Rationality problems for complete reducibility of subgroups of reductive algebraic groups

by

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In loving memory of Cinda Uchiyama (2007-2016), who made me smile everyday for 10 years.

Abstract

Let k be a field. Let G be a connected reductive algebraic group defined over k. Following Serre, a closed subgroup H is of G is called G-completely reducible over k (G-cr over k for short) if whenever H is contained in a k-defined parabolic subgroup P of G, H is contained in a k-defined Levi subgroup of P. This thesis is a compilation of four independent papers concerning rationality problems for complete reducibility of subgroups of G and various related problems. Here is a list of problems that we consider. Let $\overline{k}$ be an algebraic closure of k.

Problem 1. Let H be a k-subgroup of G. Suppose that H is G-cr over $\overline{k}$. Then is H necessarily G-cr over k?

Problem 2. Let H be a k-subgroup of G. Suppose that H is G-cr over k. Then is H always G-cr over $\overline{k}$?

Problem 3. Let $H < M < G$ be a triple of reductive $\overline{k}$-groups. Then if H is G-cr over $\overline{k}$, is H necessarily M-cr over $\overline{k}$?

Problem 4 (Külshammer). Let Γ be a finite group. Let Γ_p be a Sylow p-subgroup of Γ. Let $k = \overline{k}$ be of characteristic p. Let $\rho_p \in \mathrm{Hom}(\Gamma_p, G)$. Then are there only finitely many representations $\rho \in \mathrm{Hom}(\Gamma, G)$ such that $\rho\mid_{\Gamma_p}$ is G-conjugate to ρ_p?

Problem 5. Let H be a k-subgroup of G. Suppose that H is G-cr over k. Then is the centralizer of H in G G-cr over k?

We obtain various general results concerning complete reducibility over an arbitrary k via geometric invariant theory (GIT for short) and the theory of spherical building (in particular the recently proved center conjecture of Tits). GIT and the center conjecture give a very short proof for many results. We also consider non-connected G, and obtain analogous results. Various open problems concerning complete reducibility and related problems are discussed.

Contents

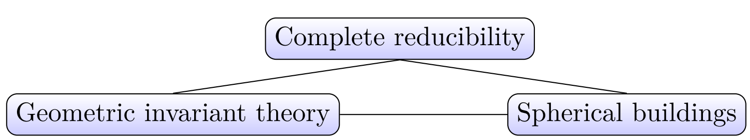
Introduction

Let k be a field. Let G/k be a connected reductive algebraic group. J. Tits invented the notion of the spherical building $\Delta(G)$ of G [25]; that is a simplicial complex on which $G(k)$ acts by permuting simplices of $\Delta(G)$. The following is the 50-year-old *center conjecture of Tits* (see [19] and [24]), which was recently proved by Tits, Mühlherr, Leeb, and Ramos-Cuevas [13], [15], [18]: if X is a convex contractible subcomplex of $\Delta(G)$, then there exists a simplex in X which is stabilized by all automorphisms of $\Delta(G)$ stabilizing X. The center conjecture has far-reaching consequences in many areas of mathematics; see [5], [16], [18] for example.

We are interested in the subgroup structure of G. Representation theory has been very useful in this respect [11]. In [19], generalizing of the notion of complete reduciblity in representation theory, Serre defined: a closed subgroup H of G is *G-completely reducible over k* (G-cr over k for short) if whenever H is contained in a k-defined parabolic subgroup P of G, H is contained in some k-defined Levi subgroup of P. In particular, if H is not contained in any k-defined parabolic subgroup of G, H is *G-irreducible over k* (G-ir over k for short).

The notion of complete reducibility has been much studied [3], [14], [19], [22], but most work assume $k = \overline{k}$ where $\overline{k}$ is an algebraic closure of k. In this thesis, we investigate the unexplored area of complete reducibility over an arbitrary k.

In [4], it was shown that if k is perfect (for example if k is finite or the characteristic of k is 0), a subgroup H of G is G-cr over $\overline{k}$ if and only if H is G-cr over k. So our problem is interesting only when k is nonperfect. In particular, we show that there exist subgroups H of G such that H are G-cr over $\overline{k}$ but not G-cr over k, and vice versa; see, [27], [28], [29]. The key there is to find *non-separable subgroups*. Recall that a subgroup H of G is called *non-separable* if the scheme-theoretic centralizer of H is G is not smooth [6]. Non-separability is crucial to construct various interesting examples concerning complete reducibility over $\overline{k}$, the number of conjugacy classes, representations of finite groups (Külshammer's question), etc., in [2], [6], [27], [28], [29].



Although traditional representation-theoretic methods give detailed information on subgroup structure of G, their arguments tend to be long and depend on a complicated case-by-case analysis; see [14], [21], [22], [23]. Instead, we use *geometric invariant theory* (GIT for short) and the theory of spherical buildings, in particular, *the center conjecture* which enabled a short and uniform proof in many results concerning complete reducibility and other various related problems in [1], [3], [7], [28].

The center conjecture comes into play in the study of complete reducibility via the following argument. First, we can identify each simplex of $\Delta(G)$ with a proper k-parabolic subgroup of G. Second, Serre has shown that if a subgroup H of G is not G-cr over k, then the fixed point subcomplex $\Delta(G)^H$ is contractible [19]. This means, roughly speaking, that the center conjecture gives an *optimal* (in the spirit of Kempf [12]) k-parabolic subgroup P of G such that P *witnesses* the non-G-complete reducibility of H.

Now, we turn the attention to GIT. Let V/k be an affine variety on which G acts. A central problem in GIT is to understand the structure of the set of orbits of G on V [17]. GIT is related to complete reducibility in the following way. Let H be a subgroup of G such that $H := \langle h_1, h_2, \cdots, h_N\rangle$ for some natural number N. Suppose that G acts on G^N by simultaneous conjugation. Bate et al. [7] showed that H is G-cr over k if and only if the $G(k)$-orbit $G(k)\cdot(h_1, h_2, \cdots, h_N)$ is closed in G^N. Thus, we can turn a problem concerning complete reducibility into a problem concerning $G(k)$-orbits. This technique is used in the proofs of [28, Prop. 3.6] and [29, Thm. 6.4] for example.

This thesis consists of four independent papers [26], [27], [28], [29] on G-complete reducibility and various related problems. Our basic references for algebraic groups are [8], [9], [10], [20], and we follow the notation therein. In particular, we denote by G a (possibly non-connected) reductive algebraic group defined over $\overline{k}$ with a k-structure in the sense of Borel [8]. Here is a brief sketch of each paper in the thesis.

• T. Uchiyama, Separability and complete reducibility of subgroups of the Weyl group of a simple algebraic group of type E_7, J. Algebra, 422:357-372, 2015.

Let G be a connected reductive algebraic group defined over an algebraically closed field k. In this paper, we present a method to find triples (G, M, H) with the following three properties. Property 1: G is simple and k has characteristic 2. Property 2: H and M are closed reductive subgroups of G such that $H < M < G$, and (G, M) is a reductive pair (see the sentence after [28, Prop. 1.4] for the definition of a reductive pair). Property 3: H is G-completely reducible, but not M-completely reducible. We exhibit our method by presenting a new example of such a triple in $G = E_7$. Then we consider a rationality problem and a problem concerning conjugacy classes as important application of our construction. Our later work ([27] and [29]) was motivated by the construction in this paper.

Note: Theorem 1.1 appeared in author's MSc thesis: Separability and complete reducibility of subgroups of the Weyl group of a simple algebraic group, University of Canterbury, (2012).
http://ir.canterbury.ac.nz/handle/10092/7150
Other main results (Theorems 1.10, 1.12) are new.

• T. Uchiyama, Non-separability and complete reducibility: E_n examples with an application to a question of Külshammer, submitted, arXiv:1510.00997, 2015.

This paper supplements our previous work [28]. Let G be a simple algebraic group of type $E_n(n = 6, 7, 8)$ defined over an algebraically closed field k of characteristic 2. We present examples of triples of closed reductive groups $H < M < G$ such that H is G-completely reducible, but not M-completely reducible. As an application, we consider a question of Külshammer on representations of finite groups in reductive groups. We also consider a rationality problem for G-complete reducibility and a problem concerning conjugacy classes. We have used the computer software Magma for computations.

• T. Uchiyama, Complete reducibility of subgroups of reductive algebraic groups over nonperfect fields I, J. Algebra, 463:168-187, 2016.

This and the next papers exclusively deal with the rationality problems for G-complete reducibility. Let k be a nonperfect field of characteristic 2. Let G be a k-split simple algebraic group of type E_6 (or G_2) defined over k. In this paper, we present the first examples of nonabelian non-G-completely reducible k-subgroups of G which are G-completely reducible over k. Our construction is based on that of subgroups of G acting non-separably on the unipotent radical of a proper parabolic subgroup of G in our previous work. We also present examples with the same property for a non-connected reductive group G. Along the way, several general results concerning complete reducibility over nonperfect fields are proved using the recently proved Tits center conjecture for spherical buildings. In particular, we show that under mild conditions a k-subgroup of G is pseudo-reductive if it is G-completely reducible over k.

• T. Uchiyama, Complete reducibility of subgroups of reductive algebraic groups over nonperfect fields II, submitted, arXiv:1512.04616, 2015.

Let k be a separably closed field. Let G be a reductive algebraic k-group. In this paper, we study Serre's notion of complete reducibility of subgroups of G over k. In particular, using the recently proved center conjecture of Tits, we show that the centralizer of a k-subgroup H of G is G-completely reducible over k if it is reductive and H is G-completely reducible over k. We also show that a regular reductive k-subgroup of G is G-completely reducible over k (see [3, Sec. 2.1] for the definition of a regular subgroup). Various open problems concerning complete reducibility are discussed. We present examples where the number of overgroups of irreducible subgroups and the number of $G(k)$-conjugacy classes of unipotent elements are infinite. This paper complements author's previous work on rationality problems for complete reducibility.

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Separability and complete reducibility of subgroups of the Weyl group of a simple algebraic group of type E_7

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Abstract

Let G be a connected reductive algebraic group defined over an algebraically closed field k. The aim of this paper is to present a method to find triples (G, M, H) with the following three properties. Property 1: G is simple and k has characteristic 2. Property 2: H and M are closed reductive subgroups of G such that $H < M < G$, and (G, M) is a reductive pair. Property 3: H is G-completely reducible, but not M-completely reducible. We exhibit our method by presenting a new example of such a triple in $G = E_7$. Then we consider a rationality problem and a problem concerning conjugacy classes as important applications of our construction.

Keywords: algebraic groups, separable subgroups, complete reducibility

1 Introduction

Let G be a connected reductive algebraic group defined over an algebraically closed field k of characteristic p. In [15, Sec. 3], J.P. Serre defined that a closed subgroup H of G is *G-completely reducible* (G-cr for short) if whenever H is contained in a parabolic subgroup P of G, H is contained in a Levi subgroup L of P. This is a faithful generalization of the notion of semisimplicity in representation theory since if $G = GL_n(k)$, a subgroup H of G is G-cr if and only if H acts complete reducibly on k^n [15, Ex. 3.2.2(a)]. It is known that if a closed subgroup H of G is G-cr, then H is reductive [15, Prop. 4.1]. Moreover, if $p = 0$, the converse holds [15, Prop. 4.2]. Therefore the notion of G-complete reducibility is not interesting if $p = 0$. In this paper, we assume that $p > 0$.

Completely reducible subgroups of connected reductive algebraic groups have been much studied [9], [10], [15]. Recently, studies of complete reducibility via Geometric Invariant Theory (GIT for short) have been fruitful [1], [2], [3]. In this paper, we see another application of GIT to complete reducibility (Proposition 3.6).

Here is the main problem we consider. Let H and M be closed reductive subgroups of G such that $H \leq M \leq G$. It is natural to ask whether H being M-cr implies that H is G-cr and vice versa. It is not difficult to find a counterexample for the forward direction. For example, take $H = M = PGL_2(k)$ and $G = SL_3(k)$ where $p = 2$ and H sits inside G via the adjoint representation. Another such example is [1, Ex. 3.45]. However, it is hard to get a counterexample for the reverse direction, and it necessarily involves a small p. In [3, Sec. 7], Bate et al. presented the only known counterexample for the reverse direction where $p = 2$, $H \cong S_3$, $M \cong A_1 A_1$, and $G = G_2$, which we call "the G_2 example". The aim of this paper is to prove the following.

Theorem 1.1. *Let G be a simple algebraic group of type E_7 defined over k of characteristic $p = 2$. Then there exists a connected reductive subgroup M of type A_7 of G and a reductive subgroup $H \cong D_{14}$ (the dihedral group of order 14) of M such that (G, M) is a reductive pair and H is G-cr but not M-cr.*

Our work is motivated by [3]. We recall a few relevant definitions and results here. We denote the Lie algebra of G by $\operatorname{Lie} \mathrm{G} = \mathfrak{g}$. From now on, by a subgroup of G, we always mean a closed subgroup of G.

Definition 1.2. Let H be a subgroup of G acting on G by inner automorphisms. Let H act on $\mathfrak{g}$ by the corresponding adjoint action. Then H is called *separable* if $\operatorname{Lie} C_G(H) = \mathfrak{c}_{\mathfrak{g}}(H)$.

Recall that we always have $\operatorname{Lie} C_G(H) \subseteq c_{\mathfrak{g}}(H)$. In [3], Bate et al. investigated the relationship between G-complete reducibility and separability, and showed the following [3, Thm. 1.2, Thm. 1.4].

Proposition 1.3. *Suppose that p is very good for G. Then any subgroup of G is separable in G.*

Proposition 1.4. *Suppose that (G, M) is a reductive pair. Let H be a subgroup of M such that H is a separable subgroup of G. If H is G-cr, then it is also M-cr.*

Recall that a pair of reductive groups G and M is called a *reductive pair* if $\operatorname{Lie} M$ is an M-module direct summand of $\mathfrak{g}$. This is automatically satisfied if $p = 0$. Propositions 1.3 and 1.4 imply that the subgroup H in Theorem 1.1 must be non-separable, which is possible for small p only.

Now, we introduce the key notion of *separable action*, which is a slight generalization of the notion of a separable subgroup.

Definition 1.5. Let H and N be subgroups of G where H acts on N by group automorphisms. The action of H is called *separable* in N if $\operatorname{Lie} C_N(H) = \mathfrak{c}_{\operatorname{Lie} N}(H)$. Note that the condition means that the fixed points of H acting on N, taken with their natural scheme structure, are smooth.

Here is a brief sketch of our method. **Note that in our construction, p needs to be 2.**

1. Pick a parabolic subgroup P of G with a Levi subgroup L of P. Find a subgroup K of L such that K acts non-separably on the unipotent radical $R_u(P)$ of P. In our case, K is generated by elements corresponding to certain reflections in the Weyl group of G.

2. Conjugate K by a suitable element v of $R_u(P)$, and set $H = vKv^{-1}$. Then choose a connected reductive subgroup M of G such that H is not M-cr. Use a recent result from GIT (Proposition 2.4) to show that H is not M-cr. Note that K is M-cr in our case.

3. Prove that H is G-cr.

Remark 1.6. It can be shown using [17, Thm. 13.4.2] that K in Step 1 is a non-separable subgroup of G.

First of all, for Step 1, p cannot be very good for G by Proposition 1.3 and 1.4. It is known that 2 and 3 are bad for E_7. We explain the reason why we choose $p = 2$, not $p = 3$ (Remark 2.9). Remember that the non-separable action on $R_u(P)$ was the key ingredient for the G_2 example to work. Since K is isomorphic to a subgroup of the Weyl group of G, we are able to turn a problem of non-separability into a purely combinatorial problem involving the root system of G (Section 3.1). Regarding Step 2, we explain the reason of our choice of v and M explicitly (Remarks 3.4, 3.5). Our use of Proposition 2.4 gives an improved way for checking G-complete reducibility (Remark 3.7). Finally, Step 3 is easy.

In the G_2 and E_7 examples, the G-cr and non-M-cr subgroups H are finite. The following is the only known example of a triple (G, M, H) with positive dimensional H such that H is G-cr but not M-cr. It is obtained by modifying [1, Ex. 3.45].

Example 1.7. Let $p = 2$, $m \geq 4$ be even, and $(G, M) = (GL_{2m}(k), Sp_{2m}(k))$. Let H be a copy of $Sp_m(k)$ diagonally embedded in $Sp_m(k) \times Sp_m(k)$. Then H is not M-cr by the argument in [1, Ex. 3.45]. But H is G-cr since H is $GL_m(k) \times GL_m(k)$-cr by [1, Lem. 2.12]. Also note that any subgroup of $GL(k)$ is separable in $GL(k)$ (cf. [1, Ex. 3.28]), so (G, M) is not a reductive pair by Proposition 1.4.

In view of this, it is natural to ask:

Open Problem 1.8. Is there a triple $H < M < G$ of connected reductive algebraic groups such that (G, M) is a reductive pair, H is non-separable in G, and H is G-cr but not M-cr?

Beyond its intrinsic interest, our E_7 example has some important consequences and applications. For example, in Section 4, we consider a rationality problem concerning complete reducibility. We need a definition first to explain our result there.

Definition 1.9. Let k_0 be a subfield of an algebraically closed field k. Let H be a k_0-defined closed subgroup of a k_0-defined reductive algebraic group G. Then H is called *G-cr over k_0* if whenever H is contained in a k_0-defined parabolic subgroup P of G, it is contained in some k_0-defined Levi subgroup of P.

Note that if k_0 is algebraically closed then G-cr over k_0 means G-cr in the usual sense. Here is the main result of Section 4.

Theorem 1.10. *Let k_0 be a nonperfect field of charecteristic $p = 2$, and let G be a k_0-defined split simple algebraic group of type E_7. Then there exists a k_0-defined subgroup H of G such that H is G-cr over k, but not G-cr over k_0.*

As another application of the E_7 example, we consider a problem concerning conjugacy classes. Given $n \in \mathbb{N}$, we let G act on G^n by simultaneous conjugation:

$$g \cdot (g_1, g_2, \ldots, g_n) = (gg_1g^{-1}, gg_2g^{-1}, \ldots, gg_ng^{-1}).$$

In [16], Slodowy proved the following fundamental result applying Richardson's tangent space argument, [12, Sec. 3], [13, Lem. 3.1].

Proposition 1.11. *Let M be a reductive subgroup of a reductive algebraic group G defined over k. Let $n \in \mathbb{N}$, let $(m_1, \ldots, m_n) \in M^n$ and let H be the subgroup of M generated by $m_1, \ldots, m_n$. Suppose that (G, M) is a reductive pair and that H is separable in G. Then the intersection $G \cdot (m_1, \ldots, m_n) \cap M^n$ is a finite union of M-conjugacy classes.*

Proposition 1.11 has many consequences. See [1], [16], and [18, Sec. 3] for example. In [3, Ex. 7.15], Bate et al. found a counterexample for $G = G_2$ showing that Proposition 1.11 fails without the separability hypothesis. In Section 5, we present a new counterexample to Proposition 1.11 without the separability hypothesis. Here is the main result of Section 5.

Theorem 1.12. *Let G be a simple algebraic group of type E_7 defined over an algebraically closed k of characteristic $p = 2$. Let M be the connected reductive subsystem subgroup of type A_7. Then there exists $n \in \mathbb{N}$ and a tuple $\mathbf{m} \in M^n$ such that $G \cdot \mathbf{m} \cap M^n$ is an infinite union of M-conjugacy classes. Note that (G, M) is a reductive pair in this case.*

Now, we give an outline of the paper. In Section 2, we fix our notation which follows [4], [8], and [17]. Also, we recall some preliminary results, in particular, Proposition 2.4 from GIT. After that, in Section 3, we prove our main result, Theorem 1.1. Then in Section 4, we consider a rationality problem, and prove Theorem 1.10. Finally, in Section 5, we discuss a problem concerning conjugacy classes, and prove Theorem 1.12.

2 Preliminaries

2.1 Notation

Throughout the paper, we denote by k an algebraically closed field of positive characteristic p. We denote the multiplicative group of k by k^*. We use a capital roman letter, G, H, K, etc., to represent an algebraic group, and the corresponding lowercase gothic letter, $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{k}$, etc., to represent its Lie algebra. We sometimes use another notation for Lie algebras: $\operatorname{Lie} G$, $\operatorname{Lie} H$, and $\operatorname{Lie} K$ are the Lie algebras of G, H, and K respectively.

We denote the identity component of G by G°. We write $[G, G]$ for the derived group of G. The *unipotent radical* of G is denoted by $R_u(G)$. An algebraic group G is *reductive* if $R_u(G) = \{1\}$. In particular, G is *simple* as an algebraic group if G is connected and all proper normal subgroups of G are finite.

In this paper, when a subgroup H of G acts on G, H always acts on G by inner automorphisms. The adjoint representation of G is denoted by $\operatorname{Ad}_{\mathfrak{g}}$ or just Ad if no confusion arises. We write $C_G(H)$ and $\mathfrak{c}_{\mathfrak{g}}(H)$ for the global and the infinitesimal centralizers of H in G and $\mathfrak{g}$ respectively. We write $X(G)$ and $Y(G)$ for the set of characters and cocharacters of G respectively.

2.2 Complete reducibility and GIT

Let G be a connected reductive algebraic group. We recall Richardson's formalism [14, Sec. 2.1–2.3] for the characterization of a parabolic subgroup P of G, a Levi subgroup L of P, and the unipotent radical $R_u(P)$ of P in terms of a cocharacter of G and state a result from GIT (Proposition 2.4).

Definition 2.1. Let X be an affine variety. Let $\phi : k^* \to X$ be a morphism of algebraic varieties. We say that $\lim\limits_{a\to 0} \phi(a)$ exists if there exists a morphism $\hat{\phi} : k \to X$ (necessarily unique) whose restriction to k^* is ϕ. If this limit exists, we set $\lim\limits_{a\to 0} \phi(a) = \hat{\phi}(0)$.

Definition 2.2. Let λ be a cocharacter of G. Define $P_\lambda := \{g \in G \mid \lim\limits_{a\to 0} \lambda(a)g\lambda(a)^{-1} \text{ exists}\}$, $L_\lambda := \{g \in G \mid \lim\limits_{a\to 0} \lambda(a)g\lambda(a)^{-1} = g\}$, $R_u(P_\lambda) := \{g \in G \mid \lim\limits_{a\to 0} \lambda(a)g\lambda(a)^{-1} = 1\}$.

Note that P_λ is a parabolic subgroup of G, L_λ is a Levi subgroup of P_λ, and $R_u(P_\lambda)$ is a unipotent radical of P_λ [14, Sec. 2.1-2.3]. By [17, Prop. 8.4.5], any parabolic subgroup P of G, any Levi subgroup L of P, and any unipotent radical $R_u(P)$ of P can be expressed in this form. It is well known that $L_\lambda = C_G(\lambda(k^*))$.

Let M be a reductive subgroup of G. Then, there is a natural inclusion $Y(M) \subseteq Y(G)$ of cocharacter groups. Let $\lambda \in Y(M)$. We write $P_\lambda(G)$ or just P_λ for the parabolic subgroup of G corresponding to λ, and $P_\lambda(M)$ for the parabolic subgroup of M corresponding to λ. It is obvious that $P_\lambda(M) = P_\lambda(G) \cap M$ and $R_u(P_\lambda(M)) = R_u(P_\lambda(G)) \cap M$.

Definition 2.3. Let $\lambda \in Y(G)$. Define a map $c_\lambda : P_\lambda \to L_\lambda$ by $c_\lambda(g) := \lim\limits_{a\to 0} \lambda(a)g\lambda(a)^{-1}$.

Note that the map c_λ is the usual canonical projection from P_λ to $L_\lambda \cong P_\lambda/R_u(P_\lambda)$. Now, we state a result from GIT (see [1, Lem. 2.17, Thm. 3.1], [2, Thm. 3.3]).

Proposition 2.4. *Let H be a subgroup of G. Let λ be a cocharacter of G with $H \subseteq P_\lambda$. If H is G-cr, there exists $v \in R_u(P_\lambda)$ such that $c_\lambda(h) = vhv^{-1}$ for every $h \in H$.*

2.3 Root subgroups and root subspaces

Let G be a connected reductive algebraic group. Fix a maximal torus T of G. Let $\Psi(G,T)$ denote the set of roots of G with respect to T. We sometimes write $\Psi(G)$ for $\Psi(G,T)$. Fix a Borel subgroup B containing T. Then $\Psi(B,T) = \Psi^+(G)$ is the set of positive roots of G defined by B. Let $\Sigma(G,B) = \Sigma$ denote the set of simple roots of G defined by B. Let $\zeta \in \Psi(G)$. We write U_ζ for the corresponding root subgroup of G and $\mathfrak{u}_\zeta$ for the Lie algebra of U_ζ. We define $G_\zeta := \langle U_\zeta, U_{-\zeta}\rangle$.

Let H be a subgroup of G normalized by some maximal torus T of G. Consider the adjoint representation of T on $\mathfrak{h}$. The root spaces of $\mathfrak{h}$ with respect to T are also root spaces of $\mathfrak{g}$ with respect to T, and the set of roots of H relative to T, $\Psi(H,T) = \Psi(H) = \{\zeta \in \Psi(G) \mid \mathfrak{g}_\zeta \subseteq \mathfrak{h}\}$, is a subset of $\Psi(G)$.

Let $\zeta, \xi \in \Psi(G)$. Let $\xi^\vee$ be the coroot corresponding to ξ. Then $\zeta \circ \xi^\vee : k^* \to k^*$ is a homomorphism such that $(\zeta \circ \xi^\vee)(a) = a^n$ for some $n \in \mathbb{Z}$. We define $\langle \zeta, \xi^\vee\rangle := n$. Let s_ξ denote the reflection corresponding to ξ in the Weyl group of G. Each s_ξ acts on the set of roots $\Psi(G)$ by the following formula [17, Lem. 7.1.8]: $s_\xi \cdot \zeta = \zeta - \langle \zeta, \xi^\vee\rangle \xi$. By [5, Prop. 6.4.2, Lem. 7.2.1], we can choose homomorphisms $\epsilon_\zeta : k \to U_\zeta$ so that

$$n_\xi \epsilon_\zeta(a) n_\xi^{-1} = \epsilon_{s_\xi \cdot \zeta}(\pm a), \text{ where } n_\xi = \epsilon_\xi(1)\epsilon_{-\xi}(-1)\epsilon_\xi(1). \tag{2.1}$$

We define $e_\zeta := \epsilon'_\zeta(0)$. Then we have

$$\mathrm{Ad}(n_\xi) e_\zeta = \pm e_{s_\xi \cdot \zeta}. \tag{2.2}$$

Now, we list four lemmas which we need in our calculations. The first one is [17, Prop. 8.2.1].

Lemma 2.5. *Let P be a parabolic subgroup of G. Any element u in $R_u(P)$ can be expressed uniquely as*

$$u = \prod_{i \in \Psi(R_u(P))} \epsilon_i(a_i), \text{ for some } a_i \in k,$$

where the product is taken with respect to a fixed ordering of $\Psi(R_u(P))$.

The next two lemmas [8, Lem. 32.5 and Lem. 33.3] are used to calculate $C_{R_u(P)}(K)$.

Lemma 2.6. *Let $\xi, \zeta \in \Psi(G)$. If no positive integral linear combination of ξ and ζ is a root of G, then*

$$\epsilon_\xi(a)\epsilon_\zeta(b) = \epsilon_\zeta(b)\epsilon_\xi(a).$$

Lemma 2.7. *Let Ψ be the root system of type A_2 spanned by roots ξ and ζ. Then*

$$\epsilon_\xi(a)\epsilon_\zeta(b) = \epsilon_\zeta(b)\epsilon_\xi(a)\epsilon_{\xi+\zeta}(\pm ab).$$

The last result is used to calculate $\mathfrak{c}_{\mathrm{Lie}\,(R_u(P))}(K)$.

Lemma 2.8. *Suppose that $p = 2$. Let W be a subgroup of G generated by all the n_ξ where $\xi \in \Psi(G)$ (the group W is isomorphic to the Weyl group of G). Let K be a subgroup of W. Let $\{O_i \mid i = 1 \cdots m\}$ be the set of orbits of the action of K on $\Psi(R_u(P))$. Then,*

$$\mathfrak{c}_{\mathrm{Lie}(R_u(P))}(K) = \left\{ \sum_{i=1}^{m} a_i \sum_{\zeta \in O_i} e_\zeta \,\middle|\, a_i \in k \right\}.$$

Proof. When $p = 2$, (2.2) yields $\mathrm{Ad}(n_\xi)e_\zeta = e_{n_\xi \cdot \zeta}$. Then an easy calculation gives the desired result. $\square$

Remark 2.9. Lemma 2.8 holds in $p = 2$ but fails in $p = 3$.

3 The E_7 example

3.1 Step 1

Let G be a simple algebraic group of type E_7 defined over k of characteristic 2. Fix a maximal torus T of G. Fix a Borel subgroup B of G containing T. Let $\Sigma = \{\alpha, \beta, \gamma, \delta, \epsilon, \eta, \sigma\}$ be the set of simple roots of G. Figure 1 defines how each simple root of G corresponds to each node in the Dynkin diagram of E_7.

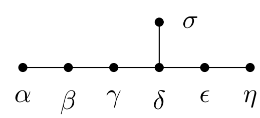

Figure 1: Dynkin diagram of E_7

From [6, Appendix, Table B], one knows the coefficients of all positive roots of G. We label all positive roots of G in Table 1 in the Appendix. Our ordering of roots is different from [6, Appendix, Table B], which will be convenient later on.

The set of positive roots is $\Psi^+(G) = \{1, 2, \cdots, 63\}$. Note that $\{1, \cdots, 35\}$ and $\{36, \cdots, 42\}$ are precisely the roots of G such that the coefficient of σ is 1 and 2 respectively. We call the roots of the first type *weight-1 roots*, and the second type *weight-2 roots*. Define

$$L_{\alpha\beta\gamma\delta\epsilon\eta} := \langle T, G_{43}, \cdots, G_{63} \rangle,\ P_{\alpha\beta\gamma\delta\epsilon\eta} := \langle L_{\alpha\beta\gamma\delta\epsilon\eta}, U_1, \cdots, U_{42} \rangle.$$

Then $P_{\alpha\beta\gamma\delta\epsilon\eta}$ is a parabolic subgroup of G, and $L_{\alpha\beta\gamma\delta\epsilon\eta}$ is a Levi subgroup of $P_{\alpha\beta\gamma\delta\epsilon\eta}$. Note that $L_{\alpha\beta\gamma\delta\epsilon\eta}$ is of type A_6. We have $\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})) = \{1, \cdots, 42\}$. Define

$$q_1 := n_\epsilon n_\beta n_\gamma n_\alpha n_\beta,\ q_2 := n_\epsilon n_\beta n_\gamma n_\alpha n_\beta n_\eta n_\delta n_\beta,\ K := \langle q_1, q_2 \rangle.$$

Let ζ_1, ζ_2 be simple roots of G. From the Cartan matrix of E_7 [7, Sec. 11.4] we have

$$\langle \zeta_1, \zeta_2 \rangle = \begin{cases} 2, & \text{if } \zeta_1 = \zeta_2. \\ -1, & \text{if } \zeta_1 \text{ is adjacent to } \zeta_2 \text{ in the Dynkin diagram.} \\ 0, & \text{otherwise.} \end{cases}$$

From this, it is not difficult to calculate $\langle \xi, \zeta^\vee \rangle$ for all $\xi \in \Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))$ and for all $\zeta \in \Sigma$. These calculations show how $n_\alpha, n_\beta, n_\gamma, n_\delta, n_\epsilon$, and n_η act on $\Psi(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))$. Let π :

$\langle n_\alpha, n_\beta, n_\gamma, n_\delta, n_\epsilon, n_\eta\rangle \to \mathrm{Sym}\left(\Psi\left(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})\right)\right) \cong S_{42}$ be the corresponding homomorphism. Then we have

$$\begin{aligned}\pi(q_1) =&(1\ 2)(3\ 6)(4\ 7)(9\ 10)(11\ 12)(13\ 14)(15\ 20)(16\ 17)(18\ 21)(19\ 23)(22\ 25)(24\ 26)\\ &(27\ 28)(29\ 32)(31\ 33)(34\ 35)(36\ 38)(37\ 39)(40\ 41),\\ \pi(q_2) =&(1\ 6\ 7\ 5\ 4\ 3\ 2)(8\ 10\ 12\ 14\ 13\ 11\ 9)(15\ 16\ 21\ 23\ 26\ 27\ 22)(17\ 20\ 25\ 28\ 24\ 19\ 18)\\ &(29\ 30\ 32\ 33\ 35\ 34\ 31)(36\ 38\ 39\ 41\ 42\ 40\ 37).\end{aligned}$$

It is easy to see that $K \cong D_{14}$. The orbits of K in $\Psi\left(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})\right)$ are

$$\begin{aligned}O_1 =&\{1, \cdots, 7\}, O_8 = \{8, \cdots, 14\}, O_{15} = \{15, \cdots, 28\}, O_{29} =\{29, \cdots, 35\},\\ O_{36} =&\{36, \cdots, 42\}.\end{aligned}$$

Thus Lemma 2.8 yields

Proposition 3.1.

$$\begin{aligned}\mathfrak{c}_{\mathrm{Lie}(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))}(K) =&\Bigg\{a\left(\sum_{\lambda\in O_1} e_\lambda\right) + b\left(\sum_{\lambda\in O_8} e_\lambda\right) + c\left(\sum_{\lambda\in O_{15}} e_\lambda\right) + d\left(\sum_{\lambda\in O_{29}} e_\lambda\right)\\ &+ m\left(\sum_{\lambda\in O_{36}} e_\lambda\right)\Bigg|\, a, b, c, d, m \in k\Bigg\}.\end{aligned}$$

The following is the most important technical result in this paper.

Proposition 3.2. *Let $u \in C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})}(K)$. Then u must have the form,*

$$u = \prod_{i=1}^{7}\epsilon_i(a)\prod_{i=8}^{14}\epsilon_i(b)\prod_{i=15}^{28}\epsilon_i(c)\prod_{i=29}^{35}\epsilon_i(a+b+c)\prod_{i=36}^{42}\epsilon_i(a_i) \text{ for some } a, b, c, a_i \in k.$$

Proof. By Lemma 2.5, u can be expressed uniquely as $u = \prod_{i=1}^{42}\epsilon_i(b_i)$ for some $b_i \in k$. By (2.1), we have $n_\xi\epsilon_\zeta(a)n_\xi^{-1} = \epsilon_{s_\xi\cdot\zeta}(a)$ for any $a \in k$ and $\xi, \zeta \in \Psi(G)$. Thus we have

$$\begin{aligned}q_1uq_1^{-1} &= q_1\left(\prod_{i=1}^{42}\epsilon_i(b_i)\right)q_1^{-1}\\ &= \left(\prod_{i=1}^{7}\epsilon_{q_1\cdot i}(b_i)\right)\left(\prod_{i=8}^{14}\epsilon_{q_1\cdot i}(b_i)\right)\left(\prod_{i=15}^{28}\epsilon_{q_1\cdot i}(b_i)\right)\left(\prod_{i=29}^{35}\epsilon_{q_1\cdot i}(b_i)\right)\\ &\quad\left(\prod_{i=36}^{42}\epsilon_{q_1\cdot i}(b_i)\right).\end{aligned}\tag{3.1}$$

A calculation using the commutator relations (Lemma 2.6 and Lemma 2.7) shows that

$$\begin{aligned}q_1uq_1^{-1} =& \epsilon_1(b_2)\epsilon_2(b_1)\epsilon_3(b_6)\epsilon_4(b_7)\epsilon_5(b_5)\epsilon_6(b_3)\epsilon_7(b_4)\epsilon_8(b_8)\epsilon_9(b_{10})\epsilon_{10}(b_9)\epsilon_{11}(b_{12})\epsilon_{12}(b_{11})\epsilon_{13}(b_{14})\\ &\epsilon_{14}(b_{13})\epsilon_{15}(b_{20})\epsilon_{16}(b_{17})\epsilon_{17}(b_{16})\epsilon_{18}(b_{21})\epsilon_{19}(b_{23})\epsilon_{20}(b_{15})\epsilon_{21}(b_{18})\epsilon_{22}(b_{25})\epsilon_{23}(b_{19})\epsilon_{24}(b_{26})\\ &\epsilon_{25}(b_{22})\epsilon_{26}(b_{24})\epsilon_{27}(b_{28})\epsilon_{28}(b_{27})\epsilon_{29}(b_{32})\epsilon_{30}(b_{30})\epsilon_{31}(b_{33})\epsilon_{32}(b_{29})\epsilon_{33}(b_{31})\epsilon_{34}(b_{35})\epsilon_{35}(b_{34})\\ &\left(\prod_{i=36}^{41}\epsilon_i(a_i)\right)\epsilon_{42}(b_4b_7 + b_{11}b_{12} + b_{22}b_{25} + b_{34}b_{35} + b_{42}) \text{ for some } a_i \in k.\end{aligned}\tag{3.2}$$

Since q_1 and q_2 centralize u, we have $b_1 = \cdots = b_7,\ b_8 = \cdots = b_{14},\ b_{15} = \cdots = b_{28},\ b_{29} = \cdots = b_{35}$. Set $b_1 = a,\ b_8 = b,\ b_{15} = c,\ b_{29} = d$. Then (3.2) simplifies to

$$q_1 u q_1^{-1} = \prod_{i=1}^{7} \epsilon_i(a) \prod_{i=8}^{14} \epsilon_i(b) \prod_{i=15}^{28} \epsilon_i(c) \prod_{i=29}^{35} \epsilon_i(d) \left(\prod_{i=36}^{41} \epsilon_i(a_i) \right) \epsilon_{42}(a^2 + b^2 + c^2 + d^2 + b_{42}).$$

Since q_1 centralizes u, comparing the arguments of the ϵ_{42} term on both sides, we must have

$$b_{42} = a^2 + b^2 + c^2 + d^2 + b_{42},$$

which is equivalent to $a + b + c + d = 0$. Then we obtain the desired result. $\square$

Proposition 3.3. *K acts non-separably on $R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})$.*

Proof. In view of Proposition 3.1, it suffices to show that $e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 \notin \operatorname{Lie} C_{R_u(P_\lambda)}(K)$. Suppose the contrary. Since by [17, Cor. 14.2.7] $C_{R_u(P_\lambda)}(K)^\circ$ is isomorphic as a variety to k^n for some $n \in \mathbb{N}$, there exists a morphism of varieties $v : k \to C_{R_u(P_\lambda)}(K)^\circ$ such that $v(0) = 1$ and $v'(0) = e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7$. By Lemma 2.5, $v(a)$ can be expressed uniquely as $v(a) = \prod_{i=1}^{42} \epsilon_i(f_i(a))$ for some $f_i \in k[X]$. Differentiating the last equation, and evaluating at $a = 0$, we obtain $v'(0) = \sum_{i \in \{1,\cdots,42\}} (f_i)'(0) e_i$. Since $v'(0) = \sum_{i \in O_1} e_i$, we have

$$(f_i)'(0) = \begin{cases} 1 & \text{if } i \in O_1, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$f_i(a) = \begin{cases} a + g_i(a) & \text{if } i \in O_1, \\ g_i(a) & \text{otherwise,} \end{cases}$$

where $g_i \in k[X]$ has no constant or linear term.

Then from Proposition 3.2, we obtain $(a + g_1(a)) + g_8(a) + g_{15}(a) = g_{29}(a)$. This is a contradiction. $\square$

3.2 Step 2

Let $C_1 := \left\{ \prod_{i=1}^{7} \epsilon_i(a) \mid a \in k \right\}$, pick any $a \in k^*$, and let $v(a) := \prod_{i=1}^{7} \epsilon_i(a)$. Now, set

$$\begin{aligned} H :=& v(a) K v(a)^{-1} = \langle q_1 \epsilon_{40}(a^2) \epsilon_{41}(a^2) \epsilon_{42}(a^2), q_2 \epsilon_{36}(a^2) \epsilon_{39}(a^2) \rangle, \\ M :=& \langle L_{\alpha\beta\gamma\delta\epsilon\eta}, G_{36}, \cdots, G_{42} \rangle. \end{aligned}$$

Remark 3.4. By Proposition 3.1 and Proposition 3.2, the tangent space of C_1 at the identity, $T_1(C_1)$, is contained in $\mathfrak{c}_{\operatorname{Lie}(R_u(P_{\alpha\beta\gamma\delta\epsilon\eta}))}(K)$ but not contained in $\operatorname{Lie}(C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})}(K))$. The element $v(a)$ can be any non-trivial element in C_1.

Remark 3.5. In this case σ is the unique simple root not contained in $\Psi(L_{\alpha\beta\gamma\delta\epsilon\eta})$. M was chosen so that M is generated by a Levi subgroup $L_{\alpha\beta\gamma\delta\epsilon\eta}$ containing K and all root subgroups of σ-weight 2.

We have $H \subset M, H \not\subset L_{\alpha\beta\gamma\delta\epsilon\eta}$. Note that $\Psi(M) = \{\pm 36, \cdots, \pm 63\}$. Since M is generated by all root subgroups of even σ-weight, it is easy to see that $\Psi(M)$ is a closed subsystem of $\Psi(G)$, thus M is reductive by [3, Lem. 3.9]. Note that M is of type A_7.

Proposition 3.6. *H is not M-cr.*

Proof. Let $\lambda = 3\alpha^\vee + 6\beta^\vee + 9\gamma^\vee + 12\delta^\vee + 8\epsilon^\vee + 4\eta^\vee + 7\sigma^\vee$. We have

$$\begin{aligned}
&\langle \alpha, \lambda\rangle = 0, \langle \beta, \lambda\rangle = 0, \langle \gamma, \lambda\rangle = 0, \langle \delta, \lambda\rangle = 0,\\
&\langle \epsilon, \lambda\rangle = 0, \langle \eta, \lambda\rangle = 0, \langle \sigma, \lambda\rangle = 2.
\end{aligned}$$

So $L_{\alpha\beta\gamma\delta\epsilon\eta} = L_\lambda$, $P_{\alpha\beta\gamma\delta\epsilon\eta} = P_\lambda$.

It is easy to see that L_λ is of type A_6, so $[L_\lambda, L_\lambda]$ is isomorphic to either SL_7 or PGL_7. We rule out the latter. Pick $x \in k^*$ such that $x \neq 1, x^7 = 1$. Then $\lambda(x) \neq 1$ since $\sigma(\lambda(x)) = x^2 \neq 1$. Also, we have $\lambda(x) \in Z([L_\lambda, L_\lambda])$. Therefore $[L_\lambda, L_\lambda] \cong SL_7$. It is easy to check that the map $k^* \times [L_\lambda, L_\lambda] \to L_\lambda$ is separable, so we have $L_\lambda \cong GL_7$.

Let $c_\lambda : P_\lambda \to L_\lambda$ be the homomorphism as in Definition 2.3. In order to prove that H is not M-cr, by Theorem 2.4 it suffices to find a tuple $(h_1, h_2) \in H^2$ which is not $R_u(P_\lambda(M))$-conjugate to $c_\lambda((h_1, h_2))$. Set $h_1 := v(a)q_1v(a)^{-1}$, $h_2 := v(a)q_2v(a)^{-1}$. Then

$$\begin{aligned}
c_\lambda((h_1, h_2)) &= \lim_{x\to 0} \left(\lambda(x)q_1\epsilon_{40}(a^2)\epsilon_{41}(a^2)\epsilon_{42}(a^2)\lambda(x)^{-1},\ (\lambda(x)q_2\epsilon_{36}(a^2)\epsilon_{39}(a^2)\lambda(x)^{-1}\right)\\
&= (q_1, q_2).
\end{aligned}$$

Now suppose that (h_1, h_2) is $R_u(P_\lambda(M))$-conjugate to $c_\lambda((h_1, h_2))$. Then there exists $m \in R_u(P_\lambda(M))$ such that

$$mv(a)q_1v(a)^{-1}m^{-1} = q_1,\ mv(a)q_2v(a)^{-1}m^{-1} = q_2.$$

Thus we have $mv(a) \in C_{R_u(P_\lambda)}(K)$. Note that $\Psi(R_u(P_\lambda(M))) = \{36, \cdots, 42\}$. So, by Lemma 2.5, m can be expressed uniquely as $m := \prod_{i=36}^{42} \epsilon_i(a_i)$ for some $a_i \in k$. Then we have

$$mv(a) = \epsilon_1(a)\epsilon_2(a)\epsilon_3(a)\epsilon_4(a)\epsilon_5(a)\epsilon_6(a)\epsilon_7(a)\left(\prod_{i=36}^{42} \epsilon_i(a_i)\right) \in C_{R_u(P_\lambda)}(K).$$

This contradicts Proposition 3.2. ☐

Remark 3.7. In [3, Sec. 7, Prop .7.17], Bate et al. used [1, Lem. 2.17, Thm. 3.1] to turn a problem on M-complete reducibility into a problem involving M-conjugacy. We have used Proposition 2.4 to turn the same problem into a problem involving $R_u(P \cap M)$-conjugacy, which is easier.

Remark 3.8. Instead of using C_1 to define $v(a)$, we can take $C_8 := \left\{\prod_{i=8}^{14} \epsilon_i(a) \mid a \in k\right\}$, $C_{15} := \left\{\prod_{i=15}^{28} \epsilon_i(a) \mid a \in k\right\}$, or $C_{29} := \left\{\prod_{i=29}^{35} \epsilon_i(a) \mid a \in k\right\}$. In each case, a similar argument goes through and gives rise to a different example with the desired property.

3.3 Step 3

Proposition 3.9. *H is G-cr.*

Proof. First note that H is conjugate to K, so H is G-cr if and only if K is G-cr. Then, by [1, Lem. 2.12, Cor. 3.22], it suffices to show that K is $[L_\lambda, L_\lambda]$-cr. We can identify K with the image of the corresponding subgroup of S_7 under the permutation representation $\pi_1 : S_7 \to SL_7(k)$. It is easy to see that $K \cong D_{14}$. A quick calculation shows that this representation of D_{14} is a direct sum of a trivial 1-dimensional and 3 irreducible 2-dimensional subrepresentations. Therefore K is $[L_\lambda, L_\lambda]$-cr. ☐

4 A rationality problem

We prove Theorem 1.10. The key here is again the existence of a 1-dimensional curve C_1 such that $T_1(C_1)$ is contained in $\mathfrak{c}_{\mathrm{Lie}\,(R_u(P_\lambda))}(K)$ but not contained in $\mathrm{Lie}(C_{R_u(P_\lambda)}(K))$. The same phenomenon was seen in the G_2 example.

Proof of Theorem 1.10. Let k_0, k, and G be as in the hypothesis. We choose a k_0-defined k_0-split maximal torus T such that for each $\zeta \in \Psi(G)$ the corresponding root ζ, coroot $\zeta^\vee$, and homomorphism ϵ_ζ are defined over k_0. Since k_0 is not perfect, there exists $\tilde{a} \in k\backslash k_0$ such that $\tilde{a}^2 \in k_0$. We keep the notation $q_1, q_2, v, K, P_\lambda, L_\lambda$ of Section 3. Let

$$\begin{aligned} H &= \langle v(\tilde{a})q_1 v(\tilde{a})^{-1}, v(\tilde{a})q_2 v(\tilde{a})^{-1}\rangle \\ &= \langle q_1\epsilon_{40}(\tilde{a}^2)\epsilon_{41}(\tilde{a}^2)\epsilon_{42}(\tilde{a}^2), q_2\epsilon_{36}(\tilde{a}^2)\epsilon_{39}(\tilde{a}^2)\rangle. \end{aligned}$$

Now it is obvious that H is k_0-defined. We already know that H is G-cr by Proposition 3.9. Since G and T are k_0-split, P_λ and L_λ are k_0-defined by [4, V.20.4, V.20.5]. Suppose that there exists a k_0-Levi subgroup L' of P_λ such that L' contains H. Then there exists $w \in R_u(P_\lambda)(k_0)$ such that $L' = wL_\lambda w^{-1}$ by [4, V.20.5]. Then $w^{-1}Hw \subseteq L_\lambda$ and $v(\tilde{a})^{-1}Hv(\tilde{a}) \subseteq L_\lambda$. So we have $c_\lambda(w^{-1}hw) = w^{-1}hw$ and $c_\lambda\left(v(\tilde{a})^{-1}hv(\tilde{a})\right) = v(\tilde{a})^{-1}hv(\tilde{a})$ for any $h \in H$. We also have $c_\lambda(w) = c_\lambda\left(v(\tilde{a})\right) = 1$ since $w, v(\tilde{a}) \in R_u(P_\lambda)(k)$. Therefore we obtain $w^{-1}hw = c_\lambda(w^{-1}hw) = c_\lambda(h) = c_\lambda\left(v(\tilde{a})^{-1}hv(\tilde{a})\right) = v(\tilde{a})^{-1}hv(\tilde{a})$ for any $h \in H$. So we have $w = v(\tilde{a})z$ for some $z \in C_{R_u(P_\lambda)}(K)(k)$. By Proposition 3.2, z must have the form

$$z = \prod_{i=1}^{7}\epsilon_i(a)\prod_{i=8}^{14}\epsilon_i(b)\prod_{i=15}^{28}\epsilon_i(c)\prod_{i=29}^{35}\epsilon_i(a+b+c)\prod_{i=36}^{42}\epsilon_i(a_i) \text{ for some } a, b, c, a_i \in k.$$

Then

$$\begin{aligned} w &= \left(\prod_{i=1}^{7}\epsilon_i(\tilde{a})\right)\prod_{i=1}^{7}\epsilon_i(a)\prod_{i=8}^{14}\epsilon_i(b)\prod_{i=15}^{28}\epsilon_i(c)\prod_{i=29}^{35}\epsilon_i(a+b+c)\prod_{i=36}^{42}\epsilon_i(a_i) \\ &= \prod_{i=1}^{7}\epsilon_i(\tilde{a}+a)\prod_{i=8}^{14}\epsilon_i(b)\prod_{i=15}^{28}\epsilon_i(c)\prod_{i=29}^{35}\epsilon_i(a+b+c)\prod_{i=36}^{42}\epsilon_i(b_i) \text{ for some } b_i \in k. \end{aligned}$$

Since w is a k_0-point, b, c, and $a+b+c$ all belong to k_0, so $a \in k_0$. But $a + \tilde{a}$ belongs to k_0 as well, so $\tilde{a} \in k_0$. This is a contradiction. $\square$

Remark 4.1. As in Section 3, we can take $v(\tilde{a})$ from C_8, C_{15}, or C_{29}. In each case, a similar argument goes through, and gives rise to a different example.

Remark 4.2. [1, Ex. 5.11] shows that there is a k_0-defined subgroup of G of type A_n which is not G-cr over k even though it is G-cr over k_0. Note that this example works for any $p > 0$.

5 A problem of conjugacy classes

We prove Theorem 1.12. Here, the key is again the existence of a 1-dimensional curve C_1 such that $T_1(C_1)$ is contained in $\mathfrak{c}_{\mathrm{Lie}\,(R_u(P_\lambda))}(K)$ but not contained in $\mathrm{Lie}(C_{R_u(P_\lambda)}(K))$ as in the G_2 example. Let G, M, k be as in the hypotheses of the theorem. We keep the notation $q_1, q_2, v, K, P_\lambda, L_\lambda$ of Section 3. A calculation using the commutator relations (Lemma 2.6) shows that

$$Z(R_u(P_\lambda)) = \langle U_{36}, U_{37}, U_{38}, U_{39}, U_{40}, U_{41}, U_{42}\rangle.$$

Let $K_0 := \langle K, Z(R_u(P_\lambda))\rangle$. It is standard that there exists a finite subset $F = \{z_1, z_2, \cdots, z_{n'}\}$ of $Z(R_u(P))$ such that $C_{P_\lambda}(\langle K, F\rangle) = C_{P_\lambda}(K_0)$. Let $\mathbf{m} := (q_1, q_2, z_1, \cdots, z_{n'})$. Let $n := n' + 2$. For every $x \in k^*$, define $\mathbf{m}(x) := v(x) \cdot \mathbf{m} \in P_\lambda(M)^n$.

Lemma 5.1. $C_{P_\lambda}(K_0) = C_{R_u(P_\lambda)}(K_0)$.

Proof. It is obvious that $C_{R_u(P_\lambda)}(K_0) \subseteq C_{P_\lambda}(K_0)$. We prove the converse. Let $lu \in C_{P_\lambda}(K_0)$ for some $l \in L_\lambda$ and $u \in R_u(P_\lambda)$. Then lu centralizes $Z(R_u(P_\lambda))$, so l centralizes $Z(R_u(P_\lambda))$, since u does. It suffices to show that $l = 1$. Let $l = t\tilde{l}$ where $t \in Z(L_\lambda)^\circ = \lambda(k^*)$ and $\tilde{l} \in [L_\lambda, L_\lambda]$. We have

$$\langle i, \lambda\rangle = 4 \text{ for any } i \in \{36, \cdots, 42\}. \tag{5.1}$$

So for any $z \in Z(R_u(P_\lambda))$, there exists $\alpha \in k^*$ such that $t \cdot z = \alpha z$. Then we have $\tilde{l} \cdot z = \alpha^{-1} z$. Now define $A := \{\tilde{l} \in [L_\lambda, L_\lambda] \mid \tilde{l} \text{ acts on } Z(R_u(P_\lambda)) \text{ by multiplication by a scalar}\}$. Then it is easy to see that $A \trianglelefteq [L_\lambda, L_\lambda]$. Since $[L_\lambda, L_\lambda] \cong SL_7$ and $L_\lambda \cong GL_7$, we have $A = Z([L_\lambda, L_\lambda])$. Therefore we obtain $\tilde{l} \in A = Z([L_\lambda, L_\lambda]) \subseteq \lambda(k^*)$. So we have $l = c\tilde{l} \in \lambda(k^*)$. Then we obtain $l \in C_{\lambda(k^*)}(Z(R_u(P_\lambda)))$. By (5.1) this implies $l = 1$. $\square$

Lemma 5.2. *$G \cdot \mathbf{m} \cap P_\lambda(M)^n$ is an infinite union of $P_\lambda(M)$-conjugacy classes.*

Proof. Fix $a' \in k^*$. By Lemma 5.1, we have $C_{P_\lambda}(K_0) = C_{R_u(P_\lambda)}(K_0) \subseteq C_{R_u(P_\lambda)}(K)$. Then we obtain

$$C_{P_\lambda}(v(a')K_0 v(a')^{-1}) = v(a')C_{P_\lambda}(K_0)v(a')^{-1} \subseteq v(a')C_{R_u(P_\lambda)}(K)v(a')^{-1}. \tag{5.2}$$

Choose $b' \in k^*$ such that $\mathbf{m}(a')$ is $P_\lambda(M)$-conjugate to $\mathbf{m}(b')$. Then there exists $m \in P_\lambda(M)$ such that $m \cdot \mathbf{m}(b') = \mathbf{m}(a')$. By (5.2), we have

$$mv(b')v(a')^{-1} \in C_{P_\lambda}(v(a')K_0 v(a')^{-1}) \subseteq v(a')C_{R_u(P_\lambda)}(K)v(a')^{-1}.$$

By Proposition 3.2, we have

$$v(a')^{-1}mv(b') = \prod_{i=1}^{7}\epsilon_i(a)\prod_{i=8}^{14}\epsilon_i(b)\prod_{i=15}^{28}\epsilon_i(c)\prod_{i=29}^{35}\epsilon_i(a+b+c)\prod_{i=36}^{42}\epsilon_i(a_i), \text{ for some } a, b, c, a_i \in k.$$

This yields

$$m = \prod_{i=1}^{7}\epsilon_i(a+a'+b')\prod_{i=8}^{14}\epsilon_i(b)\prod_{i=15}^{28}\epsilon_i(c)\prod_{i=29}^{35}\epsilon_i(a+b+c)\prod_{i=36}^{42}\epsilon_i(b_i), \text{ for some } a, b, c, b_i \in k.$$

But $m \in P_\lambda(M)$, so $a + a' + b' = 0, b = 0, c = 0, a + b + c = 0$. Hence we have $a' = b'$. Thus we have shown that if $a' \neq b'$, then $\mathbf{m}(a')$ is not $P_\lambda(M)$-conjugate to $\mathbf{m}(b')$. So, in particular, $G \cdot \mathbf{m} \cap P_\lambda(M)^n$ is an infinite union of $P_\lambda(M)$-conjugacy classes. $\square$

We need the next result [11, Lem. 4.4]. We include the proof to make this paper self-contained.

Lemma 5.3. *$G \cdot \mathbf{m} \cap P_\lambda(M)^n$ is a finite union of M-conjugacy classes if and only if it is a finite union of $P_\lambda(M)$-conjugacy classes.*

Proof. Pick $\mathbf{m_1}, \mathbf{m_2} \in G \cdot \mathbf{m} \cap P_\lambda(M)^n$ such that $\mathbf{m_1}$ and $\mathbf{m_2}$ are in the same M-conjugacy class of $G \cdot \mathbf{m} \cap P_\lambda(M)^n$. Then there exists $m \in M$ such that $m \cdot \mathbf{m_1} = \mathbf{m_2}$. Let $Q = m^{-1}P_\lambda(M)m$. Then we have $\mathbf{m_1} \in (P_\lambda(M) \cap Q)^n$. Now let S be a maximal torus of M contained in $P_\lambda(M) \cap Q$.

Since S and $m^{-1}Sm$ are maximal tori of Q, they must be Q-conjugate. So there exists $q \in Q$ such that

$$qSq^{-1} = m^{-1}Sm. \tag{5.3}$$

Since $Q = m^{-1}P_\lambda(M)m$, there exists $p \in P_\lambda(M)$ such that $q = m^{-1}pm$. Then from (5.3), we obtain $pmSm^{-1}p^{-1} = S$. This implies $m^{-1}p^{-1} \in N_M(S)$. Fix a finite set $N \subseteq N_M(S)$ of coset representatives for the Weyl group $W = N_M(S)/S$. Then we have

$$m^{-1}p^{-1} = ns \text{ for some } n \in N, s \in S.$$

So we obtain $\mathbf{m_1} = m^{-1} \cdot \mathbf{m_2} = (nsp) \cdot \mathbf{m_2} \in (nP_\lambda(M)) \cdot \mathbf{m_2}$. Since N is a finite set, this shows that a M-conjugacy class in $G \cdot \mathbf{m} \cap P_\lambda(M)^n$ is a finite union of $P_\lambda(M)$-conjugacy classes. The converse is obvious. $\square$

Proof of Theorem 1.12. By Lemma 5.2 and Lemma 5.3, we conclude that $G \cdot \mathbf{m} \cap P_\lambda(M)^n$ is an infinite union of M-conjugacy classes. Now it is evident that $G \cdot \mathbf{m} \cap M^n$ is an infinite union of M-conjugacy classes. $\square$

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Appendix

1	$\begin{smallmatrix} &&&1&& \\ 0&0&1&1&1&0 \end{smallmatrix}$	2	$\begin{smallmatrix} &&&1&& \\ 1&1&1&1&0&0 \end{smallmatrix}$	3	$\begin{smallmatrix} &&&1&& \\ 0&1&1&2&1&1 \end{smallmatrix}$	4	$\begin{smallmatrix} &&&1&& \\ 0&0&1&2&2&1 \end{smallmatrix}$
5	$\begin{smallmatrix} &&&1&& \\ 1&1&2&2&1&0 \end{smallmatrix}$	6	$\begin{smallmatrix} &&&1&& \\ 0&1&1&2&2&1 \end{smallmatrix}$	7	$\begin{smallmatrix} &&&1&& \\ 1&2&2&2&1&1 \end{smallmatrix}$	8	$\begin{smallmatrix} &&&1&& \\ 0&0&0&0&0&0 \end{smallmatrix}$
9	$\begin{smallmatrix} &&&1&& \\ 0&0&0&1&0&0 \end{smallmatrix}$	10	$\begin{smallmatrix} &&&1&& \\ 0&1&1&1&1&0 \end{smallmatrix}$	11	$\begin{smallmatrix} &&&1&& \\ 0&0&1&2&1&1 \end{smallmatrix}$	12	$\begin{smallmatrix} &&&1&& \\ 1&2&2&2&2&1 \end{smallmatrix}$
13	$\begin{smallmatrix} &&&1&& \\ 1&1&2&3&2&1 \end{smallmatrix}$	14	$\begin{smallmatrix} &&&1&& \\ 1&2&3&3&2&1 \end{smallmatrix}$	15	$\begin{smallmatrix} &&&1&& \\ 0&0&1&1&0&0 \end{smallmatrix}$	16	$\begin{smallmatrix} &&&1&& \\ 0&0&0&1&1&0 \end{smallmatrix}$
17	$\begin{smallmatrix} &&&1&& \\ 0&1&1&1&0&0 \end{smallmatrix}$	18	$\begin{smallmatrix} &&&1&& \\ 0&0&0&1&1&1 \end{smallmatrix}$	19	$\begin{smallmatrix} &&&1&& \\ 0&0&1&2&1&0 \end{smallmatrix}$	20	$\begin{smallmatrix} &&&1&& \\ 1&1&1&1&1&0 \end{smallmatrix}$
21	$\begin{smallmatrix} &&&1&& \\ 0&1&1&1&1&1 \end{smallmatrix}$	22	$\begin{smallmatrix} &&&1&& \\ 1&1&1&2&1&1 \end{smallmatrix}$	23	$\begin{smallmatrix} &&&1&& \\ 1&2&2&2&1&0 \end{smallmatrix}$	24	$\begin{smallmatrix} &&&1&& \\ 1&1&2&2&1&1 \end{smallmatrix}$
25	$\begin{smallmatrix} &&&1&& \\ 0&1&2&2&2&1 \end{smallmatrix}$	26	$\begin{smallmatrix} &&&1&& \\ 1&1&2&2&2&1 \end{smallmatrix}$	27	$\begin{smallmatrix} &&&1&& \\ 0&1&2&3&2&1 \end{smallmatrix}$	28	$\begin{smallmatrix} &&&1&& \\ 1&2&2&3&2&1 \end{smallmatrix}$
29	$\begin{smallmatrix} &&&1&& \\ 0&0&1&1&1&1 \end{smallmatrix}$	30	$\begin{smallmatrix} &&&1&& \\ 0&1&1&2&1&0 \end{smallmatrix}$	31	$\begin{smallmatrix} &&&1&& \\ 1&1&1&2&1&0 \end{smallmatrix}$	32	$\begin{smallmatrix} &&&1&& \\ 1&1&1&1&1&1 \end{smallmatrix}$
33	$\begin{smallmatrix} &&&1&& \\ 0&1&2&2&1&0 \end{smallmatrix}$	34	$\begin{smallmatrix} &&&1&& \\ 0&1&2&2&1&1 \end{smallmatrix}$	35	$\begin{smallmatrix} &&&1&& \\ 1&1&1&2&2&1 \end{smallmatrix}$	36	$\begin{smallmatrix} &&&2&& \\ 0&1&2&3&2&1 \end{smallmatrix}$
37	$\begin{smallmatrix} &&&2&& \\ 1&1&2&3&2&1 \end{smallmatrix}$	38	$\begin{smallmatrix} &&&2&& \\ 1&2&2&3&2&1 \end{smallmatrix}$	39	$\begin{smallmatrix} &&&2&& \\ 1&2&3&3&2&1 \end{smallmatrix}$	40	$\begin{smallmatrix} &&&2&& \\ 1&2&3&4&2&1 \end{smallmatrix}$
41	$\begin{smallmatrix} &&&2&& \\ 1&2&3&4&3&1 \end{smallmatrix}$	42	$\begin{smallmatrix} &&&2&& \\ 1&2&3&4&3&2 \end{smallmatrix}$	43	$\begin{smallmatrix} &&&0&& \\ 1&0&0&0&0&0 \end{smallmatrix}$	44	$\begin{smallmatrix} &&&0&& \\ 0&1&0&0&0&0 \end{smallmatrix}$
45	$\begin{smallmatrix} &&&0&& \\ 0&0&1&0&0&0 \end{smallmatrix}$	46	$\begin{smallmatrix} &&&0&& \\ 0&0&0&1&0&0 \end{smallmatrix}$	47	$\begin{smallmatrix} &&&0&& \\ 0&0&0&0&1&0 \end{smallmatrix}$	48	$\begin{smallmatrix} &&&0&& \\ 0&0&0&0&0&1 \end{smallmatrix}$
49	$\begin{smallmatrix} &&&0&& \\ 1&1&0&0&0&0 \end{smallmatrix}$	50	$\begin{smallmatrix} &&&0&& \\ 0&1&1&0&0&0 \end{smallmatrix}$	51	$\begin{smallmatrix} &&&0&& \\ 0&0&1&1&0&0 \end{smallmatrix}$	52	$\begin{smallmatrix} &&&0&& \\ 0&0&0&1&1&0 \end{smallmatrix}$
53	$\begin{smallmatrix} &&&0&& \\ 0&0&0&0&1&1 \end{smallmatrix}$	54	$\begin{smallmatrix} &&&0&& \\ 1&1&1&0&0&0 \end{smallmatrix}$	55	$\begin{smallmatrix} &&&0&& \\ 0&1&1&1&0&0 \end{smallmatrix}$	56	$\begin{smallmatrix} &&&0&& \\ 0&0&1&1&1&0 \end{smallmatrix}$
57	$\begin{smallmatrix} &&&0&& \\ 0&0&0&1&1&1 \end{smallmatrix}$	58	$\begin{smallmatrix} &&&0&& \\ 1&1&1&1&0&0 \end{smallmatrix}$	59	$\begin{smallmatrix} &&&0&& \\ 0&1&1&1&1&0 \end{smallmatrix}$	60	$\begin{smallmatrix} &&&0&& \\ 0&0&1&1&1&1 \end{smallmatrix}$
61	$\begin{smallmatrix} &&&0&& \\ 1&1&1&1&1&0 \end{smallmatrix}$	62	$\begin{smallmatrix} &&&0&& \\ 0&1&1&1&1&1 \end{smallmatrix}$	63	$\begin{smallmatrix} &&&0&& \\ 1&1&1&1&1&1 \end{smallmatrix}$		

Table 1: The set of positive roots of $G = E_7$

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Non-separability and complete reducibility: E_n examples with an application to a question of Külshammer

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Abstract

Let G be a simple algebraic group of type $E_n (n = 6, 7, 8)$ defined over an algebraically closed field k of characteristic 2. We present examples of triples of closed reductive groups $H < M < G$ such that H is G-completely reducible, but not M-completely reducible. As an application, we consider a question of Külshammer on representations of finite groups in reductive groups. We also consider a rationality problem for G-complete reducibility and a problem concerning conjugacy classes.

Keywords: algebraic groups, separable subgroups, complete reducibility, representations of finite groups

1 Introduction

Let G be a connected reductive algebraic group defined over an algebraically closed field k of characteristic p. In [17, Sec. 3], J.P. Serre defined the following:

Definition 1.1. A closed subgroup H of G is *G-completely reducible* (G-cr for short) if whenever H is contained in a parabolic subgroup P of G, H is contained in a Levi subgroup L of P.

This is a faithful generalization of the notion of semisimplicity in representation theory: if $G = GL_n(k)$, a subgroup H of G is G-cr if and only if H acts semisimply on k^n [17, Ex. 3.2.2(a)]. If $p = 0$, the notion of G-complete reducibility agrees with the notion of reductivity [17, Props. 4.1, 4.2]. In this paper, we assume $p > 0$. In that case, if a subgroup H is G-cr, then H is reductive [17, Prop. 4.1], but the other direction fails: take H to be a unipotent subgroup of order p of $G = SL_2$. See [20] for examples of connected non-G-cr subgroups. In this paper, by a subgroup of G, we always mean a closed subgroup.

Completely reducible subgroups have been much studied as important ingredients to understand the subgroup structure of connected reductive algebraic groups [12], [13], [21]. Recently, studies of complete reducibility via Geometric Invariant Theory (GIT for short) have been fruitful [3], [4], [2]. In this paper, we use a recent result from GIT (Proposition 2.4).

Here is the first problem we consider in this paper. Let $H < M < G$ be a triple of reductive algebraic groups. It is known to be hard to find such a triple with H G-cr but not M-cr [3], [23]. The only known such examples are [3, Sec. 7] for $p = 2, G = G_2$ and [23] for $p = 2, G = E_7$. Recall that a pair of reductive groups G and M is called a *reductive pair* if $\operatorname{Lie} M$ is an M-module direct summand of $\mathfrak{g}$. For more on reductive pairs, see [8]. Our main result is:

Theorem 1.2. *Let G be a simple algebraic group of type E_6 (respectively E_7, E_8) of any isogeny type defined over an algebraically closed field k of characteristic 2. Then there exist reductive subgroups $H < M$ of G such that H is finite, M is semisimple of type A_5A_1 (respectively A_7, D_8), (G, M) is a reductive pair, and H is G-cr but not M-cr.*

In this paper, we present new examples with the properties of Theorem 1.2 giving an explicit description of the mechanism for generating such examples. We give 11 examples for $G = E_6$, 1 new example for $G = E_7$, and 2 examples for $G = E_8$. We use Magma [5] for our computations. Recall that G-complete reducibility is invariant under isogenies [2, Lem. 2.12]; in Sections 3,4, and 5, we do computations for simply-connected G only, but that is sufficient to prove Theorem 1.2 for G of any isogeny type.

We recall a few relevant definitions and results from [3], [23], which motivated our work. We denote the Lie algebra of G by $\mathrm{Lie}\,\mathrm{G} = \mathfrak{g}$.

Definition 1.3. Let H and N be subgroups of G where H acts on N by group automorphisms. The action of H is called *separable* in N if the global centralizer of H in N agrees with the infinitesimal centralizer of H in Lie N, that is, $C_N(H) = \mathfrak{c}_{\mathrm{Lie}\,N}(H)$. Note that the condition means that the set of fixed points of H acting on N, taken with its natural scheme structure, is smooth.

This is a slight generalization of the notion of separable subgroups. Recall that

Definition 1.4. Let H be a subgroup of G acting on G by inner automorphisms. Let H act on $\mathfrak{g}$ by the corresponding adjoint action. Then H is called *separable* if $\mathrm{Lie}\, C_G(H) = \mathfrak{c}_{\mathfrak{g}}(H)$.

Note that we always have $\mathrm{Lie}\, C_G(H) \subseteq \mathfrak{c}_{\mathfrak{g}}(H)$. In [3], Bate et al. investigated the relationship between G-complete reducibility and separability, and showed the following [3, Thm. 1.2, Thm. 1.4] (see [9] for more on separability).

Proposition 1.5. *Suppose that p is very good for G. Then any subgroup of G is separable in G.*

Proposition 1.6. *Suppose that (G, M) is a reductive pair. Let H be a subgroup of M such that H is a separable subgroup of G. If H is G-cr, then it is also M-cr.*

Propositions 1.5 and 1.6 imply that the subgroup H in Theorem 1.2 must be non-separable, which is possible for small p only.

We recap our method from [23]. Fix a maximal torus T of $G = E_6$ (respectively E_7, E_8). Fix a system of positive roots. Let L be the A_5 (respectively A_6, A_7)-Levi subgroup of G containing T. Let P be the parabolic subgroup of G containing L, and let $R_u(P)$ be the unipotent radical of P. Let W_L be the Weyl group of L. Abusing the notation, we write W_L for the group generated by canonical representatives n_ζ of reflections in W_L. (See Section 2 for the definition of n_ζ.) Now W_L is a subgroup of L.

1. Find a subgroup K' of W_L acting non-separably on $R_u(P)$.
2. If K' is G-cr, set $K := K'$ and go to the next step. Otherwise, add an element t from the maximal torus T in such a way that $K := \langle K' \cup \{t\} \rangle$ is G-cr and K still acts non-separably on $R_u(P)$.
3. Choose a suitable element $v \in R_u(P)$ in a 1-dimensional curve C such that $T_1(C)$ is contained in $\mathfrak{c}_{Lie(R_u(P))}(K)$ but not contained in $\mathrm{Lie}(C_{R_u(P)}(K))$. Set $H := vKv^{-1}$. Choose a connected reductive subgroup M of G containing H such that H is not G-cr. Show that H is not M-cr using Proposition 2.4.

As the first application of our construction, we consider a rationality problem for G-complete reducibility. We need a definition first.

Definition 1.7. Let k_0 be a subfield of k. Let H be a k_0-defined subgroup of a k_0-defined reductive algebraic group G. Then H is *G-completely reducible over k_0* (G-cr over k_0 for short) if whenever H is contained in a k_0-defined parabolic subgroup P of G, it is contained in some k_0-defined Levi subgroup of P.

Note that if k_0 is algebraically closed then G-cr over k_0 means G-cr in the usual sense. Here is the main result concerning rationality.

Theorem 1.8. *Let k_0 be a nonperfect field of characteristic 2, and let G be a k_0-defined split simple algebraic group of type $E_n (n = 6, 7, 8)$ of any isogeny type. Then there exists a k_0-defined subgroup H of G such that H is G-cr but not G-cr over k_0.*

Proof. Use the same $H = v(a)Kv(a)^{-1}$ as in the proof of Theorem 1.2 with $v := v(a)$ for $a \in k_0 \backslash k_0^2$. Then a similar method to [23, Sec. 4] shows that subgroups H have the desired properties. The crucial thing here is the existence of a 1-dimensional curve C such that $T_1(C)$ is contained in $\mathfrak{c}_{\mathrm{Lie}(R_u(P))}(K)$ but not contained in $\mathrm{Lie}(C_{R_u(P)}(K))$ (see [23, Sec. 4] for details). □

Remark 1.9. Let k_0 and $G = E_6$ be as in Theorem 1.8. Based on the construction of the E_6 examples in this paper, we found the first examples of nonabelian k_0-defined subgroups H of G such that H is G-cr over k_0 but not G-cr; see [22]. Note that G-complete reducibility over k_0 is invariant under central isogenies [22, Sec. 2].

As the second application, we consider a problem concerning conjugacy classes. Given $n \in \mathbb{N}$, we let G act on G^n by simultaneous conjugation:

$$g \cdot (g_1, g_2, \ldots, g_n) = (gg_1g^{-1}, gg_2g^{-1}, \ldots, gg_ng^{-1}).$$

In [18], Slodowy proved the following result, applying Richardson's tangent space argument [14, Sec. 3], [15, Lem. 3.1].

Proposition 1.10. *Let M be a reductive subgroup of a reductive algebraic group G defined over an algebraically closed field k. Let $N \in \mathbb{N}$, let $(m_1, \ldots, m_N) \in M^N$ and let H be the subgroup of M generated by $m_1, \ldots, m_N$. Suppose that (G, M) is a reductive pair and that H is separable in G. Then the intersection $G \cdot (m_1, \ldots, m_N) \cap M^N$ is a finite union of M-conjugacy classes.*

Proposition 1.10 has many consequences; see [2], [18], and [24, Sec. 3] for example. Here is our main result on conjugacy classes:

Theorem 1.11. *Let G be a simple algebraic group of type E_6 (respectively E_7, E_8) defined over an algebraically closed k of characteristic $p = 2$. Let M be the subsystem subgroup of type A_5A_1 (respectively A_7, D_8). Then there exists $N \in \mathbb{N}$ and a tuple $\mathbf{m} \in M^N$ such that $G \cdot \mathbf{m} \cap M^N$ is an infinite union of M-conjugacy classes.*

Proof. We give a sketch with one example for $G = E_6$ (see Section 3, case 4). Keep the same notation P_λ, K', q_1, q_2 therein. Define $K_0 := \langle K', Z(R_u(P_\lambda))\rangle$. By a standard result, there exists a finite subset $F = \{z_1, \cdots, z_n\}$ of $Z(R_u(P_\lambda))$ such that $C_{P_\lambda}(\langle K' \cup F\rangle) = C_{R_u(P_\lambda)}(K_0)$. Let $\mathbf{m} := (q_1, q_2, z_1, \cdots, z_n)$. Set $N := n+2$. Then, a similar computation to that of [23, Sec. 5] shows that the tuple $\mathbf{m} \in M^N$ has the desired properties. The existence of a 1-dimensional curve C such that $T_1(C)$ is contained in $\mathfrak{c}_{\mathrm{Lie}(R_u(P))}(K')$ but not contained in $\mathrm{Lie}(C_{R_u(P)}(K'))$ is crucial. □

Now we discuss another application of our construction with a different flavor. Here, we consider a question of Külshammer on representations of finite groups in reductive algebraic groups. Let Γ be a finite group. By a representation of Γ in a reductive algebraic group G, we mean a homomorphism from Γ to G. We write $\mathrm{Hom}(\Gamma, G)$ for the set of representations ρ of Γ in G. The group G acts on $\mathrm{Hom}(\Gamma, G)$ by conjugation. Let Γ_p be a Sylow p-subgroup of G. In [11, Sec. 2], Külshammer asked:

Question 1.12. Let G be a reductive algebraic group defined over an algebraically closed field of characteristic p. Let $\rho_p \in \mathrm{Hom}(\Gamma, G)$. Then are there only finitely many representations $\rho \in \mathrm{Hom}(\Gamma, G)$ such that $\rho\,|_{\Gamma_p}$ is G-conjugate to ρ_p?

In [1], Bate et al. presented an example where $p = 2, G = G_2$ and G has a finite subgroup Γ with Sylow 2-subgroup Γ_2 such that Γ has an infinite family of pairwise non-conjugate representations ρ whose restrictions to Γ_2 are all conjugate. In this paper, we present another example which answers Question 1.12 negatively:

Theorem 1.13. *Let G be a simple simply-connected algebraic group of type E_6 defined over an algebraically closed field k of characteristic $p = 2$. Then there exist a finite group Γ with a Sylow 2-subgroup Γ_2 and representations $\rho_a \in \mathrm{Hom}(\Gamma, G)$ for $a \in k$ such that ρ_a is not conjugate to ρ_b for $a \neq b$ but the restrictions $\rho_a\,|_{\Gamma_2}$ are pairwise conjugate for all $a \in k$.*

Note that the example of Theorem 1.13 is derived from Case 4 in the proof of Theorem 1.2. We also present an example giving a negative answer to Question 1.12 for a non-connected reductive G (this is much easier than the connected case):

Theorem 1.14. *Let k be an algebraically closed field of characteristic 2. Let $G := SL_3(k) \rtimes \langle\sigma\rangle$ where σ is the nontrivial graph automorphism of $SL_3(k)$. Let $d \geq 3$ be odd. Let D_{2d} be the dihedral group of order $2d$. Let*

$$\Gamma := D_{2d} \times C_2 = \langle r, s, z \mid r^d = s^2 = z^2 = 1, srs^{-1} = r^{-1}, [r, z] = [s, z] = 1\rangle.$$

Let $\Gamma_2 = \langle s, z\rangle$ (a Sylow 2-subgroup of Γ). Then there exist representations $\rho_a \in \mathrm{Hom}(\Gamma, G)$ for $a \in k$ such that ρ_a is not conjugate to ρ_b for $a \neq b$ but restrictions $\rho_a\,|_{\Gamma_2}$ are pairwise conjugate for all $a \in k$.

Here is the structure of this paper. In Section 2, we set out the notation and give a few preliminary results. Then in Section 3, 4, 5, we present a list of G-cr but non M-cr subgroups for $G = E_6, E_7, E_8$ respectively. This proves Theorem 1.2. Some details of our method will be explained in Section 3 using one of the examples for $G = E_6$. Finally in Section 6, we give proofs of Theorems 1.13 and 1.14.

2 Preliminaries

Throughout, we denote by k an algebraically closed field of positive characteristic p. Let G be an algebraic group defined over k. We write $R_u(G)$ for the *unipotent radical* of G, and G is called (possibly non-connected) *reductive* if $R_u(G) = \{1\}$. In particular, G is *simple* as an algebraic group if G is connected and all proper normal subgroups of G are finite. In this paper, when a subgroup H of G acts on G, we assume H acts on G by inner automorphisms. We write $C_G(H)$ and $\mathfrak{c}_{\mathfrak{g}}(H)$ for the global and the infinitesimal centralizers of H in G and $\mathfrak{g}$ respectively. We write $X(G)$ and $Y(G)$ for the set of characters and cocharacters of G respectively.

Let G be a connected reductive algebraic group. Fix a maximal torus T of G. Let $\Psi(G, T)$ denote the set of roots of G with respect to T. We sometimes write $\Psi(G)$ for $\Psi(G, T)$. Let

$\zeta \in \Psi(G)$. We write U_ζ for the corresponding root subgroup of G and $\mathfrak{u}_\zeta$ for the Lie algebra of U_ζ. We define $G_\zeta := \langle U_\zeta, U_{-\zeta}\rangle$. Let $\zeta, \xi \in \Psi(G)$. Let $\xi^\vee$ be the coroot corresponding to ξ. Then $\zeta \circ \xi^\vee : k^* \to k^*$ is a homomorphism such that $(\zeta \circ \xi^\vee)(a) = a^n$ for some $n \in \mathbb{Z}$. We define $\langle \zeta, \xi^\vee\rangle := n$. Let s_ξ denote the reflection corresponding to ξ in the Weyl group of G. Each s_ξ acts on the set of roots $\Psi(G)$ by the following formula [19, Lem. 7.1.8]: $s_\xi \cdot \zeta = \zeta - \langle \zeta, \xi^\vee\rangle \xi$. By [6, Prop. 6.4.2, Lem. 7.2.1] we can choose homomorphisms $\epsilon_\zeta : k \to U_\zeta$ so that $n_\xi \epsilon_\zeta(a) n_\xi^{-1} = \epsilon_{s_\xi \cdot \zeta}(\pm a)$ where $n_\xi = \epsilon_\xi(1)\epsilon_{-\xi}(-1)\epsilon_\xi(1)$. We define $e_\zeta := \epsilon'_\zeta(0)$.

We recall [16, Sec. 2.1–2.3] for the characterization of a parabolic subgroup P of G, a Levi subgroup L of P, and the unipotent radical $R_u(P)$ of P in terms of a cocharacter of G and state a result from GIT (Proposition 2.4).

Definition 2.1. Let X be an affine variety. Let $\phi : k^* \to X$ be a morphism of algebraic varieties. We say that $\lim_{a\to 0} \phi(a)$ exists if there exists a morphism $\hat{\phi} : k \to X$ (necessarily unique) whose restriction to k^* is ϕ. If this limit exists, we set $\lim_{a\to 0} \phi(a) = \hat{\phi}(0)$.

Definition 2.2. Let λ be a cocharacter of G. Define $P_\lambda := \{g \in G \mid \lim_{a\to 0} \lambda(a) g \lambda(a)^{-1} \text{ exists}\}$, $L_\lambda := \{g \in G \mid \lim_{a\to 0} \lambda(a) g \lambda(a)^{-1} = g\}$, $R_u(P_\lambda) := \{g \in G \mid \lim_{a\to 0} \lambda(a) g \lambda(a)^{-1} = 1\}$.

Note that P_λ is a parabolic subgroup of G, L_λ is a Levi subgroup of P_λ, and $R_u(P_\lambda)$ is the unipotent radical of P_λ [16, Sec. 2.1-2.3]. By [19, Prop. 8.4.5], any parabolic subgroup P of G, any Levi subgroup L of P, and any unipotent radical $R_u(P)$ of P can be expressed in this form. It is well known that $L_\lambda = C_G(\lambda(k^*))$.

Let M be a reductive subgroup of G. There is a natural inclusion $Y(M) \subseteq Y(G)$ of cocharacter groups. Let $\lambda \in Y(M)$. We write $P_\lambda(G)$ or just P_λ for the parabolic subgroup of G corresponding to λ, and $P_\lambda(M)$ for the parabolic subgroup of M corresponding to λ. It is obvious that $P_\lambda(M) = P_\lambda(G) \cap M$ and $R_u(P_\lambda(M)) = R_u(P_\lambda(G)) \cap M$.

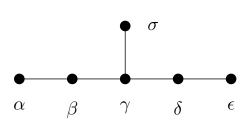
Definition 2.3. Let $\lambda \in Y(G)$. Define a map $c_\lambda : P_\lambda \to L_\lambda$ by $c_\lambda(g) := \lim_{a\to 0} \lambda(a) g \lambda(a)^{-1}$.

Note that the map c_λ is the usual canonical projection from P_λ to $L_\lambda \cong P_\lambda / R_u(P_\lambda)$. Now we state a result from GIT (see [2, Lem. 2.17, Thm. 3.1], [4, Thm. 3.3]).

Proposition 2.4. *Let H be a subgroup of G. Let λ be a cocharacter of G with $H \subseteq P_\lambda$. If H is G-cr, there exists $v \in R_u(P_\lambda)$ such that $c_\lambda(h) = vhv^{-1}$ for every $h \in H$.*

3 The E_6 examples

For the rest of the paper, we assume k is an algebraically closed field of characteristic 2. Let G be a simple algebraic group of type E_6 defined over k. Without loss, we assume that G is simply-connected. Fix a maximal torus T of G. Pick a Borel subgroup B of G containing T. Let $\Sigma = \{\alpha, \beta, \gamma, \delta, \epsilon, \sigma\}$ be the set of simple roots of G corresponding to B and T. The next figure defines how each simple root of G corresponds to each node in the Dynkin diagram of E_6. We label the positive roots of G as shown in Table 4 in the Appendix [7, Appendix, Table



B]. Define $L := \langle T, G_{22}, \cdots, G_{36}\rangle$, $P := \langle L, U_1, \cdots, U_{21}\rangle, W_L := \langle n_\alpha, n_\beta, n_\gamma, n_\delta, n_\epsilon\rangle$. Then P

is a parabolic subgroup of G, L is a Levi subgroup of P, and $\Psi(R_u(P)) = \{1, \cdots, 21\}$. Let $M = \langle L, G_{21}\rangle$. Then M is a subsystem subgroup of type A_5A_1, (G, M) is a reductive pair, and $\Psi(M) = \{\pm 21, \cdots, \pm 36\}$. Note that L is generated by T and all root subgroups with σ-weight 0, and M is generated by L and all root subgroups with σ-weight ± 2. Here, by the σ-weight of a root subgroup U_ζ, we mean the σ-coefficient of ζ.

Using Magma, we found that there are 56 subgroups of W_L up to conjugacy, and 11 of them act non-separably on $R_u(P)$. Table 1 lists these 11 subgroups K', and also gives the choice of t we use to give $K := \langle K' \cup \{t\}\rangle$. Note that $[L, L] = SL_6$ since G is simply-connected. We identify n_α, n_β, n_γ, n_δ, n_ϵ with (12), (23), (34), (45), (56) in S_6. To illustrate our method, we look at Case 4 closely.

case	generators of K'	$\lvert K'\rvert$	t	$v(a)$
1	(1 5)(2 3)(4 6)	2	$(\alpha^\vee + \epsilon^\vee)(b)$	$\epsilon_7(a)\epsilon_8(a)$
2	(1 5)(4 6), (1 4 5 6)(2 3)	4	$\alpha^\vee(b)$	$\epsilon_{10}(a)\epsilon_{13}(a)$
3	(2 4)(3 6), (1 5)(2 6)(3 4)	4	$(\alpha^\vee + \epsilon^\vee)(b)$	$\epsilon_7(a)\epsilon_8(a)$
4	(1 5)(2 3)(4 6), (1 4 2)(3 6 5)	6	$(\alpha^\vee + \epsilon^\vee)(b)$	$\epsilon_7(a)\epsilon_8(a)$
5	(1 5)(2 6)(3 4), (1 4 2)(3 6 5)	6	$(\alpha^\vee + \epsilon^\vee)(b)$	$\epsilon_7(a)\epsilon_8(a)$
6	(4 6), (1 4)(2 3)(5 6), (1 5)(4 6)	8	$\alpha^\vee(b)$	$\epsilon_{10}(a)\epsilon_{13}(a)$
7	(1 5)(2 6)(3 4), (2 4)(3 6), (1 2 4)(3 5 6)	12	$(\alpha^\vee + \epsilon^\vee)(b)$	$\epsilon_7(a)\epsilon_8(a)$
8	(1 4)(2 3)(5 6), (1 3 5)(2 4 6), (2 4 6)	18	$(\alpha^\vee + \beta^\vee)(b)$	$\epsilon_{11}(a)\epsilon_{12}(a)$
9	(1 4)(2 3)(5 6), (3 5)(4 6), (1 3 5), (2 4 6)	36	$(\alpha^\vee + \beta^\vee)(b)$	$\epsilon_{11}(a)\epsilon_{12}(a)$
10	(1 4 5 6)(2 3), (3 5)(4 6), (1 3 5), (2 4 6)	36	$(\alpha^\vee + \beta^\vee)(b)$	$\epsilon_{11}(a)\epsilon_{12}(a)$
11	(1 3), (1 4)(2 3)(5 6), (1 3)(4 6), (1 5 3), (2 6 4)	72	$(\alpha^\vee + \beta^\vee)(b)$	$\epsilon_{11}(a)\epsilon_{12}(a)$

Table 1: The E_6 examples

- Case 4:

Let $b \in k$ such that $b^3 = 1$ and $b \neq 1$. Define

$$\begin{aligned} &q_1 := n_\alpha n_\beta n_\gamma n_\beta n_\alpha n_\beta n_\gamma n_\beta n_\gamma n_\delta n_\epsilon n_\delta n_\gamma n_\epsilon,\ q_2 := n_\alpha n_\beta n_\gamma n_\delta n_\gamma n_\beta n_\alpha n_\beta n_\delta n_\epsilon n_\delta, \\ &t := (\alpha^\vee + \epsilon^\vee)(b),\ K' := \langle q_1, q_2\rangle,\ K := \langle q_1, q_2, t\rangle. \end{aligned}$$

It is easy to calculate how W_L acts on $\Psi(R_u(P))$. Let $\pi : W_L \to \text{ Sym } (\Psi(R_u(P))) \cong S_{21}$ be the corresponding homomorphism. Then we have

$$\begin{aligned} \pi(q_1) &= (1\ 5\ 4)(2\ 3\ 6)(9\ 12\ 10)(11\ 13\ 14)(15\ 16\ 17)(18\ 20\ 19), \\ \pi(q_2) &= (1\ 2)(3\ 4)(5\ 6)(7\ 8)(9\ 14)(10\ 11)(12\ 13)(15\ 18)(16\ 19)(17\ 20). \end{aligned}$$

The orbits of $\langle q_1, q_2\rangle$ are $O_1 = \{1, 2, 3, 4, 5, 6\}$, $O_7 = \{7, 8\}$, $O_9 = \{9, 10, 11, 12, 13, 14\}$, $O_{15} = \{15, 16, 17, 18, 19, 20\}$, $O_{21} = \{21\}$. Since t acts trivially on $e_7 + e_8$, [23, Lem. 2.8] yields

Proposition 3.1. $e_7 + e_8 \in \mathfrak{c}_{\mathrm{Lie}(R_u(P))}(K)$.

Proposition 3.2. *Let $u \in C_{R_u(P_{\alpha\beta\gamma\delta\epsilon\eta})}(K)$. Then u must have the form,*

$$u = \prod_{i=1}^{6} \epsilon_i(a) \prod_{i=7}^{8} \epsilon_i(b) \prod_{i=9}^{14} \epsilon_i(c) \left(\prod_{i=15}^{20} \epsilon_i(a+b+c)\right) \epsilon_{21}(a_{21}) \text{ for some } a, b, c, a_{21} \in k.$$

Proof. By [19, Prop. 8.2.1], u can be expressed uniquely as $u = \prod_{i=1}^{21} \epsilon_i(a_i)$ for some $a_i \in k$. Since $p = 2$ we have $n_\xi \epsilon_\zeta(a) n_\xi^{-1} = \epsilon_{s_\xi \cdot \zeta}(a)$ for any $a \in k$ and $\xi, \zeta \in \Psi(G)$. Then a calculation using the commutator relations ([10, Lem. 32.5, Lem. 33.3]) shows that

$$\begin{aligned} q_2uq_2^{-1} = {}&\epsilon_1(a_2)\epsilon_2(a_1)\epsilon_3(a_4)\epsilon_4(a_3)\epsilon_5(a_6)\epsilon_6(a_5)\epsilon_7(a_8)\epsilon_8(a_7)\epsilon_9(a_{14})\epsilon_{10}(a_{11})\epsilon_{11}(a_{10})\epsilon_{12}(a_{13})\epsilon_{13}(a_{12}) \\ &\epsilon_{14}(a_9)\epsilon_{15}(a_{18})\epsilon_{16}(a_{19})\epsilon_{17}(a_{20})\epsilon_{18}(a_{15})\epsilon_{19}(a_{16})\epsilon_{20}(a_{17})\epsilon_{21}(a_{18} + a_{21}). \qquad (3.1) \end{aligned}$$

Since q_1 and q_2 centralize u, we have $a_1 = \cdots = a_6,\ a_7 = a_8,\ a_9 = \cdots = a_{14},\ a_{15} = \cdots = a_{20}$. Set $a_1 = a,\ a_7 = b,\ a_9 = c,\ a_{15} = d$. Then (3.1) simplifies to

$$q_2 u q_2^{-1} = \prod_{i=1}^{6} \epsilon_i(a) \prod_{i=7}^{8} \epsilon_i(b) \prod_{i=9}^{14} \epsilon_i(c) \left(\prod_{i=15}^{20} \epsilon_i(d) \right) \epsilon_{21}(a^2 + b^2 + c^2 + d^2 + a_{21}).$$

Since q_2 centralizes u, comparing the arguments of the ϵ_{21} term on both sides, we must have

$$a_{21} = a^2 + b^2 + c^2 + d^2 + a_{21},$$

which is equivalent to $a + b + c + d = 0$. Then we obtain the desired result. $\square$

Proposition 3.3. *K acts non-separably on $R_u(P)$.*

Proof. Proposition 3.2 and a similar argument to that of the proof of [23, Prop. 3.3] show that $e_7 + e_8 \notin \operatorname{Lie} C_{R_u(P)}(K)$. Then Proposition 3.1 gives the desired result. $\square$

Remark 3.4. The following three facts are essential for the argument above:

1. The orbit O_7 contains a pair of roots corresponding to a non-commuting pair of root subgroups which get swapped by q_2; $q_2 \cdot (\epsilon_7(a)\epsilon_8(a)) = \epsilon_8(a)\epsilon_7(a) = \epsilon_7(a)\epsilon_8(a)\epsilon_{21}(a^2)$.
2. The correction term $\epsilon_{21}(a^2)$ in the last equation is contained in $Z(R_u(P))$.
3. The root 21 corresponding to the correction term is fixed by $\pi(q_2)$.

Now, let $C := \left\{ \prod_{i=7}^{8} \epsilon_i(a) \mid a \in k \right\}$, pick any $a \in k^*$, and let $v(a) := \prod_{i=7}^{8} \epsilon_i(a)$. Now set $H := v(a) K v(a)^{-1} = \langle q_1, q_2 \epsilon_{21}(a^2), t \rangle$. Note that $H \subset M, H \not\subset L$.

Proposition 3.5. *H is not M-cr.*

Proof. Let $\lambda = \alpha^\vee + 2\beta^\vee + 3\gamma^\vee + 2\delta^\vee + \epsilon^\vee + 2\sigma^\vee$. Then $L = L_\lambda$, $P = P_\lambda$. Let $c_\lambda : P_\lambda \to L_\lambda$ be the homomorphism from Definition 2.3. In order to prove that H is not M-cr, by Proposition 2.4 it suffices to find a tuple $(h_1, h_2) \in H^2$ that is not $R_u(P_\lambda(M))$-conjugate to $c_\lambda((h_1, h_2))$. Set $h_1 := v(a) q_1 v(a)^{-1},\ h_2 := v(a) q_2 v(a)^{-1}$. Then

$$c_\lambda((h_1, h_2)) = \lim_{x \to 0} \left(\lambda(x) q_1 \lambda(x)^{-1},\ \lambda(x) q_2 \epsilon_{21}(a^2) \lambda(x)^{-1} \right) = (q_1, q_2).$$

Now suppose that (h_1, h_2) is $R_u(P_\lambda(M))$-conjugate to $c_\lambda((h_1, h_2))$. Then there exists $m \in R_u(P_\lambda(M))$ such that $m v(a) q_1 v(a)^{-1} m^{-1} = q_1$, $m v(a) q_2 v(a)^{-1} m^{-1} = q_2$. Thus we have $m v(a) \in C_{R_u(P_\lambda)}(K)$. Note that $\Psi(R_u(P_\lambda(M))) = \{21\}$. Let $m = \epsilon_{21}(a_{21})$ for some $a_{21} \in k$. Then we have $m v(a) = \epsilon_7(a)\epsilon_8(a)\epsilon_{21}(a_{21}) \in C_{R_u(P_\lambda)}(K)$. This contradicts Proposition 3.2. $\square$

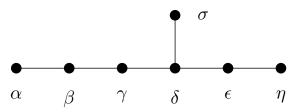
Proposition 3.6. *H is G-cr.*

Proof. Since H is G-conjugate to K, it is enough to show that K is G-cr. Since K is contained in L, by [17, Prop. 3.2] it suffices to show that K is L-cr. Then by [2, Lem. 2.12], it is enough to show that K is $[L, L]$-cr. Note that $[L, L] = SL_6$. An easy matrix computation shows that K acts semisimply on k^n, so K is G-cr by [17, Ex. 3.2.2(a)]. $\square$

It is clear that similar arguments work for the other cases. We omit proofs.

4 The E_7 examples

Let G be a simple simply-connected algebraic group of type E_7 defined over k. Fix a maximal torus T of G, and a Borel subgroup of G containing T. We define the set of simple roots $\Sigma = \{\alpha, \beta, \gamma, \delta, \epsilon, \eta, \sigma\}$ as in the following Dynkin diagram. The positive roots of G are listed in [7, Appendix, Table B].



Let L be the subgroup of G generated by T and all root subgroups of G with σ-weight 0. Let P be the subgroup of G generated by L and all root subgroups of G with σ-weight 1 or 2. Then P is a parabolic subgroup of G and L is a Levi subgroup of P. Let $W_L := \langle n_\alpha, n_\beta, n_\gamma, n_\delta, n_\epsilon, n_\eta \rangle$. Let M be the subgroup of G generated by L and all root subgroups of G with σ-weight ± 2. Then M is the subsystem subgroup of G of type A_7, and (G, M) is a reductive pair.

In the E_7 cases, we take $t = 1$ and $K' := K$; so each K is a subgroup of W_L. We use the same method as the E_6 examples, so we just give a sketch.

Using Magma, we found 95 non-trivial subgroups K of W_L up to conjugacy, and 19 of them are G-cr. Only two of them act non-separably on $R_u(P)$ (see Table 2). We determined G-complete reducibility and non-separability of K by a similar argument to that of the proof of Proposition 3.6. Note that $[L, L] = SL_7$. We identify $n_\alpha, \cdots, n_\eta$ with $(12), \cdots, (67)$ in S_7.

case	generators of K	$\lvert K \rvert$
1	(2 5)(3 7)(4 6), (1 4 3 2 5 7 6)	14
2	(2 6 7)(3 5 4), (2 5)(3 7)(4 6), (1 6 7 5 2 3 4)	42

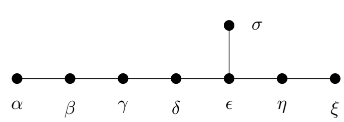
Table 2: The E_7 examples

- Case 1 was in [23, Sec. 3].
- Case 2:

Let $q_1 = n_\epsilon n_\gamma n_\alpha, q_2 = n_\alpha n_\gamma n_\alpha n_\beta n_\gamma n_\alpha n_\beta n_\gamma n_\eta n_\epsilon n_\delta n_\gamma n_\beta$, $K = \langle q_1, q_2 \rangle \cong \mathrm{Frob}_{42}$ (Frobenius group of order 42). We label some roots of G in Table 5 in Appendix. It can be calculated that K has an orbit $\{1, \cdots, 14\}$ which contains only one non-commuting pair of roots $\{2, 10\}$ contributing to a correction term that lies in U_{15}. Also, $\pi(q_1)$ swaps 2 with 10, and fixes 15. Thus K acts non-separably on $R_u(P)$ (see Remark 3.4). Now, set $v(a) = \prod_{i=1}^{14} \epsilon_i(a)$, and $H := v(a) \cdot K$. Then a similar argument to that of the proof of Proposition 3.5 show that H is not M-cr.

5 The E_8 examples

Let G be a simple simply-connected algebraic group of type E_8 defined over k. Fix a maximal torus T and a Borel subgroup B containing T. Define $\Sigma = \{\alpha, \beta, \gamma, \delta, \epsilon, \eta, \xi, \sigma\}$ by the next Dynkin diagram. All roots of G are listed in [7, Appendix, Table B]. Let L be



the subgroup of G generated by T and all root subgroups of G with σ-weight 0. Let P be the subgroup of G generated by L and all root subgroups of G with σ-weight 1, 2, or 3. Let $W_L := \langle n_\alpha, n_\beta, n_\gamma, n_\delta, n_\epsilon, n_\eta, n_\xi \rangle$. Then P is a parabolic subgroup of G, and L is a Levi subgroup of P. Let M be the subgroup of G generated by L and all root subgroups of G with σ-weight ± 2. Then M is a subsystem subgroup of type D_8, and (G, M) is a reductive pair. In the E_8 cases, we take $t = 1$ and $K' := K$; so each K is a subgroup of W_L. We use the same method as in the E_6, E_7 examples, so we just give a sketch.

With Magma, we found 295 non-trivial subgroups K of W up to conjugacy, and 31 of them are G-cr. Only two of them act non-separably on $R_u(P)$ (see Table 3). Note that $[L, L] \cong SL_8$. We identify $n_\alpha, \cdots, n_\xi$ with $(12), \cdots, (78)$ in S_8.

case	generators of K	$\lvert K\rvert$
1	(2 6)(4 5)(7 8), (1 4 2 8 7 6 5)	14
2	(1 7 5)(2 6 8), (1 2)(5 8)(6 7), (1 2 7 5 4 8 6)	42

Table 3: The E_8 examples

• Case 1:

Let $q_1 = n_\beta n_\gamma n_\delta n_\epsilon n_\delta n_\gamma n_\beta n_\delta n_\xi$, $q_2 = n_\alpha n_\beta n_\gamma n_\beta n_\alpha n_\beta n_\delta n_\epsilon n_\eta n_\xi n_\eta n_\epsilon n_\delta n_\gamma n_\beta n_\xi n_\eta n_\epsilon$, $K = \langle q_1, q_2 \rangle$.

We label some roots of G as in Table 6 in the Appendix. It can be calculated that K has an orbit $O_1 = \{1, \cdots, 7\}$ which contains only one non-commuting pair of roots $\{3, 4\}$, contributing a correction term that lies in U_8. Also $\pi(q_1)$ swaps 3 with 4, and fixes 8. So K acts nonseparably on $R_u(P)$ (see Remark 3.4). Now let $v(a) = \prod_{i=1}^{7} \epsilon_i(a)$, and define $H = v(a) \cdot K$. Then it is clear that H is not M-cr by the same argument as in the E_6 cases.

• Case 2:

Let $q_1 = n_\alpha n_\beta n_\gamma n_\delta n_\epsilon n_\eta n_\epsilon n_\delta n_\gamma n_\beta n_\alpha n_\epsilon n_\eta n_\epsilon n_\beta n_\gamma n_\delta n_\epsilon n_\delta n_\gamma n_\beta n_\eta n_\xi n_\eta$, $q_2 = n_\alpha n_\epsilon n_\eta n_\xi n_\eta n_\epsilon n_\eta, q_3 = n_\alpha n_\beta n_\gamma n_\delta n_\epsilon n_\eta n_\epsilon n_\delta n_\gamma n_\beta n_\epsilon n_\xi n_\eta n_\epsilon n_\delta n_\eta n_\xi n_\eta, K = \langle q_1, q_2, q_3 \rangle$.

We label some roots of G as in Table 7 in Appendix. It can be calculated that K has an orbit $O_1 = \{1, \cdots, 14\}$ which contains only one non-commuting pair of roots $\{4, 9\}$ contributing a correction term that lies in U_{15}. Also $\pi(q_1)$ swaps 4 with 9, and fixes 15. Let $v(a) = \prod_{i=1}^{14} \epsilon_i(a)$ and define $H := v(a) \cdot K$. It is clear that the same arguments work as in the last case.

6 On a question of Külshammer for representations of finite groups in reductive groups

6.1 The E_6 example

Proof of Theorem 1.13. Let G be a simple simply-connected algebraic group of type E_6 defined over k. We keep the notation from Sections 2 and 3. Pick $c \in k$ such that $c^3 = 1$ and $c \neq 1$. Let

$$\begin{aligned}
t_1 &:= \alpha^\vee(c),\ t_2 := \beta^\vee(c),\ t_3 := \gamma^\vee(c),\ t_4 := \delta^\vee(c),\ t_5 := \epsilon^\vee(c),\\
q_1 &:= n_\alpha n_\beta n_\gamma n_\beta n_\alpha n_\beta n_\gamma n_\beta n_\gamma n_\delta n_\epsilon n_\delta n_\gamma n_\epsilon,\\
q_2 &:= n_\alpha n_\beta n_\gamma n_\delta n_\gamma n_\beta n_\alpha n_\beta n_\delta n_\epsilon n_\delta,\\
H' &:= \langle t_1, t_2, t_3, t_4, t_5, q_1, q_2 \rangle.
\end{aligned}$$

Note that q_1 and q_2 here are the same as q_1 and q_2 in Case 4 of Section 3. Using Magma, we obtain the defining relations of H':

$$t_i^3 = 1,\ q_1^3 = 1,\ q_2^2 = 1,\ q_1 \cdot t_1 = (t_1t_2t_3)^{-1},\ q_1 \cdot t_2 = t_1t_2t_3t_4t_5,\ q_1 \cdot t_3 = (t_2t_3t_4t_5)^{-1},$$
$$q_1 \cdot t_4 = t_2,\ q_1 \cdot t_5 = t_3t_4,\ q_2 \cdot t_1 = (t_3t_4)^{-1},\ q_2 \cdot t_2 = t_2^{-1},\ q_2 \cdot t_3 = t_2t_3t_4t_5,$$
$$q_2 \cdot t_4 = (t_1t_2t_3t_4t_5)^{-1},\ q_2 \cdot t_5 = t_1t_2t_3,\ [t_i, t_j] = 1,\ (q_1^2q_2)^2 = 1.$$

Let

$$\begin{aligned}\Gamma := F \times C_2 =\langle & r_1, r_2, r_3, r_4, r_5, s_1, s_2, z \mid r_i^3 = s_1^3 = 1, s_2^2 = 1, s_1r_1s_1^{-1} = (r_1r_2r_3)^{-1},\\ & s_1r_2s_1^{-1} = r_1r_2r_3r_4r_5, s_1r_3s_1^{-1} = (r_2r_3r_4r_5)^{-1}, s_1r_4s_1^{-1} = r_2, s_1r_5s_1^{-1} = r_3r_4,\\ & s_2r_1s_2^{-1} = (r_3r_4)^{-1}, s_2r_2s_2^{-1} = r_2^{-1}, s_2r_3s_2^{-1} = r_2r_3r_4r_5, s_2r_4s_2^{-1} = (r_1r_2r_3r_4r_5)^{-1},\\ & s_2r_5s_2^{-1} = r_1r_2r_3, [r_i, r_j] = (s_1^2s_2)^2 = [r_i, z] = [s_i, z] = 1\rangle.\end{aligned}$$

Then $F \cong 3^{1+2} : 3^2 : S_3$ and $|F| = 1458 = 2 \times 3^6$. Let $\Gamma_2 := \langle s_2, z\rangle$ (a Sylow 2-subgroup of Γ). It is clear that $F \cong H'$.

For any $a \in k$ define $\rho_a \in \mathrm{Hom}(\Gamma, G)$ by

$$\rho_a(r_i) = t_i,\ \rho_a(s_1) = q_1,\ \rho_a(s_2) = q_2\epsilon_{21}(a),\ \rho_a(z) = \epsilon_{21}(1).$$

It is easily checked that this is well-defined.

Lemma 6.1. *$\rho_a|_{\Gamma_2}$ is G-conjugate to $\rho_b|_{\Gamma_2}$ for any $a, b \in k$.*

Proof. It is enough to prove that $\rho_0|_{\Gamma_2}$ is G-conjugate to $\rho_a|_{\Gamma_2}$ for any $a \in k$. Now let

$$u(\sqrt{a}) = \epsilon_7(\sqrt{a})\epsilon_8(\sqrt{a}).$$

Then an easy computation shows that

$$u(\sqrt{a}) \cdot q_2 = q_2\epsilon_{21}(a),\ u(\sqrt{a}) \cdot \epsilon_{21}(1) = \epsilon_{21}(1).$$

So we have

$$u(\sqrt{a}) \cdot (\rho_0|_{\Gamma_2}) = \rho_a|_{\Gamma_2}.$$

$\square$

Lemma 6.2. *ρ_a is not G-conjugate to ρ_b for $a \neq b$.*

Proof. Let $a, b \in k$. Suppose that there exists $g \in G$ such that $g \cdot \rho_a = \rho_b$. Since $\rho_a(r_i) = t_i$, we need $g \in C_G(t_1, t_2, t_3, t_4, t_5)$. A direct computation shows that $C_G(t_1, t_2, t_3, t_4, t_5) = TG_{21}$. So let $g = tm$ for some $t \in T$ and $m \in G_{21}$. Note that q_2 centralizes G_{21}. So,

$$\begin{aligned}(tq_2t^{-1})(tm\epsilon_{21}(a)m^{-1}t^{-1}) &= (tm)q_2\epsilon_{21}(a)(m^{-1}t^{-1})\\ &= g \cdot \rho_a(s_2)\\ &= \rho_b(s_2)\\ &= q_2\epsilon_{21}(b). \qquad (6.1)\end{aligned}$$

Note that $tq_2t^{-1} \in G_{\alpha\beta\gamma\delta\epsilon}$ and $tm\epsilon_{21}(a)m^{-1}t^{-1} \in G_{21}$. Since $[G_{\alpha\beta\gamma\delta\epsilon}, G_{21}] = 1$, it is clear that $G_{\alpha\beta\gamma\delta\epsilon} \cap G_{21} = 1$. Now (6.1) yields that $tq_2t^{-1} = q_2$. We also have

$$q_1 = \rho_b(s_1) = g \cdot \rho_a(s_1) = tm \cdot q_1 = tq_1t^{-1}.$$

So t commutes with q_1 and q_2. Then a quick calculation shows that $t \in G_{21}$. So $g \in G_{21}$. But G_{21} is a simple group of type A_1, so the pair $(q_2\epsilon_{21}(a), \epsilon_{21}(1))$ is not G_{21}-conjugate to $(q_2\epsilon_{21}(b), \epsilon_{21}(1))$ if $a \neq b$. Therefore ρ_a is not G-conjugate to ρ_b if $a \neq b$. $\square$

Now Theorem 1.13 follows from Lemmas 6.1 and 6.2. ☐

Remark 6.3. One can obtain examples with the same properties as in Theorem 1.13 for $G = E_7, E_8$ using the E_7 and E_8 examples in Sections 4 and 5.

6.2 The non-connected A_2 example

Proof of Theorem 1.14. We have $G^\circ = SL_3(k)$. Fix a maximal torus T of G°, and a Borel subgroup of G° containing T. Let $\{\alpha, \beta\}$ be the set of simple roots of G°. Let $c \in k$ such that $|c| = d$ is odd and $c \neq 1$. Define $t := (\alpha - \beta)^\vee(c)$. For each $a \in k$, define $\rho_a \in \mathrm{Hom}(\Gamma, G)$ by

$$\rho_a(r) = t,\ \rho_a(s) = \sigma\epsilon_{\alpha+\beta}(a),\ \rho_a(z) = \epsilon_{\alpha+\beta}(1).$$

An easy computation shows that this is well-defined.

Lemma 6.4. *$\rho_a \mid_{\Gamma_2}$ is G-conjugate to $\rho_b \mid_{\Gamma_2}$ for any $a, b \in k$.*

Proof. Let $u(\sqrt{a}) := \epsilon_\alpha(\sqrt{a})\epsilon_\beta(\sqrt{a})$. Then

$$u(\sqrt{a}) \cdot \sigma = \sigma\epsilon_{\alpha+\beta}(a),\ u(\sqrt{a}) \cdot \epsilon_{\alpha+\beta}(1) = \epsilon_{\alpha+\beta}(1).$$

This shows that $u(\sqrt{a}) \cdot (\rho_0 \mid_{\Gamma_2}) = \rho_a \mid_{\Gamma_2}$. ☐

Lemma 6.5. *ρ_a is not G-conjugate to ρ_b if $a \neq b$.*

Proof. Let $a, b \in k$. Suppose that there exists $g \in G$ such that $g \cdot \rho_a = \rho_b$. Since $\rho_a(r) = t$, we have $g \in C_G(t) = TG_{\alpha+\beta}$. So let $g = hm$ for some $h \in T$ and $m \in G_{\alpha+\beta}$. We compute

$$\begin{aligned}(h\sigma h^{-1})(hm\epsilon_{\alpha+\beta}(a)m^{-1}h^{-1}) &= (hm)\sigma\epsilon_{\alpha+\beta}(a)(m^{-1}h^{-1}) \\ &= g \cdot \rho_a(s) \\ &= \rho_b(s) \\ &= \sigma\epsilon_{\alpha+\beta}(b). \end{aligned} \tag{6.2}$$

Now (6.2) shows that h commutes with σ. Then h is of the form $h := (\alpha + \beta)^\vee(x)$ for some $x \in k^*$. So $h \in G_{\alpha+\beta}$. Thus $g \in G_{\alpha+\beta}$. But $G_{\alpha+\beta}$ is a simple group of type A_1, so the pair $(\sigma\epsilon_{\alpha+\beta}(a), \epsilon_{\alpha+\beta}(1))$ is not $G_{\alpha+\beta}$-conjugate to $(\sigma\epsilon_{\alpha+\beta}(b), \epsilon_{\alpha+\beta}(1))$ unless $a = b$. So ρ_a is not G-conjugate to ρ_b unless $a = b$. ☐

Theorem 1.14 follows from Lemmas 6.4 and 6.5. ☐

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Appendix

1 $\begin{smallmatrix} &&1&& \\ 0&0&0&0&0 \end{smallmatrix}$	2 $\begin{smallmatrix} &&1&& \\ 1&1&1&1&0 \end{smallmatrix}$	3 $\begin{smallmatrix} &&1&& \\ 0&1&1&1&1 \end{smallmatrix}$	4 $\begin{smallmatrix} &&1&& \\ 1&1&2&1&0 \end{smallmatrix}$	5 $\begin{smallmatrix} &&1&& \\ 0&1&2&1&1 \end{smallmatrix}$
6 $\begin{smallmatrix} &&1&& \\ 1&2&3&2&1 \end{smallmatrix}$	7 $\begin{smallmatrix} &&1&& \\ 0&0&1&0&0 \end{smallmatrix}$	8 $\begin{smallmatrix} &&1&& \\ 1&2&2&2&1 \end{smallmatrix}$	9 $\begin{smallmatrix} &&1&& \\ 0&1&1&0&0 \end{smallmatrix}$	10 $\begin{smallmatrix} &&1&& \\ 0&0&1&1&0 \end{smallmatrix}$
11 $\begin{smallmatrix} &&1&& \\ 0&1&1&1&0 \end{smallmatrix}$	12 $\begin{smallmatrix} &&1&& \\ 1&1&2&1&1 \end{smallmatrix}$	13 $\begin{smallmatrix} &&1&& \\ 1&2&2&1&1 \end{smallmatrix}$	14 $\begin{smallmatrix} &&1&& \\ 1&1&2&2&1 \end{smallmatrix}$	15 $\begin{smallmatrix} &&1&& \\ 1&1&1&0&0 \end{smallmatrix}$
16 $\begin{smallmatrix} &&1&& \\ 0&0&1&1&1 \end{smallmatrix}$	17 $\begin{smallmatrix} &&1&& \\ 0&1&2&1&0 \end{smallmatrix}$	18 $\begin{smallmatrix} &&1&& \\ 1&1&1&1&1 \end{smallmatrix}$	19 $\begin{smallmatrix} &&1&& \\ 1&2&2&1&0 \end{smallmatrix}$	20 $\begin{smallmatrix} &&1&& \\ 0&1&2&2&1 \end{smallmatrix}$
21 $\begin{smallmatrix} &&2&& \\ 1&2&3&2&1 \end{smallmatrix}$	22 $\begin{smallmatrix} &&0&& \\ 1&0&0&0&0 \end{smallmatrix}$	23 $\begin{smallmatrix} &&0&& \\ 0&1&0&0&0 \end{smallmatrix}$	24 $\begin{smallmatrix} &&0&& \\ 0&0&1&0&0 \end{smallmatrix}$	25 $\begin{smallmatrix} &&0&& \\ 0&0&0&1&0 \end{smallmatrix}$
26 $\begin{smallmatrix} &&0&& \\ 0&0&0&0&1 \end{smallmatrix}$	27 $\begin{smallmatrix} &&0&& \\ 1&1&0&0&0 \end{smallmatrix}$	28 $\begin{smallmatrix} &&0&& \\ 0&1&1&0&0 \end{smallmatrix}$	29 $\begin{smallmatrix} &&0&& \\ 0&0&1&1&0 \end{smallmatrix}$	30 $\begin{smallmatrix} &&0&& \\ 0&0&0&1&1 \end{smallmatrix}$
31 $\begin{smallmatrix} &&0&& \\ 1&1&1&0&0 \end{smallmatrix}$	32 $\begin{smallmatrix} &&0&& \\ 0&1&1&1&0 \end{smallmatrix}$	33 $\begin{smallmatrix} &&0&& \\ 0&0&1&1&1 \end{smallmatrix}$	34 $\begin{smallmatrix} &&0&& \\ 1&1&1&1&0 \end{smallmatrix}$	35 $\begin{smallmatrix} &&0&& \\ 0&1&1&1&1 \end{smallmatrix}$
36 $\begin{smallmatrix} &&0&& \\ 1&1&1&1&1 \end{smallmatrix}$				

Table 4: The set of positive roots of E_6

1 $\begin{smallmatrix} &&&1&& \\ 1&1&1&1&0&0 \end{smallmatrix}$	2 $\begin{smallmatrix} &&&1&& \\ 0&1&1&2&1&1 \end{smallmatrix}$	3 $\begin{smallmatrix} &&&1&& \\ 1&2&2&2&1&1 \end{smallmatrix}$	4 $\begin{smallmatrix} &&&1&& \\ 0&1&1&1&1&0 \end{smallmatrix}$
5 $\begin{smallmatrix} &&&1&& \\ 1&2&2&2&2&1 \end{smallmatrix}$	6 $\begin{smallmatrix} &&&1&& \\ 1&1&2&3&2&1 \end{smallmatrix}$	7 $\begin{smallmatrix} &&&1&& \\ 0&0&1&1&0&0 \end{smallmatrix}$	8 $\begin{smallmatrix} &&&1&& \\ 0&0&0&1&1&0 \end{smallmatrix}$
9 $\begin{smallmatrix} &&&1&& \\ 0&0&0&1&1&1 \end{smallmatrix}$	10 $\begin{smallmatrix} &&&1&& \\ 1&1&2&2&2&1 \end{smallmatrix}$	11 $\begin{smallmatrix} &&&1&& \\ 0&1&2&3&2&1 \end{smallmatrix}$	12 $\begin{smallmatrix} &&&1&& \\ 0&0&1&1&1&1 \end{smallmatrix}$
13 $\begin{smallmatrix} &&&1&& \\ 1&1&1&2&1&0 \end{smallmatrix}$	14 $\begin{smallmatrix} &&&1&& \\ 0&1&2&2&1&0 \end{smallmatrix}$	15 $\begin{smallmatrix} &&&2&& \\ 1&2&3&4&3&2 \end{smallmatrix}$	

Table 5: Case 2 (E_7)

1 $\begin{smallmatrix} &&&&1&& \\ 0&0&0&0&0&0&0 \end{smallmatrix}$	2 $\begin{smallmatrix} &&&&1&& \\ 0&1&1&1&1&0&0 \end{smallmatrix}$	3 $\begin{smallmatrix} &&&&1&& \\ 0&0&0&0&1&1&1 \end{smallmatrix}$	4 $\begin{smallmatrix} &&&&1&& \\ 0&1&1&2&2&1&0 \end{smallmatrix}$
5 $\begin{smallmatrix} &&&&1&& \\ 1&1&1&1&2&2&1 \end{smallmatrix}$	6 $\begin{smallmatrix} &&&&1&& \\ 1&1&1&2&3&2&1 \end{smallmatrix}$	7 $\begin{smallmatrix} &&&&1&& \\ 1&2&2&3&3&2&1 \end{smallmatrix}$	8 $\begin{smallmatrix} &&&&2&& \\ 0&1&1&2&3&2&1 \end{smallmatrix}$

Table 6: Case 1 (E_8)

1 $\begin{smallmatrix} &&&&1&& \\ 0&0&0&0&1&0&0 \end{smallmatrix}$	2 $\begin{smallmatrix} &&&&1&& \\ 0&0&0&1&1&0&0 \end{smallmatrix}$	3 $\begin{smallmatrix} &&&&1&& \\ 0&0&0&0&1&1&0 \end{smallmatrix}$	4 $\begin{smallmatrix} &&&&1&& \\ 0&1&1&1&1&1&0 \end{smallmatrix}$
5 $\begin{smallmatrix} &&&&1&& \\ 0&1&1&1&1&1&1 \end{smallmatrix}$	6 $\begin{smallmatrix} &&&&1&& \\ 0&0&0&1&2&2&1 \end{smallmatrix}$	7 $\begin{smallmatrix} &&&&1&& \\ 1&1&1&1&1&1&1 \end{smallmatrix}$	8 $\begin{smallmatrix} &&&&1&& \\ 1&1&1&2&2&1&0 \end{smallmatrix}$
9 $\begin{smallmatrix} &&&&1&& \\ 1&1&1&1&2&1&1 \end{smallmatrix}$	10 $\begin{smallmatrix} &&&&1&& \\ 0&1&1&2&2&1&1 \end{smallmatrix}$	11 $\begin{smallmatrix} &&&&1&& \\ 1&2&2&2&2&1&0 \end{smallmatrix}$	12 $\begin{smallmatrix} &&&&1&& \\ 1&1&1&2&2&2&1 \end{smallmatrix}$
13 $\begin{smallmatrix} &&&&1&& \\ 0&1&1&2&3&2&1 \end{smallmatrix}$	14 $\begin{smallmatrix} &&&&1&& \\ 1&2&2&2&3&2&1 \end{smallmatrix}$	15 $\begin{smallmatrix} &&&&2&& \\ 1&2&2&2&3&2&1 \end{smallmatrix}$	

Table 7: Case 2 (E_8)

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Complete reducibility of subgroups of reductive algebraic groups over nonperfect fields I

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Abstract

Let k be a nonperfect field of characteristic 2. Let G be a k-split simple algebraic group of type E_6 (or G_2) defined over k. In this paper, we present the first examples of nonabelian non-G-completely reducible k-subgroups of G which are G-completely reducible over k. Our construction is based on that of subgroups of G acting non-separably on the unipotent radical of a proper parabolic subgroup of G in our previous work. We also present examples with the same property for a non-connected reductive group G. Along the way, several general results concerning complete reducibility over nonperfect fields are proved using the recently proved Tits center conjecture for spherical buildings. In particular, we show that under mild conditions a connected k-subgroup of G is pseudo-reductive if it is G-completely reducible over k.

Keywords: algebraic groups, complete reducibility, separability, spherical buildings

1 Introduction

Let k be an arbitrary field. We write $\bar{k}$ for an algebraic closure of k. Let G/k be a connected reductive algebraic group defined over k: we regard G as a $\bar{k}$-defined algebraic group together with a choice of k-structure [9, AG.11]. Following Serre [23], define:

Definition 1.1. A closed subgroup H of G is *G-completely reducible over k* (*G-cr over k* for short) if whenever H is contained in a k-parabolic subgroup P of G, H is contained in some k-Levi subgroup L of P. In particular, if H is not contained in any proper k-parabolic subgroup of G, H is *G-irreducible over k* (*G-ir over k* for short). Note that we do not require H to be k-defined.

Our definition is a slight generalization of Serre's original definition in [23], where H is assumed to be k-defined. This generalized definition was used in [1] and [2].

The notion of G-complete reducibility over k is a natural generalization of that of complete reducibility in representation theory: if $G = GL(V)$ for some finite dimensional k-vector space V, a subgroup H of G acts on V semisimply over k if and only if H is G-complete reducible over k [23, Sec. 1.3]. We say that a subgroup H of G is G-cr (G-ir) if H is G-cr over $\bar{k}$ (G-ir over $\bar{k}$) regarding G to be defined over $\bar{k}$. By a subgroup H of G, we always mean a closed subgroup of G. Any algebraic group in this paper is smooth and affine unless otherwise stated.

Complete reducible subgroups are much studied, but most studies so far considered complete reducibility over $\bar{k}$ only; see [4], [20], [25]. Not much is known about completely reducible subgroups over arbitrary k except for a few results and important examples in [1], [2], [4, Sec. 6.5], [6], [7, Sec. 7], [32, Thm. 1.8], [34, Sec. 4]. We write k_s for a separable closure of k. The main result of this paper is the following:

Theorem 1.2. *Let $k = k_s$ be a nonperfect field of characteristic 2. Let G/k be a simple algebraic group of type E_6 (or G_2). Then there exists a nonabelian k-subgroup H of G such that H is G-cr over k, but not G-cr.*

Several examples of an abelian subgroup $H < G$ such that H is G-cr over k but not G-cr are known; see Example 3.10, [15], [28], [29]. Note that in these examples, $H < G$ is generated by a *k-anisotropic unipotent element* [28].

Definition 1.3. Let G/k be a reductive algebraic group. A unipotent element u of G is *k-nonplongeable unipotent* if u is not contained in the k-unipotent radical of any proper k-parabolic subgroup of G. In particular, if u is not contained in any proper k-parabolic subgroup of G, u is *k-anisotropic unipotent*.

By the k-unipotent radical of an affine k-group N, we mean the maximal connected unipotent normal k-subgroup of N. It is clear that a subgroup H of G generated by a k-anisotropic unipotent element is G-ir over k. Since H is unipotent, the classical result of Borel-Tits [11, Prop. 3.1] shows that H is not G-cr; see Example 3.10.

The next result [6, Thm. 1.1] shows that the nonperfectness assumption of k in Theorem 1.2 is necessary. Recall that if k is perfect, we have $k_s = \bar{k}$.

Proposition 1.4. *Let k be an arbitrary field. Let G/k be a connected reductive algebraic group. Then a subgroup H of G is G-cr over k if and only if H is G-cr over k_s.*

The forward direction of Proposition 1.4 holds for a non-connected reductive group G in an appropriate sense (see Definition 2.3). The reverse direction depends on the *Tits center conjecture* (Theorem 3.1), but this method does not work for non-connected G; see [31].

In Section 3, we present an example of a subgroup H for $G = E_6$ (or G_2) satisfying the properties of Theorem 1.2. The key to our construction is the notion of a *non-separable action* [34, Def. 1.5].

Definition 1.5. Let H and N be affine algebraic groups. Suppose that H acts on N by group automorphisms. The action of H is called *separable* in N if $\mathrm{Lie}\, C_N(H) = \mathfrak{c}_{\mathrm{Lie} N}(H)$ where $C_N(H)$ is the centralizer of H in N in the sense of [9, Sec. 1.7]. Note that the condition means that the scheme-theoretic centralizer of H in N (in the sense of [13, Def. A.1.9]) is smooth.

Note that the notion of a separable action is a slight generalization of that of a separable subgroup [7, Def. 1.1]. See [7] and [16] for more on separability. It is known that if the characteristic p of k is very good for G, every subgroup of G is separable [7, Thm. 1.2]. This suggests that we need to work in small p. Proper non-separable subgroups are hard to find. Only a handful of such examples are known [7, Sec. 7], [32], [34].

Remark 1.6. The examples of subgroups H of G in Section 3 are G-ir over k but not G-cr. So, we can regard these examples as a generalization of k-anisotropic unipotent elements.

Next, we consider a non-connected case. Again, non-separability is the key to our construction, but the computations are much simpler than in the connected cases. See Definition 2.3 for the definition of G-complete reducibility for non-connected G.

Theorem 1.7. *Let $k = k_s$ be a nonperfect field of characteristic 2. Let $\tilde{G}/k$ be a simple algebraic group of type A_4. Let $G := \tilde{G} \rtimes \langle \sigma \rangle$ where σ is the non-trivial graph automorphism of $\tilde{G}$. Then there exists a nonabelian k-subgroup H of G such that H is G-cr over k, but not G-cr.*

In Section 2 we extend several existing results concerning complete reducibility over $\bar{k}$ to a nonperfect k. Most arguments are based on [4] and the Tits center conjecture (Theorem 3.1) in spherical buildings. We also consider the relationship between complete reducibility over k and *pseudo-reductivity* [13]. Recall:

Definition 1.8. Let k be a field. Let G/k be a connected affine algebraic group. If the k-unipotent radical $R_{u,k}(G)$ of G is trivial, G is called *pseudo-reductive.*

Note that if k is perfect, pseudo-reductive groups are reductive. Our main result on pseudo-reductivity is the following:

Theorem 1.9. *Let $k = k_s$ be a field. Let G/k be a semisimple simply connected algebraic group. Assume that $[k : k^p] \leq p$. If a connected k-subgroup H of G is G-cr over k, then H is pseudo-reductive.*

Let G/k be connected reductive. A standard argument [23, Prop. 4.1] (which depends on [11, Prop. 3.1]) shows that a G-cr subgroup of G is reductive, hence pseudo-reductive. However when k is nonperfect we have:

Proposition 1.10. *Let k be a nonperfect field of characteristic 2. Let $G = PGL_2$. Then there exists a connected k-subgroup H of G such that H is G-cr over k, but not pseudo-reductive.*

We extend [4, Lem. 2.12] using the notion of a *central isogeny.* Recall [9, Sec. 22.3]:

Definition 1.11. Let k be a field. Let G_1/k and G_2/k be connected reductive. A k-isogeny $f : G_1 \rightarrow G_2$ is *central* if ker df_1 is central in $\mathfrak{g}_1$ where df_1 is the differential of f at the identity of G_1.

Proposition 1.12. *Let k be a field. Let G_1/k and G_2/k be connected reductive. Let H_1 and H_2 be (not necessarily k-defined) subgroups of G_1 and G_2 respectively. Let $f : G_1 \rightarrow G_2$ be a central k-isogeny.*

1. *If H_1 is G_1-cr over k, then $f(H_1)$ is G_2-cr over k.*

2. *If H_2 is G_2-cr over k, then $f^{-1}(H_2)$ is G_1-cr over k.*

Here is the structure of the paper. In Section 2, we set out the notation. Then, in Section 3, we prove various general results including Theorem 1.9, Proposition 1.10, Proposition 1.12. In Section 4, we prove Theorem 1.2. Then, in Section 5, we consider non-connected G, and prove Theorem 1.7. Finally, in Section 6, we consider further applications of non-separable actions for non-connected G, and prove Theorem 6.2 and Theorem 6.4.

2 Preliminaries

Throughout, we denote by k a separably closed field unless otherwise stated. Although some results hold for an arbitrary field, our assumption on k makes the exposition cleaner. Our references for algebraic groups are [9], [10], [18], and [24].

Let G/k be a (possibly non-connected) affine algebraic group defined over k. By a k-group G, we mean a $\bar{k}$-defined affine algebraic group with a k-structure [9, AG.11]. We write $G(k)$ for the set of k-points of G. The *unipotent radical* of G is denoted by $R_u(G)$, and G is called *reductive* if $R_u(G) = \{1\}$. A reductive group G is called *simple* as an algebraic group if G is connected and all proper normal subgroups of G are finite. We write $X_k(G)$ and $Y_k(G)$ for the set of k-characters and k-cocharacters of G respectively.

Let G/k be reductive. Fix a k-split maximal torus T of G (such a T exists by [9, Cor. 18.8]). Let $\Psi(G,T)$ denote the set of roots of G with respect to T. We sometimes write $\Psi(G)$ for $\Psi(G,T)$. Let $\zeta \in \Psi(G)$. We write U_ζ for the corresponding root subgroup of G. We define $G_\zeta := \langle U_\zeta, U_{-\zeta}\rangle$. Let $\zeta, \xi \in \Psi(G)$. Let $\xi^\vee$ be the coroot corresponding to ξ. Then $\zeta \circ \xi^\vee : \bar{k}^* \to \bar{k}^*$ is a k-homomorphism such that $(\zeta \circ \xi^\vee)(a) = a^n$ for some $n \in \mathbb{Z}$. Let s_ξ denote the reflection corresponding to ξ in the Weyl group of G. Each s_ξ acts on the set of roots $\Psi(G)$ by the following formula [24, Lem. 7.1.8]: $s_\xi \cdot \zeta = \zeta - \langle \zeta, \xi^\vee\rangle \xi$. By [12, Prop. 6.4.2, Lem. 7.2.1] we can choose k-homomorphisms $\epsilon_\zeta : \bar{k} \to U_\zeta$ so that $n_\xi \epsilon_\zeta(a) n_\xi^{-1} = \epsilon_{s_\xi \cdot \zeta}(\pm a)$ where $n_\xi = \epsilon_\xi(1)\epsilon_{-\xi}(-1)\epsilon_\xi(1)$.

We recall the notions of R-parabolic subgroups and R-Levi subgroups from [8, Sec. 2.2]. These notions are essential to define G-complete reducibility for subgroups of non-connected reductive groups; see [3] and [4, Sec. 6].

Definition 2.1. Let X/k be a k-affine variety. Let $\phi : \bar{k}^* \to X$ be a k-morphism of k-affine varieties. We say that $\lim\limits_{a\to 0} \phi(a)$ exists if there exists a k-morphism $\hat{\phi} : \bar{k} \to X$ (necessarily unique) whose restriction to $\bar{k}^*$ is ϕ. If this limit exists, we set $\lim\limits_{a\to 0} \phi(a) = \hat{\phi}(0)$.

Definition 2.2. Let $\lambda \in Y_k(G)$. Define $P_\lambda := \{g \in G \mid \lim\limits_{a\to 0} \lambda(a) g \lambda(a)^{-1} \text{ exists}\}$, $L_\lambda := \{g \in G \mid \lim\limits_{a\to 0} \lambda(a) g \lambda(a)^{-1} = g\}$, $R_u(P_\lambda) := \{g \in G \mid \lim\limits_{a\to 0} \lambda(a) g \lambda(a)^{-1} = 1\}$.

We call P_λ an R-parabolic subgroup of G, L_λ an R-Levi subgroup of P_λ. Note that $R_u(P_\lambda)$ is the unipotent radical of P_λ [8, Sec. 2.2]. If λ is k-defined, P_λ, L_λ, and $R_u(P_\lambda)$ are k-defined [8, Sec. 2.2]. If G is connected, R-parabolic subgroups and R-Levi subgroups are parabolic subgroups and Levi subgroups in the usual sense [24, Prop. 8.4.5]. It is well known that $L_\lambda = C_G(\lambda(\bar{k}^*))$.

Let M/k be a reductive subgroup of G. Then, there is a natural inclusion $Y_k(M) \subseteq Y_k(G)$ of k-cocharacter groups. Let $\lambda \in Y_k(M)$. We write $P_\lambda(G)$ or just P_λ for the k-parabolic subgroup of G corresponding to λ, and $P_\lambda(M)$ for the k-parabolic subgroup of M corresponding to λ. It is clear that $P_\lambda(M) = P_\lambda(G) \cap M$ and $R_u(P_\lambda(M)) = R_u(P_\lambda(G)) \cap M$. Now we define:

Definition 2.3. Let G/k be a (possibly non-connected) reductive algebraic group. A subgroup H of G is G-cr over k if whenever H is contained in a k-defined R-parabolic subgroup P_λ, H is contained in a k-defined R-Levi subgroup of P_λ.

3 General results

3.1 The Tits center conjecture

Let G/k be connected reductive. We write $\Delta(G)$ for the Tits spherical building of G [27]. Recall that each simplex in $\Delta(G)$ corresponds to a proper k-parabolic subgroup of G, and the conjugation action of $G(k)$ on itself induces building automorphisms of $\Delta(G)$. The following is the so-called *Tits center conjecture* ([23, Sec. 2.4] and [26, Lem. 1.2]), which was recently proved by Tits, Mühlherr, Leeb, and Ramos-Cuevas [19], [21], [22]:

Theorem 3.1. *Let X be a convex contractible subcomplex of $\Delta(G)$. Then there exists a simplex in X that is stabilized by all automorphisms of $\Delta(G)$ stabilizing X.*

In [23, Def. 2.2.1] Serre defined that a convex subcomplex X of $\Delta(G)$ is $\Delta(G)$-*completely reducible over k* ($\Delta(G)$-cr over k for short) if for every simplex $x \in X$, there exists a simplex $x' \in X$ opposite to x in X. Serre showed [23, Thm. 2]:

Proposition 3.2. *Let X be a convex subcomplex of $\Delta(G)$. Then X is $\Delta(G)$-cr over k if and only if X is not contractible.*

Combining Theorem 3.1 with Proposition 3.2, and translating the result into the language of algebraic groups we obtain

Proposition 3.3. *Let H be a (not necessarily k-defined) subgroup of G/k. If H is not G-cr over k, then there exists a proper k-parabolic subgroup P of G such that P contains H and $N_G(H)(k)$ where $N_G(H)(k) := G(k) \cap N_G(H)$.*

Proof. Let $\Delta(G)^H$ be the set of all k-parabolic subgroups of G containing H. Then $\Delta(G)^H$ is a convex subcomplex of $\Delta(G)$ by [23, Prop. 3.1]. Since H is not G-cr over k, there exists a proper k-parabolic subgroup of G containing H such that H is not contained in any opposite of P. So, $\Delta(G)^H$ is contractible by Proposition 3.2. It is clear that $N_G(H)(k)$ induces automorphisms of $\Delta(G)$ stabilizing $\Delta(G)^H$. By Theorem 3.1, there exists a simplex s_P in $\Delta(G)^H$ stabilized by automorphisms induced by $N_G(H)(k)$. Since parabolic subgroups are self-normalizing, we have $N_G(H)(k) < P$. □

Note that under the assumption of Proposition 3.3, $N_G(H)$ is not necessarily k-defined even when H is k-defined. So, we might not have a proper k-parabolic subgroup containing H and $N_G(H)$.

Many problems concerning complete reducibility over nonperfect fields are still open. For example:

Open Problem 3.4. *Let k be a field. Let G/k be connected reductive. Suppose that a k-subgroup H of G is G-cr over k. Is the centralizer $C_G(H)$ of H in G G-cr over k?*

See [31] for more on this problem and other related open problems. It is known that if $k = \bar{k}$, the answer to Open Problem 3.4 is yes; see [4, Cor. 3.17].

Proposition 3.5. *Let G/k be connected reductive. Let H be a k-subgroup of G. Suppose that H is G-ir over k. Then $C_G(H)$ is G-cr over k.*

Proof of Proposition 3.5. Suppose that $C_G(H)$ is not G-cr over k. Since H normalizes $C_G(H)$, by Proposition 3.3, there exists a proper k-parabolic subgroup of P of G containing $H(k)$. Since $k = k_s$, $H(k)$ is dense in H by [9, AG.13.3]. So $H \leq P$. This is a contradiction since H is G-ir over k. □

3.2 Complete reducibility and pseudo-reductivity

The main task in this section is to prove Theorem 1.9. Before that, we need some preparations:

Lemma 3.6. *Let G/k be connected reductive, and let H be a (not necessarily k-defined) subgroup of G. Let L be a k-Levi subgroup of G containing H. Then H is G-cr over k if and only if H is L-cr over k.*

Proof. This is [1, Thm. 1.4]. □

The next result is a slight generalization of [23, Prop. 2.9], where Serre assumed the subgroup N is k-defined. Note that Serre's argument assumed that Theorem 3.1 holds, but this was not known at the time. We have translated Serre's building-theoretic argument into a group-theoretic one.

Proposition 3.7. *Let G/k be connected reductive. Let H/k be a subgroup of G such that H is G-cr over k. If N is a (not necessarily k-defined) normal subgroup of H, then N is G-cr over k.*

Proof. Let P be a minimal k-parabolic subgroup of G containing H. Since H is G-cr over k, there exists a k-Levi subgroup L of P containing H. If N is L-cr over k, by Lemma 3.6, we are done. So suppose that N is not L-cr over k. Let $\Delta(L)$ be the spherical building corresponding to L. Let $\Delta(L)^N$ be the set of all k-parabolic subgroups of L containing N. Since $N \trianglelefteq H \leq L$ and N is not L-cr over k, by Proposition 3.3, there exists a proper k-parabolic subgroup P_L of L containing N and $H(k)$. Since $k = k_s$, $H(k)$ is dense in H by [9, AG.13.3]. So $H \leq P_L < L$. Then $H \leq P_L \ltimes R_u(P) < L \ltimes R_u(P) = P$. Since $P_L \ltimes R_u(P)$ is a k-parabolic subgroup of G by [10, Sec. 4.4(c)], this is a contradiction by the minimality of P. □

We also need the following deep result which was conjectured by Tits [29] and proved by Gille [14].

Proposition 3.8. *Let G/k be a semisimple simply connected algebraic group. If $[k : k^p] \leq p$, then every unipotent subgroup of $G(k)$ is k-plongeable.*

Now we are ready:

Proof of Theorem 1.9. Since $R_{u,k}(H)(k)$ is a unipotent subgroup of $G(k)$, by Proposition 3.8, there exists a k-parabolic subgroup P of G such that $R_{u,k}(H)(k) \leq R_u(P)$. Then $R_{u,k}(H) \leq R_u(P)$ since k-points are dense in $R_{u,k}(H)$ (because we assumed $k = k_s$). Since $R_{u,k}(H)$ is a normal subgroup of H, and H is G-cr over k, $R_{u,k}(H)$ is G-cr over k by Proposition 3.7. So $R_{u,k}(H)$ is contained in some k-Levi subgroup of P. Thus $R_{u,k}(H) = 1$. We are done. □

Note that in Proposition 3.8, the condition $[k : k^p] \leq p$ was necessary since Tits showed the following [30, Thm. 7].

Proposition 3.9. *Let G/k be a simple simply connected algebraic group. If $[k : k^p] \geq p^2$ and p is bad for G, then $G(k)$ has a k-nonplongeable unipotent element.*

We quickly review an example of abelian $H < G$ such that H is G-cr over k but not G-cr. Although this example is known, it has not been interpreted in the context of G-complete reducibility.

Example 3.10. Let k be a nonperfect field of characteristic $p = 2$. Let $a \in k \backslash k^2$. Let $G/k = PGL_2$. We write $\bar{A}$ for the image in PGL_2 of $A \in GL_2$. Set $u = \overline{\begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix}} \in G(k)$. Let $U := \langle u \rangle$. Then U is unipotent, so by the classical result of Borel-Tits [11, Prop. 3.1] U is contained in the unipotent radical of a proper parabolic subgroup of G. So U is not G-cr. However U is not contained in any proper k-parabolic subgroup of G since there is no nontrivial k-defined flag of $\mathbb{P}^1_k$ stabilized by U. So U is G-ir over k, hence G-cr over k. Note that this example shows that [11, Prop. 3.1] fails over a nonperfect k.

Proof of Proposition 1.10. Let k be a nonperfect field of characteristic 2. Let $a \in k \backslash k^2$. Let $G = PGL_2$ and $H := \left\{ \overline{\begin{bmatrix} x & ay \\ y & x \end{bmatrix}} \in PGL_2(\bar{k}) \mid x, y \in \bar{k} \right\}$. Then H is a connected k-defined unipotent subgroup of G. Therefore H is not pseudo-reductive. It is clear that H contains a k-anisotropic unipotent element $\overline{\begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix}}$ of G. So H is G-ir over k. □

Remark 3.11. Let k, a, G, H be as in the proof of Proposition 1.10. Note that the subgroup H is the centralizer of the subgroup $U := \left\langle \overline{\begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix}} \right\rangle$ of G. So without the perfectness assumption of k we have a counterexample to [4, Prop. 3.12] which states that the centralizer of a G-cr over k subgroup is reductive. Reducitivity of the centralizer was a key ingredient in the proof of [4, Cor. 3.17]. Although our example does not give a negative answer to Open Problem 3.4, it suggests that the answer is no.

3.3 Complete reducibility under isogenies

Proof of Proposition 1.12. Suppose that $f(H_1)$ is contained in a k-parabolic subgroup P_μ of G_2 where $\mu \in Y_k(G_2)$. Then $H_1 < f^{-1}(P_\mu) = P_\lambda$ for some $\lambda \in Y_k(G_1)$ since $f^{-1}(P_\mu)$ is a k-defined parabolic subgroup of G_1 by [9, Thm. 22.6]. So $P_\mu = f(P_\lambda) = P_{f\circ\lambda}$ by [4, Lem. 2.11]. Since H_1 is G_1-cr over k, there exists a k-Levi subgroup L of P_λ containing H_1. We can set $L := u\cdot L_\lambda$ for some $u \in R_u(P_\lambda)(k)$ since k-Levi subgroups of P_λ are $R_u(P_\lambda)(k)$-conjugate by [9, Prop. 20.5]. Then $f(H_1) < f(u)\cdot f(L_\lambda) = f(u)\cdot L_{f\circ\lambda}$ by [4, Lem. 2.11]. Since $f\circ\lambda$ is a k-cocharacter of G_2 and $f(u)$ is a k-point of $f(R_u(P_\lambda)) = R_u(P_{f\circ\lambda})$ ([4, Lem. 2.11]), $f(u)\cdot L_{f\circ\lambda}$ is a k-Levi subgroup of $P_{f\circ\lambda} = P_\mu$ containing $f(H_1)$. So we have the first part of the proposition.

Now, suppose that there exists a k-parabolic subgroup $P_{\lambda'}$ of G_1 containing $f^{-1}(H_2)$ where $\lambda' \in Y_k(G_1)$. Then there exists some $\mu' \in Y_k(G_2)$ such that $P_{\lambda'} = f^{-1}(P_{\mu'})$ since every k-parabolic subgroup of G_1 is the inverse image of a k-parabolic subgroup of G_2 by [9, Thm. 22.6]. So $H_2 < P_{\mu'}$. Since H_2 is G_2-cr over k, there exists a k-Levi subgroup L' of $P_{\mu'}$ containing H_2. By the same argument as in the last paragraph, set $L' := u'\cdot L_{\mu'} = L_{u'\cdot\mu'}$ for some $u' \in R_u(P_{\mu'})(k)$. Then $f^{-1}(H_2) < f^{-1}(L_{u'\cdot\mu'}) < f^{-1}(P_{u'\cdot\mu'}) = f^{-1}(P_{\mu'}) = P_{\lambda'}$. Note that $f^{-1}(L_{u'\cdot\mu'})$ is a Levi subgroup of $f^{-1}(P_{u'\cdot\mu'}) = P_{\lambda'}$ by [4, Lem. 2.11], and it is k-defined by [9, Cor. 22.5] since $L_{u'\cdot\mu'}$ is a k-defined subgroup of G_2 containing a maximal torus of G_2. We are done. $\square$

Note that if $k = \bar{k}$, Proposition 1.12 holds without assuming f central, but if k is nonperfect, the next example shows that the first part of Proposition 1.12 does not necessarily hold:

Example 3.12. Let k be a nonperfect field of characteristic 2. Let $a \in k\backslash k^2$. Let $G_1 = G_2 = PGL_2$, and f be the Frobenius map. Let $h_1 = \overline{\begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix}}$. Then it is clear that $H_1 := \langle h_1 \rangle$ is G_1-ir over k, but $H_2 := \langle f(h_1) \rangle = \left\langle \overline{\begin{bmatrix} 0 & a^2 \\ 1 & 0 \end{bmatrix}} \right\rangle$ is not G_2-cr over k; H_2 acts on $\mathbb{P}^1_k$ with a k-defined H_2-invariant subspace spanned by $[a, 1]$ which has no k-defined H_2-invariant complementary subspace.

Remark 3.13. Let $f : SL_2 \to PGL_2$ be the canonical projection. Take the same H_1 as in Example 3.12. Then $f^{-1}(H_1) = \left\langle \begin{bmatrix} 0 & \sqrt{a} \\ \sqrt{a}^{-1} & 0 \end{bmatrix} \right\rangle$ is not k-defined, but $f^{-1}(H_1)$ is G-ir over k.

Open Problem 3.14. *Does the second part of Proposition 1.12 hold without assuming f central?*

4 Proof of Theorem 1.2

For the rest of the paper, we assume $k = k_s$ is a nonperfect field of characteristic 2 and $a \in k \backslash k^2$.

4.1 The G_2 example

Let G/k be a simple algebraic group of type G_2. Fix a k-split maximal torus T of G and a k-Borel subgroup of G containing T. Let $\Sigma = \{\alpha, \beta\}$ be the set of simple roots corresponding to B and T where α is short and β is long. Then the set of roots of G is $\Psi = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta), \pm(2\alpha+\beta), \pm(3\alpha+\beta), \pm(3\alpha+2\beta)\}$. Let $b \in k^*$ such that $b^3 = 1$ and $b \neq 1$. Let $n_\alpha := \epsilon_{-\alpha}(1)\epsilon_\alpha(1)\epsilon_{-\alpha}(1)$, and $t := \alpha^\vee(b)$. Let $L_\alpha := \langle T, G_\alpha \rangle$ and $P_\alpha := \langle L_\alpha, U_\beta, U_{\alpha+\beta}, U_{2\alpha+\beta}, U_{3\alpha+\beta}, U_{3\alpha+2\beta} \rangle = P_{(3\alpha+2\beta)^\vee}$.

In the following computation, we use the commutation relations for root subgroups of G; see [18, Sec. 33.5]. Define

$$K := \langle n_\alpha \rangle,\ v(\sqrt{a}) := \epsilon_{-\beta}(\sqrt{a})\epsilon_{-3\alpha-\beta}(\sqrt{a}),\ M := \langle n_\alpha, t \rangle.$$

Let

$$H := \langle v(\sqrt{a}) \cdot M,\ \epsilon_{2\alpha+\beta}(1) \rangle = \langle n_\alpha \epsilon_{-3\alpha-2\beta}(a), t, \epsilon_{2\alpha+\beta}(1) \rangle.$$

Proposition 4.1. *H is G-ir over k.*

Proof. Let

$$\widetilde{H} := v(\sqrt{a})^{-1} \cdot H = \langle n_\alpha, t, \epsilon_{2\alpha+\beta}(1)\epsilon_{3\alpha+\beta}(a)\epsilon_{3\alpha+2\beta}(\sqrt{a})\epsilon_{\alpha+\beta}(\sqrt{a})\epsilon_{-\alpha}(\sqrt{a}) \rangle.$$

It is clear that L_α is a Levi subgroup of G containing M. Since M is not contained in any Borel subgroup of L_α, M is L-ir. So M is G-cr by Lemma 3.6.

We see that P_α is a proper parabolic subgroup of G containing $\widetilde{H}$. Let P be a proper parabolic subgroup of G containing $\widetilde{H}$. We show that $P = P_\alpha$. Let $\lambda \in Y_{\bar{k}}(G)$ such that $P_\lambda = P$. Then P contains M. Since M is G-cr, M is contained in some Levi subgroup L of P. Since any Levi subgroup L of P can be expressed as $L = C_G(u \cdot \lambda(\bar{k}^*))$ for some $u \in R_u(P_\lambda)$, we may assume that $\lambda(\bar{k}^*)$ centralizes M. From [7, Lem. 7.10], we know that $C_G(M) = G_{3\alpha+2\beta}$. So we can write λ as $\lambda = g \cdot (3\alpha+2\beta)^\vee$ for some $g \in G_{3\alpha+2\beta}$. By the Bruhat decomposition, g is in one of the following forms:

$$\begin{aligned}
&(1)\ g = (3\alpha+2\beta)^\vee(s)\epsilon_{3\alpha+2\beta}(x_1),\\
&(2)\ g = \epsilon_{3\alpha+2\beta}(x_1)n_{3\alpha+2\beta}(3\alpha+2\beta)^\vee(s)\epsilon_{3\alpha+2\beta}(x_2)\\
&\ \text{for some } s \in \bar{k}^*, x_1, x_2 \in \bar{k}.
\end{aligned}$$

We rule out the second case. Suppose that g is in form (2). Since $\widetilde{H} < P_\lambda = P_{g\cdot(3\alpha+2\beta)^\vee} = g \cdot P_{(3\alpha+2\beta)^\vee} = g \cdot P_\alpha$, it is enough to show that $g^{-1} \cdot \widetilde{H} \not\subset P_\alpha$. Let

$$h := \epsilon_{2\alpha+\beta}(1)\epsilon_{3\alpha+\beta}(a)\epsilon_{3\alpha+2\beta}(\sqrt{a})\epsilon_{\alpha+\beta}(\sqrt{a})\epsilon_{-\alpha}(\sqrt{a}) \in \widetilde{H}.$$

We show that $g^{-1} \cdot h \notin P_\alpha$. Since h centralizes $U_{3\alpha+2\beta}$ and $(3\alpha+2\beta)^\vee(s)\epsilon_{3\alpha+2\beta}(x_2)$ belongs to P_α for any $s \in \bar{k}^*$, $x_2 \in \bar{k}$, without loss, we assume $g = n_{3\alpha+2\beta}$. We compute

$$\begin{aligned}
n_{3\alpha+2\beta}^{-1} \cdot h =& (n_\beta n_\alpha n_\beta n_\alpha n_\beta) \cdot h\\
=& \epsilon_{-\alpha-\beta}(1)\epsilon_{-\beta}(a)\epsilon_{-3\alpha-2\beta}(\sqrt{a})\epsilon_{-2\alpha-\beta}(\sqrt{a})\epsilon_{-\alpha}(\sqrt{a}) \notin P_\alpha.
\end{aligned}$$

So g must be in form (1) above. Then $g \in P_\alpha$ and $P_\lambda = P_\alpha$. Thus we have shown that P_α is the unique proper parabolic subgroup of G containing $\widetilde{H}$. Since $H < v(\sqrt{a}) \cdot P_\alpha$, we have

Lemma 4.2. *$v(\sqrt{a}) \cdot P_\alpha$ is the unique proper parabolic subgroup containing H.*

Lemma 4.3. *$v(\sqrt{a}) \cdot P_\alpha$ is not k-defined.*

Proof. Suppose that $v(\sqrt{a}) \cdot P_\alpha$ is k-defined. Since P_α is k-defined, $v(\sqrt{a}) \cdot P_\alpha$ is $G(k)$-conjugate to P_α by [9, Thm. 20.9]. So we can write $gv(\sqrt{a}) \cdot P_\alpha = P_\alpha$ for some $g \in G(k)$. Then $gv(\sqrt{a}) \in P_\alpha$ since parabolic subgroups are self-normalizing. Thus $g = pv(\sqrt{a})^{-1}$ for some $p \in P_\alpha$. So g is a k-point of $P_\alpha R_u(P_\alpha^-)$. By the rational version of the Bruhat decomposition [9, Thm. 21.15], there exist a unique $p' \in P_\alpha$ and a unique $u' \in R_u(P_\alpha^-)$ such that $g = p'u'$; moreover p' and u' are k-points. This is a contradiction since $v(\sqrt{a})^{-1} \notin R_u(P_\alpha^-)(k)$. ☐

Lemmas 4.2 and 4.3 yield Proposition 4.1. ☐

Proposition 4.4. *H is not G-cr.*

Proof. Recall that $C_G(M) = G_{3\alpha+2\beta}$. Then $C_G(\widetilde{H}) < G_{3\alpha+2\beta}$ since $M < \widetilde{H}$. Using the commutation relations, we see that $U_{3\alpha+2\beta} < C_G(\widetilde{H})$. Since $\langle 3\alpha + 2\beta, (3\alpha + 2\beta)^\vee \rangle = 2$, $(3\alpha + 2\beta)^\vee(s)$ does not commute with $h \in \widetilde{H}$ for any $s \in \bar{k}^* \backslash \{1\}$. Then $C_G(\widetilde{H}) = U_{3\alpha+2\beta}$ since $G_{3\alpha+2\beta} = SL_2$. Thus $C_G(H) = v(\sqrt{a}) \cdot U_{3\alpha+2\beta}$ which is unipotent. So by [11, Prop. 3.1], $C_G(H)$ is not G-cr. Then [4, Cor. 3.17] shows that H is not G-cr. ☐

By Propositions 4.1 and 4.4 we are done.

Remark 4.5. In the proof of Proposition 4.1, K acts non-separably on $R_u(P_\alpha^-)$. This non-separable action was essential to make $v(\sqrt{a}) \cdot K$ k-defined; see [7, Sec. 7] for details.

Remark 4.6. Note that

$$\begin{aligned}
C_G(H) &= \{v(\sqrt{a}) \cdot \epsilon_{3\alpha+2\beta}(x) \mid x \in \bar{k}\} \\
&= \{\epsilon_{3\alpha+2\beta}(x)\epsilon_{3\alpha+\beta}(\sqrt{a}x)\epsilon_{-\beta}(\sqrt{a})\epsilon_\beta(\sqrt{a}x)\epsilon_{-\beta}(\sqrt{a}) \mid x \in \bar{k}\} \\
&= \{\epsilon_{3\alpha+2\beta}(a^{-1}) \cdot (\epsilon_{-\beta}(\sqrt{a})\epsilon_\beta(\sqrt{a}x)\epsilon_{-\beta}(\sqrt{a}) \mid x \in \bar{k}\}
\end{aligned}$$

We can identify $\epsilon_{-\beta}(\sqrt{a})\epsilon_\beta(\sqrt{a}x)\epsilon_{-\beta}(\sqrt{a})$ with the product of 2×2 matrices in $L_\beta = SL_2$:

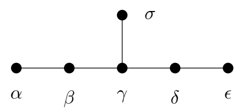
$$\begin{aligned}
\epsilon_{-\beta}(\sqrt{a})\epsilon_\beta(\sqrt{a}x)\epsilon_{-\beta}(\sqrt{a}) &= \begin{bmatrix} 1 & 0 \\ \sqrt{a} & 1 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{a}x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \sqrt{a} & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 + ax & \sqrt{a}x \\ a\sqrt{a}x & 1 + ax \end{bmatrix}.
\end{aligned}$$

Then $C := \{\epsilon_{-\beta}(\sqrt{a})\epsilon_\beta(\sqrt{a}x)\epsilon_{-\beta}(\sqrt{a}) \mid x \in \bar{k}\}$ is L_β-ir over k since C contains a k-anisotropic unipotent element $\begin{bmatrix} 1 + a & \sqrt{a} \\ a\sqrt{a} & 1 + a \end{bmatrix}$. Thus C is G-cr over k by Lemma 3.6. Since $C_G(H)$ is $G(k)$-conjugate to C, it is G-cr over k. Note that this agrees with Proposition 3.5.

4.2 The E_6 example

Let G/k be a simple algebraic group of type E_6. By Proposition 1.12, we may assume G is simply-connected. Fix a maximal k-split torus T of G and a k-Borel subgroup B of G containing T. Let $\Sigma = \{\alpha, \beta, \gamma, \delta, \epsilon, \sigma\}$ be the set of simple roots of G corresponding to B and T. The next figure defines how each simple root of G corresponds to each node in the Dynkin diagram of E_6.

We label all positive roots of G in Table 1 in Appendix. The labeling for the negative roots follows in the obvious way. Let $L := L_{\alpha\beta\gamma\delta\epsilon} = \langle T, U_i \mid i \in \{\pm 22, \cdots, \pm 36\} \rangle$. $P := P_{\alpha\beta\gamma\delta\epsilon} =$



$\langle L, U_i \mid i \in \{1, \cdots, 21\}\rangle$. Then P is a parabolic subgroup of G and L is a Levi subgroup of P. Since our argument is similar to that of the G_2 example, we just give a sketch. We use the commutation relations [18, Lem. 32.5 and Prop. 33.3] repeatedly. Let

$$q_1 := n_\alpha n_\beta n_\alpha,\ q_2 := n_\alpha n_\beta n_\gamma n_\beta n_\alpha n_\beta n_\epsilon,\ q_3 := n_\alpha n_\beta n_\alpha n_\delta n_\epsilon n_\delta,$$
$$q_4 := n_\alpha n_\beta n_\gamma n_\delta n_\gamma n_\beta n_\alpha n_\gamma n_\delta n_\gamma,\ q_5 := n_\beta n_\gamma n_\delta n_\epsilon n_\delta n_\gamma n_\beta n_\delta n_\epsilon n_\delta,$$
$$K := \langle q_1, q_2, q_3, q_4, q_5\rangle < L,\ |K| = 72.$$

We took $q_1, \cdots, q_5$ from Table 1 (case 11) in [32, Sec. 3]. From the Cartan matrix of E_6 [17, Sec. 11.4], we see how $n_\alpha, \cdots, n_\epsilon$ act on $\Psi(R_u(P))$. Let $\pi : \langle n_\alpha, \cdots, n_\epsilon\rangle \to \mathrm{Sym}(\Psi(R_u(P))) \cong S_{21}$ be the corresponding homomorphism. Then

$$\begin{aligned}
\pi(q_1) &= (2\ 5)(4\ 8)(7\ 11)(10\ 15)(13\ 17)(16\ 19),\\
\pi(q_2) &= (1\ 5)(2\ 3)(4\ 17)(6\ 14)(7\ 15)(8\ 11)(9\ 12)(10\ 13)(16\ 20)(18\ 19),\\
\pi(q_3) &= (2\ 11)(3\ 9)(4\ 8)(5\ 7)(10\ 19)(12\ 18)(13\ 17)(15\ 16),\\
\pi(q_4) &= (1\ 4\ 8)(2\ 12\ 5)(3\ 10\ 15)(7\ 18\ 11)(9\ 16\ 19)(13\ 20\ 17),\\
\pi(q_5) &= (1\ 9\ 3)(2\ 7\ 13)(4\ 16\ 10)(5\ 11\ 17)(8\ 19\ 15)(12\ 18\ 20).
\end{aligned} \tag{4.1}$$

The orbits of K in $\Psi(R_u(P))$ are

$$O_1 = \{21\},\ O_2 = \{6, 14\},\ O_3 = \Psi(R_u(P))\backslash\{6, 14, 21\}.$$

Let

$$M' := \langle U_i \mid i \in \{\pm 27, \pm 28, \pm 29, \pm 30\}\rangle < L,$$
$$M := \langle K, M'\rangle,\ v(\sqrt{a}) := \epsilon_{-6}(\sqrt{a})\epsilon_{-14}(\sqrt{a}).$$

Note that $v(\sqrt{a})$ centralizes M'. Define

$$H := \langle v(\sqrt{a}) \cdot M, \epsilon_2(1)\rangle = \langle q_1, q_2\epsilon_{-21}(a), q_3, q_4, q_5, M', \epsilon_2(1)\rangle.$$

Proposition 4.7. *H is G-ir over k.*

Proof. Let

$$\widetilde{H} := v(\sqrt{a})^{-1} \cdot H = \langle M, v(\sqrt{a})^{-1} \cdot \epsilon_2(1)\rangle = \langle M, \epsilon_2(1)\epsilon_{-36}(\sqrt{a})\rangle.$$

Since $U_{-36} < L$, we see that P contains $\widetilde{H}$. Thus $v(\sqrt{a}) \cdot P$ contains H.

Lemma 4.8. *$v(\sqrt{a}) \cdot P$ is the unique proper parabolic subgroup of G containing H.*

Proof. It is clear that M is contained in L. Note that $[L, L] = SL_6$. We identify $n_\alpha, n_\beta, n_\gamma, n_\delta, n_\epsilon$ with $(1\ 2)$, $(2\ 3)$, $(3\ 4)$, $(4\ 5)$, $(5\ 6)$ in S_6. Then $q_1 = (1\ 3)$, $q_2 = (1\ 4)(2\ 3)(5\ 6)$, $q_3 = (1\ 3)(4\ 6)$, $q_4 = (1\ 5\ 3)$ and $q_5 = (2\ 6\ 4)$. Let $T_1 := (\alpha+\beta)^\vee(\bar{k}^*), T_2 := (\beta+\gamma)^\vee(\bar{k}^*), T_3 := (\gamma+\delta)(\bar{k}^*)$, and $T_4 := (\delta+\epsilon)(\bar{k}^*)$. Then T_i is a maximal torus of G_i for $i = 27, 28, 29$ and 30 respectively. So $\langle T_1, T_2, T_3, T_4\rangle < M$. Now a simple matrix calculation shows that M is $[L, L]$-ir, hence L-cr by [5, Prop. 2.8]. Thus M is G-cr by Lemma 3.6.

Let P_λ be a proper parabolic subgroup of G containing $\widetilde{H}$. Then P_λ contains M. Since M is G-cr, without loss we may assume that $\lambda(\bar{k}^*)$ centralizes M. Recall that by [24, Thm. 13.4.2], $C_{R_u(P)}(M)^\circ \times C_L(M)^\circ \times C_{R_u(P^-)}(M)^\circ$ is an open set of $C_G(M)^\circ$ where P^- is the opposite of P containing L.

Lemma 4.9. $C_G(M)^\circ = G_{21}$.

Proof. First of all, from equations (4.1), we see that K centralizes G_{21}. Using the commutation relations [18, Lem. 32.5 and Prop. 33.3], M' centralizes G_{21}. So M centralizes G_{21}. By [24, Prop. 8.2.1], we write an arbitrary element u of $R_u(P)$ as $u = \prod_{i=1}^{21} \epsilon_i(x_i)$ for some $x_i \in \bar{k}$. It is not hard to show that if $u \in C_{R_u(P)}(T_1, T_2, T_3, T_4)$, u must be of the form

$$u = \epsilon_6(x_6)\epsilon_{14}(x_{14})\epsilon_{21}(x_{21}) \text{ for some } x_i \in \bar{k}.$$

Then

$$\begin{aligned} q_2 \cdot u =& \epsilon_{14}(x_6)\epsilon_6(x_{14})\epsilon_{21}(x_{21}) \\ =& \epsilon_6(x_{14})\epsilon_{14}(x_6)\epsilon_{21}(x_6 x_{14} + x_{21}). \end{aligned}$$

So, for $u \in C_{R_u(P)}(M)$, $x_6 = x_{14} = 0$. Thus $C_{R_u(P)}(M) = U_{21}$. Likewise $C_{R_u(P^-)}(M) = U_{-21}$. Note that $C_L(M) < C_L(T_1, T_2, T_3, T_4)$. We find by direct computations that $C_L(T_1, T_2, T_3, T_4) = T$ and $C_T(K) = (\alpha + 2\beta + 3\gamma + 2\delta + \epsilon + 2\sigma)^\vee(\bar{k}^*) < G_{21}$. So we are done. ☐

Now we have $\lambda(\bar{k}^*) < G_{21}$. Without loss, set $\lambda = g \cdot (\alpha + 2\beta + 3\gamma + 2\delta + \epsilon + 2\sigma)^\vee$ for some $g \in G_{21}$. By the Bruhat decomposition, g is in one of the following forms:

$$\begin{aligned} &(1)\ g = (\alpha + 2\beta + 3\gamma + 2\delta + \epsilon + 2\sigma)^\vee(s)\epsilon_{21}(x_1), \\ &(2)\ g = \epsilon_{21}(x_1)n_{21}(\alpha + 2\beta + 3\gamma + 2\delta + \epsilon + 2\sigma)^\vee(s)\epsilon_{21}(x_2) \\ &\ \text{for some } x_1, x_2 \in \bar{k}, s \in \bar{k}^*. \end{aligned}$$

By the similar argument to that of the G_2 case, if we rule out the second case we are done. Suppose that g is in form (2). Let $h := \epsilon_2(1)\epsilon_{-36}(\sqrt{a}) \in \widetilde{H}$. It is enough to show that $g^{-1} \cdot h \not\subset P_{(\alpha+2\beta+3\gamma+2\delta+\epsilon+2\sigma)^\vee}$. Since h centralizes U_{21} and $\epsilon_{21}(x_2)(\alpha+2\beta+3\gamma+2\delta+\epsilon+2\sigma)^\vee(s)$ belongs to $P_{(\alpha+2\beta+3\gamma+2\delta+\epsilon+2\sigma)^\vee}$ for any $x_2 \in \bar{k}, s \in \bar{k}^*$, we may assume $g = n_{21}$. We have

$$\begin{aligned} n_{21} =& n_\epsilon n_\sigma n_\delta n_\epsilon n_\gamma n_\sigma n_\delta n_\epsilon n_\gamma n_\delta n_\beta n_\gamma n_\sigma n_\delta n_\epsilon n_\gamma n_\delta n_\beta n_\gamma n_\sigma n_\alpha n_\beta n_\gamma n_\sigma n_\delta n_\epsilon n_\gamma n_\delta n_\beta n_\gamma \\ & n_\sigma n_\alpha n_\beta n_\gamma n_\delta n_\epsilon \text{ (the longest element in the Weyl group of } E_6). \end{aligned}$$

A quick calculation shows $n_{21} \cdot U_2 = U_{-2}$ and $n_{21} \cdot U_{-36} = U_{36}$. Then

$$n_{21}^{-1} \cdot (\epsilon_2(1)\epsilon_{-36}(\sqrt{a})) = \epsilon_{-2}(1)\epsilon_{36}(\sqrt{a}) \notin P_{(\alpha+2\beta+3\gamma+2\delta+\epsilon+2\sigma)^\vee}.$$

So we are done. ☐

Lemma 4.10. *$v(\sqrt{a}) \cdot P$ is not k-defined.*

Proof. This is similar to Lemma 4.3. ☐

☐

Proposition 4.11. *H is not G-cr.*

Proof. This is similar to Proposition 4.4. Since $M < v(\sqrt{a})^{-1} \cdot H$, Lemma 4.9 yields $C_G(H)^\circ < v(\sqrt{a}) \cdot G_{21}$. Using the commutation relations, $v(\sqrt{a}) \cdot U_{21} < C_G(H)$. Note that $\langle -2, (\alpha + 2\beta + 3\gamma + 2\delta + \epsilon + 2\sigma)^\vee \rangle = -1$. So, $(\alpha + 2\beta + 3\gamma + 2\delta + \epsilon + 2\sigma)^\vee(s)$ does not commute with h for any $s \in \bar{k}^* \backslash \{1\}$. A similar argument to that of the G_2 case shows that $C_G(H)^\circ = v(\sqrt{a}) \cdot U_{21}$ which is unipotent. So by [11, Prop. 3.1], $C_G(H)^\circ$ is not G-cr. Then $C_G(H)$ is not G-cr by Proposition 3.7 since $C_G(H)^\circ$ is a normal subgroup of $C_G(H)$. Now [4, Cor. 3.17] shows that H is not G-cr. □

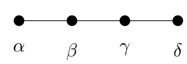
By Propositions 4.7 and 4.11, we are done.

Remark 4.12. Note that $C_G(H)^\circ = \{\epsilon_{21}(a^{-1}) \cdot (\epsilon_{-6}(\sqrt{a})\epsilon_6(\sqrt{a}x)\epsilon_{-6}(\sqrt{a})) \mid x \in \bar{k}\}$ which is G-cr over k by the same argument as that of the G_2 example.

Remark 4.13. One can obtain more examples satisfying Theorem 1.2 using nonseparable subgroups in [32, Sec. 3,4,5] for $G = E_6, E_7$, and E_8; see [33].

5 Proof of Theorem 1.7

Let $\tilde{G}/k$ be a simple algebraic group of type A_4. Let $G := \tilde{G} \rtimes \langle \sigma \rangle$ where σ is the non-trivial graph automorphism of $\tilde{G}$. Fix a maximal k-split torus T and k-Borel subgroup B of G containing T. Define the set of simple roots $\{\alpha, \beta, \gamma, \delta\}$ of G as in the following Dynkin diagram. Let $\lambda = (\alpha + \beta + \gamma + \delta)^\vee$. Then



$$L_\lambda = \langle T, G_\beta, G_\gamma, G_{\beta+\gamma}, \sigma \rangle,$$
$$P_\lambda = \langle L, U_i \mid i \in \{\alpha, \delta, \alpha+\beta, \gamma+\delta, \alpha+\beta+\gamma, \beta+\gamma+\delta, \alpha+\beta+\gamma+\delta\} \rangle.$$

Let

$$K := \langle \sigma \rangle,\ v(\sqrt{a}) := \epsilon_{-\alpha-\beta}(\sqrt{a})\epsilon_{-\gamma-\delta}(\sqrt{a}),$$
$$M := \langle K, G_{\beta+\gamma} \rangle < L_\lambda.$$

Note that $v(\sqrt{a})$ centralizes $G_{\beta+\gamma}$. Define

$$H := \langle v(\sqrt{a}) \cdot M, \epsilon_{\beta+\gamma+\delta}(1) \rangle.$$

Proposition 5.1. *H is G-cr over k, but not G-cr.*

Proof. We have

$$\begin{aligned}\widetilde{H} := v(\sqrt{a})^{-1} \cdot H :=& \langle \sigma, v(\sqrt{a})^{-1} \cdot G_{\beta+\gamma}, v(\sqrt{a})^{-1} \cdot \epsilon_{\beta+\gamma+\delta}(1) \rangle \\ =& \langle \sigma, G_{\beta+\gamma}, \epsilon_{\beta+\gamma+\delta}(1)\epsilon_\beta(\sqrt{a}) \rangle < P_\lambda.\end{aligned}$$

Let $M' := \langle \sigma, (\beta+\gamma)^\vee(\bar{k}^*) \rangle < M$. We show that M' is L_λ-cr. We have

$$M' < L_{(\beta+\gamma)^\vee}(L_\lambda) = C_{L_\lambda}((\beta+\gamma)(\bar{k}^*)) = \langle \sigma, T \rangle.$$

Clearly, M' is $L_{(\beta+\gamma)^\vee}(L_\lambda)$-ir. Then, by [4, Cor. 3.5 and Sec. 6.3], M' is L_λ-cr. Now we show that M is L_λ-cr. Suppose not; then there exists a proper R-parabolic subgroup P_L of L_λ

containing M. Let $P_L = P_\mu(L_\lambda)$ for some cocharacter μ of L_λ. Since M' is L_λ-cr and M' is contained in $P_\mu(L_\lambda)$, there exists an R-Levi subgroup of $P_\mu(L_\lambda)$ containing M'. So, without loss, we assume that $\mu(\bar{k}^*)$ is contained in $C_{L_\lambda}(M') = \langle \sigma, T \rangle$. Then $\mu = c\alpha^\vee + d\beta^\vee + d\gamma^\vee + c\delta^\vee$ for some $c, d \in \mathbb{Q}$ since σ centralizes μ. Note that $P_\mu(L_\lambda)$ contains $G_{\beta+\gamma}$, so we have $\langle \beta+\gamma, \mu \rangle = 0$. Then $\mu = (\alpha+\beta+\gamma+\delta)^\vee$ up to a positive scaler multiple. But then $P_\mu(L_\lambda) = L_\lambda$. This is a contradiction. So, M is L_λ-cr, and it is G-cr by [4, Cor. 3.5 and Sec. 6.3].

Let P_μ be a proper R-parabolic subgroup of G containing $\widetilde{H}$. Since M is G-cr, without loss we can assume that M is centralized by μ. It is clear that $G_{\alpha+\beta+\gamma+\delta} < C_G(M)^\circ$. Note that $C_G(M)^\circ < C_G(\sigma, (\beta+\gamma)^\vee(\bar{k}^*)) = G_{\alpha+\beta+\gamma+\delta}$. So, $C_G(M) = G_{\alpha+\beta+\gamma+\delta}$. Thus $\mu(\bar{k}^*) < G_{\alpha+\beta+\gamma+\delta}$. Set

$$\mu := g \cdot \lambda \text{ for some } g \in G_{\alpha+\beta+\gamma+\delta}.$$

Let $n_{\alpha+\beta+\gamma+\delta} := n_\alpha n_\beta n_\gamma n_\delta n_\gamma n_\beta n_\alpha$. By the Bruhat decomposition, any element g of $G_{\alpha+\beta+\gamma+\delta}$ can be expressed as

(1) $g = \lambda(s)\epsilon_{\alpha+\beta+\gamma+\delta}(y_1)$ or

(2) $g = \epsilon_{\alpha+\beta+\gamma+\delta}(y_1) n_{\alpha+\beta+\gamma+\delta} \lambda(s) \epsilon_{\alpha+\beta+\gamma+\delta}(y_2)$ for some $s \in \bar{k}^*, y_1, y_2 \in \bar{k}$.

We rule out the second case. Suppose g is in form (2). Since $\widetilde{H} \leq P_\mu = P_{g\cdot\lambda}$, it is enough to show that $g^{-1} \cdot \widetilde{H} \not\subset P_\lambda$. Let

$$h := \epsilon_{\beta+\gamma+\delta}(1)\epsilon_\beta(\sqrt{a}) \in \widetilde{H}.$$

Since h centralizes $U_{\alpha+\beta+\gamma+\delta}$ and $\epsilon_{\alpha+\beta+\gamma+\delta}(y_2)\lambda(s)$ belongs to P_λ for any $y_2 \in \bar{k}, s \in \bar{k}^*$, without loss, we assume $g = n_{\alpha+\beta+\gamma+\delta}$. Then

$$g^{-1} \cdot h = \epsilon_{-\alpha}(1)\epsilon_\beta(\sqrt{a}) \notin P_\lambda.$$

Thus g is in form (1) and $g \in P_\lambda$, so P_λ is the unique proper R-parabolic subgroup of G containing $\widetilde{H}$. A similar argument to the G_2 and the E_6 cases shows that $v(\sqrt{a}) \cdot P_\lambda$ is not k-defined. Thus H is G-ir over k.

We find by a direct computation that $C_G(H)^\circ = v(\sqrt{a}) \cdot U_{\alpha+\beta+\gamma+\delta}$, which is unipotent. Then [11, Prop. 3.1] yields that $C_G(H)^\circ$ is not G°-cr. Thus $C_G(H)^\circ$ is not G-cr by [4, Lem. 6.12]. Suppose that H is G-cr. Then $C_G(H)$ is G-cr by [4, Thm. 3.14 and Sec. 6.3]. But $C_G(H)^\circ$ is a normal subgroup of $C_G(H)$, so $C_G(H)^\circ$ is G-cr by [8, Ex. 5.20]. This is a contradiction. □

Remark 5.2. In the proof of Theorem 1.7, K acts non-separably on $R_u(P_\lambda^-)$.

6 Related results

The following was shown in [7, Sec. 7], [34, Sec. 4], [32, Thm. 1.8]. The key to the construction in the proofs was again non-separability.

Theorem 6.1. *Let $k = k_s$ be a nonperfect field of characteristic 2. Let G/k be a simple algebraic group of type E_n (or G_2). Then there exists a k-subgroup H of G such that H is G-cr, but not G-cr over k.*

Note that this is the opposite direction of Theorem 1.2. We now show the following. The point is that if we allow G to be non-connected, computations become much simpler than the connected cases.

Theorem 6.2. *Let $k = k_s$ be a nonperfect field of characteristic 2. Let $\tilde{G}/k$ be a simple algebraic group of type A_2. Let $G := \tilde{G} \rtimes \langle\sigma\rangle$ where σ is the non-trivial graph automorphism of $\tilde{G}$. Then there exists a k-subgroup H of G such that H is G-cr but not G-cr over k.*

Proof. Let G be as in the hypotheses. Fix a maximal k-split torus T of G and a k-Borel subgroup containing T. Let $\{\alpha, \beta\}$ be the set of simple roots of G corresponding to T and B. Let $\lambda := (\alpha+\beta)^\vee$. Then $P_\lambda = \langle B, \sigma\rangle$ is a k-defined R-parabolic subgroup of G and $L_\lambda = \langle T, \sigma\rangle = C_G((\alpha+\beta)^\vee(\bar{k}^*))$ is a k-defined R-Levi subgroup of P_λ. Let $K := \langle\sigma\rangle$ and $v(\sqrt{a}) := \epsilon_\alpha(\sqrt{a})\epsilon_\beta(\sqrt{a})$. Define

$$H := v(\sqrt{a}) \cdot K = \langle\sigma\epsilon_{\alpha+\beta}(a)\rangle.$$

First, we prove that H is G-cr. It is enough to show that K is G-cr since H is G-conjugate to K. It is clear that K is contained in L_λ and K is L_λ-ir, so K is G-cr by [4, Sec. 6.3].

Now we show that H is not G-cr over k. Suppose the contrary. It is clear that P_λ contains H. Then there exists a k-defined R-Levi subgroup L of P_λ containing H. By [8, Lem. 2.5(iii)], there exists $u \in R_u(P_\lambda)(k)$ such that $L = u \cdot L_\lambda$. Then $u^{-1}\cdot H \leq L_\lambda$. It is obvious that $v(\sqrt{a})^{-1} \cdot H \leq L_\lambda$. Let $\pi_\lambda : P_\lambda \to L_\lambda$ be the canonical projection. For any $s \in H$, we have

$$u^{-1}\cdot s = \pi_\lambda(u^{-1}\cdot s) = \pi_\lambda(s) = \pi_\lambda(v(\sqrt{a})^{-1}\cdot s) = v(\sqrt{a})^{-1}\cdot s.$$

So, $u = v(\sqrt{a})z$ for some $z \in C_{R_u(P_\lambda)}(K)(k)$. We compute $C_{R_u(P_\lambda)}(K) = U_{\alpha+\beta}$. So, $u = v(\sqrt{a})\epsilon_{\alpha+\beta}(x) = \epsilon_\alpha(\sqrt{a})\epsilon_\beta(\sqrt{a})\epsilon_{\alpha+\beta}(x)$ for some $x \in k$. This is a contradiction since u is a k-point. Thus H is not G-cr over k. □

To finish the paper, we consider another application of non-separability for non-connected G with a slightly different flavor. In [7, Sec. 7], [34, Sec. 3], [32, Thm. 1.2], it was shown that

Theorem 6.3. *Let k be an algebraically closed field of characteristic 2. Let G/k be a simple algebraic group of type E_n (or G_2). Then there exists a pair of reductive subgroups $H < M$ of G such that H is G-cr but not M-cr.*

The following is much easier to prove than the connected cases.

Theorem 6.4. *Let k be an algebraically closed field of characteristic 2. Let $\tilde{G}/k$ be a simple algebraic group of type A_2. Let $G := \tilde{G} \rtimes \langle\sigma\rangle$ where σ is the non-trivial graph automorphism of $\tilde{G}$. Then there exists a pair of reductive subgroups $H < M$ of G such that H is G-cr but not M-cr.*

Proof. Use the same $H = \langle\sigma\epsilon_{\alpha+\beta}(a)\rangle$ as in the proof of Theorem 6.2. Then H is G-cr. We show that H is not M-cr. Set $M := \langle\sigma, G_{\alpha+\beta}\rangle$. Let $\lambda := (\alpha+\beta)^\vee$. Then the image of $\sigma\epsilon_{\alpha+\beta}(a)$ under the canonical projection $\pi_\lambda : P_\lambda \to L_\lambda$ is σ. By a recent result in Geometric Invariant Theory ([4, Lem. 2.17, Thm. 3.1] and [8, Thm. 3.3]), it is enough to show that σ is not $R_u((P_\lambda)(M))$-conjugate to $\sigma\epsilon_{\alpha+\beta}(a)$. This is easy since $R_u((P_\lambda)(M)) = U_{\alpha+\beta}$ which is centralized by σ. We are done. □

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Appendix

(1) $\begin{smallmatrix} & & 1 & & \\ 0 & 0 & 0 & 0 & 0 \end{smallmatrix}$	(2) $\begin{smallmatrix} & & 1 & & \\ 0 & 0 & 1 & 0 & 0 \end{smallmatrix}$	(3) $\begin{smallmatrix} & & 1 & & \\ 0 & 1 & 1 & 0 & 0 \end{smallmatrix}$	(4) $\begin{smallmatrix} & & 1 & & \\ 0 & 0 & 1 & 1 & 0 \end{smallmatrix}$
(5) $\begin{smallmatrix} & & 1 & & \\ 1 & 1 & 1 & 0 & 0 \end{smallmatrix}$	(6) $\begin{smallmatrix} & & 1 & & \\ 0 & 1 & 1 & 1 & 0 \end{smallmatrix}$	(7) $\begin{smallmatrix} & & 1 & & \\ 0 & 0 & 1 & 1 & 1 \end{smallmatrix}$	(8) $\begin{smallmatrix} & & 1 & & \\ 1 & 1 & 1 & 1 & 0 \end{smallmatrix}$
(9) $\begin{smallmatrix} & & 1 & & \\ 0 & 1 & 1 & 1 & 1 \end{smallmatrix}$	(10) $\begin{smallmatrix} & & 1 & & \\ 0 & 1 & 2 & 1 & 0 \end{smallmatrix}$	(11) $\begin{smallmatrix} & & 1 & & \\ 1 & 1 & 1 & 1 & 1 \end{smallmatrix}$	(12) $\begin{smallmatrix} & & 1 & & \\ 1 & 1 & 2 & 1 & 0 \end{smallmatrix}$
(13) $\begin{smallmatrix} & & 1 & & \\ 0 & 1 & 2 & 1 & 1 \end{smallmatrix}$	(14) $\begin{smallmatrix} & & 1 & & \\ 1 & 1 & 2 & 1 & 1 \end{smallmatrix}$	(15) $\begin{smallmatrix} & & 1 & & \\ 1 & 2 & 2 & 1 & 0 \end{smallmatrix}$	(16) $\begin{smallmatrix} & & 1 & & \\ 0 & 1 & 2 & 2 & 1 \end{smallmatrix}$
(17) $\begin{smallmatrix} & & 1 & & \\ 1 & 2 & 2 & 1 & 1 \end{smallmatrix}$	(18) $\begin{smallmatrix} & & 1 & & \\ 1 & 1 & 2 & 2 & 1 \end{smallmatrix}$	(19) $\begin{smallmatrix} & & 1 & & \\ 1 & 2 & 2 & 2 & 1 \end{smallmatrix}$	(20) $\begin{smallmatrix} & & 1 & & \\ 1 & 2 & 3 & 2 & 1 \end{smallmatrix}$
(21) $\begin{smallmatrix} & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{smallmatrix}$	(22) $\begin{smallmatrix} & & 0 & & \\ 1 & 0 & 0 & 0 & 0 \end{smallmatrix}$	(23) $\begin{smallmatrix} & & 0 & & \\ 0 & 1 & 0 & 0 & 0 \end{smallmatrix}$	(24) $\begin{smallmatrix} & & 0 & & \\ 0 & 0 & 1 & 0 & 0 \end{smallmatrix}$
(25) $\begin{smallmatrix} & & 0 & & \\ 0 & 0 & 0 & 1 & 0 \end{smallmatrix}$	(26) $\begin{smallmatrix} & & 0 & & \\ 0 & 0 & 0 & 0 & 1 \end{smallmatrix}$	(27) $\begin{smallmatrix} & & 0 & & \\ 1 & 1 & 0 & 0 & 0 \end{smallmatrix}$	(28) $\begin{smallmatrix} & & 0 & & \\ 0 & 1 & 1 & 0 & 0 \end{smallmatrix}$
(29) $\begin{smallmatrix} & & 0 & & \\ 0 & 0 & 1 & 1 & 0 \end{smallmatrix}$	(30) $\begin{smallmatrix} & & 0 & & \\ 0 & 0 & 0 & 1 & 1 \end{smallmatrix}$	(31) $\begin{smallmatrix} & & 0 & & \\ 1 & 1 & 1 & 0 & 0 \end{smallmatrix}$	(32) $\begin{smallmatrix} & & 0 & & \\ 0 & 1 & 1 & 1 & 0 \end{smallmatrix}$
(33) $\begin{smallmatrix} & & 0 & & \\ 0 & 0 & 1 & 1 & 1 \end{smallmatrix}$	(34) $\begin{smallmatrix} & & 0 & & \\ 1 & 1 & 1 & 1 & 0 \end{smallmatrix}$	(35) $\begin{smallmatrix} & & 0 & & \\ 0 & 1 & 1 & 1 & 1 \end{smallmatrix}$	(36) $\begin{smallmatrix} & & 0 & & \\ 1 & 1 & 1 & 1 & 1 \end{smallmatrix}$

Table 1: The set of positive roots of E_6

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Complete reducibility of subgroups of reductive algebraic groups over nonperfect fields II

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Abstract

Let k be a separably closed field. Let G be a reductive algebraic k-group. In this paper, we study Serre's notion of complete reducibility of subgroups of G over k. In particular, using the recently proved center conjecture of Tits, we show that the centralizer of a k-subgroup H of G is G-completely reducible over k if it is reductive and H is G-completely reducible over k. We also show that a regular reductive k-subgroup of G is G-completely reducible over k. Various open problems concerning complete reducibility are discussed. We present examples where the number of overgroups of irreducible subgroups and the number of $G(k)$-conjugacy classes of unipotent elements are infinite (this cannot happen if k is algebraically closed). This paper complements the author's previous work on rationality problems for complete reducibility.

Keywords: algebraic groups, complete reducibility, pseudo-reductivity, spherical buildings

1 Introduction

Let k be an arbitrary field. Let $\overline{k}$ be an algebraic closure of k. Let H be a (possibly non-connected) affine algebraic k-group, that is a (possibly non-connected) $\overline{k}$-defined affine algebraic group with a k-structure in the sense of Borel [9, AG.12.1]. We write $R_{u,k}(H)$ for the unique maximal smooth connected unipotent normal k-subgroup of H. An affine algebraic k-group H is *pseudo-reductive* if $R_{u,k}(H) = 1$ [12, Def. 1.1.1], and *reductive* if the unipotent radical $R_u(H) = 1$. Throughout, we write G for a (possibly non-connected) reductive algebraic k-group. By a subgroup of G, we always mean a closed subgroup of G. Generalizing Serre [25, Sec. 3], define

Definition 1.1. A (possibly non-k-defined) closed subgroup $H < G$ is *G-completely reducible over k* (G-cr over k for short) if whenever H is contained in a k-defined R-parabolic subgroup P of G, it is contained in some k-defined R-Levi subgroup of P. In particular, if H is not contained in any k-defined proper R-parabolic subgroup, H is *G-irreducible* over k (G-ir over k for short).

For the definition of R-parabolic subgroups and R-Levi subgroups, see Definition 2.2. If G is connected, R-parabolic subgroups and R-Levi subgroups are parabolic subgroups and Levi subgroups in the usual sense. Definition 1.1 extends usual Serre's definition in the following sense: 1. $H < G$ is not necessarily k-defined, 2. G is not necessarily connected. Definition 1.1 was used in [3] and [32].

The notion of complete reducibility generalizes that of complete reducibility in representation theory, and it has been much studied. However most studies assume $k = \overline{k}$ and G is connected; see [5],[16],[28] for example. We say that $H < G$ is G-cr when it is G-cr over $\overline{k}$. Not much is known about complete reducibility over an arbitrary k except a few general results and important examples in [3], [6], [7, Sec. 7], [8], [32], [33, Thm. 1.8], [34, Sec. 4].

Let k_s be a separable closure of k. Recall that if k is perfect, we have $k_s = \overline{k}$. The following result [6, Thm. 1.1] shows that if k is perfect and G is connected, most results in this paper just reduce to the algebraically closed case.

Proposition 1.2. *Let k be a field. Let G be connected. Then a k-subgroup H of G is G-cr over k if and only if H is G-cr over k_s.*

We write $G(k)$ for the set of k-points of G. For $H < G$, we write $H(k) := G(k) \cap H$. By $\overline{H}$, we mean the Zariski closure of H. We write $C_G(H)$ for the set-theoretic centralizer of H in G. Centralizers of subgroups of G are important to understand the subgroup structure of G [1], [2] [18], [27]. Recall the following [5, Prop. 3.12, Cor. 3.17]:

Proposition 1.3. *Let $k = \overline{k}$. Suppose that a subgroup H of G is G-cr. Then $C_G(H)$ is reductive, and moreover it is G-cr.*

Note that any G-cr subgroup of G is reductive [25, Prop. 4.1]. It is natural to ask (cf. [32, Open Problem 3.4]):

Open Problem 1.4. *Let k be a field. Suppose that a k-subgroup H of G is G-cr over k. Is $C_G(H)$ G-cr over k? Is $\overline{C_G(H)(k_s)}$ G-cr over k?*

Even if H is k-defined, $C_G(H)$ is not necessarily k-defined; see [32, Theorem 1.2] for examples of non-k-defined $C_G(H)$. See Lemma 6.9 and [3, Prop. 7.4] for some k-definability criteria for $C_G(H)$. Let $\Gamma := \mathrm{Gal}(k_s/k) = \mathrm{Gal}(\overline{k}/k)$. Note that $\overline{C_G(H)(k_s)}$ is the unique maximal k-defined subgroup of $C_G(H)$; it is k-defined by [9, Prop. 14.2] since it is k_s-defined and Γ-stable. Our principal result is the following.

Theorem 1.5. *Let $k = k_s$. Let G be connected. Suppose that a k-subgroup H of G is G-cr over k.*

1. *If $\overline{C_G(H)(k)}$ is pseudo-reductive, then it is G-cr over k,*
2. *If $C_G(H)$ is reductive, then it is G-cr over k.*

Recall the following [32, Prop. 1.14]:

Proposition 1.6. *Let $k = k_s$. Let G be connected. Let H be a k-subgroup of G. If H is G-ir over k, then $C_G(H)$ is G-cr over k.*

Theorem 1.5 and Proposition 1.6 give a partial affirmative answer to Open problem 1.4. However, in [32, Rem. 3.11], it was shown that there exists a k-subgroup H of G with the following properties: 1. H is G-cr over k, 2. $C_G(H)$ is k-defined, 3. $C_G(H)$ is G-cr over k, 4. $C_G(H)$ is not pseudo-reductive. This result suggests a negative answer to Open problem 1.4 since reductivity of $C_G(H)$ was crucial to show that $C_G(H)$ is G-cr in the proof of Proposition 1.3.

In this paper, we extend various other results concerning complete reducibility in [5] to an arbitrary k. First, we extend the notion of *strong reductivity* [24, Def. 16.1].

Definition 1.7. Let $k = k_s$. Let H be a (possibly non-k-defined) subgroup of G. Then H is *strongly reductive over k in G* if H is not contained in any proper k-defined R-parabolic subgroup of the reductive k-group $C_G(S)$, where S is a maximal k-torus of $C_G(H)$.

Note that this definition does not depend on the choice of S. We generalize [5, Thm. 3.1], which was the main result of [5].

Theorem 1.8. *Let $k = k_s$. Let H be a (possibly non-k-defined) subgroup of G. Then H is G-cr over k if and only if H is strongly reductive over k in G.*

Next, generalizing the notion of a *regular subgroup* of G [16], [17], define:

Definition 1.9. A (possibly non-k-defined) subgroup H of G is *k-regular* if H is normalized by a maximal k-torus of G.

We extend [5, Prop. 3.20].

Theorem 1.10. *Let $k = k_s$. Let G be connected. Let H be a k-regular reductive k-subgroup of G. Then H is G-cr over k.*

Note that in Propositions 1.3, 1.6 and Theorems 1.5, 1.10 we assumed G to be connected. This is because the proofs of these results depend on the following (Theorem 1.11) that is a consequence of the recently proved *center conjecture of Tits* [13], [21], [23], [25, Sec. 2.4], [29, Lem. 1.2], [32, Sec. 3.1].

Theorem 1.11. *Let k be a field. Let G be connected. Let $\Delta(G)$ be the spherical building of G. Let H be a (possibly non-k-defined) subgroup of G that is not G-cr over k. Let $\Delta(G)^H$ be the fixed point subcomplex of $\Delta(G)$. Then there exists a simplex in $\Delta(G)^H$ that is fixed by all building automorphisms of $\Delta(G)$ stabilizing $\Delta(G)^H$.*

Recall that each simplex of $\Delta(G)$ is identified with a proper k-parabolic subgroup of G [30, Thm. 5.2], and $\Delta(G)^H$ is identified with the set of k-parabolic subgroups containing H [25, Sec. 2]. Now let G be non-connected reductive. Note that $\Delta(G) = \Delta(G^\circ)$ by definition. Let $\Lambda(G)$ be the set of k-defined R-parabolic subgroups of G. Let $\Lambda(G)^H$ be the set of k-defined R-parabolic subgroups of G containing H.

Theorem 1.12. *Let k be a field. Let $\tilde{G} = SL_3$. Let $G = \tilde{G} \rtimes \langle\sigma\rangle$ where σ is the non-trivial graph automorphism of $\tilde{G}$. Then there exists a k-subgroup H of G such that H is not G-cr over k and $\Lambda(G)^H$ is not a subset of $\Delta(G)$. Moreover, $\Lambda(G)$ ordered by reversed inclusion does not form a simplicial complex in the sense of [30, Thm. 5.2].*

Theorem 1.12 shows that we cannot use Theorem 1.11 to extend Propositions 1.3, 1.6 and Theorems 1.5, 1.10 to non-connected G. Before finishing this section, we consider a problem with a slightly different flavor. In [19, Thm. 1], Liebeck and Testerman showed that:

Proposition 1.13. *Let $k = \overline{k}$. Let G be a semisimple. Suppose that H is a connected subgroup of G and H is G-ir. Then H has only finitely many overgroups.*

We show that:

Theorem 1.14. *Let k be a nonperfect field of characteristic 2. Let $G := PGL_4$. Then there exists a connected k-subgroup H of G such that H is G-ir over k and H has infinitely many (non-k-defined) overgroups.*

Here is the structure of the paper. In Section 2, we set out the notation. In Sections 3 and 4, we prove Theorems 1.8 and 1.10, respectively. Then in Section 5, we discuss some relationships between complete reducibility and linear reductivity. In Section 6, we prove Theorem 1.5. In Sections 7 and 8, we prove Theorems 1.12 and 1.14, respectively. Finally, in Section 9, we present an example where the number of $G(k)$-conjugacy classes of unipotent elements of $G(k)$ is infinite.

2 Preliminaries

Throughout, we denote by k an arbitrary field. Our basic references for algebraic groups are [9], [10], [12], and [26]. We write G for a (possibly non-connected) reductive k-group. We write $X_k(G)$ and $Y_k(G)$ for the set of k-characters and k-cocharacters of G respectively. For an algebraic group H, we write H° for the identity component of H.

Fix a maximal k-split torus T of G. We write $\Psi_k(G,T)$ for the set of k-roots of G with respect to T [9, 21.1]. We sometimes write $\Psi_k(G)$ for $\Psi_k(G,T)$. Let $\zeta \in \Psi_k(G)$. We write U_ζ for the corresponding root subgroup of G. Let $\zeta, \xi \in \Psi_k(G)$. Let $\xi^\vee$ be the coroot corresponding to ξ. Then $\zeta \circ \xi^\vee : \overline{k}^* \to \overline{k}^*$ is a k-homomorphism such that $(\zeta \circ \xi^\vee)(a) = a^n$ for some $n \in \mathbb{Z}$. Let s_ξ denote the reflection corresponding to ξ in the Weyl group W_k relative to T. Each s_ξ acts on the set of roots $\Psi_k(G)$ by the following formula [26, Lem. 7.1.8]: $s_\xi \cdot \zeta = \zeta - \langle \zeta, \xi^\vee \rangle \xi$. By [11, Prop. 6.4.2, Lem. 7.2.1] we can choose k-homomorphisms $\epsilon_\zeta : \overline{k} \to U_\zeta$ so that $n_\xi \epsilon_\zeta(a) n_\xi^{-1} = \epsilon_{s_\xi \cdot \zeta}(\pm a)$ where $n_\xi = \epsilon_\xi(1)\epsilon_{-\xi}(-1)\epsilon_\xi(1)$.

We recall the notions of R-parabolic subgroups and R-Levi subgroups from [24, Sec. 2.1–2.3]. These notions are essential to define G-complete reducibility for subgroups of non-connected reductive groups; see [4] and [5, Sec. 6].

Definition 2.1. Let X be a k-affine variety. Let $\phi : \overline{k}^* \to X$ be a k-morphism of k-affine varieties. We say that $\lim_{a\to 0} \phi(a)$ exists if there exists a k-morphism $\hat{\phi} : \overline{k} \to X$ (necessarily unique) whose restriction to $\overline{k}^*$ is ϕ. If this limit exists, we set $\lim_{a\to 0} \phi(a) = \hat{\phi}(0)$.

Definition 2.2. Let $\lambda \in Y_k(G)$. Define $P_\lambda := \{g \in G \mid \lim_{a\to 0} \lambda(a) g \lambda(a)^{-1} \text{ exists}\}$, $L_\lambda := \{g \in G \mid \lim_{a\to 0} \lambda(a) g \lambda(a)^{-1} = g\}$, $R_u(P_\lambda) := \{g \in G \mid \lim_{a\to 0} \lambda(a) g \lambda(a)^{-1} = 1\}$. We call P_λ an R-parabolic subgroup of G, L_λ an R-Levi subgroup of P_λ. Note that $R_u(P_\lambda)$ is the unipotent radical of P_λ.

If λ is k-defined, P_λ, L_λ, and $R_u(P_\lambda)$ are k-defined [8, Lem. 2.5]. It is well known that $L_\lambda = C_G(\lambda(\overline{k}^*))$. Note that if $k = k_s$, for a k-defined R-parabolic subgroup P of G and a k-defined R-Levi subgroup L of P there exists $\lambda \in Y_k(G)$ such that $P = P_\lambda$ and $L = L_\lambda$ [8, Lem. 2.5, Cor. 2.6].

Let M be a reductive k-subgroup of G. Then there is a natural inclusion $Y_k(M) \subseteq Y_k(G)$ of k-cocharacter groups. Let $\lambda \in Y_k(M)$. We write $P_\lambda(G)$ or just P_λ for the k-defined R-parabolic subgroup of G corresponding to λ, and $P_\lambda(M)$ for the k-defined R-parabolic subgroup of M corresponding to λ. It is clear that $P_\lambda(M) = P_\lambda(G) \cap M$ and $R_u(P_\lambda(M)) = R_u(P_\lambda(G)) \cap M$.

The next result is a consequence of Theorem 1.11, and we use it repeatedly.

Proposition 2.3. *Let $k = k_s$. Let G be connected. Suppose that a (possibly non-k-defined) subgroup H of G is not G-cr over k. If a k-subgroup N of G normalizes H, then there exist a proper k-parabolic subgroup of G containing H and N.*

Proof. By [32, Prop. 3.3], there exists a proper k-parabolic subgroup P containing H and $N(k)$. Since the k_s-points are dense in N by [9, AG.13.3], P contains H and N. ∎

3 Complete reducibility and strong reductivity

Our proof is similar to [5, Thm. 3.1].

Proof of Theorem 1.8. Suppose that H is G-cr over k. Let S be a maximal k-defined torus of $C_G(H)$. Suppose that S is central in G. Then $C_G(S) = G$. Suppose that H is contained in a proper k-defined R-parabolic subgroup P of G. Then there exists a k-defined R-Levi subgroup L of P containing H since H is G-cr over k. Since $k = k_s$, we can set $L = L_\lambda$ and $P = P_\lambda$ for some $\lambda \in Y_k(G)$. Then $\lambda(\overline{k}^*) < \overline{C_G(H)}$ and $\lambda(\overline{k}^*)$ is a connected commutative non-central k-subgroup of G. Now let $C := \overline{C_G(H)(k_s)}$. Then C is the unique maximal k-defined subgroup of $C_G(H)$. Since $k = k_s$, maximal k-tori of C are $G(k)$-conjugate by [9, Thm. 20.9]. Then $\lambda(\overline{k}^*) < S$ since S is central in G. This is a contradiction. So H cannot be contained in a proper k-defined R-parabolic subgroup of G. Therefore H is strongly reductive over k.

Now we assume S is non-central in G. Suppose that H is contained in a k-defined proper R-parabolic subgroup Q of $C_G(S)$. Note that $C_G(S)$ is a k-defined R-Levi subgroup of G ([5, Cor. 6.10]). Then by the rational version of [5, Lem. 6.2(ii)] (note that [5, Lem. 6.2(ii)] is for $k = \overline{k}$, but the same proof works work by word if we set $P = P_\lambda$, $P' = P'_\mu$, and $L = L_\lambda$ for $\lambda \in Y_k(G)$ in the proof), there exists a k-defined proper R-parabolic subgroup P_μ of G such that $Q = C_G(S) \cap P_\mu$. It is clear that $S < Q < P_\mu$. Since H is G-cr over k, there exists a k-defined R-Levi subgroup L of P_μ containing H. Without loss we set $L = L_\mu$. Then $\mu(\overline{k}^*)$ is a k-torus in $C_{P_\mu}(H)$. Since S is contained in P_μ and S is a maximal k-torus of $C_G(H)$, S is a maximal k-torus of $C_{P_\mu}(H)$. Since $k = k_s$, by the same argument as in the first paragraph, we have $g\mu(\overline{k}^*)g^{-1} < S$ for some $g \in P_\mu(k)$. Then $C_G(S) < C_G(g\mu(\overline{k}^*)g^{-1}) = gL_\mu g^{-1} < P_\mu$. Therefore $Q = C_G(S)$, which is a contradiction.

Now suppose that H is strongly reductive over k. Let S be a maximal k-torus of $C_G(H)$. Then H is not contained in any proper k-defined R-parabolic subgroup of $C_G(S)$. Let $L := C_G(S)$. Let Q be a k-defined R-parabolic subgroup of G containing L as a k-Levi subgroup. Then by the rational version of [5, Lem. 6.2(ii)], Q is minimal among all k-defined R-parabolic subgroups of G containing H. Let P be a k-defined R-parabolic subgroup of G containing H. Our goal is to find a k-defined R-Levi subgroup of P containing H.

If P' is a k-defined R-parabolic subgroup of G such that $P' < P$ and M' is a k-defined R-Levi subgroup of P', then, by [5, Cor. 6.6], there exists a unique $\overline{k}$-defined R-Levi subgroup M'' of P containing M'. But M'' is k-defined by [8, Lem. 2.5(iii)]. So, we assume that P is minimal among all k-defined R-parabolic subgroups of G containing H. By [9, Prop. 20.7], $P \cap Q$ contains a maximal k-torus T of G. We see that there exists a (possibly non-k-defined) common R-Levi subgroup M of P and Q containing T by a similar argument to that in the proof of [5, Thm. 3.1] (use [5, Lem. 6.2] where necessary). Since T is k-defined, M is k-defined by [8, Lem. 2.5(iii)].

Let P^- be the unique opposite of P such that $M = P \cap P^-$. Since $R_u(Q)$ is a product of root subgroups in any prescribed order, we have

$$R_u(Q) = (R_u(Q) \cap M)(R_u(Q) \cap R_u(P^-))(R_u(Q) \cap R_u(P)).$$

It is clear that $R_u(Q) \cap M$ is trivial. Since L and M are k-defined R-Levi subgroups of Q, there exists $u' \in R_u(Q)(k)$ such that $u'Mu'^{-1} = L$ by [8, Lem. 2.5(iii)]. Using the rational version of the Bruhat decomposition ([9, Thm. 21.15]), we can express u' as $u' = yz$ where $y \in (R_u(Q) \cap R_u(P^-))(k)$ and $z \in (R_u(Q) \cap R_u(P))(k)$. Note that zMz^{-1} is a k-defined common R-Levi subgroup of P and Q because $z \in (R_u(Q) \cap R_u(P))(k)$. So, without loss, we may assume $z = 1$. Then $yMy^{-1} = L$ and $L < P^-$. Since L contains H, we have $H < P \cap P^- = M$. We are done.

□

4 Complete reducibility and regular subgroups

We extend [5, Prop. 3.19].

Lemma 4.1. *Let $k = k_s$. Let H be a reductive k-subgroup of G. Let K be a (possibly non-k-defined) subgroup of H. Suppose that H contains a maximal k-torus of $C_G(K)$ and that K is G-cr over k. Then K is H-cr over k and H is G-cr over k.*

Proof. Let S be a maximal k-torus of $C_G(K)$ contained in H. Then S is a maximal k-torus of $C_H(K)$. Since K is G-cr over k, K is $C_G(S)$-ir over k by Theorem 1.8. Note that $K < C_H(S) < C_G(S)$. So K is $C_H(S)$-ir over k. Thus K is H-cr over k by Theorem 1.8.

Let P be a k-defined R-parabolic subgroup of G containing H. Then P contains K. Since K is G-cr over k, by the same argument as in the second paragraph of the proof of Theorem 1.8, there exists $\lambda \in Y_k(G)$ such that $C_G(S) < L_\lambda$ and $P = P_\lambda$. Thus we have $\lambda(\overline{k}^*) < C_G(C_G(S))^\circ = Z(C_G(S))^\circ$. It is clear that $S < Z(C_G(S))^\circ$. We have $Z(C_G(S))^\circ < C_G(K)$. By [9, Thm. 18.2], $Z(C_G(S))^\circ$ is k-defined. Since S is a maximal k-torus of $C_G(K)$, we have $S = Z(C_G(S))^\circ$. Thus $\lambda(\overline{k}^*) < S < H$. So $\lambda \in Y_k(H)$. Thus we have $P_\lambda(H) = P_\lambda \cap H = H$. So $\lambda \in Y_k(Z(H))$. Then $H < C_G(\lambda(\overline{k}^*)) = L_\lambda$, and we are done. □

Proof of Theorem 1.10. Let T be a maximal k-torus of G normalizing H. Then if we show that TH is G-cr over k, by [32, Prop. 3.5] we are done since H is a normal subgroup of the k-group TH. Note that $C_G(T) = T$ and T is G-cr over k by [3, Cor. 9.8]. Applying Lemma 4.1 to $T < TH < G$, we obtain the desired result. □

5 Complete reducibility and linear reductivity

In this section we assume that G is connected. Recall that a (possibly non-k-defined) subgroup H of G is called *linearly reductive* if every rational representation of H is completely reducible. It is known that a (possibly non-k-defined) linearly reductive subgroup H of G is G-cr [5, Lem. 2.6] and if H is k-defined, it is G-cr over k [3, Cor. 9.8].

It is clear that the converse of Proposition 1.3 is false; take H to be a Borel subgroup of G. However we have the following partial converse [5, Cor. 3.18]:

Lemma 5.1. *Let $k = \overline{k}$. Let H be a subgroup of G. If $C_G(H)$ is G-ir, then H is linearly reductive. In particular, H is G-cr.*

The previous lemma was a consequence of the following [5, Lem. 3.38]:

Lemma 5.2. *Let $k = \overline{k}$. Let H be a subgroup of G. If H is G-ir, then $C_G(H)$ is linearly reductive.*

Remark 5.3. A natural analogue of Lemmas 5.1 and 5.2 for an arbitrary field k is false even if H is k-defined. Let k be a nonperfect field of characteristic 2. Let $a \in k \backslash k^2$. Let $G = PGL_2$. We write $\overline{A}$ for the image in PGL_2 of $A \in GL_2$. Let $H := \left\{ \overline{\begin{bmatrix} x & ay \\ y & x \end{bmatrix}} \in PGL_2(\overline{k}) \mid x, y \in \overline{k} \right\}$. Then H is G-ir over k since H contains the element $\overline{\begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix}}$, which is a k-anisotropic unipotent element; see [32, Ex. 3.10]. It is clear that $C_G(H) = H$ is not a torus. So, by [22, Thm. 2], $C_G(H) = H$ is not linearly reductive. We see that H is not G-cr since H is unipotent.

Definition 5.4. A unipotent element $u \in G(k)$ is called *k-plongeable* [31] if u belongs to the unipotent radical of some proper k-parabolic subgroup of G.

Proposition 5.5. *Let $k = k_s$. Suppose that a k-subgroup H of G is G-ir over k. Let $C := \overline{C_G(H)(k)}$ (or $C_G(H)$). If every unipotent element of $C(k)$ is k-plongeable, then every element of $C(k)$ is semisimple.*

Proof. Let $C := \overline{C_G(H)(k)}$. Suppose that there exists a non-trivial unipotent element $u \in C(k)$. Since we assumed that every unipotent element of $C(k)$ is k-plongeable, there exists a proper k-parabolic subgroup P of G such that $u \in R_u(P)$. Then, it is clear that the subgroup $U := \langle u \rangle$ is not G-cr over k. Since H normalizes U and H is k-defined, by Proposition 2.3, there exists a proper k-parabolic subgroup P' of G containing U and H. This is a contradiction. Therefore every element of $C(k)$ is semisimple. The same argument works for $C := C_G(H)$. $\square$

Proposition 5.6. *Let $k = k_s$. Let H be a (possibly non-k-defined) subgroup of G. Suppose that $C := \overline{C_G(H)(k)}$ is G-ir over k. If every unipotent element of $H(k)$ is k-plongeable, then every element of $H(k)$ is semisimple.*

Proof. Swap the roles of H and $\overline{C_G(H)(k)}$ in the proof of Proposition 5.5. $\square$

Remark 5.7. We do not know whether C in Proposition 5.5 (or H in Proposition 5.6) is linearly reductive. For that purpose we need to know whether every element of $C(\overline{k})$ (or $H(\overline{k})$) is semisimple [22, Thm. 2].

Proposition 5.8. *Suppose that a k-subgroup H of G is linearly reductive. Then $C_G(H)$ is k-defined and G-cr over k.*

Proof. Since H is linearly reductive, it is G-cr [5, Lem. 2.6]. So $C_G(H)$ is reductive by [5, Prop. 3.12]. Then $C_G(H)$ is G-cr over k by Theorem 1.5. Note that if H is linearly reductive, H is separable in G (see the sentence just after Lemma 6.9 for the definition of a separable subgroup of G). So, by the proof of [3, Prop. 7.4], $C_G(H)$ is k-defined since H is k-defined. $\square$

6 Centralizers of completely reducible subgroups

Proof of Theorem 1.5. We start with the first part of the theorem. Let $C := \overline{C_G(H)(k)}$. Let P be a k-parabolic subgroup of G such that $HC < P$. Since H is G-cr over k, there exists a k-Levi subgroup L of P with $H < L$. Let λ be a k-cocharacter of G such that $P = P_\lambda$ and $L = L_\lambda$. Then λ is a cocharacter of C. We have $C = (C \cap L)(C \cap R_u(P))$. Since λ normalizes C, by [3, Prop. 2.2] $C \cap R_u(P)$ is k-defined. For $u \in (C \cap R_u(P))(k)$, we define a k-morphism $\phi_u : \overline{k} \to C \cap R_u(P)$ by $\phi_u(0) = 1$ and $\phi_u(t) = \lambda(t)u\lambda(t)^{-1}$ for $t \in \overline{k}^*$. Then the image of ϕ_u is a connected k-subvariety of $C \cap R_u(P)$ containing 1 and $\phi_u(1) = u$. Then $C \cap R_u(P)$ must be trivial since C is pseudo-reductive. Thus $C < L$. Therefore HC is G-cr over k. Note that C is a normal subgroup of the k-defined subgroup HC. So by [32, Prop. 3.7], C is G-cr over k.

For the second part, the same argument shows that $HC_G(H)$ is G-cr over k since we assumed that $C_G(H)$ is reductive. However, if $C_G(H)$ is not k-defined we cannot apply [32, Prop. 3.7] to conclude that $C_G(H)$ is G-cr over k. We need a different argument. Let P' be a minimal k-parabolic subgroup of G containing $HC_G(H)$. Since $HC_G(H)$ is G-cr over k, there exists a k-Levi subgroup L' of P' containing $HC_G(H)$. If $C_G(H)$ is L'-cr over k, it is G-cr over k by [32, Lem. 3.6]. Otherwise, by [32, Lem. 3.6] and Proposition 2.3, there exist a proper k-parabolic subgroup $P_{L'}$ of L' containing $C_G(H)$ and H since H is k-defined and H normalizes $C_G(H)$. Note that $P_{L'} \ltimes R_u(P')$ is a k-parabolic subgroup of G by [10, Prop. 4.4(c)] and it is properly contained in P'. This contradicts the minimality of P'. $\square$

The following are further consequences of Theorem 1.11, and they all deal with special cases of Open Problem 1.4.

Proposition 6.1. *Let $k = k_s$. Let G be connected. If a k-subgroup H of G is G-cr over k and if $\overline{C_G(H)(k)}$ (or $C_G(H)$) is unipotent, then $\overline{C_G(H)(k)}$ (or $C_G(H)$) is G-cr over k.*

Proof. Let $C := C_G(H)$. Suppose that C is not G-cr over k. By Proposition 2.3 there exists a k-defined proper parabolic subgroup P_λ of G containing H and C. Then there exists a k-Levi subgroup L of P_λ containing H since H is G-cr over k. Without loss, we assume $L = L_\lambda$. Then $\lambda(\overline{k}^*) < C$. So λ must be trivial since C is unipotent. This is a contradiction. The other case can be shown in the same way. □

Remark 6.2. See [32, Sec. 4,5] for examples of a k-subgroup H of connected G (or non-connected G) such that: 1. H is G-cr over k, 2. $C_G(H)$ (or $\overline{C_G(H)(k_s)}$) is unipotent.

Corollary 6.3. *Let $k = k_s$. Let G be connected. Let H be a k-subgroup of G. If H is G-ir over k, then $\overline{C_G(H)(k)}$ is G-cr over k.*

Proof. By Proposition 2.3, there exists a proper k-parabolic subgroup of G containing H and $\overline{C_G(H)(k)}$. This is a contradiction since H is G-ir over k. □

Corollary 6.4. *Let $k = k_s$. Let G be connected. Let H be a (possibly non-k-defined) subgroup of G. If $\overline{C_G(H)(k)}$ is G-ir over k, then H is G-cr over k.*

Proof. The same proof as that of Corollary 6.3 works. □

Remark 6.5. In Corollary 6.4, we cannot replace $\overline{C_G(H)(k)}$ by $C_G(H)$ even if H is k-defined; if $C_G(H)$ is not k-defined, there might not be any proper k-parabolic subgroup containing H and $C_G(H)$. It would be interesting to know whether such examples exist.

By a similar argument to that in the proof of Corollary 6.3, we obtain:

Corollary 6.6. *Let $k = k_s$. Let G be connected. If a k-subgroup H of G is G-ir over k, then $N_G(H)$ and $\overline{N_G(H)(k)}$ are G-cr over k.*

Corollary 6.7. *Let $k = k_s$. Let G be connected. Let H be a (possibly non-k-defined) subgroup of G. If $\overline{N_G(H)(k)}$ is G-ir over k, then H is G-cr over k.*

It is natural to ask:

Open Problem 6.8. *Let k be a field. Suppose that a k-subgroup H of G is G-cr over k. Is $N_G(H)$ G-cr over k? Is $\overline{N_G(H)(k_s)}$ G-cr over k?*

Propositions 6.10 and 6.11 below show that if we allow H to be non-k-defined, the answer to Open Problem 1.4 is no. First, we need [3, Prop. 7.4]:

Lemma 6.9. *Let k be a field. If a k-subgroup H of G is separable in G, then $C_G(H)^\circ$ is k-defined.*

Recall that [5, Def. 3.27], a subgroup H of G is called *separable* if the scheme theoretic centralizer of H in G (in the sense of [12, Def. A.1.9]) is smooth. It is known that every subgroup of GL_n is separable [5, Ex. 3.28].

Proposition 6.10. *Let k be a nonperfect field of characteristic 3. Let $a \in k \backslash k^3$. Let $G = GL_4$. Then there exists a subgroup H of G such that H is G-cr over k but $C_G(H)$ is not G-cr over k.*

Proof. Let $h_1 = \begin{bmatrix} 0 & 0 & a & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a^{1/3} \end{bmatrix}$, $h_2 = \begin{bmatrix} 1 & a^{1/3} & 2a^{2/3} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Set $H := \langle h_1, h_2 \rangle$.

A simple matrix computation shows $C_G(H) = \left\{ \begin{bmatrix} s & 0 & 0 & x_1 \\ 0 & s & 0 & a^{-1/3}x_1 \\ 0 & 0 & s & a^{-2/3}x_1 \\ 0 & 0 & 0 & t \end{bmatrix} \in GL_4(\overline{k}) \middle| x_1, s, t \in \overline{k} \right\}$.

Note that a subgroup H of $G = GL_n(V)$ is G-cr over k if and only if H acts k-semisimply on V [25, Ex. 3.2.2(a)]. We find by a direct computation that H acts semisimply on a 4-dimensional k-vector space in the usual way with k-irreducible summands $V_1 := \begin{bmatrix} * \\ * \\ * \\ 0 \end{bmatrix}$ and $V_2 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ * \end{bmatrix}$.

Hence H is G-cr over k. It is clear that $C_G(H)$ is not G-cr over k since V_1 is a k-defined 3-dimensional $C_G(H)$-stable subspace with no $C_G(H)$ stable complement.

We show that H is not k-defined. First, we see that $C_G(H)^\circ(k_s)$ is not dense in $C_G(H)^\circ$, so $C_G(H)^\circ$ is not k-defined by [9, AG. 13.3]. We conclude that by Lemma 6.9, H cannot be k-defined since H is separable in G. $\square$

Proposition 6.11. *Let k be a nonperfect field of characteristic p. Let $a \in k \backslash k^p$. Let $G = GL_4$. Then there exists a subgroup H of G such that H is G-cr over k, but $\overline{C_G(H)(k_s)}$ is not G-cr over k.*

Proof. Let $h_1 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $h_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ a^{1/p} & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Define $H := \langle h_1, h_2 \rangle$. Then

$$C_G(H) = \left\{ \begin{bmatrix} s & 0 & 0 & 0 \\ 0 & t & a^{\frac{p-1}{p}}(s-t) & x_2 \\ 0 & s-t & (1-a^{\frac{p-1}{p}})s + a^{\frac{p-1}{p}}t & -x_2 \\ 0 & y_2 & a^{\frac{p-1}{p}}y_2 & w \end{bmatrix} \in GL_4(\overline{k}) \middle| x_2, y_2, s, t, w \in \overline{k} \right\}.$$

So, $\overline{C_G(H)(k_s)} = \left\{ \begin{bmatrix} s & 0 & 0 & 0 \\ 0 & s & 0 & x_2 \\ 0 & 0 & s & -x_2 \\ 0 & 0 & 0 & w \end{bmatrix} \in GL_4(\overline{k}) \middle| x_2, s, w \in \overline{k} \right\}$. A similar argument to that in the proof of Proposition 6.10 shows that H is G-cr over k and H is not k-defined. It is clear that $\overline{C_G(H)(k_s)}$ is not G-cr over k. $\square$

7 On the structure of the set of R-parabolic subgroups

Proof of Theorem 1.12. Let $\tilde{G} = SL_3$. Let $G = \tilde{G} \rtimes \langle \sigma \rangle$ where σ is the nontrivial graph automorphism group of $\tilde{G}$. Fix a k-split maximal torus T, and a k-Borel subgroup B of G containing T. Let α, β be the simple roots of G corresponding to T and B. Let n_α, n_β be the canonical reflections corresponding to α and β respectively. Let f be the automorphism of G such that $f(A) = (A^T)^{-1}$. Then we obtain

$$\sigma(P_i) = (n_\alpha n_\beta n_\alpha) f(P_i) (n_\alpha n_\beta n_\alpha)^{-1} \text{ for } i \in \{\alpha, \beta\}. \tag{7.1}$$

In particular, we have $\sigma(P_\alpha) = P_\beta$, $\sigma(P_\beta) = P_\alpha$. Let W_k be the Weyl group of G°. Then $W_k \cong S_3$. We list all canonical representatives of W_k: $1, n_\alpha, n_\beta, n_\alpha n_\beta, n_\beta n_\alpha, n_\alpha n_\beta n_\alpha$. Taking all W_k-conjugates of P_α and P_β, we obtain all proper maximal k-parabolic subgroups of G° containing T: P_α, P_β, $P_{-\alpha}$, $P_{-\beta}$, $n_\beta \cdot P_\alpha$, $n_\alpha \cdot P_\beta$. Using (7.1), we find that none of these proper maximal k-parabolic subgroups of G° is normalized by σ. So, by [5, Prop. 6.1] they are k-defined proper maximal R-parabolic subgroups of G containing T.

Now we look at k-Borel subgroups of G°. By taking all W_k-conjugates of B, we obtain all k-Borel subgroups of G° containing T: B, $n_\alpha \cdot B$, $n_\beta B$, $n_\alpha n_\beta \cdot B$, $n_\beta n_\alpha \cdot B$, $n_\alpha n_\beta n_\alpha \cdot B$.

Lemma 7.1. $C_{Y_k(T)}(\sigma) = a(\alpha^\vee + \beta^\vee)$, *where* $a \in \mathbb{Z}$.

Proof. Let $\lambda = x\alpha^\vee + y\beta^\vee$. Using (7.1), we obtain $\sigma(\lambda) = n_\alpha n_\beta n_\alpha \cdot (-x\alpha^\vee - y\beta^\vee) = y\alpha^\vee + x\beta^\vee$. Then, for $\lambda \in C_{Y_k(T)}(\sigma)$ we must have $x = y$. ☐

Lemma 7.2. *$H := \langle \sigma, B \rangle$ is a k-defined R-parabolic subgroup of G.*

Proof. Let $\lambda = \alpha^\vee + \beta^\vee$. By Lemma 7.1, we have $\sigma \in L_\lambda$. An easy calculation shows that $P_\lambda = H$. ☐

It is clear that H is not G-cr over k. Note that $\Lambda(G)^H = \{H\}$, and H is not a simplex in $\Delta(G) = \Delta(G^0)$. This gives the first part of the theorem. Consider the set of k-defined R-parabolic subgroups of G containing T. We have

$$B < B, B < P_\alpha, B < P_\beta, B < P_\lambda, B < G.$$

Thus the cardinality of the set of R-parabolic subgroups of G containing B is 5, which is not a power of 2. So $\Lambda(G)$ ordered by reverse inclusion is not a simplicial complex (in the sense of [30, Thm. 5.2]) since it cannot be isomorphic to the partially ordered set of subsets of some finite set; see Figure 1 where vertices (edges) correspond to k-defined maximal (minimal) R-parabolic subgroups of G containing T. ☐

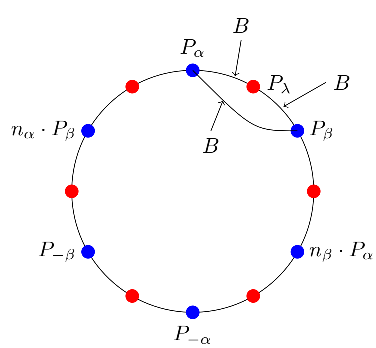

Figure 1: The set of R-parabolic subgroups of $G = SL_3$ containing T

Open Problem 7.3. *Let k be a field. Let G be non-connected. Suppose that a (possibly non-k-defined) subgroup H of G is not G-cr over k. If a k-subgroup N of G normalizes H, does there exist a k-defined proper R-parabolic subgroup of G containing H and N?*

8 The number of overgroups of G-ir subgroups

Recall that Proposition 1.13 depended on the following [19, Lem. 2.1]:

Lemma 8.1. *Let k, G, H be as in the hypotheses of Proposition 1.13. Then $C_G(H)$ is finite.*

However if k is nonperfect, we have

Proposition 8.2. *Let k be a nonperfect field of characteristic 2. Let $G = PGL_2$. Then there exists a connected k-subgroup H of G such that H is G-ir over k but $C_G(H)$ is infinite.*

Proof. Let $a \in k \backslash k^2$. Let $H := \left\{ \overline{\begin{bmatrix} x & ay \\ y & x \end{bmatrix}} \in PGL_2(\overline{k}) \mid x, y \in \overline{k} \right\}$. Then H is connected and G-ir over k; see Remark 5.3. We have $H = C_G(H)$, and $C_G(H)$ is infinite. $\square$

Proof of Theorem 1.14. Let $a \in k \backslash k^2$. Define
$H := \left\{ \overline{\begin{bmatrix} x & ay & az & aw \\ w & x & ay & az \\ z & w & x & ay \\ y & z & w & x \end{bmatrix}} \in PGL_4(\overline{k}) \middle| x, y, z, w \in \overline{k} \right\}$. Note that H is the centralizer of a k-anisotropic unipotent element $\overline{\begin{bmatrix} 0 & 0 & 0 & a \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}$ of $PGL_4(k)$. Then H is connected and G-ir over k. Note that H is 3-dimensional. Since $h^4 = 1$ for any $h \in H$, H is a unipotent group. So, for an appropriate element $g \in G(\overline{k})$, gHg^{-1} is a (possibly non-k-defined) subgroup of the 6-dimensional group U of upper unitriangular matrices of G. A computation by Magma shows that there exist some $g \in G(\overline{k})$ such that
$g \cdot H = \left\{ \overline{\begin{bmatrix} X & Y & Z & W \\ 0 & X & Y & Z \\ 0 & 0 & X & Y \\ 0 & 0 & 0 & X \end{bmatrix}} \in PGL_4(\overline{k}) \middle| X, Y, Z, W \in \overline{k} \right\}$. Now for each $b \in \overline{k}$, define
$H_b := \left\langle g \cdot H, \overline{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \right\rangle$. Then a quick computation shows that the groups H_b are distinct. So the groups $g^{-1} \cdot H_b$ are infinitely many overgroups of H of index 2. $\square$

Open Problem 8.3. *Let k be a field. Let G be semisimple algebraic group. Suppose that a k-subgroup H of G is connected and G-ir over k. Then does H have only finitely many k-defined overgroups?*

9 The number of conjugacy classes of unipotent elements

Let $k = \overline{k}$. Let G/k be conncected reductive. The following are known.

1. There are only finitely many conjugacy classes of unipotent elements in G [20, Thm. 13].
2. There is only a finite number c_N of G-conjugacy classes of G-cr subgroups of fixed order N [5, Cor. 3.8].

3. Moreover, if G is simple, there is a uniform bound on c_N that depends only on N and the type of G but not on k [15, Prop. 2.1].

Now let k be nonperfect, and let G/k be connected reductive. In this section, we show that the natural analogue of the above results 1, 2, 3 fail over a nonperfect k.

Proposition 9.1. *Let $\mathbb{F}_2$ be the finite field with 2 elements. Let $k := \mathbb{F}_2(x)$ be the field of rational functions in one variable over $\mathbb{F}_2$. Let $G = PGL_2$. Then there exist infinitely many $G(k)$-conjugacy classes of k-anisotropic unipotent elements in G.*

Proof. Let $p_n(x) = x^{2\cdot 3^n} + x^{3^n} + 1 \in k$ for $n \in \mathbb{N}$. Then each $p_n(x)$ is irreducible over $\mathbb{F}_2$ by [14, Ex. 3.96]. Let $u_n = \begin{bmatrix} 0 & p_n(x) \\ 1 & 0 \end{bmatrix}$. It is clear that u_n is a unipotent element of order 2. Let $U_n := \langle u_n \rangle$. Let U_n act on a 2-dimensional vector space V in the usual way. Since no eigenvalue of u_n belongs to k, there is no k-defined U_n-invariant subspace of V. Thus u_n is k-anisotropic.

Suppose that u_i is $PGL_2(k)$-conjugate to u_j for some $j \neq i$. Then there exist $m \in GL_2(k)$ and $d \in k$ such that $mu_jm^{-1} = \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix} u_i$. Taking determinants on both sides, we obtain $p_j(x) = d^2 p_i(x)$. Let $d := d_1/d_2$ where $d_1, d_2 \in \mathbb{F}_2[x]$ and d_1, d_2 have no nontrivial common factor in $\mathbb{F}_2[x]$. Then $p_j(x)d_2^2 = p_i(x)d_1^2$. So d_1^2 divides $p_j(x)$, but this is a impossible unless $d_1 = 1$ since $p_j(x)$ is irreducible. If $d_1 = 1$, we have $p_j(x)d_2^2 = p_i(x)$. Then $d_2 = 1$ by the same argument. This is a contradiction since $p_i(x) \neq p_j(x)$. $\square$

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