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Comprehensive Analysis of Gaussian Discrete Truncated Random Variables

by

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Abstract

There are, maybe, the largest number of random variables and stochastic processes in nature, which are described in science and engineering, that have Gaussian continuous distribution and density function. However, the theoretical analysis and application of these processes, in particular when the random variable is both discrete and truncated, is not completed. For the sake of completeness, this Report presents a brief review of Gaussian continuous density functions, both un-truncated and truncated, which are well described in existing literature. Then the discrete density function is derived and expressed in terms of Dirac’s delta functions and related mean and variance are derived and analyzed. The necessity of having truncated discrete density function, from the application point of view in communication systems, for example, is explained and related density and distribution functions are derived. For these functions, the mean and variance are expressed as functions of the length of the defined truncation interval and compared with related moments of the continuous truncated density function. The important advancement is achieved by deriving the truncated discrete density functions and expressing them in terms of Dirac’s delta and unit step functions. In this way it becomes possible to solve integrals which contain these density and distribution functions.
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1. Continuous Gaussian density function

The density and distribution function of a Gaussian continuous random variable $\tau$ can be expressed as

$$f_c(\tau) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\tau-\eta)^2}{2\sigma^2}},$$

(1)

and graphically presented, as shown in Fig. 1, for the mean value $\eta$ equal zero and the finite variance $\sigma^2$.

![Figure 1 Continuous Gaussian density function](image)

**Definitions for standard normal functions $N(0,1)$**

The density and distribution functions of the standard normal distribution $N(0, 1)$ can be expressed using $fai$ function, $\phi$ and $\Phi$, expressed in the following forms and related to $Q$ and $erfc$ functions:

$$\phi(\tau) = \frac{1}{\sqrt{2\pi}} e^{-\tau^2},$$

(3)

$$\Phi(\tau) = \int_{-\infty}^{\tau} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{1}{2} \text{erfc}(\tau / \sqrt{2}) = \frac{1}{2} (1 + \text{erf}(\tau / \sqrt{2})) = 1 - Q(\tau)$$

(4)

$$\Phi(\tau) = \int_{-\infty}^{\tau} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{1}{2} \text{erfc}(\tau / \sqrt{2}) = \frac{1}{2} (2 - \text{erfc}(\tau / \sqrt{2})) = 1 - \frac{1}{2} \text{erfc}(\tau / \sqrt{2})$$

(5)

$$Q(\tau) = 1 - \Phi(\tau) = \frac{1}{2} (1 + \text{erf}(\tau / \sqrt{2})) = \frac{1}{2} - \frac{1}{2} \text{erf}(\tau / \sqrt{2}) = \frac{1}{2} \text{erfc}(\tau / \sqrt{2})$$

(6)

The integral limits defining the $fai$ and $erf$ functions are presented in Fig. 2.
Figure 2 Definition of  $f_{ai}$ and $erf$ functions for continuous Gaussian density function.

**Definition of error functions ($erf$)**

Error function and its complementary error function are defined as follows

\[
erf(\tau) = \frac{1}{\sqrt{\pi}} \int_{-\tau}^{\tau} e^{-x^2} dx = \frac{2}{\sqrt{\pi}} \int_{0}^{\tau} e^{-x^2} dx = 1 - \text{erfc}(\tau) = 2\Phi(\tau\sqrt{2}) - 1
\]  

(7)

\[
erfc(\tau) = \int_{\tau}^{\infty} \frac{2}{\sqrt{\pi}} e^{-x^2} dx = 1 - \text{erf}(\tau) = 2(1 - \Phi(\tau\sqrt{2}))
\]  

(8)

\[
erf(-\tau) = -\text{erf}(\tau)
\]  

(9)

\[
erfc(-\tau) = 2 - \text{erfc}(\tau)
\]  

(10)

Graphs of $erf$ and $erfc$ are presented in Fig. 3.

Figure 3 Graphs of $erf$ and $erfc$ function for continuous Gaussian random variable.
2. Truncated continuous Gaussian density function

2.1 Truncated continuous Gaussian density function with a finite mean value

Density presentation with \( \phi \) (\( \phi \) and \( \Phi \)) functions: If the expected delay is finite and can have values in a limited interval \( 2T \), the continuous truncated density function needs to be found. It is well known fact that the truncated continuous Gaussian density function is shown in Fig. 4 and expressed as

\[
 f_t(\tau) = \begin{cases} 
 \frac{1}{\sqrt{2\pi}\sigma^2} e^{-\frac{(\tau-\eta)^2}{2\sigma^2}} & a \leq \tau < b \\
 0 & \text{otherwise} 
\end{cases}
\]

\( \Phi(\frac{b-\eta}{\sigma}) - \Phi(\frac{a-\eta}{\sigma}) \)

where \( P(T) \) depends on the width of the truncation interval \( 2T = (b - a) \) and is expressed as

\[
 C(T) = \frac{1}{\Phi(\frac{b-\eta}{\sigma}) - \Phi(\frac{a-\eta}{\sigma})}. 
\]

The mean value of this density is

\[
 \eta_t = \eta + \sigma C(T) \left( \phi(\frac{a-\eta}{\sigma}) - \phi(\frac{b-\eta}{\sigma}) \right) 
\]

where

\[
 \phi(\frac{a-\eta}{\sigma}) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(a-\eta)^2}{2\sigma^2}}. 
\]
Here, $2T = (b - a)$ is defined as truncation interval in which the truncated function exists. Beyond this interval the function has zero values. The variance of the truncated function is

$$
\sigma_i^2 = \sigma^2 \left[ 1 + C(T) \left( \frac{a - \eta}{\sigma} \phi\left( \frac{a - \eta}{\sigma} \right) - \frac{b - \eta}{\sigma} \phi\left( \frac{b - \eta}{\sigma} \right) \right) \right] - \sigma^2 C^2(T) \left( \phi\left( \frac{a - \eta}{\sigma} \right) - \phi\left( \frac{b - \eta}{\sigma} \right) \right)^2 \tag{15}
$$

**Density presentation in terms of erfc-functions**: The truncated function can also be expressed in terms of erfc-functions.

**Proof**: Because

$$
\Phi(\tau) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \frac{1}{2} \text{erfc}(-\tau / \sqrt{2}) = \frac{1}{2} \left( 2 - \text{erfc}(\tau / \sqrt{2}) \right) = 1 - \frac{1}{2} \text{erfc}(\tau / \sqrt{2}) \tag{16}
$$

The fai-function is

$$
\Phi\left( \frac{b - \eta}{\sigma} \right) = \frac{1}{2} \left( 1 + \text{erf} \left( \frac{b - \eta}{\sqrt{2\sigma^2}} \right) \right) = 1 - \frac{1}{2} \text{erfc} \left( \frac{b - \eta}{\sqrt{2\sigma^2}} \right) \tag{17}
$$

And $\phi$ and $P$ functions are

$$
\phi\left( \frac{a - \eta}{\sigma} \right) = \frac{1}{\sqrt{2\pi}} e^{- \left( \frac{a - \eta}{\sigma} \right)^2 / 2\sigma^2} \tag{18}
$$

$$
C(T) = \frac{1}{\Phi\left( \frac{b - \eta}{\sigma} \right) - \Phi\left( \frac{a - \eta}{\sigma} \right)} = \frac{1}{1 - \frac{1}{2} \text{erfc} \left( \frac{b - \eta}{\sqrt{2\sigma^2}} \right) - 1 + \frac{1}{2} \text{erfc} \left( \frac{a - \eta}{\sqrt{2\sigma^2}} \right)} = 2 \left( \frac{\text{erfc} \left( \frac{a - \eta}{\sqrt{2\sigma^2}} \right) - \text{erfc} \left( \frac{b - \eta}{\sqrt{2\sigma^2}} \right)}{\sqrt{2\sigma^2}} \right)^2 \tag{19}
$$

The continuous truncated density function is

$$
f_{ct}(\tau) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{- \left( \frac{\tau - \eta}{\sigma} \right)^2 / 2\sigma^2} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{- \left( \frac{\tau - \eta}{\sigma} \right)^2 / 2\sigma^2} \tag{20}
$$

The mean value can be calculated as
\[ \eta_i = \eta + C(T)\sigma \left( \frac{a-\eta}{\sigma} - \phi\left(\frac{b-\eta}{\sigma}\right) \right) \]
\[ = \eta + \frac{2\sigma}{\text{erfc} \frac{a-\eta}{\sqrt{2\sigma^2}} - \text{erfc} \frac{b-\eta}{\sqrt{2\sigma^2}}} \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{(a-\eta)^2}{2\sigma^2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(b-\eta)^2}{2\sigma^2}} \right), \]
\[ = \eta + C(T)\sigma \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{(a-\eta)^2}{2\sigma^2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(b-\eta)^2}{2\sigma^2}} \right) \]  
\[ \text{(21)} \]

and the variance is
\[ \sigma_i^2 = \sigma^2 \left[ 1 + C(T) \left( \frac{a-\eta}{\sigma} - \phi\left(\frac{b-\eta}{\sigma}\right) \right) - \sigma^2 C^2(T) \left( \phi\left(\frac{a-\eta}{\sigma}\right) - \phi\left(\frac{b-\eta}{\sigma}\right) \right)^2 \right] \]
\[ = \sigma^2 \left[ 1 + C(T) \left( \frac{a-\eta}{\sqrt{2\pi}} e^{-\frac{(a-\eta)^2}{2\sigma^2}} - \frac{b-\eta}{\sqrt{2\pi}} e^{-\frac{(b-\eta)^2}{2\sigma^2}} \right) - \sigma^2 C^2(T) \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{(a-\eta)^2}{2\sigma^2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(b-\eta)^2}{2\sigma^2}} \right)^2 \right] \]
\[ = \sigma^2 \left[ 1 + C(T) \left( \frac{a-\eta}{\sqrt{2\pi\sigma^2}} - \frac{b-\eta}{\sqrt{2\pi\sigma^2}} \right) - \sigma^2 C^2(T) \left( \frac{1}{\sqrt{2\pi}} e^{-a^2/2\sigma^2} - \frac{1}{\sqrt{2\pi}} e^{-b^2/2\sigma^2} \right)^2 \right] \]  
\[ \text{(22)} \]

2.2 Truncated continuous Gaussian density function with zero mean value

**Derivatives of the truncated density function for continuous asymmetric density function:**

In the case when the continuous non-truncated variable has zero mean, i.e. \( \eta = 0 \), and the density is asymmetric in respect to zero and defined in the interval \( 2T = (a, b) \) for \( |a| \neq |b| \), we may derive the following expressions for the density, mean and variance of the truncated continuous random variable as follows

\[ C(T) = \frac{2}{\text{erfc} \frac{a \sigma}{\sqrt{2\sigma^2}} - \text{erfc} \frac{b \sigma}{\sqrt{2\sigma^2}}} = \frac{2}{\text{erfc} \frac{a}{\sqrt{2\sigma^2}} - \text{erfc} \frac{b}{\sqrt{2\sigma^2}}} \]  
\[ \text{(23)} \]

\[ f_\tau(t) = C(T) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\tau)^2}{2\sigma^2}} = C(T) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-t^2/2\sigma^2}, \quad a \leq \tau < b, \]  
\[ \text{(24)} \]

\[ \eta_i = \eta + P(T)\sigma \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{(a-\eta)^2}{2\sigma^2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(b-\eta)^2}{2\sigma^2}} \right) \]
\[ = P(T)\sigma \left( \frac{1}{\sqrt{2\pi}} e^{-a^2/2\sigma^2} - \frac{1}{\sqrt{2\pi}} e^{-b^2/2\sigma^2} \right) \]  
\[ \text{(25)} \]
Derivative of the truncated density function for continuous symmetric density function:

In the case when the random variable has zero mean and symmetric density defined in the interval \((a, b)\) with the width \((b - a) = T - (-T) = 2T\) we can derive the related density function and its parameters. Having in mind the relation between \(\text{erf}\) and \(\text{erfc}\), expressed as

\[
(26) \quad \text{erfc}(\tau) = 2 - \text{erfc}(\tau)
\]

we may find

\[
(27) \quad C(T) = \frac{2}{\text{erfc}\left(\frac{a}{\sqrt{2}\sigma}\right) - \text{erfc}\left(\frac{b}{\sqrt{2}\sigma}\right)} = \frac{2}{\text{erfc}\left(\frac{-T}{\sqrt{2}\sigma}\right) - \text{erfc}\left(\frac{T}{\sqrt{2}\sigma}\right)} = \frac{2}{1 - \text{erfc}T / \sqrt{2}\sigma^2}
\]

and the density function expressed as

\[
(28) \quad f_\alpha(\tau) = C(T) \frac{1}{\sqrt{2\pi}\sigma^3} e^{-(\tau - \eta)/2\sigma^2} = C(T) \frac{1}{\sqrt{2\pi}\sigma^3} e^{-\tau^2/2\sigma^2}, \quad a \leq \tau < b.
\]

Due to the symmetry of the density function the mean is zero

\[
(29) \quad \eta = \eta + C(T)\sigma \left( \frac{1}{\sqrt{2\pi}} e^{-a^2/2\sigma^2} - \frac{1}{\sqrt{2\pi}} e^{-(b - \eta)^2/2\sigma^2} \right) = C(T)\sigma \left( \frac{1}{\sqrt{2\pi}} e^{-T^2/2\sigma^2} - \frac{1}{\sqrt{2\pi}} e^{-T^2/2\sigma^2} \right) = 0
\]

and the variance is

\[
(30) \quad \sigma^2_i = \sigma^2 \left[ 1 + C(T) \left( \frac{a}{\sqrt{2\pi}\sigma^3} e^{-a^2/2\sigma^2} - \frac{b}{\sqrt{2\pi}\sigma^3} e^{-b^2/2\sigma^2} \right) \right] - \sigma^2 C^2(T) \left( \frac{1}{\sqrt{2\pi}} e^{-a^2/2\sigma^2} - \frac{1}{\sqrt{2\pi}} e^{-b^2/2\sigma^2} \right)^2
\]

\[
= \sigma^2 \left[ 1 + \frac{1}{1 - \text{erfc}T / \sqrt{2}\sigma^2} \left( \frac{-T}{\sqrt{2\pi}\sigma^3} e^{-T^2/2\sigma^2} - \frac{T}{\sqrt{2\pi}\sigma^3} e^{-T^2/2\sigma^2} \right) \right] + 0
\]

\[
= \sigma^2 \left[ 1 - \frac{1}{1 - \text{erfc}T / \sqrt{2}\sigma^2} \frac{2T}{\sqrt{2\pi}\sigma^3} e^{-T^2/2\sigma^2} \right]
\]

We may present this variance as a function of the variance for continuous density \(\sigma^2\) for fixed \(T\) values, as shown in Fig. 5a). Alongside this graph the dependence of \(\sigma^2\) on \(\sigma^2\) is presented for the sake of comparison. In that case the truncated variance values will be smaller and they will increase slower than the values for variance \(\sigma^2\). Secondly, we may express variance of the truncated density as a function of \(T\) for fixed variance of the continuous function \(\sigma^2\), as shown in Fig. 5a). When \(T\) increases the variance truncated increases and goes to saturation for
reasonable high $T$ and tends to the variance of continuous variable when $T$ tends to infinity. That makes sense because for extremely high $T$ the density function is very close to the continuous density.

\[
\sigma_t^2 = \sigma^2 \left[ 1 - \frac{1}{1 - \text{erfc}T / \sqrt{2}\sigma^2} \frac{2T}{\sqrt{2\pi\sigma^2}} e^{-\frac{s^2}{2}} \right] = \left[ 1 - \frac{1}{1 - \text{erfc}T / \sqrt{2} \sqrt{2\pi}} \frac{2T}{\sqrt{2\pi} e^{-\frac{s^2}{2}}} \right]
\]

(31)

Figure 5  

a) Relations between variances of continuous density and truncated density. 

b) Dependence of the truncated variance on the truncation interval $T$.

For standard normal distribution, $\sigma = 1$, we may have

\[
\sigma_t^2 = \sigma^2 \left[ 1 - \frac{1}{1 - \text{erfc}T / \sqrt{2}\sigma^2} \frac{2T}{\sqrt{2\pi\sigma^2}} e^{-\frac{s^2}{2}} \right] = \left[ 1 - \frac{1}{1 - \text{erfc}T / \sqrt{2} \sqrt{2\pi}} \frac{2T}{\sqrt{2\pi} e^{-\frac{s^2}{2}}} \right]
\]

(31)

Figure 6   

a) Graphs of both $N(0, 1)$ inside interval $2C = (-5, 5)$ and truncated $N(0, 0.8796)$ inside interval $2T = (-2, +2)$.

b) Continuous density function and its truncated density for the truncation interval $T$ as a parameter.
3. **Discrete Gaussian density function**

In this chapter the discrete Gaussian density function will be presented and expressed in terms of delta functions. The, in the next chapter, the truncated density functions will be derived.

3.1 **Discrete density with zero mean**

The procedure of discretizing a continuous Gaussian density function is illustrated in Fig. 7.

![Fig. 7 Discretisation of Gaussian density function](image)

**Proposition:** The discrete density function of Gaussian random variable, having the values at the uniformly spaced instants $T_s$ of a random variable $\tau$, can be expressed as

$$f_d(\tau) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left( \text{erfc} \left( \frac{(2n-1)T_s/2}{\sqrt{2}\sigma} \right) - \text{erfc} \left( \frac{(2n+1)T_s/2}{\sqrt{2}\sigma} \right) \right) \delta(\tau - nT_s),$$  \hspace{1cm} (32)

where $\delta(\cdot)$ are Dirac’s delta functions. For a defined unit interval $T_s = 1$, this function is

$$f_d(\tau) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left( \text{erfc} \left( \frac{(2n-1)T_s}{\sqrt{8}\sigma^2} \right) - \text{erfc} \left( \frac{(2n+1)T_s}{\sqrt{8}\sigma^2} \right) \right) \delta(\tau - n).$$  \hspace{1cm} (33)

**Proof:** If this density is uniformly discretized in respect to $\tau$, with the interval of discretisation of $T_s$, the probability value in each interval can be calculated as follows. The probability value inside the interval around zero can be calculated as
\[
P(-T_s/2 \leq \tau < T_s/2) = \int_{-T_s/2}^{T_s/2} f_s(\tau) d\tau = \int_{-T_s/2}^{T_s/2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\tau^2}{2\sigma^2}} d\tau
\]

\[
= \int_{-T_s/2}^{T_s/2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\tau^2}{2\sigma^2}} d\tau - \int_{-T_s/2}^{T_s/2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\tau^2}{2\sigma^2}} d\tau = \frac{1}{2} \text{erfc} \left( \frac{T_s/2}{\sqrt{2\sigma^2}} \right) - \frac{1}{2} \text{erfc} \left( \frac{-T_s/2}{\sqrt{2\sigma^2}} \right)
\]  

(34)

The probability the random variable is in the first positive interval \((n=1)\) is

\[
P(T_s/2 \leq \tau < 3T_s/2) = \int_{T_s/2}^{3T_s/2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\tau^2}{2\sigma^2}} d\tau - \int_{-T_s/2}^{T_s/2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\tau^2}{2\sigma^2}} d\tau = \frac{1}{2} \text{erfc} \left( \frac{T_s/2}{\sqrt{2\sigma^2}} \right) - \frac{1}{2} \text{erfc} \left( \frac{3T_s/2}{\sqrt{2\sigma^2}} \right).
\]

(35)

and for any discrete interval defined by \(n\), the related probabilities can be expressed as

\[
P((2n-1)T_s/2 \leq \tau < (2n+1)T_s/2) = \frac{1}{2} \text{erfc} \left( \frac{(2n-1)T_s/2}{\sqrt{2\sigma^2}} \right) - \frac{1}{2} \text{erfc} \left( \frac{(2n+1)T_s/2}{\sqrt{2\sigma^2}} \right).
\]

(36)

The discrete probability values for negative \(n\), i.e., \(n < 0\), which are symmetrically placed in respect to \(n = 0\), are expressed as

\[
P((2n-1)T_s/2 \leq \tau < (2n+1)T_s/2) = \frac{1}{2} \text{erfc} \left( \frac{(2n-1)T_s/2}{\sqrt{2\sigma^2}} \right) - \frac{1}{2} \text{erfc} \left( \frac{(2n+1)T_s/2}{\sqrt{2\sigma^2}} \right).
\]

(37)

If the calculated probabilities are assigned as the weights to the Dirac’s delta functions that are defined at discrete instants \(\tau = nT_s\), then the obtained function represents the discrete Gaussian density function expressed as

\[
f_s(\tau) = \frac{1}{2} \sum_{n=-\infty}^{\infty} P((2n-1)T_s/2 \leq \tau < (2n+1)/2) \delta(\tau - nT_s)
\]

\[
= \frac{1}{2} \sum_{n=-\infty}^{\infty} \left( \text{erfc} \left( \frac{(2n-1)T_s/2}{\sqrt{2\sigma^2}} \right) - \text{erfc} \left( \frac{(2n+1)T_s/2}{\sqrt{2\sigma^2}} \right) \delta(\tau - nT_s)
\]

(38)

For the unit interval, \(T_s = 1\), this density is

\[
f_s(\tau) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left( \text{erfc} \left( \frac{(2n-1)}{\sqrt{8\sigma^2}} \right) - \text{erfc} \left( \frac{(2n+1)}{\sqrt{8\sigma^2}} \right) \delta(\tau - n) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \text{Erfc}(n) \delta(\tau - n),
\]

where the \(\text{Erfc}(n)\) is defined as

\[
\text{Erfc}(n) = \text{erfc} \left( \frac{(2n-1)}{\sqrt{8\sigma^2}} \right) - \text{erfc} \left( \frac{(2n+1)}{\sqrt{8\sigma^2}} \right),
\]

(40)

which completes our proof. The continuous and discrete density functions for two different variances are presented in Fig. 8.
Figure 8 Discrete Gaussian density functions.

**Proposition:** The integral of the discrete density function (39), calculated for all \( n \) values, and is one.

\[
\int_{\tau=-\infty}^{\infty} f_d(\tau) d\tau = 1
\]

**Proof:**

\[
\int_{\tau=-\infty}^{\infty} f_d(\tau) d\tau = \int_{\tau=0}^{0} \frac{1}{2} \sum_{n=-\infty}^{\infty} \text{Erfc}(n) \delta(\tau-n) d\tau = \frac{1}{2} \sum_{n=-\infty}^{\infty} \text{Erfc}(n) \int_{\tau=-\infty}^{\infty} \delta(\tau-n) d\tau
\]

\[
= \frac{1}{2} \sum_{n=-\infty}^{\infty} \left( \text{erfc} \left( \frac{2n-1}{\sqrt{8}\sigma^2} \right) - \text{erfc} \left( \frac{2n+1}{\sqrt{8}\sigma^2} \right) \right) \cdot 1 = \sum_{n=-\infty}^{\infty} \int_{\tau=(2n+1)/2}^{(2n-1)/2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\tau^2/2\sigma^2} d\tau
\]

\[
= \cdots + \int_{-3/2}^{-1/2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\tau^2/2\sigma^2} d\tau + \int_{-1/2}^{1/2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\tau^2/2\sigma^2} d\tau + \int_{1/2}^{3/2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\tau^2/2\sigma^2} d\tau + ...
\]

The sum of integrals is equivalent to the integral of the density on the continuum of \( \tau \) values from minus to plus infinity resulting in

\[
\int_{\tau=-\infty}^{\infty} f_d(\tau) d\tau = \int_{\tau=-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\tau^2/2\sigma^2} d\tau = 1
\]

(41)

which completes our proof.

**Proposition:** The mean of the discrete Gaussian variable is zero, i.e.,
\[ \int_{\tau=-\infty}^{\infty} \tau f_d (\tau) d\tau = 0. \]

**Proof:**

\[ \int_{\tau=-\infty}^{\infty} \tau \cdot f_d (\tau) d\tau = \int_{\tau=-\infty}^{\infty} \frac{1}{2} \sum_{n=0}^{\infty} \tau \cdot \text{Erfc}(n) \delta(\tau-n) d\tau = \frac{1}{2} \sum_{n=0}^{\infty} \text{Erfc}(n) \int_{\tau=-\infty}^{\infty} \tau \cdot \delta(\tau-n) d\tau \]

\[ = \frac{1}{2} \sum_{n=0}^{\infty} n \cdot \text{Erfc}(n) = \frac{1}{2} \sum_{n=0}^{\infty} n \cdot \text{Erfc}(n) + \frac{1}{2} \sum_{n=1}^{\infty} n \cdot \text{Erfc}(n) \]

(43)

because the terms in the sum with negative \( n \) are equal to the corresponding terms in the sum with positive \( n \) we may have

\[ \int_{\tau=-\infty}^{\infty} f_d (\tau) d\tau = \frac{1}{2} \sum_{n=1}^{\infty} (-n) \cdot \text{erfc}(n) + \frac{1}{2} \sum_{n=1}^{\infty} n \cdot \text{erfc}(n) = 0, \]

(44)

which completes our proof.

The variance of the discrete density can be calculated as follows,

\[ \sigma_d^2 = \int_{\tau=-\infty}^{\infty} (\tau-\eta)^2 f_d (\tau) d\tau = \int_{\tau=-\infty}^{\infty} (\tau-\eta)^2 \frac{1}{2} \sum_{n=0}^{\infty} \text{Erfc}(n) \delta(\tau-n) d\tau \]

\[ = \frac{1}{2} \sum_{n=0}^{\infty} \text{Erfc}(n) \int_{\tau=-\infty}^{\infty} \tau^2 \delta(\tau-n) d\tau \]

\[ = \frac{1}{2} \sum_{n=0}^{\infty} n^2 \text{Erfc}(n) + \frac{1}{2} \sum_{n=0}^{\infty} n \left( \text{erfc} \left( \frac{2n-1}{\sqrt{8\sigma^2}} \right) - \text{erfc} \left( \frac{2n+1}{\sqrt{8\sigma^2}} \right) \right) \]

(45)

\[ = \sum_{n=0}^{\infty} n^2 \int_{(2n-1)/2}^{(2n+1)/2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} dx \leq \sigma^2 \]

The last integral is the probability value defined for every discrete interval. Because the discrete interval for \( n = 0 \) does not contribute to the value of this variance the total sum will give the discrete variance that is smaller or equal to the variance of the continuous variable.

**Proposition:** If the contribution to the variance is calculated using integral in each interval, the final value is equal to the variance of the continuous variable.

**Proof:**
\[ \sigma_d^2 = \sum_{n=0}^{\infty} \left[ \int_{-\tau/2}^{\tau/2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\tau^2/2\sigma^2} d\tau \right] \]

\[ = \ldots + \int_{-\tau/2}^{\tau/2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\tau^2/2\sigma^2} d\tau + \int_{-\tau/2}^{\tau/2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\tau^2/2\sigma^2} d\tau + \int_{\tau/2}^{3\tau/2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\tau^2/2\sigma^2} d\tau + \ldots \]  

(46)

Due to the assumed constant values of the discrete density function inside the intervals of integration and zero value for \( n = 0 \), this real value of the variance for discrete density function will be smaller than the value of the continuous variable, as we said before.
3.2 Discrete density function with finite mean

In the case when the mean value of the discrete truncated Gaussian density function is different from zero, the expressions for the density and distribution function can be derived in the same way as for the case when the mean was zero. The discretisation of the density function is shown in Fig. 9.

\[ f_d(\tau) \]

\[ P(\eta - T_s / 2 \leq \tau < \eta + T_s / 2) \]

Figure 9 Discretisation of the Gaussian density function with finite mean.

**Proposition:** The discrete density function, having the values at the uniformly spaced instants \( T_s \) of a random variable, is expressed as

\[
f_d(\tau) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left( \text{erfc} \left( \frac{2n-1)T_s}{2\sqrt{2}\sigma^2} \right) - \text{erfc} \left( \frac{(2n+1)T_s}{2\sqrt{2}\sigma^2} \right) \right) \delta(\tau - (\eta + nT_s)),
\]

where \( \delta(.) \) are Dirac’s delta functions. For the defined unit interval \( T_s = 1 \), this function is

\[
f_d(\tau) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left( \text{erfc} \left( \frac{2n-1}{\sqrt{8}\sigma^2} \right) - \text{erfc} \left( \frac{(2n+1)}{\sqrt{8}\sigma^2} \right) \right) \delta(\tau - (\eta + n)).
\]

**Proof:** If this density is uniformly discretized in respect to \( \tau \), with the interval of discretisation of \( T_s \), the probability value in each interval can be calculated as follows. For the interval defined by \( n = 0 \), we may have

\[
P_0 = P(\eta - T_s / 2 \leq \tau < \eta + T_s / 2) = \int_{\eta - T_s / 2}^{\eta + T_s / 2} f_d(\tau) d\tau = \int_{\eta - T_s / 2}^{\eta + T_s / 2} \frac{1}{\sqrt{2\pi}\sigma^2} e^{-(\tau - \eta)^2/2\sigma^2} d\tau
\]

\[
= \int_{-\infty}^{\eta - T_s / 2} \frac{1}{\sqrt{2\pi}\sigma^2} e^{-(\tau - \eta)^2/2\sigma^2} d\tau - \int_{\eta - T_s / 2}^{\eta + T_s / 2} \frac{1}{\sqrt{2\pi}\sigma^2} e^{-(\tau - \eta)^2/2\sigma^2} d\tau = 1 - 2 \int_{\eta - T_s / 2}^{\eta + T_s / 2} \frac{1}{\sqrt{2\pi}\sigma^2} e^{-(\tau - \eta)^2/2\sigma^2} d\tau
\]

If we do the following replacement of variable \( \tau \), \( z^2 = (\tau - \eta)^2 / 2\sigma^2 \), we can find the limits of integration and express the integral in this form
for the complementary error function, \( \text{erfc} \), defined as in the above integral. The probability that the random variable is in the interval defined for \( n = 1 \) can be calculated as

\[
P_1 = P(\eta + T_s / 2 \leq \tau < \eta + 3T_s / 2) = \int_{\eta + T_s / 2}^{\eta + 3T_s / 2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\tau - \mu)^2}{2\sigma^2}} d\tau = \frac{1}{2} \left[ \text{erfc} \left( \frac{T_s / 2}{\sqrt{2\sigma^2}} \right) - \text{erfc} \left( \frac{3T_s / 2}{\sqrt{2\sigma^2}} \right) \right]
\]

The same probability can be obtained in the interval defined for \( n = -1 \) as

\[
P_{-1} = P(\eta - 3T_s / 2 \leq \tau < \eta - T_s / 2) = \int_{\eta - 3T_s / 2}^{\eta - T_s / 2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\tau - \mu)^2}{2\sigma^2}} d\tau - \int_{\eta - T_s / 2}^{\eta + T_s / 2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(\tau - \mu)^2}{2\sigma^2}} d\tau = \frac{1}{2} \left[ \text{erfc} \left( \frac{-3T_s / 2}{\sqrt{2\sigma^2}} \right) - \text{erfc} \left( \frac{-T_s / 2}{\sqrt{2\sigma^2}} \right) \right] = \frac{1}{2} \left[ 2 - \text{erfc} \left( \frac{3T_s / 2}{\sqrt{2\sigma^2}} \right) - \text{erfc} \left( \frac{T_s / 2}{\sqrt{2\sigma^2}} \right) \right]
\]

Therefore, for any discrete interval defined for zero, negative and positive values of \( n \), the discrete probabilities are

\[
P_n = P(\eta - (2n-1)T_s / 2 \leq \tau < \eta + (2n+1)T_s / 2) = \frac{1}{2} \text{erfc} \left( \frac{(2n-1)T_s / 2}{\sqrt{2\sigma^2}} \right) - \text{erfc} \left( \frac{(2n+1)T_s / 2}{\sqrt{2\sigma^2}} \right).
\]

Thus, the discrete Gaussian density function of discrete random variable can be represented by a series of Dirac’s delta function at discrete instants \( \tau = nT_s \) having the weights equal to the calculated probabilities and expressed as

\[
f_d(\tau) = \frac{1}{2} \sum_{n=-\infty}^{\infty} P((2n-1)T_s / 2 \leq \tau < (2n+1)T_s / 2) \delta(\tau - nT_s)
\]

\[
= \frac{1}{2} \sum_{n=-\infty}^{\infty} \left[ \text{erfc} \left( \frac{(2n-1)T_s / 2}{\sqrt{2\sigma^2}} \right) - \text{erfc} \left( \frac{(2n+1)T_s / 2}{\sqrt{2\sigma^2}} \right) \right] \delta(\tau - (\eta + nT_s))
\]

and for \( T_s = 1 \) the discrete density is

\[
f_d(\tau) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left[ \text{erfc} \left( \frac{2n-1}{\sqrt{8\sigma^2}} \right) - \text{erfc} \left( \frac{2n+1}{\sqrt{8\sigma^2}} \right) \right] \delta(\tau - (\eta + n)) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \text{Erfc}(n) \delta(\tau - (\eta + n)),
\]

where the \( \text{Erfc}(n) \) is defined as

\[\text{Erfc}(n) = 1 - \text{erfc} \left( \frac{n}{\sqrt{2\sigma^2}} \right)\]
\[
\text{Erfc}(n) = \text{erfc}\left(\frac{2n-1}{\sqrt{8\sigma^2}}\right) - \text{erfc}\left(\frac{2n+1}{\sqrt{8\sigma^2}}\right),
\]

which completes our proof.

**Proposition:** The integral of function (55) for all \(n\) is one.

**Proof:**

\[
\int_{\tau=-\infty}^{\infty} f_d(\tau)d\tau = \frac{1}{2} \sum_{n=-\infty}^{\infty} \text{Erfc}(n) \delta(\tau-(\eta+n))d\tau = \frac{1}{2} \sum_{n=-\infty}^{\infty} \text{Erfc}(n) \int_{\tau=-\infty}^{\infty} \delta(\tau-(\eta+n))d\tau
\]

\[
= \frac{1}{2} \sum_{n=-\infty}^{\infty} \left( \text{erfc}\left(\frac{2n-1}{\sqrt{8\sigma^2}}\right) - \text{erfc}\left(\frac{2n+1}{\sqrt{8\sigma^2}}\right) \right) \cdot 1 = \sum_{n=-\infty}^{\infty} \int_{(2n-1)/2}^{(2n+1)/2} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\tau^2}{2\sigma^2}} d\tau
\]

The sum of integrals is equivalent to the integral on the continuum of \(\tau\) values from minus to plus infinity resulting in

\[
\int_{\tau=-\infty}^{\infty} f_d(\tau)d\tau = \int_{x=-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} dx = 1
\]

which completes our proof.

**Proposition:** The mean of the discrete Gaussian random variable is equal to the mean \(\eta\) of the continuous variable.

**Proof:** Having in mind the definition of the mean and property of the density function we may have

\[
\eta_d = \int_{\tau=-\infty}^{\infty} \tau \cdot f_d(\tau)d\tau = \int_{\tau=-\infty}^{\infty} \frac{1}{2} \sum_{n=-\infty}^{\infty} \tau \cdot \text{Erfc}(n) \delta(\tau-(\eta+n))d\tau = \frac{1}{2} \sum_{n=-\infty}^{\infty} \text{Erfc}(n) \int_{\tau=-\infty}^{\infty} \tau \cdot \delta(\tau-(\eta+n))d\tau
\]

\[
= \frac{1}{2} \sum_{n=-\infty}^{\infty} (\eta+n) \cdot \text{Erfc}(n) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \eta \cdot \text{Erfc}(n) + \frac{1}{2} \sum_{n=-\infty}^{\infty} n \cdot \text{Erfc}(n) = \frac{1}{2} \eta + \frac{1}{2} \eta_d
\]

which results in

\[
\eta_d = \eta
\]

and completes our proof.

The variance of the discrete density can be calculated as follows
\[
\sigma_d^2 = \int_{\tau=-\infty}^{\infty} (\tau - \eta_d)^2 f_d(\tau) d\tau .
\] (61)

The last integral is the probability value defined for every discrete interval. Because the discrete interval for \( n = 0 \) does not contribute to the value of this variance the total sum will give the discrete variance that is slightly smaller than the variance of the continuous variable.

**Proposition:** If the contribution to the variance is calculated using integral in each interval, the final value is equal to the variance of the continuous variable.

**Proof:**

\[
\sigma_d^2 = \sum_{n=-\infty}^{\infty} \int_{(2n+1)T/2}^{(2n+1)T/2} \tau^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\tau^2/2\sigma^2} d\tau - \eta_d^2
\]

\[
= \ldots + \int_{-3T/2}^{-T/2} \tau^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\tau^2/2\sigma^2} d\tau + \int_{-T/2}^{T/2} \tau^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\tau^2/2\sigma^2} d\tau + \int_{T/2}^{3T/2} \tau^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\tau^2/2\sigma^2} d\tau + \ldots - \eta^2
\] (62)

\[
= \int_{-\infty}^{\infty} \tau^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\tau^2/2\sigma^2} d\tau - \eta^2 = E(\tau^2) - \eta^2 = \sigma^2
\]
4. Truncated discrete Gaussian density and distribution function

4.1 Discrete truncated density with zero mean

In practical application the discrete delays are taking values in a limited interval of, say, $S$ possible values. Therefore, the function which describes the delay distribution is truncated to $S$, as shown in Fig. 10.

![Fig. 10 Discrete truncated Gaussian density function presented using Dirac’s delta functions](image)

In order to find this truncated function, the whole domain of possible discrete values $\tau$ from 0 to infinity, for the already derived functions, need to be divided into intervals containing $S$ discrete values. All corresponding values in these intervals, starting with $1^\text{st}$, $(S+1)$th, $(2S+1)$th, etc. terms, need to be added to obtain the truncated density function values for $n = 1, 2, ..., S$. However, in this case this method cannot give us the expression of density function in a closed form. For this reason, a method based on the definition of the truncated density function as the conditional density function will be used. According to this method the truncated density function of the truncated random variable $\tau$ is defined as the conditional density function in the interval $(-S, S)$ and expressed as

$$f_d(\tau) = f_d(\tau | \tau \leq S) = \frac{f_d(\tau)}{P(\tau \geq -S) - P(\tau > S)}$$

$$= \frac{1}{2} \sum_{n=-S}^{n=S} \left( \text{erfc} \left( \frac{2n-1}{\sqrt{8\sigma^2}} \right) - \text{erfc} \left( \frac{2n+1}{\sqrt{8\sigma^2}} \right) \right) \delta(\tau - n)$$

$$= \frac{1}{2} \sum_{n=-S}^{n=S} \left( \text{erfc} \left( \frac{2n-1}{\sqrt{8\sigma^2}} \right) - \text{erfc} \left( \frac{2n+1}{\sqrt{8\sigma^2}} \right) \right) \delta(\tau - n)$$

$$= P(S) \sum_{n=-S}^{n=S} \text{erfc}(n) \cdot \delta(\tau - n)$$

(63)

Where

$$P(S) = \sum_{n=-S}^{n=S} \left( \text{erfc} \left( \frac{2n-1}{\sqrt{8\sigma^2}} \right) - \text{erfc} \left( \frac{2n+1}{\sqrt{8\sigma^2}} \right) \right)$$

(64)

and
\[ \text{Erfc}(n) = \text{erfc} \left( \frac{(2n - 1)}{\sqrt{8\sigma^2}} \right) - \text{erfc} \left( \frac{(2n + 1)}{\sqrt{8\sigma^2}} \right), \]  

(65)

are functions defined in the truncation interval. The truncated density function, for different values of the truncation interval \( S \), is presented in Fig. 11 alongside with the continuous and discrete density functions. When the truncation interval increases the truncated variance increases and is always smaller that the variance of continuous density. The SD is the domain of discrete function before truncation and \( S \) defines the truncation interval, i.e., the domain of the truncated variable.

Figure 11 Continuous, discrete and discrete truncated Gaussian densities.

**Proposition:** The mean of the truncated discrete density function is zero.

**Proof:** By definition

\[
\eta_{dt} = \int_{-\infty}^{\infty} \tau f_\tau(\tau) d\tau = \int_{-\infty}^{\infty} \tau P(S) \sum_{n=-S}^{n=S} \text{Erfc}(n) \delta(\tau - n) d\tau
\]

\[= P(S) \sum_{n=-S}^{n=S} \text{Erfc}(n) \int_{-\infty}^{\infty} \tau \cdot \delta(\tau - n) d\tau = P(S) \sum_{n=-S}^{n=S} n \cdot \text{Erfc}(n), \]  

(66)

The term of the sum for \( n = 0 \) is zero. The corresponding terms for negative and positive \( n \) are cancelling each other. Then, we may have

\[
\eta_{dt} = P(S) \sum_{n=-S}^{n=S} n \cdot \text{Erfc}(n) = P(S) \sum_{n=-S}^{n=1} n \cdot \text{Erfc}(n) - P(S) \sum_{n=1}^{n=S} n \cdot \text{Erfc}(n) = 0 ,
\]

(67)

which completes our proof.

**Proposition:** The variance of this density is
\[ \sigma_{d}^2 = P(S) \sum_{n=1}^{\infty} n^2 \text{Erfc}(n) \]  

(68)

**Proof:** By definition

\[ \sigma_{d}^2 = \int_{-\infty}^{\infty} \tau^2 f_{\tau}(\tau) d\tau = \int_{-\infty}^{\infty} \tau^2 P(S) \sum_{n=-\infty}^{\infty} \text{Erfc}(n) \cdot \delta(\tau - n) d\tau \]

\[ = P(S) \sum_{n=-\infty}^{\infty} \text{Erfc}(n) \cdot \int_{-\infty}^{\infty} \tau^2 \delta(\tau - n) d\tau = P(S) \sum_{n=-\infty}^{\infty} n^2 \text{Erfc}(n) \]  

(69)

The term for \( n = 0 \) is zero. The corresponding terms for negative and positive \( n \) are added to each other. In addition, the expression \( P(S) \leq 1 \), thus, having in mind the variance for the discrete Gaussian random variable, we may have

\[ \sigma_{d}^2 = P(S) \sum_{n=-\infty}^{\infty} n^2 \text{Erfc}(n) = 2P(S) \sum_{n=1}^{\infty} n^2 \text{Erfc}(n) \leq \sigma^2 \]

(70)

which completes our proof.

**Proposition:** A good approximation of the mean and variance for discrete truncated Gaussian density can be obtained from the variance of continuous truncated Gaussian density.

**Proof:** Due to fact \( a = -b = -T \) and \( \text{erfc}(-x) = 2 - \text{erfc}(x) \), we may have, for \( T = (2S+1)/2 \),

\[ P(T) = \frac{2}{\text{erfc} \frac{a}{\sqrt{2\sigma^2}} - \text{erfc} \frac{b}{\sqrt{2\sigma^2}}} = \frac{2}{\text{erfc} \frac{-(2S+1)/2}{\sqrt{2\sigma^2}} - \text{erfc} \frac{(2S+1)/2}{\sqrt{2\sigma^2}}} \]

\[ = \frac{2}{2 - \text{erfc} \frac{(2S+1)/2}{\sqrt{2\sigma^2}} - \text{erfc} \frac{(2S+1)/2}{\sqrt{2\sigma^2}}} = \frac{1}{1 - \text{erfc} \frac{(2S+1)/2}{\sqrt{2\sigma^2}}} = \frac{1}{1 - \text{erfc}(2S+1)/\sqrt{8\sigma^2}} \]  

(71)

For the limits

\[ a = -(2S+1)/2 = -T, \]

(72)

\[ b = +(2S+1)/2 = T, \] and \( \eta = 0 \),

(73)

we may find the mean is zero, i.e.,

\[ \eta_{d} = \eta + P(T)\sigma \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{(a-\eta)^2}{2\sigma^2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(b-\eta)^2}{2\sigma^2}} \right) \]

\[ = \frac{1}{1 - \text{erfc}(2S+1)/\sqrt{8\sigma^2}} \sigma \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{-(2S+1)^2}{2\sigma^2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(2S+1)^2}{2\sigma^2}} \right) = 0 \]  

(74)
and variance can be expressed as

\[ \sigma_{dt}^2 = \sigma^2 \left[ 1 + P(T) \left( \frac{a - b}{\sigma \sqrt{2\pi}} e^{-\frac{a^2}{2\sigma^2}} - \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{b^2}{2\sigma^2}} \right) \right] - \sigma^2 P^2(T) \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{a^2}{2\sigma^2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{b^2}{2\sigma^2}} \right)^2 \]

\[ = \sigma^2 \left[ 1 + P(T) \left( \frac{(2S + 1)/2}{\sigma \sqrt{2\pi}} e^{-\frac{(2S + 1)^2/2\sigma^2}{2\sigma^2}} - \frac{(2S + 1)/2}{\sigma \sqrt{2\pi}} e^{-\frac{(2S + 1)^2/2\sigma^2}{2\sigma^2}} \right) \right] + 0 \]

\[ = \sigma^2 \left[ 1 + P(T) \left( \frac{(2S + 1)/2}{\sigma \sqrt{2\pi}} e^{-\frac{(2S + 1)^2/2\sigma^2}{2\sigma^2}} - \frac{(2S + 1)/2}{\sigma \sqrt{2\pi}} e^{-\frac{(2S + 1)^2/2\sigma^2}{2\sigma^2}} \right) \right] \]

\[ = \sigma^2 \left[ 1 - \frac{2}{1 - \text{erfc} \left( \frac{(2S + 1)/2}{\sqrt{2\sigma^2}} \right)} \right] \]

\[ = \sigma^2 \left[ 1 - \frac{1}{1 - \text{erfc} \left( \frac{(2S + 1)}{\sqrt{8\sigma^2}} \right)} \right] \]

which completes the proof and shows that the variance of the truncated density is smaller than the variance of the continuous density without truncation and close to the variance of the continuous truncated density that is expressed in (30) as

\[ \sigma_i^2 = \sigma^2 \left[ 1 - \frac{1}{1 - \text{erfc} T / \sqrt{2\sigma^2}} \frac{2T}{\sqrt{2\pi\sigma^2}} e^{-\frac{T^2}{2\sigma^2}} \right] \quad \text{(76)} \]

---

**Figure 12** Discrete truncated Gaussian density function

a) Density presented using Dirac’s delta functions

b) The weight of Dirac’s delta functions
4.2 Discrete truncated density with a finite mean

The discrete density function, having the values at the uniformly spaced instants $T_s$ of a random variable, was expressed as

$$f_{d}(\tau) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left[ \text{erfc} \left( \frac{2(n-1)T_s/2}{\sqrt{2} \sigma^2} \right) - \text{erfc} \left( \frac{2(n+1)T_s/2}{\sqrt{2} \sigma^2} \right) \right] \delta(\tau - (\eta + n)),$$

and

where $\delta(.)$ are Dirac’s delta functions. For the defined unit interval $T_s = 1$, this function was expressed as

$$f_{d}(\tau) = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left[ \text{erfc} \left( \frac{2(n-1)}{\sqrt{8} \sigma^2} \right) - \text{erfc} \left( \frac{2(n+1)}{\sqrt{8} \sigma^2} \right) \right] \delta(\tau - (\eta + n)).$$

Following the definition of the truncated discrete function with zero mean, we may say that the truncated density function can be defined as in (63), i.e.,

$$f_{d}(\tau) = \frac{1}{2} \sum_{n=-S}^{S} \left[ \text{erfc} \left( \frac{2n-1}{\sqrt{8} \sigma^2} \right) - \text{erfc} \left( \frac{2n+1}{\sqrt{8} \sigma^2} \right) \right] \delta(\tau - (\eta + n))$$

$$= P(S) \sum_{n=-S}^{S} \left[ \text{erfc} \left( \frac{2n-1}{\sqrt{8} \sigma^2} \right) - \text{erfc} \left( \frac{2n+1}{\sqrt{8} \sigma^2} \right) \right] \delta(\tau - (\eta + n)),$$

where

$$P(S) = \left( \sum_{n=-S}^{S} \left[ \text{erfc}(2n-1)/\sqrt{8} \sigma^2 - \text{erfc}(2n+1)/\sqrt{8} \sigma^2 \right] \right)^{-1},$$

as defined in (64).
5. Conclusions

In this Report a Gaussian discrete truncated density function is derived and investigated. The function is expressed in closed form in terms of Dirac’s delta functions. Expressions for the first and second moments are derived. If complete discretisation is necessary, it is possible to express the density and distribution function in terms of Kronecker’s delta function and discrete unit step functions.
Appendix - Notation used in the Report

2C – domain of continuous Gaussian random variable
2T – truncation interval, domain of continuous truncated Gaussian random variable
2S – truncation interval for discrete truncated density, domain of discrete truncated Gaussian random variable

\( f_c(\tau) \) – Gaussian density function of continuous variable (continuous density)
\( f_{ct}(\tau) \) – Gaussian density function of continuous truncated variable (truncated density)
\( f_d(\tau) \) – Gaussian density function of discrete variable (discrete density)
\( f_{dt}(\tau) \) – Gaussian density function of discrete truncated variable (discrete truncated density)

\( C(T) \) – probability function defined in the truncation interval 2T for continuous variable
\( P(S) \) – probability function defined in the truncation interval 2S for discrete variable
\( P(T) \) – probability function defined in the truncation interval (2S+1)/2 for discrete variable

\( \eta \) – mean for the Gaussian continuous variable
\( \eta_d \) – mean for the Gaussian discrete variable
\( \eta_{dt} \) – mean for the Gaussian discrete truncated variable

\( \sigma, \sigma^2 \) – standard deviation and variance of Gaussian continuous random variable
\( \sigma_t, \sigma_t^2 \) – standard deviation and variance of truncated Gaussian continuous random variable
\( \sigma_d, \sigma_d^2 \) – standard deviation and variance of Gaussian discrete random variable