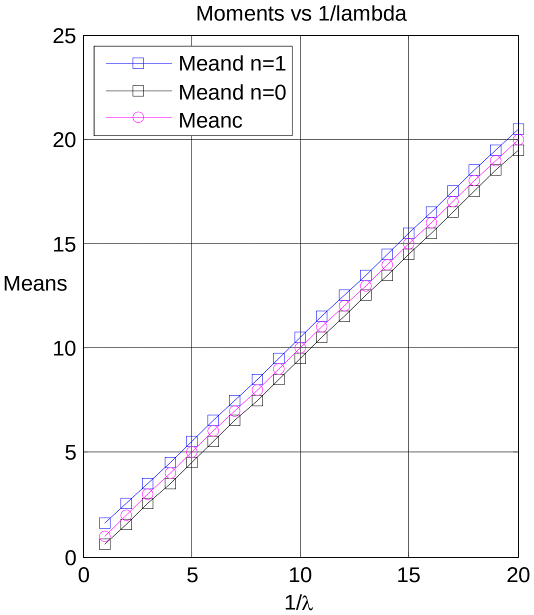

Figure 10 The mean values for continuous and discrete density functions.

Proofs: In the case when the density function starts with $n = 0$, the mean of the discrete density function is

$$\eta_{d0} = \int_{\tau=-\infty}^{\infty} \tau \cdot f_d(\tau) d\tau = \int_{\tau=0}^{\infty} \tau \cdot \sum_{n=0}^{\infty} (e^{\lambda} - 1) e^{-(n+1)\lambda} \delta(\tau - n) d\tau \; = \sum_{n=0}^{\infty} (e^{\lambda} - 1) e^{-\lambda(n+1)} \int_{\tau=T_s}^{\infty} \tau \cdot \delta(\tau - n) d\tau$$

$$= (e^{\lambda} - 1) e^{-\lambda} \sum_{n=0}^{\infty} n \cdot e^{-\lambda n} = (1 - e^{-\lambda}) \cdot \Sigma_1$$

where

$$\Sigma_1 = \sum_{n=0}^{\infty} n \cdot e^{-\lambda n} = e^{-\lambda} + 2e^{-2\lambda} + 3e^{-3\lambda} + \ldots$$

$$e^{\lambda}\Sigma_1 - \Sigma_1 = (1 + 2e^{-\lambda} + 3e^{-2\lambda} + \ldots) - (e^{-\lambda} + 2e^{-2\lambda} + 3e^{-3\lambda} + \ldots) = 1 + e^{-\lambda} + e^{-2\lambda} + e^{-3\lambda} + \ldots$$

$$= \sum_{n=0}^{\infty} (e^{-\lambda})^n = \frac{1}{1 - e^{-\lambda}} = \frac{e^{\lambda}}{e^{\lambda} - 1} \quad ,$$

$$\Sigma_1 = \frac{e^{\lambda}}{(e^{\lambda} - 1)^2}$$

Therefore, the mean is

$$\eta_{d0} = (1-e^{-\lambda})\cdot\Sigma_1 = (1-e^{-\lambda})\cdot\frac{e^{\lambda}}{(e^{\lambda}-1)^2} = \frac{(e^{\lambda}-1)}{(e^{\lambda}-1)^2} = \frac{1}{e^{\lambda}-1}. \quad (35)$$

The difference between this and previously calculated mean is

$$\eta_d - \eta_{d0} = \frac{e^{\lambda}}{e^{\lambda}-1} - \frac{1}{e^{\lambda}-1} = 1 \quad (36)$$

as we stated in the proposition.

4. Truncated discrete exponential density and distribution function

4.1 Derivation of the density and distribution function

In practical application the discrete delays are taking values in a limited interval of, say, S possible discrete values. Therefore, the function which describes the delay distribution is truncated to S, as shown in Fig. 11.

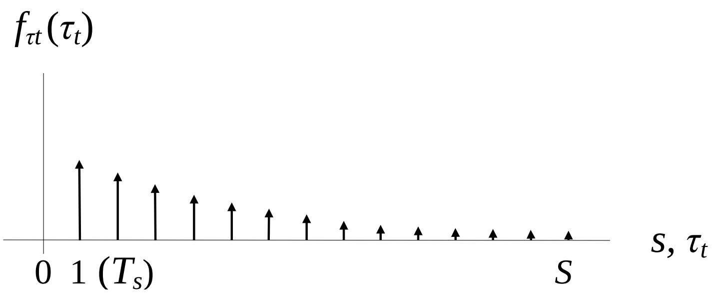

Fig. 11 Discrete truncated exponential density function presented using Dirac's delta functions.

Proposition: The density and distribution functions are given by these expressions in closed form

$$f_{\tau_t}(\tau_t) = \sum_{s=1}^{S} \frac{(e^{\lambda}-1)}{1-e^{-S\lambda}} e^{-s\lambda} \delta(\tau_t - s) \tag{37}$$

$$F_{\tau_t}(\tau_t) = \sum_{s=1}^{\tau_t} \frac{(e^{\lambda}-1)}{1-e^{-S\lambda}} e^{-s\lambda} U(\tau_t - s)\,. \tag{38}$$

Proof: In order to find this truncated function, the whole domain of possible τ_t values from 0 to infinity, for the already analysed discrete density function, will be divided into intervals containing S values. All corresponding values in these intervals, starting with 1st, (S+1)th, (2S+1)th, etc. terms will be added to obtain the truncated density function values for s = 1, 2, …., S. Using expression for the density function with T_s = 1, the first values of the density function in the 1st, 2nd and m-th interval of S values will be

$$f_\tau(\tau = 1) = (e^\lambda - 1)e^{-\lambda},$$
$$f_\tau(\tau = S+1) = (e^\lambda - 1)e^{-(S+1)\lambda} = (e^\lambda - 1)e^{-\lambda}e^{-S\lambda},...,\qquad(39)$$
$$f_\tau(\tau = mS) = (e^\lambda - 1)e^{-\lambda}e^{-mS\lambda}$$

The sum of these values, when *m* tends to infinity, will give the first truncated value defined for unit delay $\tau_t = 1$, i.e.,

$$f_{\tau_t}(\tau_t = 1) = \frac{(e^\lambda - 1)}{1 - e^{-S\lambda}}e^{-\lambda}$$

and the value for density function for any delay *s* will be

$$f_{\tau_t}(\tau_t = s) = \frac{(e^\lambda - 1)}{1 - e^{-S\lambda}}e^{-s\lambda}\ .\qquad(40)$$

The sum of these values in the whole interval of possible truncated values *S* will specify the truncated density function as

$$f_{\tau_t}(\tau_t) = \sum_{s=1}^{S}\frac{(e^\lambda - 1)}{1 - e^{-S\lambda}}e^{-s\lambda}\delta(\tau_t - s) = C\sum_{s=1}^{S}e^{-s\lambda}\delta(\tau_t - s)\qquad(41)$$

And the related distribution function can be obtained by integration the last expression as

$$F_{\tau_t}(\tau_t) = \sum_{s=1}^{\tau_t}\frac{(e^\lambda - 1)}{1 - e^{-S\lambda}}e^{-s\lambda}U(\tau_t - s)\ .\qquad(42)$$

4.2 Mean and variance of the truncated discrete exponential density function

In the process of density function discretisation, the discrete probability values were calculated at time instants sT_s, starting with s = 1. This could be also done starting with the value s = 0. However, this will not be the same, because it will result in different mean values of the delay and imprecise simulation of the discrete delays, as we will show. For the case when the discrete values of the density function start at s = 0, the mean value is

$$\eta_{t0} = \sum_{s=0}^{S-1} (sT_s) \cdot f_{\tau_t}((s+1)T_s) = 0 \cdot f_{\tau_t}(1 \cdot T_s) + T_s \cdot f_{\tau_t}(2 \cdot T_s) + 2T_s \cdot f_{\tau_t}(3 \cdot T_s) + \ldots + ST_s \cdot f_{\tau_t}(ST_s)\,. \quad (43)$$

When the first density value is settled at s = 1 the mean value will be

$$\eta_{t1} = \sum_{s=1}^{S} (sT_s) \cdot f_{\tau_t}(sT_s)\,.$$

These two mean values are not the same. Their difference can be found as follows

$$\begin{aligned} \eta_{t1} = \sum_{s=1}^{S} (sT_s) \cdot f_{\tau_t}(sT_s) &= T_s \cdot f_{\tau_t}(T_s) + 2T_s \cdot f_{\tau_t}(2T_s) + 3T_s \cdot f_{\tau_t}(3T_s) + \ldots (ST_s) \cdot f_{\tau_t}(ST_s) \\ &= T_s \cdot f_{\tau_t}(T_s) + T_s \cdot f_{\tau_t}(2T_s) + T_s \cdot f_{\tau_t}(3T_s) + \ldots T_s \cdot f_{\tau_t}(ST_s) \\ &+ T_s \cdot f_{\tau_t}(2T_s) + 2T_s \cdot f_{\tau_t}(3T_s) + \ldots (S-1)T_s \cdot f_{\tau_t}(ST_s) \\ &= T_s \cdot \sum_{s=1}^{S} \cdot f_{\tau_t}(sT_s) + \eta_0 = T_s + \eta_{t0} \end{aligned} \quad (44)$$

Therefore, the mean value depends on the starting value of the discrete values of the density function, as we said in the previous chapter, and can vary with the duration of discrete interval T_s. It is important to have this difference in mind especially in the case when we are doing simulation of the discrete delay values. Namely, in the case when the first density value is defined for s = 0, and the discrete variable values (variates) need to be generated, then the density values in the first interval from 0 to T_s will be equated with zero, and the values from T_s

to 2T_s will be equated with T_s and so on, until (S-1)T_s is reached. Therefore, if it is **not** important to notify and take into account all the delays inside the first T_s interval this presentation of the density function will be used. However, if all delays in the first T_s interval need to be taken into account the first sample of the density function will be at the first discrete time instant T_s.

Proposition: The mean and variance of the discrete truncated random variable, when the first discrete value with the first density value at s = 1, are expressed in this form

$$\eta_{t1} = \frac{1}{1-e^{-S\lambda}} \frac{Se^{-(S+1)\lambda} - (S+1)e^{-S\lambda} + 1}{1-e^{-\lambda}} . \qquad (45)$$

This value is for one greater than the mean value η_{t0} of the discrete density with the first density value at s = 0. Therefore we can find the mean value η_{t0} in closed form as

$$\eta_{t0} = \frac{e^{-\lambda}}{1-e^{-S\lambda}} \frac{(S-1)e^{-S\lambda} - Se^{-(S-1)\lambda} + 1}{1-e^{-\lambda}} . \qquad (46)$$

Thus, in practical applications, for the defined mean value 1/λ of the continuous exponential distribution, the mean value of discrete exponential η_t can be found and compared.

Proof: Based on the expression (37) for the discrete density function we may have

$$\begin{aligned} \eta_{t1} &= \int_{-\infty}^{\infty} \tau_t f_{\tau_t}(\tau_t) d\tau_t = \int_{-\infty}^{\infty} \tau_t \sum_{s=1}^{S} \frac{(e^{\lambda}-1)}{1-e^{-S\lambda}} e^{-s\lambda} \delta(\tau_t - s) d\tau_t \\ &= \frac{(e^{\lambda}-1)}{1-e^{-S\lambda}} \sum_{s=1}^{S} e^{-s\lambda} \int_{-\infty}^{\infty} \tau_t \cdot \delta(\tau_t - s) d\tau_t = \frac{(e^{\lambda}-1)}{1-e^{-S\lambda}} \sum_{s=1}^{S} s \cdot e^{-s\lambda} = C \cdot \Sigma_S \end{aligned} \qquad (47)$$

where constant C is defined by

$$C = \frac{(e^{\lambda} - 1)}{1 - e^{-S\lambda}} \tag{48}$$

The sum in (47) can be found in a closed form as follows

$$\Sigma_S = \sum_{s=1}^{S} s e^{-s\lambda} = e^{-\lambda} + 2e^{-2\lambda} + 3e^{-3\lambda} + ... + (S-1)e^{-(S-1)\lambda} + Se^{-S\lambda}$$

$$\Sigma_S e^{-\lambda} = \sum_{s=1}^{S} s e^{-\lambda} e^{-s\lambda} = \sum_{s=1}^{S} s e^{-\lambda(1+s)}$$

$$= e^{-2\lambda} + 2e^{-3\lambda} + 3e^{-4\lambda} + ... + (S-2)e^{-(S-1)\lambda} + (S-1)e^{-S\lambda} + Se^{-(S+1)\lambda}$$

$$\Sigma_S - \Sigma_S e^{-\lambda} = \Sigma_S(1 - e^{-\lambda}) = e^{-\lambda} + e^{-2\lambda} + e^{-3\lambda} + ... + e^{-(S-1)\lambda} + e^{-S\lambda} - Se^{-(S+1)\lambda}$$

$$= \sum_{s=1}^{S} (e^{-\lambda})^s - Se^{-(S+1)\lambda} = e^{-\lambda} \frac{1 - e^{-\lambda S}}{1 - e^{-\lambda}} - Se^{-(S+1)\lambda}$$

$$\Sigma_S = e^{-\lambda} \frac{1 - e^{-\lambda S}}{(1 - e^{-\lambda})^2} - \frac{Se^{-(S+1)\lambda}}{1 - e^{-\lambda}} \tag{49}$$

Inserting (48) and (49) in (47) we may calculate the mean value as

$$\begin{aligned}
\eta_{t1} = C \cdot \Sigma_S &= \frac{(e^{\lambda} - 1)}{1 - e^{-S\lambda}} e^{-\lambda} \frac{1 - e^{-\lambda S}}{(1 - e^{-\lambda})^2} - \frac{(e^{\lambda} - 1)}{1 - e^{-S\lambda}} \frac{Se^{-(S+1)\lambda}}{1 - e^{-\lambda}} \\
&= \frac{1}{(1 - e^{-\lambda})} - \frac{e^{-\lambda}(e^{\lambda} - 1)}{1 - e^{-S\lambda}} \frac{Se^{-(S+1)\lambda}}{e^{-\lambda}(1 - e^{-\lambda})} = \frac{1}{(1 - e^{-\lambda})} - \frac{1}{1 - e^{-S\lambda}} \frac{Se^{-(S+1)\lambda}}{e^{-\lambda}} \\
&= \frac{1}{(1 - e^{-\lambda})} - \frac{e^{\lambda} Se^{-(S+1)\lambda}}{1 - e^{-S\lambda}} = \frac{1 - e^{-S\lambda}}{(1 - e^{-S\lambda})(1 - e^{-\lambda})} - \frac{(1 - e^{-\lambda})e^{\lambda} Se^{-(S+1)\lambda}}{(1 - e^{-\lambda})(1 - e^{-S\lambda})} \\
&= \frac{1 - e^{-S\lambda}}{(1 - e^{-S\lambda})(1 - e^{-\lambda})} - \frac{(e^{\lambda} - 1)Se^{-(S+1)\lambda}}{(1 - e^{-\lambda})(1 - e^{-S\lambda})} = \frac{1 - e^{-S\lambda} - Se^{\lambda} e^{-(S+1)\lambda} + Se^{-(S+1)\lambda}}{(1 - e^{-S\lambda})(1 - e^{-\lambda})} \\
&= \frac{1 - e^{-S\lambda} - Se^{-S\lambda} + Se^{-(S+1)\lambda}}{(1 - e^{-S\lambda})(1 - e^{-\lambda})} = \frac{1 - e^{-S\lambda}(1 + S) + Se^{-(S+1)\lambda}}{(1 - e^{-S\lambda})(1 - e^{-\lambda})} \\
&= \frac{1}{1 - e^{-S\lambda}} \frac{Se^{-(S+1)\lambda} - (S+1)e^{-S\lambda} + 1}{1 - e^{-\lambda}}
\end{aligned} \tag{50}$$

Using (44) for T_s = 1, we may derive the expression for the mean of the random variable that starts with n = 0, as follows

$$\eta_{t0}=\eta_{t1}-1=\frac{e^{-\lambda}}{1-e^{-S\lambda}}\frac{(S-1)e^{-S\lambda}-Se^{-(S-1)\lambda}+1}{1-e^{-\lambda}} \tag{51}$$

The mean value can be obtained as a solution of this integral

$$E\{\tau_t^2\}=\int_{-\infty}^{\infty}\tau_t^2 f_{\tau_t}(\tau_t)d\tau_t=\frac{(e^{\lambda}-1)}{1-e^{-S\lambda}}\cdot\sum_{s=1}^{S}e^{-s\lambda}\int_{-\infty}^{\infty}\tau_t^2\delta(\tau_t-s)d\tau_t=\frac{(e^{\lambda}-1)}{1-e^{-S\lambda}}\cdot\sum_{s=1}^{S}n^2e^{-s\lambda}=C\cdot\Sigma_{S2}$$

$$\Sigma_{S2}=\sum_{s=1}^{S}n^2e^{-s\lambda}=e^{-1\lambda}+4e^{-2\lambda}+9e^{-3\lambda}+16e^{-4\lambda}+...+(S-1)^2e^{-(S-1)\lambda}+S^2e^{-S\lambda}$$

$$e^{-\lambda}\Sigma_{S2}=e^{-2\lambda}+4e^{-3\lambda}+9e^{-4\lambda}+16e^{-5\lambda}+...+(S-1)^2e^{-S\lambda}+S^2e^{-(S+1)\lambda}$$

$$(1-e^{-\lambda})\Sigma_{S2}=e^{-1\lambda}+3e^{-2\lambda}+5e^{-3\lambda}+7e^{-4\lambda}+9e^{-5\lambda}+...+(2S-1)e^{-S\lambda}-S^2e^{-(S+1)\lambda}$$

$$e^{-\lambda}\Sigma_{S2}=e^{-2\lambda}+4e^{-3\lambda}+9e^{-4\lambda}+16e^{-5\lambda}+...+(S-1)^2e^{-S\lambda}+S^2e^{-(S+1)\lambda}$$

$$(1-e^{-\lambda})\Sigma_{S2}=e^{-1\lambda}+3e^{-2\lambda}+5e^{-3\lambda}+7e^{-4\lambda}+9e^{-5\lambda}+...+(2S-1)e^{-S\lambda}-S^2e^{-(S+1)\lambda}$$

$$\Sigma_S=\sum_{s=1}^{S}se^{-s\lambda}=e^{-\lambda}+2e^{-2\lambda}+3e^{-3\lambda}+...+(S-1)e^{-(S-1)\lambda}+Se^{-S\lambda}$$

$$\Sigma_Se^{-\lambda}=\sum_{s=1}^{S}se^{-\lambda(1+s)}=e^{-2\lambda}+2e^{-3\lambda}+3e^{-4\lambda}+...+(S-2)e^{-(S-1)\lambda}+(S-1)e^{-S\lambda}+Se^{-(S+1)\lambda}$$

$$\Sigma_S-\Sigma_Se^{-\lambda}=e^{-\lambda}+e^{-2\lambda}+e^{-3\lambda}+...+e^{-(S-1)\lambda}+e^{-S\lambda}-Se^{-(S+1)\lambda}=\sum_{s=1}^{S}(e^{-\lambda})^s-Se^{-(S+1)\lambda}$$

$$=e^{-\lambda}\frac{1-e^{-\lambda S}}{1-e^{-\lambda}}-Se^{-(S+1)\lambda}$$

The sum can be expressed as

$$\Sigma_S = e^{-\lambda}\frac{1-e^{-\lambda S}}{(1-e^{-\lambda})^2} - \frac{Se^{-(S+1)\lambda}}{1-e^{-\lambda}}$$

The mean square value can be calculated as

$$E\{\tau_t^2\} = \eta_{t1}^2 + \frac{e^{-2\lambda}}{1-e^{-S\lambda}}\left[\frac{2(1-e^{-S\lambda})}{(1-e^{-\lambda})^2} - \frac{2Se^{-(S-1)\lambda}}{1-e^{-\lambda}} - S(S-1)e^{-(S-2)\lambda}\right] \tag{52}$$

and the variance is

$$\sigma_{t1}^2 = E\{\tau_t^2\} - \mu_{t1}^2, \tag{53}$$

which completes our proofs.

The three graphs, the mean, mean square value and variance as a function of parameter $1/\lambda$ starting with $1/\lambda = 1$ and finishing with $1/\lambda = 20$, for $S = 40$, are presented in Fig. 10. Alongside with these graphs, the graphs of the mean, mean square value and variance for discrete non-truncated density function, continuous (non-truncated) density function and continuous truncated density function are presented.

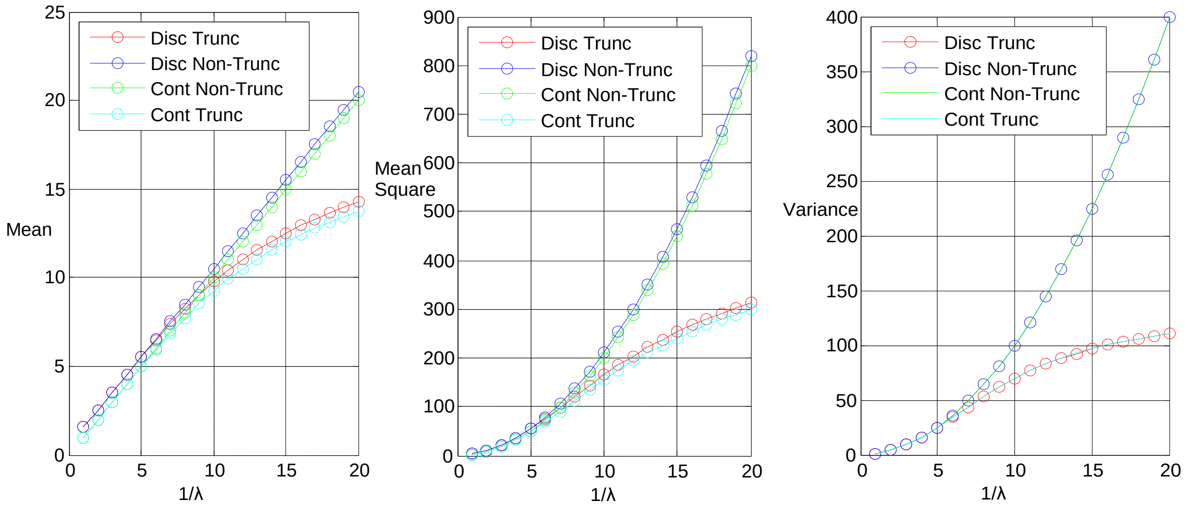

Figure 12 The mean, mean square value and variance.

5. Theory and practice of generating exponential random variates

Various methods can be used to generate the values of a random variable that belong to a particular distribution function. Often used method is the inverse transformation method (ITM). According to ITM method the inverse function $F^{-1}(x)$ of the distribution function $F(x)$ is to be found, and then the variates of that distribution can be generated using a uniform random number generator.

5.1 Generating of variates of a continuous exponential distribution

For the exponential distribution expressed as

$$F_{\tau}(\tau) = 1 - e^{-\lambda\tau}\,. \tag{54}$$

the inverse function is

$$\tau = -\frac{1}{\lambda}\ln(1 - F_{\tau}(\tau))\,, \tag{55}$$

which can be, for the purpose of generating exponential variates, expressed as

$$\tau_v = -\frac{1}{\lambda}\ln(1 - F)\,, \tag{56}$$

where F is the value of distribution function that corresponds to the particular value τ_v of the random variable τ. Therefore if random values of a uniform distribution are generated inside the interval (0,1), which correspond to the values of F, the variates τ_v can be calculated. In the case mentioned, these variates will be distributed according to the exponential distribution with parameter λ. If sufficient number of variates is generated the empirical density and distribution functions can be obtained which are approximating the underlying theoretical density and distribution functions. The problem of defining the term “sufficient number” of random variates is beyond this analysis.

5.2 Generating variates of the discrete truncated exponential distribution

According to the ***inverse transformation method***, a variate of a uniform distribution F will be generated, continuous delay value τ will be calculated and then a discrete variate value τ_v will be assigned. The value of the distribution function for particular delay $\tau_v = \tau$ was calculated for T_s =1 and expressed in this form

$$F_{\tau_t}(\tau_t) = \frac{1}{1-e^{-S\lambda}}(1-e^{-\tau_t\lambda}) = F \ . \tag{57}$$

Then the delay is expressed as a function of the distribution function value as

$$\tau_t = -\frac{1}{\lambda}\ln(1-F/A) \ , \tag{58}$$

where A is a constant, $A = 1/(1-e^{-S\lambda})$. According to the inverse transformation method, we need to generate uniform continuous valued variates F and calculate the delay values. Because these calculated delay values will be real numbers, they need to be equated to the integer which ***is not smaller*** than the real number in the argument of τ_t, i.e.,

$$\tau_v = \left\lceil -\frac{1}{\lambda}\ln(1-F/A) \right\rceil . \tag{59}$$

The discrete values, calculated in this way, represent the variates τ_v of an exponential truncated discrete density function that has the first discrete value at s = 1. The generated variates will have an empirical density function that corresponds to the theoretical density.

If the first discrete value is to be at s = 0, the discrete truncated density and distribution functions are

$$f_{\tau_t}(\tau_t) = \sum_{s=0}^{S-1} \frac{(e^{\lambda}-1)}{1-e^{-S\lambda}} e^{-(s+1)\lambda} \delta(\tau_t - s), \; 0 \le \tau_t \le S\text{-}1, \text{ and} \tag{60}$$

$$F_{\tau_t}(\tau_t) = \sum_{s=0}^{\tau_t} \frac{(e^{\lambda}-1)}{1-e^{-S\lambda}} e^{-(s+1)\lambda} U(\tau_t - s). \tag{61}$$

For this case the calculated delay values as real numbers, should be equated to the integer which ***is not greater*** than the real number in the argument of (56), i.e.,

$$\tau_v = \left\lfloor -\frac{1}{\lambda} \ln(1 - F / A) \right\rfloor.$$

Example:

a) Generate 1000, 10,000 and 1,000,000 variates τ_v of a discrete truncated exponential distribution function and find the related empirical density for the known $\lambda = 1$.

b) Using the method of the mean square error values find the mean square error of the empirical discrete truncated probability density function (pdfdt). Compare the level of these error measures as a function of the number of generated variates.

c) Calculate the theoretical mean and variance. Find the empirical mean and variance for the three cases and present them in tabular form.

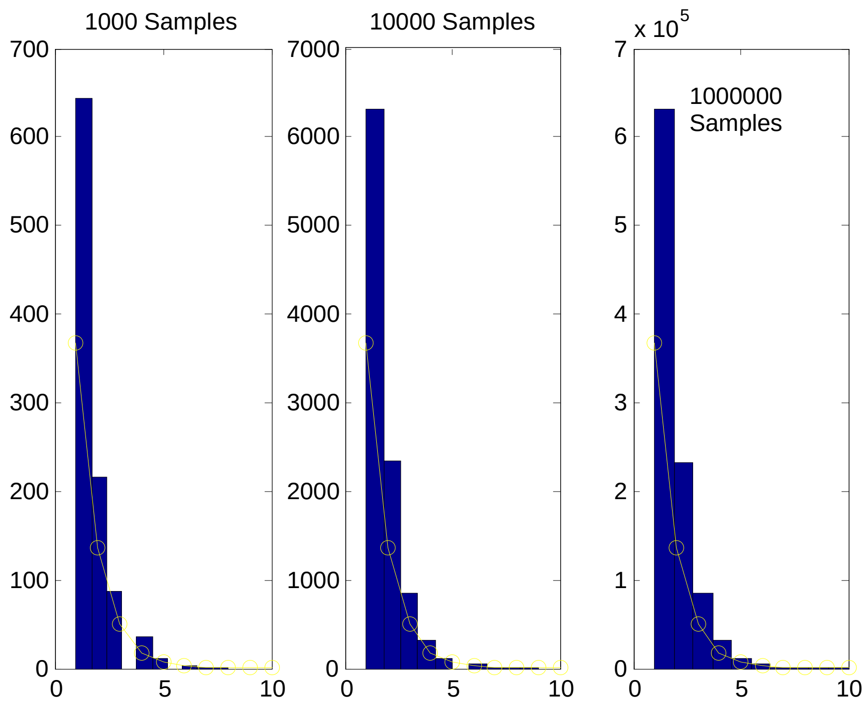

Figure 13 The empirical density function plotted alongside with the continuous expected theoretical density function.

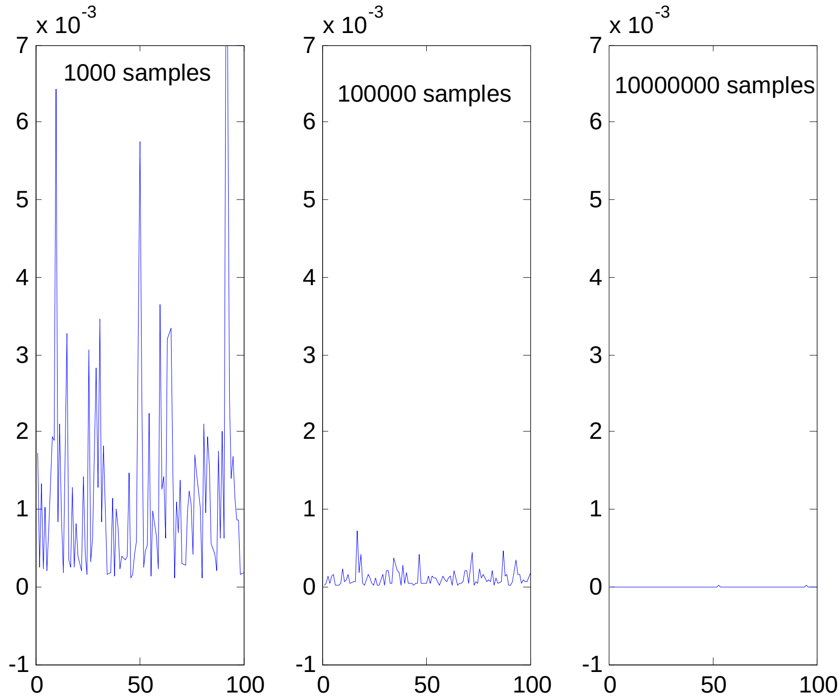

Figure 14 The mean square error for the three empirical density functions.

The theoretical mean value of 1.5815 and variance of 0.9161 are calculated as follows

$$\eta_{\tau_t} = \int_{-\infty}^{\infty} \tau_t \sum_{s=1}^{10} \frac{(e^{\lambda}-1)}{1-e^{-10\lambda}} e^{-s\lambda} \delta(\tau_t - s) d\tau_t = \sum_{s=1}^{10} s \frac{(e^{\lambda}-1)}{1-e^{-10\lambda}} e^{-s\lambda} = 1.5815,$$

$$E(\tau_t^2) = \int_{-\infty}^{\infty} \tau_t^2 \sum_{s=1}^{10} \frac{(e^{\lambda}-1)}{1-e^{-10\lambda}} e^{-s\lambda} \delta(\tau_t - s) d\tau_t = \sum_{s=1}^{10} s^2 \frac{(e^{\lambda}-1)}{1-e^{-10\lambda}} e^{-s\lambda} = 3.4173, \text{ and}$$

$$\sigma_{\tau_t}^2 = E(\tau_t^2) - \eta_{\tau_t}^2 = 3.4173 - 2.5011 = 0.91615.$$

The moments are estimated using generated variates. The more variates we generate, the more precisely the moments' values are estimated. Table 1 presents theoretical and empirical moments. The moments tend to the theoretical values when the number of variates increases. The more variates we generate, the closer empirical moments are to the theoretically expected values.

Table 1 Moments calculated for given samples.

Number of samples	Mean	Variance
1000	1.6	0.979
10,000	1.587	0.896
1,000,000	1,582	0.917
Theoretical values	1.5815	0.9161

6. Stochastic discrete truncated exponential processes

Stochastic truncated exponential process $T(n)$ can be constructed by defining the set of elementary events (outcomes) τ_t and associated probabilities. Then the realisations of this stochastic process $\tau_t(n)$ can be generated which are represented as discrete time functions showing dependence of the variates' values τ_v as a function of discrete random delay n. The realisations will form an assemble of the stochastic process $T(n)$.

Example:

Suppose the discrete truncated exponential density function is defined by $\lambda = 1$ and $S = 10$. Define the set of elementary events (outcomes) *Se* and calculate their discrete probabilities. On this set of elementary events a stochastic process $T(n)$ is defined as a time series showing the discrete delays τ_v as a function of the time n. Generate an assemble of a 1000 realisations of this stochastic process, each containing 10 realisations of the random experiment defined on the set *Se* for $n = 1, 2, 3, \ldots, 10$. Having available empirical data, investigate the stationarity of the process.

Solutions:

Using the expression

$$P_s = P(\tau_t = s) = \frac{(e^{\lambda} - 1)}{1 - e^{-S\lambda}} e^{-(s+1)\lambda},$$

the probabilities presented in Table 2 are obtained by calculations.

Table 2 Calculated discrete theoretical probabilities

Value	Probability
1	0.6321
2	0.2326
3	0.0856
4	0.0315
5	0.0116
6	0.0043
7	0.0016
8	0.00058
9	0.000212
10	7.801e-05

We can take all values across the ensemble defined for n = 1 to n = 10 and calculate the mean values and variances, as presented in Table 3. The mean values are more or less the same. It is expected this process is stationary.

Table 3 Empirical mean and variance for n = 1 to n = 10.

n	Mean	Variance
1	1.562	0.8990
2	1.595	1.0239
3	1.605	0.9979
4	1.552	0.9001
5	1.597	0.899
6	1.565	0.88
7	1.610	0.982
8	1.595	0.989
9	1.572	1.0838
10	1.542	0.8463
Average	1.5795	0.9501

Example:

Define a new stochastic process $X(n) = T(n) + T(n\text{-}1)$. Find the theoretical mean and variances of this process.

Solution:

$$E\{X(n)\} = E\{T(n)\} + E\{T(n\text{-}1)\} = 2\,E\{T(n)\} = 3.1630\text{, independence.}$$

$$E\{X^2(n)\} = E\{T^2(n) + 2T(n)T(n-1) + T^2(n-1)\} = E\{T^2\} + 2E^2\{T\}$$

$$Var(X(n)) = E\{X^2(n)\} - E^2\{X(n)\} = 2E\{T^2\} + 2E^2\{T\} - 4E^2\{T\}$$
$$= \quad 2E\{T^2\} - 2E^2\{T\} = 2Var(T(n)) = 1.8323$$

7. Conclusions

The expressions of truncated discrete density and distribution functions are derived and expressed in closed form in terms of delta functions. The functions are compared with related functions of continuous exponential random variables. The first and second moments are derived and numerically analysed in relation with the moments of continuous random variable. Practical development of a discrete exponential truncated stochastic process is illustrated using simulation examples.

References

[1] M. Ansanullah, “A Characteristic Property of the Exponential Distribution”, The Annals of Statistics, Vol. 5, No. 3, pp. 580 – 583.

[2] G. T. Freitas De Abreu, ”On the Generation of Tikhonov Variates”, IEEE Trans. Comm., Vol 56, No. 7, July 2008, pp. 1157-1168.

[3] J. I. Seo, S. B. Kang, “Notes on the exponentiated half logistic distribution”, Applied Mathematical Modelling, 39, 2015, 6491 – 6500.

[4] S. A. Eftekhari, “A note on mathematical treatment of the Dirac-delta function in the differential quadrature bending a forced vibration analysis of beams and rectangular plates subjected to concentrated loads”, Applied Mathematical Modelling, 39, Elsevier, 2015, 6223 – 6242.

[5] S. M. Berber, “Probability of Error Derivatives for Binary and Chaos-Based CDMA Systems in Wide-Band Channels”, IEEE Trans. on Wireless Communications, Vol. 13, NO. 10, October 2014, pp. 5596-5606.

[6] S. Boucheron, “A Bernstein-Von Mises Theorem for discrete probability distributions”, Electronic Journal of Statistics, Vol. 3, 2009, pp. 114 – 148.

[7] H. Sato, M. Ikota, A. Sugimoto, H. Masuda, "A new defect distribution metrology with a consistent discrete exponential formula and its applications," in Semiconductor Manufacturing, IEEE Transactions on , vol.12, no.4, Nov 1999, pp. 409-418.

[8] K. Ho and K.-H. Cheng, “A two-dimensional Fibonacci buddy system for dynamic resource management in a partitionable mesh,” in Aerospace and Electronics Conference, NAECON, Proceedings of the IEEE, 1997, vol. 1, pp. 195–201.

[9] R. Dilip, “Discrete Rayleigh Distribution”, IEEE Trans. on Reliability, Vol. 53, NO. 2, June 2014, pp. 5596-5606.

[10] R. Dilip, “A Discretizing Approach for Evaluating Reliability of Complex Systems Under Stress-Strength Model”, IEEE Trans. on Reliability, Vol. 50, NO. 2, June 2011, pp. 145-150.

[11] R. Dilip, “The Discrete Normal Distribution”, Communications in Statistics – Theory and Methods, 32:10, 2003, pp. 1871 – 1883.

[12] R. Dilip, “On Classifications of Multivariate Life Distribution in the Discrete Setup”, Microelectron. Reliabil., Vol. 37, No. 2, 1996, pp. 361 – 366.

[13] E. Xekalaki, “Hazard Functions and Life Distributions in Discrete Time,” Communications in Statistics – Theory and Methods, 12:21, 1983, pp. 25 – 2509.