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A dynamical systems approach to understanding the interplay between delayed feedback and seasonal forcing in the El Niño Southern Oscillation

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Abstract

In contrast to very detailed highly computational forecasting models, conceptual climate models focus on fundamental relationships and can help us understand the roles of underlying processes of a climate system. We consider such a model, introduced by Tziperman et al., for the El Niño Southern Oscillation (ENSO) system, which takes the form of a delay differential equation (DDE). It describes the interactions of both a positive and a negative delayed feedback, created by an ocean-atmosphere coupling, as well as seasonal forcing. In the past, investigations into this model, and variations of it, have been conducted by conventional simulation methods only. This hindered and limited the exploration of more complicated behaviour, including multistabilities and transients.

In this thesis, we take the different approach of applying state-of-the-art continuation software for DDEs in conjunction with concepts from bifurcation theory. This allows us to deliver a comprehensive overview of the possible dynamical behaviour of the ENSO model across large ranges of relevant parameters. We begin by conducting a bifurcation analysis of a simpler ENSO model, without positive delayed feedback and without asymmetry in the ocean-atmosphere coupling, first studied by Ghil et al. We explain and expand on the existence of unstable solutions, multistability and chaos as found in previous publications and show that the organisation of resonance tongues in parameter space plays a vital role. We then transition to the full version of the ENSO model of Tziperman et al. by gradually introducing the positive delayed feedback and asymmetry in the coupling function and analysing their effects. The asymmetry in the coupling is found to be particularly important for reproducing characteristic features more realistically. We find that the chaotic behaviour seen in previous studies appears through the emergence of period-doubling cascades within overlapping resonance tongues. Different routes to chaos are also found depending on the path taken through parameter space. Furthermore, we find that the observed behaviour is sensitive to changes of the delay associated with the negative feedback, but not the positive feedback. Finally, we study fold bifurcations of invariant tori, where so-called Chenciner bubbles are known from theory to be involved in transforming a stable torus into a torus of saddle-type. We detect the corresponding bifurcation structure of Chenciner bubbles in the ENSO DDE model in agreement with theory. We then briefly discuss how this bifurcation phenomenon may be interpreted as a form of climate tipping.
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<th>Nature of contribution by PhD candidate</th>
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Certification by Co-Authors

The undersigned hereby certify that:

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Introduction

Every few years there is a world-wide flurry of media interest in the climate phenomenon known as El Niño. An El Niño event, such as the recent 2015–16 event, is characterised by unseasonably warm sea-surface temperatures of the Pacific Ocean off the equatorial South American coast. El Niño events have been well-known to humans for hundreds of years, as the water temperatures have a direct and negative impact on the quality of the fishing in those areas. Furthermore, El Niño is of relevance on a global scale: for example, it has been shown to trigger droughts in Australia and South-East Asia [2, 24, 77] and it also seems to be coupled with weather behaviour across the Indian Ocean [3] and even the Atlantic Ocean [35].

El Niño events have a large impact on people and environments, causing economic hardship and endangering biodiversity; for example, see [67, 72, 106]. There are also benefits in some locations: [22] reports that in the northern United States there were fewer lives lost to bad winter weather during the 1997–98 El Niño, and that there was an observed reduction in heating costs.

Figure 1.1(a) is a colour map showing sea-surface temperature (SST) anomalies during the 1997–98 El Niño event. It is clear that there is an extremely large anomaly of warmer water in the eastern equatorial Pacific off the coast of Peru. In 1969, Jacob Bjerknes first suggested that El Niño and the Southern Oscillation, which refers to surface air pressure fluctuations between the eastern and western tropical Pacific, are in fact the ocean and atmosphere components, respectively, of a coupled system [11], nowadays known as the El Niño Southern Oscillation or ENSO. Evidence for this coupling is seen in Fig. 1.1(b). The blue curve displays the NINO3 index, which is the average SST anomaly in the ocean region across 5°N–5°S and 150°W–90°W, and the red curve displays the Southern Oscillation Index (SOI), which is the normalized surface air pressure difference between Tahiti and Darwin, Australia; both time series are shown throughout the years 1964–2014. It is evident in panel (b) that the blue and red time series are generally in anti-phase synchronization, so that periods of high NINO3 index coincide with low SOI, and vice-versa. Large peaks in the NINO3 index correspond to El Niño events, the warm phase of ENSO, while large dips correspond to the
Figure 1.1: Illustration of El Niño. Panel (a) shows the sea-surface temperature (SST) anomalies (maximal departures from a long-term average) during the period November 1997 to January 1998; these are based on a 1981–2010 long-term average and are calculated using data from the OISSTv2 dataset. Image courtesy of the US National Oceanic and Atmospheric Administration (NOAA). Panel (b) shows the monthly NINO3 index and SOI deviations from 1964–2014 as blue and red curves, respectively; the NINO3 data is from NOAA and the SOI data is from the Climatic Research Unit, University of East Anglia.

cool phase known as La Niña. Note that the large peak in the blue time series at 1997–98 is the El Niño event depicted in panel (a).

Models that aim to either forecast or investigate the dynamics of the ENSO system attempt to reproduce, at least to some degree, the dynamics represented by the measured blue time series in Fig. 1.1(b). Therefore, we now briefly discuss some of its important properties. Firstly, there are many maxima in the time series that occur on a small, intra-seasonal time-scale, but it is only the large peaks in the blue curve that represent El Niño events. They generally occur every four to seven years in a sporadic fashion with significant variability in the strengths of the El Niño events. On a larger time-scale, interdecadal variability is seen in the time series. These features make it difficult to forecast El Niño events. However, although
1.1 The delayed action oscillator

In this thesis, we focus on the delayed action oscillator description of ENSO variability. This description begins with the Bjerknes hypothesis that a positive feedback exists between the Pacific Ocean and the atmosphere, which causes an internal instability capable of producing a large positive SST anomaly in the eastern equatorial Pacific; corresponding to the beginning of an El Niño event. The basic concept of the delayed action oscillator is that a delayed negative feedback also exists, which slows down and eventually reverses the effect of the positive feedback, bringing about the demise of an El Niño event.
1.1.1 Background on physical processes

We now give some details of properties of ENSO and a brief description of the physical processes that are relevant to the delayed action oscillator theory. For further details on the associated climate processes we refer to [26].

The thermocline is a relatively thin oceanic layer between the deep cold waters (below about 4°C) and the warmer well-mixed waters above (above about 13°C). The depth of this layer is different in different regions of the ocean; in the eastern Pacific Ocean it has an average depth of about 50m. Let \( h \) denote deviations from the mean thermocline depth at the eastern boundary. The quantity \( h \) is measured downwards from the surface, so an increase in \( h \) means an increase in the depth of the thermocline away from the ocean surface. The thermocline depth is often used as a proxy for the regional SST, since a deeper thermocline means less upwelling (i.e. vertical transport of colder waters towards the surface) and, hence, a higher SST. However, the exact relationship itself between the two is non-trivial and includes delays [112].

Figure 1.2 illustrates the interactions via the ocean-atmosphere coupling that influence the thermocline depth at the eastern Pacific. In equilibrium, the thermocline is typically deeper in the west of the Pacific Ocean and shallower in the east. An atmospheric convection loop exists above the equatorial ocean as a result of this difference, where hot air rises in the west and cold air sinks in the east, giving rise to the easterly trade winds (easterly meaning blowing from east to west). As shown by the arrows in the atmosphere component of Fig. 1.2, a positive perturbation in \( h \) slows down these winds, creating westerly wind anomalies (i.e. deviations from the mean) over the equatorial Pacific Ocean. These anomalies together with the effect of the so-called Ekman transport phenomenon\(^1\) cause surface water in the central part of the Pacific basin (where the ocean-atmosphere coupling is strongest) to move towards the equator, shown by the arrows of the ocean component in Fig. 1.2. This, in turn, induces two sets of equatorial waves, which are waves trapped near the equator by the Coriolis force. A surplus of warm surface water builds at the equator, which increases the depth of the thermocline. This positive perturbation of the thermocline travels eastward in the form of equatorial Kelvin waves. After a time delay of about a month the Kelvin wave arrives back at the eastern boundary to provide a positive feedback mechanism that increases \( h \) further. Simultaneously, a deficit of warmer surface water in the off-equatorial central Pacific Ocean decreases the depth of the thermocline, which decreases the SST due to more upwelling. This negative signal propagates westward and towards the equator as a so-called Rossby wave;

---

\(^1\)Because of the Coriolis force the surface flow generated by the wind is at 45° to the wind direction (to the left/right in the southern/northern hemisphere). However, dividing the body of water into thin layers, this angle shifts further for each deeper layer since the drag force is not from the wind itself but the layer above. The spiral form of flow shifting in direction and gradually becoming weaker for deeper layers is known as the Ekman spiral. Integrating over the Ekman spiral gives a net water transportation 90° to the left/right of the surface wind in the southern/northern hemisphere — this is known as Ekman transport.
1.1. The delayed action oscillator

Figure 1.2: The variable $h$ represents deviations from the mean thermocline depth at the eastern boundary of the equatorial Pacific Ocean. The coupling between ocean and atmosphere allows for the creation of negative and positive feedback mechanisms as indicated by the arrows (see text for details).

see Fig. 1.2. At the western boundary of the Pacific Ocean the Rossby wave is reflected and travels back eastward as a negative Kelvin wave (it is again the Coriolis force that allows Kelvin waves to only travel in an eastward direction). After a certain time delay, the negative signal finally arrives back at the eastern boundary of the Pacific Ocean, where it leads to a decrease of $h$. This process provides the negative feedback mechanism. The time needed for the disturbance in the thermocline to propagate westward as a Rossby and eastward as a Kelvin wave back to the eastern boundary of the ocean is about six months.

Besides the feedback mechanisms of the delayed action oscillator, the literature [21, 39, 50, 96] indicates that the seasonal forcing and subsequent resonances play an important role in describing the dynamics of ENSO. In fact, the very name ‘El Niño’, referring to the timing of the warming events around Christmas, suggests that resonance effects with the seasons are present.

### 1.1.2 Modelling the delayed action oscillator

The use of DDEs to model the delayed action oscillator began in the late 1980s. This led to a series of simple conceptual models that would reproduce increasingly more dynamical features seen in the ENSO system or in more sophisticated models. Battisti and Hirst derived a linear autonomous DDE [10] from a much more sophisticated nonlinear coupled ocean-atmosphere
ENSO model [111], such that it still preserves essential physical processes. It takes the form:

\[
\dot{T}(t) = T - bT(t - \tau)
\]  

(1.1)

with constants \( b > 0 \) and \( \tau > 0 \). The first and second terms represent a positive delayed feedback and a negative feedback, respectively. In Eq. (1.1), \( T \) is the average sea-surface temperature over the eastern equatorial Pacific, \( b \) is an amplification factor and \( \tau \) is the delay time associated with the negative feedback. The model can produce oscillatory solutions that allow it to act as a “remarkably good proxy” [10] to the more sophisticated model from which it is derived.

A nonlinear version of the simple model was introduced in [10], also simultaneously by Suarez and Schopf [92], and has the form:

\[
\dot{T}(t) = T - T^3 - bT(t - \tau)
\]  

(1.2)

with \( 0 < b < 1 \) and \( \tau > 0 \). The nonlinearity is shown in [10, 92] to influence both the amplitude and frequency of the model solutions.

These early attempts at modelling the ENSO system with DDEs provided a simple explanation for the oscillatory nature of the system. However, these models generally produce periodic solutions, which is not a realistic representation of ENSO behaviour. Furthermore, they offer no insight into the phase locking that is apparent between the observed ENSO dynamics and the seasons.

In 1994, Tziperman, Stone, Cane and Jarosh [98] introduced a DDE model of the form:

\[
\dot{h}(t) = aA(h(t - \tau_p)) - bA(h(t - \tau_n)) + c \cos(2\pi t),
\]

(1.3)

where

\[
A(h) = \begin{cases} 
  d_u \tanh(\frac{\kappa}{d_u} h) & \text{if } h \geq 0, \\
  d_l \tanh(\frac{\kappa}{d_l} h) & \text{if } h < 0
\end{cases}
\]

is the ocean-atmosphere coupling function. We will refer to Eq. (1.3) as the TSCJ model. It describes the change of the thermocline depth \( h \) at the eastern boundary of the Pacific Ocean (more specifically, its deviation from the annual mean), which depends on time measured in years. Since, as mentioned earlier, the thermocline depth can be considered a proxy for the SST, a large increase in \( h \) relates to a warming of the eastern equatorial Pacific that is characteristic of an El Niño event. The first and second terms of Eq. (1.3) represent the positive and negative delayed feedback mechanisms, respectively, that are described in Section 1.1.1. The third term of Eq. (1.3) reflects the periodic forcing effect of the annual cycle of the seasons with a period of one year.

The parameters \( a, b \) and \( c \) of Eq. (1.3) are the amplification factors of positive feedback, negative feedback and seasonal forcing, respectively. Further, \( \tau_p \) is the delay time for the
positive feedback mechanism and \( \tau_p \) that of the negative one; in particular, \( \tau_p \) and \( \tau_n \) represent the time taken by the oceanic waves to travel across the Pacific Ocean to close the respective feedback loop. The function \( A(h) \) specifies the ocean-atmosphere coupling, where \( d_u > 0 \) and \( d_l < 0 \) are the upper and lower horizontal asymptotes, respectively, and \( \kappa \) is the coupling strength between the ocean and atmosphere, represented by the slope of \( A(h) \) at \( h = 0 \); see Fig. 1.3. This form of the ocean-atmosphere coupling function \( A(h) \) was used in [98]. It is based on a form justified in [66], where a sigmoid nonlinearity was found to be more realistic than a cubic nonlinearity. As explained in [66], the effects of the ocean-atmosphere coupling appear asymmetric in the real world, so that \(|d_l| \neq |d_u|\). This is because a positive perturbation in the thermocline can move the Pacific intertropical convergence zone (ITCZ)\(^2\) towards the equator, reducing the easterlies in the central Pacific. However, a negative thermocline perturbation has a weaker effect, translating to \(|d_l| < |d_u|\) in the TSCJ model.

For \( a = c = 0 \) in Eq. (1.3), we are left with a scalar DDE that has been studied analytically in the past. Of specific interest here is the result that for \( \tau_n \) above the critical delay time \( \tau_c = \pi/(2\kappa) \) the zero solution \( h \equiv 0 \) loses stability in a Hopf bifurcation. For \( \tau_n \geq \tau_c \) there exists a set of stable periodic solutions of period \( T = 4\tau \) [20, 25, 70]. This means that self-sustained oscillations exist due to the delayed feedback alone. Therefore, the addition of periodic forcing introduces a second frequency; this implies the possibility of dynamics on an invariant torus, which may be locked or unlocked depending on the frequencies involved. If the two frequencies of the observed dynamics have an irrational ratio then any trajectory will not close and the solution is quasi-periodic. If the two frequencies have a rational ratio then

\(^2\)The ITCZ, known as the “doldrums” to sailors, is the area where the north-east and south-east trade winds converge.
there is a pair of periodic orbits (one stable and one unstable) on the torus with a finite period; one speaks of locked dynamics. The regions in the parameter space where the dynamics on the torus is locked are known as resonance tongues; these are a well-studied phenomenon (for example, see [41, 59, 65, 87]) and they are discussed in the context of Eq. (1.3) throughout the thesis.

The TSCJ model is significant because in [98] Tziperman et al. demonstrated that the irregularity that is characteristic of ENSO could be reproduced as chaotic behaviour of this simple DDE. This is in contrast to another school of thought that suggests that such irregularity is driven by noise, in particular, by local weather in the form of small-scale, high-frequency stochastic forcing; for example, see [69].

A simplified version of the TSCJ model, or rather a special case of the TSCJ model, which we will refer to as the GZT model, was introduced by Ghil, Zaliapin and Thompson in [40]. This simpler model considers only the interactions of the negative delayed feedback with the seasonal forcing. More precisely, it is the TSCJ model without the positive feedback mechanism and without the asymmetry in the ocean-atmosphere coupling function; that is, when $|d_u| = |d_l|$ in Eq. (1.3). A follow-up investigation to [40] was presented in [110]. The GZT model is given by Eq. (1.3) when letting $a = 0$, $d_u = 1$ and $d_l = -1$, which is:

$$\dot{h}(t) = -b A(h(t - \tau_n)) + c \cos(2\pi t)$$

$$= -b \tanh(\kappa h(t - \tau_n)) + c \cos(2\pi t).$$

The authors of [40, 110] calculated so-called maximum maps, where the maxima of simulated solutions for a single fixed initial condition are illustrated as a function of two parameters, these are discussed in detail later. These maximum maps showed that the GZT model has very complicated dynamics, despite its relatively simple form; moreover, it does reproduce certain important ENSO features, including intraseasonal oscillations, interdecadal variability, frequency locking while changing parameters (observed as a Devil’s staircase), as well as phase locking with the seasonal forcing so that extrema occur at a certain time of the year.

The investigations of the GZT model in [40, 110] and the TSCJ model in [98] were conducted with simulations by numerical integration of the respective DDEs. Only a very small and specific range of parameter values was considered in [98]. Generally, it is difficult to have an estimate or even a physical interpretation for some parameters in conceptual models, so it is necessary to gain insight into the behaviour of the model across a much larger range of its parameter space. However, conventional numerical integration to explore the dynamics across a large range of parameters is impractical, because one has to deal with multistabilities, transients and many possible initial conditions. These factors can also make it challenging to correctly interpret the observed dynamics or understand how typical certain stable solutions are for the model and which mechanisms in the model are responsible for their existence.
1.2 Numerical methods for DDEs

The analysis of DDEs is generally quite difficult. Formally, the TSCJ model is a DDE of the form

\[ \dot{x}(t) = f(t, x(t - \tau_p), x(t - \tau_n), \mu) \]  

(1.5)

with \( \tau_p < \tau_n \) and where \( x \in \mathbb{R}^N \) and \( \mu \in \mathbb{R}^M \) consists of \( M \) parameters. The model is described by the function

\[ f : \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^M \rightarrow \mathbb{R}^N. \]  

(1.6)

For our purposes, \( N = 1 \), \( M = 8 \) and \( t \) is present in the form of seasonal forcing, with time measured in years, so that \( f(t, \cdot, \cdot, \cdot) = f(t + 1, \cdot, \cdot, \cdot) \). Because \( f \) depends explicitly on \( t \), Eq. (1.5) is non-autonomous. Furthermore, in contrast to ordinary differential equations, the phase space of Eq. (1.5) is not \( \mathbb{R}^N \), but the space \( C([-\tau_n, 0]; \mathbb{R}^N) \times \mathbb{R} \), where \( C([-\tau_n, 0]; \mathbb{R}^N) \) is the infinite-dimensional space of continuous functions over the delay interval \([-\tau_n, 0]\) with values in \( \mathbb{R}^N \), and \( t \in \mathbb{R} \) represents time. Practically, this means that in order to define an initial-value problem, a DDE, such as Eq. (1.3) with two fixed delays (or any number of fixed delays), requires a whole function segment over the time interval \([-\tau_n, 0]\) as an initial condition, known as an initial history. A popular choice of initial history is a constant value across the whole history interval, which is, however, only a one-dimensional subspace of the function space. When running numerical computations, the continuous function chosen as the initial history is discretised onto a finite mesh. Therefore, the discretised phase space is no longer infinite-dimensional, but generally still of a very high dimension. For details on the general theory of DDEs, see, for example, [32, 44, 89, 90].

In this thesis, we utilise state-of-the-art continuation software, specifically the package DDE-Biftool [36, 86], to investigate the Eq. (1.3) and its special case Eq. (1.4). Because of the periodic forcing in Eq. (1.3), there are no equilibria and the simplest invariant solutions are periodic. For our purposes, the basic functionality of DDE-Biftool is that it numerically continues (or tracks) periodic solutions of Eq. (1.3) while certain parameters are varied. The software can then calculate their Floquet multipliers to give the stability of the periodic solutions, which is used to identify bifurcations. The bifurcation types of interest are those of periodic orbits: we encounter saddle-node bifurcations of periodic orbits, period-doubling bifurcations and torus (or Neimark-Sacker) bifurcations. These codimension-one bifurcations can be continued numerically as curves in two-dimensional parameter space by fixing constraints on the Floquet multipliers of solutions. Details about the numerical methods implemented in DDE-Biftool can be found in [36, 80].

Equation (1.3) is a non-autonomous DDE because of the periodic forcing. However, DDE-Biftool accommodates only autonomous DDEs, so it is necessary to rewrite Eq. (1.3) in an autonomous form. This can be done by increasing the dimension of the system by one by letting \( t \equiv z \in \mathbb{S}^1 \) and \( dz/dt = 1 \). To this end, we introduce two additional variables, \( y_1 \) and
$y_2$, to implement the periodic forcing as:

$$
\dot{h}(t) = aA(h(t - \tau_p)) - bA(h(t - \tau_n)) + cy_1(t),
$$

$$
\dot{y}_1(t) = \lambda y_1(t) - \omega y_2(t) - y_1(t)(y_1^2(t) + y_2^2(t)),
$$

$$
\dot{y}_2(t) = \omega y_1(t) + \lambda y_2(t) - y_2(t)(y_1^2(t) + y_2^2(t)).
$$

(1.7)

Here $\lambda = 1$ and $\omega = 2\pi$ to obtain a stable periodic orbit on the unit circle in the $(y_1, y_2)$-plane with a period of one, which drives the seasonal forcing of the TSCJ model. This provides us with the required seasonal forcing in an autonomous form that embeds $\mathbb{S}^1$ into $\mathbb{R}^2$, which can be implemented in DDE-Biftool.

### 1.3 Motivation and goals

A primary goal of this thesis is to demonstrate the usefulness of a dynamical systems approach for the study of climate models. We apply concepts of bifurcation theory and state-of-the-art numerical methods to conceptual climate models; more generally, we wish to help bridge the gap between applied dynamical systems and climate science. We choose to focus on the ENSO system, which is a well studied system, especially since the dramatic 1982–83 El Niño event [101], and there exists a comprehensive hierarchy of models, including conceptual DDE models. A secondary goal of this thesis is to evaluate the suitability of DDEs as conceptual climate models. Specifically, we investigate the DDE models (1.3) and (1.4). These models have been studied in the past simply by simulation, which offers only a glimpse into possible model behaviour and often produces results that, as we will see later, can be misinterpreted.

By conducting a bifurcation analysis, we can effectively divide the parameter space into regions of different solution types and efficiently gain an overview of possible model behaviour in dependence on the model parameters. This is achieved by means of dedicated continuation software and reveals boundaries for the existence and stability of certain solution types. It is valuable and interesting to discover how different dynamical behaviour is organised in parameter space. We are then able to, first, understand why certain dynamics have been observed in past studies of these ENSO models, second, investigate how certain dynamical properties of observed behaviour depends on certain features of the model and, third, detect new phenomena that occur due to the interacting delayed feedback and seasonal forcing. Finding stability boundaries also serves as a sensitivity analysis, providing information on how sensitive the behaviour is to changes in parameters. Sensitivity analysis is especially valuable for conceptual models, since there are often large parameter uncertainties compared to more sophisticated models with clearly defined values of the parameters.
1.4 Outline of thesis

The thesis is organised as follows. Chapter 2 contains results from the investigation into the simpler GZT model, given by Eq. (1.4) and focusing on the interplay between the negative delayed feedback and seasonal forcing. We aim to explain observations that were made in [40, 110] and to give a comprehensive overview of what dynamics may be observed in this model. We begin in section 2.1 by presenting examples of stable solutions calculated by simulations and then show that there are bistabilities by considering one-parameter bifurcation diagrams for the seasonal forcing strength parameter. In order to address open questions left by the investigations of [40, 110], we focus our study on the influence of the delay time and the seasonal forcing strength. We calculate so-called maximum maps, which plot the maximum of attractors as a function of two parameters by simulating across a range of fixed delay times and gradually increasing or decreasing the seasonal forcing strength without resetting the initial history. Comparing maximum maps of different scanning directions, in this case, for increasing and decreasing seasonal forcing strength, reveals parameter regions of bistability of significant size in the GZT model. Similar maximum maps had been calculated in [40] for the GZT model, but with a fixed constant initial history, which showed sharp interfaces (or jumps) in the maxima of the observed solutions. In section 2.2 both the sharp interfaces and bistabilities are explained by conducting a bifurcation analysis with the continuation software DDE-Biftool, which involves the calculation of bifurcation curves of periodic orbits; in particular, saddle-node, period-doubling and torus bifurcations. Overall, in agreement with general theory [59], our bifurcation analysis describes the dynamics across the parameter plane as organised into resonance tongues and curves of quasi-periodic solutions that bridge two curves of torus bifurcations. Resonance tongues are found to be a prominent feature in the bifurcation diagrams and they are responsible for a high degree of multistability in the model. We uncover surprisingly complicated behaviour involving the interplay between seasonal forcing and delay-induced dynamics. We take a closer look at a parameter set that was suggested in [110] to harbour chaotic behaviour. Our bifurcation analysis clarifies that the observed behaviour is actually the result of quasi-periodic (or high-period) solutions. We find a generic bistability that exists in certain resonance tongues due to a symmetry property of the governing model equation then discuss the role of the changing criticality of the torus bifurcation curve in understanding the observed dynamics. Not all sharp interfaces seen in the maximum maps that are described by bifurcation curves of periodic orbits. Therefore, we investigate co-existing stable tori in section 2.3. We show that these tori are connected by quasi-periodic fold bifurcations and that these can be approximated by considering the folding of nearby resonance tongues. So-called Chenciner bubbles are known to form inside the resonance tongues near where they fold [8], which explains why these particular sharp interfaces in the maximum maps appear to be non-smooth. Finally, we demonstrate that there
are small parameter regions where some resonance tongues ‘bunch up’ to such an extent that the resulting complicated dynamics becomes chaotic. Chaotic behaviour is therefore possible in the GZT model, but it is only found in small pockets of parameter space.

In chapter 3 we aim to understand, firstly, how the behaviour of the model changes when the simpler GZT model is transformed into the full TSCJ model given by Eq. (1.3). Secondly, we elucidate the origins of the chaotic behaviour observed in [98] and determine how typical such behaviour is in the TSCJ model. We begin with the GZT model and transition to the full TSCJ model in section 3.1 by gradually introducing the additional model features — positive delayed feedback and asymmetry in the coupling function. Regarding which additional model feature is introduced first, we consider both options. During these transitions, we conduct bifurcation analyses in relevant parameters to see the individual and combined effects of these model features. Generally, as the additional model features are introduced, there is an increase in the overlapping of resonance tongues, which results in the appearance of cascades of period-doubling bifurcations in the physically relevant region of parameter space. We find that the asymmetry in the coupling alone leads the resonance tongues to overlap to a greater extent than the positive delayed feedback alone. At each step of the transformation we also present some exemplary stable solutions. When interpreting these solutions in terms of the ENSO system, we find that the model behaviour does indeed become more realistic across large parameter regions. In section 3.2 we conduct a bifurcation analysis of the full TSCJ model. We review the chaotic transition observed in [98] as the ocean-atmosphere coupling strength is increased and determine that it is, as suggested in [98], due to overlapping resonances. More specifically, the chaotic behaviour is due to period-doubling cascades that develop inside the overlapping resonance tongue and not due to erratic jumping between resonances. The transition to chaos is presented as a bifurcation diagram in the parameter of the coupling strength. Moreover, we find other routes to chaos, including an intermittency route, depending on the parameter path taken. By calculating bifurcation sets in the two-parameter plane of coupling strength and seasonal forcing strength, we show that these transitions to chaos are robust against the seasonal forcing strength. These results demonstrate that chaotic behaviour is a prominent feature of the TSCJ model. Finally, we address parameter uncertainty and show how sensitive the model behaviour is to changes of the delay times within a range of realistic values. We find that its behaviour is sensitive to the delay of the negative feedback loop, but not to that of the positive.

Chapter 4 is a detailed study of quasi-periodic fold bifurcations of tori and their potential interpretation as climate tipping. In contrast to bifurcations of equilibria or periodic orbits, bifurcations of tori have not been considered in the past in the context of climate tipping. This is despite the fact that many climate systems involve forcing of multiple frequencies, such that the resulting dynamics may exist on tori. Compared to the case of equilibria or periodic orbits, folds of tori do not exist as such, but rather are associated with very complicated dynamics. In
section 4.1 we begin by reviewing what is known theoretically about the minimal bifurcation structure inside a Chenciner bubble as presented in [8], which allows a stable torus to lose smoothness and transform into a torus of saddle-type, and vice-versa. In section 4.2 we revisit the bifurcations of tori found in the GZT model in chapter 2. For this DDE we determine the complicated bifurcation structure within the Chenciner bubble near the fold point of the 2:7 resonance tongue. Some of the features of the Chenciner bubbles are difficult to detect and cannot simply be continued with DDE-Biftool. We provide details for how we determine the criticality of the torus bifurcation curve and how we detect the bifurcations that involve tori. We compare our findings with the theoretical bifurcation structure suggested in [8], and they agree very well; the only difference is that the torus bifurcation is subcritical in the ENSO DDE model and not supercritical as assumed in [8]. We finish with a discussion in section 4.3 of the role that the additional bifurcation structure could play when interpreting the quasi-periodic fold bifurcation as a kind of tipping event.

Chapter 5 concludes the thesis with a general discussion of the main results and presents suggestions for future work.
2

Negative delayed feedback and seasonal forcing

The results in this chapter have appeared as [53].

In this chapter, we consider a simple conceptual model for the El Niño Southern Oscillation (ENSO) system and conduct a bifurcation analysis of the model. This allows us to provide an overview of possible dynamical behaviour, as well as explain observations that were made in [40, 110] with conventional simulation methods only. In order to keep this chapter self-contained, we now briefly recall the model from section 1.1.2. The model is a scalar delay differential equation (DDE); it was introduced by Tziperman, Stone, Cane and Jarosh [98] and then simplified by Ghil, Zaliapin and Thompson [40] to a form that focuses only on the interplay between the negative feedback via ocean-atmosphere coupling and the seasonal forcing. We first investigate the simplified model of Ghil et al. in this chapter, before continuing to the more detailed model of Tziperman et al. in chapter 3. The DDE model of Ghil et al., which we refer to as the GZT model, takes the form

\[
\dot{h}(t) = -b \tanh \left( \kappa h(t - \tau_n) \right) + c \cos (2\pi t).
\]  

(2.1)

It describes the evolution of the thermocline depth \( h \) at the eastern boundary of the Pacific Ocean (more specifically, its deviation from the annual mean) as a function of time measured in years. The first term of Eq. (2.1) is a nonlinear delayed negative feedback. Parameters \( b \) and \( \kappa \) represent the negative feedback amplification factor and the ocean-atmosphere coupling strength, respectively, and \( \tau_n \) is the delay time in years needed for the propagation of oceanic waves across the Pacific Ocean that form the negative feedback mechanism. The second term of Eq. (2.1) is periodic seasonal forcing with a period of one year to reflect the annual cycle of the seasons, where \( c \) is the forcing amplitude. Further details about the model are provided in chapter 1.
In [40, 110] it was shown that the two mechanisms of ocean-atmosphere feedback and seasonal forcing are both essential for ENSO variability and sufficient for creating rich behaviour, and for mimicking important features seen in real-world observations and more sophisticated high-end models. However, due to the limitations of their techniques, it was not entirely clear why certain observations were made and how the observed dynamics for different parameters relate to each other and to possible co-existing stable solutions.

Here, we take a dynamical systems point of view and conduct a bifurcation analysis of Eq. (2.1), where the technique of investigation includes the use of the state-of-the-art continuation software DDE-Biftool [36, 86]. Throughout this chapter we focus our investigation on the \((c, \tau_n)\)-plane, in order to explain features of the model observed in [40, 110]. For the same reasons, we set the parameters \(b = 1\) and \(\kappa = 11\), which are the values that were used and justified in previous investigations [40, 110]. More generally, due to uncertainties and ambiguities in how these parameters relate to the observable world, we are interested in the model sensitivity to changes in these parameters.

We begin this work by presenting some examples of stable periodic solutions that show evidence of multistability in model (2.1). We then illustrate bistability clearly in the form of one-parameter bifurcation diagrams, obtained by tracking solutions for both increasing and decreasing parameter \(c\), while using the previous solution as an initial condition history. This allows us to calculate maximum maps in the \((c, \tau_n)\)-plane for both increasing and decreasing \(c\) to map out regions of bistability. The maximum maps presented in [40] used a single fixed initial condition and, as such, did not show the parameter regions where bistability is present. The maximum maps that we show here reveal (as noted in [40]) sharp interfaces that represent rapid transitions (or jumps) in the observed maxima for varying parameters. We then overlay the maximum maps with bifurcation curves calculated with DDE-Biftool. Overall, our analysis and general theory [59] describe a parameter plane divided by curves of torus bifurcations, which are bridged by an infinite number of resonance tongues and smooth curves of quasi-periodic solutions. The computed bifurcation curves agree well with the sharp interfaces seen in the maximum maps and allow for a detailed interpretation of numerical simulations. We compare our bifurcation curves with simulation results from [110] to show that the dynamics for a parameter set that was believed to be chaotic actually consists of quasi-periodic (or high-period) solutions. The bifurcation analysis reveals resonance tongues as a prominent feature in the parameter plane. We discuss and demonstrate the role they play for multistability; in particular, we identify a symmetry property in Eq. (2.1) to be a source of bistability within \(p:q\) resonance tongues of even \(p\) or \(q\). Some of the sharp interfaces in the maximum maps cannot be explained by the bifurcation curves calculated with DDE-Biftool, which leads us to a discussion about the changing criticality along a curve of torus bifurcations and then a detailed study of bifurcations of invariant tori in the system. We provide evidence for the presence of fold bifurcations of tori and associated resonance tongues. This includes
bifurcations of quasi-periodic solutions that differ from bifurcations of periodic orbits, since the two invariant tori involved mutually destroy each other before they reach the fold locus. We show how two such bifurcations can be connected by a branch of solutions on an invariant torus of saddle-type by locating resonant solutions along the branch.

The chapter is organised as follows. Section 2.1 contains results obtained by numerical integration of Eq. (2.1), including time series of sample periodic solutions in section 2.1.1, one-parameter bifurcation diagrams in section 2.1.2 and maximum maps in section 2.1.3. The overall bifurcation set in the \((c, \tau)\)-plane is presented in section 2.2. In section 2.2.1 we focus on the role of resonance tongues for the multistability of the system. Further results concerning bistability within certain resonance tongues are presented in section 2.2.2. Section 2.2.3 addresses the changing criticality of a prominent torus bifurcation in the \((c, \tau)\)-plane and how it affects numerical observations. The properties of tori and their bifurcations are discussed in section 2.3 with a focus on how the bifurcations involving tori relate to resonance tongues and Chenciner bubbles in section 2.3.1. Finally, in section 2.4 we draw some conclusions regarding the GZT model.

2.1 Stable solutions and maximum maps

Numerical integration of the GZT model offers some initial insight into the behaviour of the system. A time series is the most intuitive way of representing solutions in the context of the observable ENSO system: maxima represent warming El Niño events and minima cooling La Niña events. The strength of the event is indicated by the magnitude of the maximum or minimum. Concerning the dynamics in a more abstract sense, phase space projections can give us an idea of the shape of the attractor and are useful for identifying attractor types, as we will see in the examples that follow. We follow the common choice of projection onto the \((h(t - \tau), h(t))\)-plane or the \((h(t - \tau), h(t - \tau/2), h(t))\)-space.

2.1.1 Time series and phase space diagrams

Figure 2.1 shows five examples of stable solutions. They are obtained by numerical integration, for which we use the Euler method. Indeed, one can use higher-order methods for the integration of DDEs, such as Matlab’s dde23 or radar5, but we opted for the Euler method because this order-one method is known to be numerically stable. Moreover, it is fast because the fixed stepsize in time avoids the need to interpolate the history function at each time step when obtaining the delayed variable values. It is also easily parallelised across high-performance computing clusters, which is important when calculating attractors on a very large mesh of parameter values.

Fixed initial conditions of either \(h \equiv 0\) (for rows (a), (b) and (d)) or \(h \equiv 1\) (for rows (c)
Figure 2.1: Stable solutions of Eq. (2.1) obtained by numerical integration after discarding sufficiently long transients, shown as time series in panels (a1)–(e1) and as projections onto the \((h(t - \tau_n), h(t))\)-plane in panels (a2)–(e2); throughout \(b = 1, \kappa = 11\) and \(\tau_n = 1.2, c = 0\) for (a), \(\tau_n = 1.2, c = 3\) for (b) and (c), and \(\tau_n = 0.62, c = 3\) for (d) and (e).
and (e)) are used. All solutions, including those in later sections, are excluding transients, i.e. the solutions shown are the trajectories after they have had sufficient time (up to hundreds of years, although mostly 30-40 years is adequate) to approach and reach a stable attractor. Panels (a1)–(e1) are examples of solutions shown as time series; they are represented in two-dimensional projections of the phase space in panels (a2)–(e2).

Row (a) displays the solution for \( c = 0 \) and \( \tau_n = 1.2 \). As seen in panel (a1), the solution is periodic with an almost zigzag form and a period \( T = 4\tau = 4.8 \) years. The phase space projection onto the \((h(t - \tau_n), h(t))\)-plane in panel (a2) shows the solution as a closed loop. An interpretation of this solution in row (a) in the context of the El Niño phenomenon yields a case where there is no seasonal forcing and the oceanic waves that produce the negative feedback mechanism take 1.2 years to reach the eastern boundary of the Pacific (cf. section 1.1). The El Niño event then occurs every 4.8 years.

By contrast, row (b) displays a solution for \( \tau_n = 1.2 \), but for \( c = 3 \), where we see a periodic solution of period \( T = 1 \) with what appears to be a sinusoidal form. A closed loop is seen in panel (b2). In this case, where the dynamics is influenced by both the internal feedback mechanism and the seasonal forcing, the solution is dominated by the seasonal forcing, which is why it has a period of one.

Row (c) is for the same parameter values as row (b), but with a different initial history. Panel (c1) reveals a different solution from that of panel (b1): a periodic solution of period \( T = 5 \). This solution has a more complicated trajectory in the phase space projection in panel (c2) compared to the previous example in panel (b2). Note that the self-intersections seen in panels (c2)–(e2) are a result of the projection of the trajectory onto two dimensions. An interpretation of the example shown in row (c) would indicate multiple El Niño events of varying strength, with the largest occurring every 5 years. This time series is a case where two frequencies (from the delayed feedback and the seasonal forcing) have a clear influence.

Row (d) for \( c = 3 \) and \( \tau_n = 0.62 \) gives an example where the stable solution is quasi-periodic (or of a very high period). The quasi-periodic behaviour can be seen particularly well in the phase space projection in panel (d2), where the trajectory over 50 years traces out a torus. When interpreting the parameters compared to the last example, the strength of the seasonal forcing is the same, but the oceanic waves now travel across the Pacific in just above half the time.

The final example in row (e) is calculated for the same parameters as row (d), but with a different initial history, which results in a periodic solution of period \( T = 3 \). As in the last two examples, both time series and phase space projection shows the influence of two frequencies on the dynamics.

Comparing rows (d) and (e), we find a case of bistability between a quasi-periodic and a periodic solution for the same values of the parameters. Hence, the solution that the system converges towards depends on the initial history. Similarly, bistability is seen when
comparing rows (b) and (c). This clearly shows that there are regions in the \((c, \tau_n)\)-plane where bistability (possibly multistability) exists.

### 2.1.2 One-parameter bifurcation diagrams

To further investigate the bistabilities observed in Figs. 2.1(b)–(c) and Figs. 2.1(d)–(e), we calculate one-parameter bifurcation diagrams. Figure 2.2 shows the overall maxima of solutions for a range of \(c\) values for \(\tau_n = 1.2\) in panel (a) and for \(\tau_n = 0.62\) in panel (b). A maximum is simply taken as the largest value from the time series; because the solutions may not necessarily be periodic, the length of time from which the maximum is obtained must be sufficiently long (it was typically about 100 years). The diagrams in Fig. 2.2 show maxima of solutions calculated while scanning both up and down in the parameter \(c\). This is done by setting the initial history used to calculate a solution as the previous solution (i.e. that of a slightly lower or higher value of \(c\), depending on the direction that \(c\) is being changed). The black arrows indicate the direction in which \(c\) is changed and where there are jumps (or rapid transitions) in the maxima obtained.

In both Figs. 2.2(a) and (b) there exist an upper and a lower branch of maxima when increasing and decreasing \(c\), respectively. In both cases we see an overlapping range of \(c\) values for which stable solutions from both branches exist, yielding the hysteresis loops indicated by the black vertical arrows in Fig. 2.2.

In Figs. 2.2(a) and (b) the upper branches originate from solutions dominated by the internal feedback mechanism for low values of \(c\). On the lower branches one sees maxima of solutions that are initially dominated by the seasonal forcing for large values of \(c\).
these branches overlap there is bistability between them. Note that the solution seen for 
\((c, \tau_n) = (3, 1.2)\) in Fig. 2.1(b) with period \(T = 1\) lies on the lower branch of Fig. 2.2(a), while
the solution seen in Fig. 2.1(c) with period \(T = 5\) lies on the upper branch. Similarly, the
quasi-periodic and periodic of \(T = 3\) solutions seen in Figs. 2.1(d) and (e) are found on the
lower and upper branch of Fig. 2.2(b), respectively.

The graph of \(\max(h(t))\) in Fig. 2.2 features a number of small sudden dips, colloquially
referred to as ‘kinks’. Such kinks represent locked tori for the following reason. In contrast
to an unlocked quasi-periodic solution, a locked periodic solution does not cover the entire
torus and, therefore, will generally not attain the overall maximum on the torus. Hence,
these kinks reveal the location of resonance tongues. For example, the small kink seen in
Fig. 2.2(b) represents the solution as it passes through a sufficiently large resonance tongue
at \(c \approx 0.3\). Such kinks are a feature of numerical simulation when the maximum of the
solution is plotted. To understand how the associated resonance tongues are organised, we
will employ numerical continuation in section 2.2.

2.1.3 Maximum maps

A maximum map plots the maximum of attractors as a function of two parameters, where
the maximum of each solution \(\max(h(t))\) is displayed according to a colour scheme. This
provides a quick overview of some features of the dynamics. As mentioned in section 1.1.2,
maximum maps in the \((c, \tau_n)\)-plane were calculated in [40] for a single fixed initial history. In
Figs. 2.3(a) and (b), we instead show two maximum maps where, for each row of fixed delay
\(\tau_n\), the parameter \(c\) is scanned up and down (using previous solutions as initial histories in
the same fashion as for Fig. 2.2), as is indicated by the arrows. Rather than starting the
simulations from many different initial histories to detect bistabilities, scanning up and down
in \(c\) offers a convenient and systematic approach. It means that the simulation stays on a
branch of solutions while \(c\) is slowly varied (cf. upper and lower branches of Figs. 2.2) until
there is a jump to a different branch when certain bifurcations are encountered. This scanning
approach is in the spirit of the continuation method used in section 2.2 and onwards.

Note that the calculation of such a maximum map is very time consuming. For example,
the maximum map in Fig. 2.3(a) involves running simulations of the GZT model across 4000
different values of \(c\), each for about 500 years to remove all transients and then another 500
years to analyse the maximum value of \(h(t)\). Such simulations take about 12 hours to run
and, in the case of the maximum map in Fig. 2.3, are parallelised for 4000 different values of
fixed \(\tau_n\) using the computing cluster of the New Zealand eScience Infrastructure.

In both panels of Fig. 2.3 one can identify two regimes — one in the upper-left and one in
the lower-right of the \((c, \tau_n)\)-plane. They are divided by a sharp interface that runs from the
bottom-left corner to the upper-right corner. There are also elongated shapes, particularly
in the upper-left corner of the plane. The sharp interfaces that form these structures and the
Figure 2.3: Maximum maps displaying the maximum value of $h(t)$ according to the colour scheme as $c$ is increased (a) and decreased (b); here $b = 1$ and $\kappa = 11$.

The dividing curve represent where there are rapid transitions in $\max(h(t))$ (for example, those seen in Fig. 2.2). Sharp interfaces in $\max(h(t))$ were also noted in maximum maps by the authors of [40].

For sufficiently large values of $c$, the solutions represented in Fig. 2.2 are dominated by the seasonal forcing and have a period of $T = 1$. In Figs. 2.3 (a) and (b), these forcing-dominated solutions can be found in the lower-right half of the plane.

Comparing panels (a) and (b) of Fig. 2.3 one notices clear differences in the interface dividing the two main regions, which are the result of bistabilities, or perhaps even multistabilities. The solution that the system converges to depends on the direction in which $c$ is varied, that is, it depends on the initial history used. The two maximum maps in Fig. 2.3 hence reflect the bistabilities seen in the one-parameter bifurcation diagrams of Fig. 2.2. For example, the upper and lower branches in Fig. 2.2(a) coincide with the maxima at $\tau_n = 1.2$ in panels (a) and (b), respectively, of Fig. 2.3.

### 2.2 The bifurcation set in the $(c, \tau_n)$-plane

We now investigate the dynamics causing the sharp interfaces in $\max(h(t))$ and the associated structures in the maximum maps. Figure 2.4 shows the maximum maps from Fig. 2.3 in gray-scale together with bifurcation curves found with DDE-Biftool, namely: saddle-node bifurcations of periodic orbits (blue), period-doubling bifurcations (black) and torus bifurcations (red). As indicated by the arrows, $c$ is increased in panel (a) and decreased in panel (b).

The bifurcation curves in Fig. 2.4 divide the $(c, \tau_n)$-plane into regions of qualitatively different solution types, which allows us to explain the features seen in the maximum maps.
2.2. The bifurcation set in the \((c, \tau_n)\)-plane

Figure 2.4: Maximum maps of Fig. 2.3 overlaid with curves of saddle-node bifurcations of periodic orbits (SN), period-doubling (PD) and torus bifurcations (T), which are drawn in blue, black and red, respectively. Several frequency ratios of resonance tongues are indicated; here \(b = 1\) and \(\kappa = 11\).
In both panels (a) and (b) closed curves of saddle-node bifurcations agree well with the elongated shapes; also compare with Fig. 2.3. Furthermore, closed curves of period-doubling bifurcations are found within some of the closed curves of saddle-node bifurcations of periodic orbits.

It is in Fig. 2.4(a), where \( c \) is being increased, that the curves of saddle-node bifurcations of periodic orbits agree to a larger extent with some of the sharp interfaces that form the elongated shapes. Except for small values of \( \tau_n \), the curve T of torus bifurcations (in red) does not agree well with the sharp interfaces seen in panel (a), suggesting that the solution undergoing the torus bifurcation is not the one being followed while increasing \( c \).

On the other hand, in Fig. 2.4(b), where \( c \) is being decreased, the curve T agrees well with the sharp interface in \( \max(h(t)) \) that divides the parameter plane. Regarding this large sharp interface, we know that the smaller maxima seen for larger \( c \) values (i.e. to the right of the large sharp interface) represent the solutions dominated by the seasonal forcing. This implies that, as \( c \) decreases, these solutions undergo a torus bifurcation at the curve T and become unstable. This is the reason why the curve T agrees well with the sharp interface in the case of decreasing \( c \) seen in Fig. 2.4(b). There are, however, some ranges of \( \tau_n \) where the sharp transitions do not occur exactly at the curve T; the reason for this is discussed in section 2.2.3.

In both panels (a) and (b) there remain some sharp interfaces that do not coincide with any bifurcation curve. This is because these sharp transitions in \( \max(h(t)) \) are due to bifurcations that cannot be readily continued numerically; this is discussed in section 2.3.

The elongated shapes bounded by curves of saddle-node bifurcations of periodic orbits are in fact resonance tongues. Numerical simulation confirms that they contain stable frequency locked solutions, meaning that all solutions inside each resonance tongue have the same fixed frequency ratio. The resonance tongues shown here are a selection of those present in the system: there are actually infinitely many resonance tongues. The resonance tongues are rooted on the line of zero forcing (where \( c = 0 \)) and/or on the curve of torus bifurcations at points of \( p:q \) resonance. They become very thin in the parameter plane for larger \( q \). General theory [59] tells us that along the torus bifurcation curve, the rotation number of the emerging invariant tori is changing continuously with the parameters \( (c, \tau_n) \). If the rotation number is a rational number, the bifurcating solution is locked on the torus and a resonance tongue will branch off at this point. So for every rational rotation number \( p/q \), the resonance tongue will contain a family of \( p:q \) resonant periodic orbits that are locked to the forcing. Such periodic orbits form \( p:q \) torus knots as they wind around the torus.

The zero-forcing line \( (c = 0) \) is a straight curve of torus bifurcations for \( \tau_n > \pi/(2\kappa) \), where delay-induced oscillations exist: once the seasonal forcing is switched on (i.e. \( c > 0 \)), a second frequency is introduced into the dynamics and an invariant torus is formed. Because \( T = 4\tau \) as mentioned in section 1.1.2, we see in the bifurcation set that \( p:q \) resonance
tongues are rooted along the zero-forcing line at $\tau_n = q/4p$. Examples of these, a 4:3, 3:7 and a 3:8 resonance tongue, are included in Fig. 2.4 branching off at $\tau_n = 3/16, 7/12$ and $8/12$, respectively, from the zero-forcing line. General theory also tells us that for an irrational value of $\tau_n$ along the zero-forcing line, or for an irrational rotation number along the curve of torus bifurcations, the location of the solution will be the starting point of a smooth curve of unlocked quasi-periodic solutions that exist on a torus [59].

An example of another source of resonance is shown in Fig. 2.4: the smaller resonance tongue that branches off at the point $(c, \tau_n) = (0, 1.25)$ has the same shape as the resonance tongue that branches off at $(c, \tau_n) = (0, 0.25)$ with each tongue containing solutions of period $T = 1$. This is due to the repeating nature of periodic solutions of DDEs. The idea is detailed in [109], where the basic concept is that, given a periodic solution of period $T$ to a certain DDE with delay time $\tau_n$, another solution with an identical time series will exist for delay time $\tau'_n = \tau_n + T$. Because the solution is periodic, when the feedback term of the DDE calls on $h(t - \tau_n - T)$ it is receiving exactly the same input as for $h(t - \tau_n)$. Particularly at larger $\tau_n$ values this will contribute to an increasing number of resonance tongues.

Note that, besides periodic (locked) and quasi-periodic (unlocked) behaviour, there may be small domains in the parameter space where chaos exists. However, chaotic behaviour does not seem to be a significant feature in the model for the parameter range investigated here.

### 2.2.1 Transition through resonance tongues

Figures 2.5 (a1) and (a2) are reproduced from [110] and display the local maxima and minima in $h(t)$ of stable solutions found by numerical integration of Eq. (2.1) for $c = 2$ and corresponding values of $\tau_n$ with a fixed initial condition $h \equiv 1$. Panel (a1) shows alternating regions of small finite numbers of local maxima and minima with regions of very large (possibly representing infinite) numbers of local maxima and minima. The differences in the numbers of local maxima and minima indicate different solution types: a periodic solution will have a finite number of local maxima and minima, while an aperiodic solution could have an infinite number of local maxima and minima over time.

Panel (a2) is an enlargement of (a1) showing only local maxima for a smaller range of $\tau_n \in [0.50, 0.59]$ values and $c = 2$. Here, most of the range of $\tau_n$ corresponds to solutions with a very large (possibly infinite) number of local maxima with windows of small finite numbers of local maxima. The authors of [110] suggested that chaos is present here between windows of periodic solutions.

The different numbers of local maxima and minima seen in panels (a1)–(a2) can be understood from the bifurcation analysis. Panel (b1) is a transposed version of Fig. 2.4 (b). The green line at $c = 2$ indicates the position of the parameter section represented in Fig. 2.5(a1). Along the green line the solution types in the bifurcation analysis coincides very well with
Figure 2.5: Panel (a1) displays the local maxima and minima of simulated solutions for $c = 2$ and $\tau_n \in [0, 2]$ and panel (a2) shows the same maxima for $\tau_n \in [0.5, 0.59]$; these two figures are reproduced from I. Zaliapin and M. Ghil, A delay differential model of ENSO variability – part 2: Phase locking, multiple solutions and dynamics of extrema, Nonlinear Processes in Geophysics, 17 (2010), pp. 123–135 under CC-BY licence. Panel (b1) is a transposed version of Fig. 2.4(b) with a green line indicating $c = 2$. Panel (b2) shows the maximum map with bifurcation curves over the same $\tau_n$-range as (a2) with some resonances indicated. Here $b = 1$ and $\kappa = 11$.

the local minima and maxima shown above in panel (a1). Beginning with small values of $\tau_n$ in panel (b1), the solution is dominated by the seasonal forcing until a torus bifurcation at $\tau_n \approx 0.51$ (red curve). For the same values of $\tau_n$ in panel (a1), there is just one set of minima and maxima, reflecting the case of seasonal forcing domination. After the torus bifurcation at curve T, a stable invariant torus is born. We now see an infinite (or very large) number of local minima and maxima in panel (a1), since these solutions are quasi-periodic or of a very high period. As $\tau_n$ increases in panel (b1), the solutions alternate between being locked (periodic) and unlocked (quasi-periodic) on the torus, depending on whether the given $\tau_n$ value lies within a resonance tongue or not. For values of $\tau_n$ for which the solution is within a resonance tongue, there is a small finite number of local minima and maxima. Due to
the nature of the seasonal forcing, there is one local maximum every year; for example, the solutions with a period of three years (in the 1:3 resonance tongue) have three local maxima. Inside some resonance tongues are period-doubling bifurcations (see Fig. 2.5(b1)), which is why the local minima and maxima sometimes split into two, for example, for values close to \( \tau_n = 1 \) in panel (a1).

Fig. 2.5(b2) is an enlargement of panel (b1), where some example resonance tongues are shown, which are bounded by the blue curves of saddle-node bifurcations of periodic orbits. Again, the green line indicates the position of the parameter section shown in panel (a2). By comparison with panel (b2), we see that panel (a2) shows in finer detail the formation of the stable invariant torus from the torus bifurcation at \( \tau_n \approx 0.51 \), after which both locked and unlocked solutions exist. Therefore, as seen in the context of the bifurcation investigation carried out above, the behaviour observed throughout the \( \tau_n \)-range considered in panel (a2) is not chaos but quasi-periodic behaviour. At some \( \tau_n \) values there are windows of smaller finite numbers of local maxima. These represent solutions that lie within thin resonance tongues, some of which can be seen in panel (b2), including a 4:9, 3:7, 2:5 and 3:8 resonance tongue. The agreement between Fig. 2.5(a2) and (b2) is very good, but there may be small discrepancies that arise because the set of local maxima from [110] were calculated by numerical integration from the same fixed initial condition for each value of \( \tau_n \), whereas the set of solutions represented in panel (b2) were calculated by scanning the parameter plane.

### 2.2.2 Bistability within resonance tongues

We observe that some resonance tongues in the maximum maps appear to be striped with alternating horizontal lines; an example is the 2:5 resonance tongue in Fig. 2.5(b2). A clear example is also the 1:2 resonance tongue in Fig. 2.6, which is an enlarged version of part of Fig. 2.4(b) with a different colour scheme; note that increasing \( c \) produces a qualitatively similar map. The resonance tongue in Fig. 2.6 is bounded by curves of saddle-node bifurcations of periodic orbits; it is rooted on the zero-forcing line at one end and on the (red) curve of torus bifurcations at the other end. Inside the tongue there are stripes representing solutions of both larger and smaller maxima.

Normally, within a resonance tongue there is one stable and one unstable solution that approach each other as parameters are varied, then coincide and disappear at the boundary of the resonance tongue in a saddle-node bifurcation of periodic orbits. However, Fig. 2.6 suggests that there are two sets of stable periodic solutions within the tongue. More specifically, for a given \( \tau_n \), as \( c \) is increased or decreased, the solution reaches one of the two solutions depending on the initial condition when the tongue is entered, leading to the visible horizontal stripes as a result of scanning in \( c \). Note that producing this figure by varying \( \tau_n \) for fixed \( c \) would result in vertical stripes.

To explain this phenomenon, Figs. 2.7(a)–(b) show two stable (blue) solutions and two
Figure 2.6: Maximum map for decreasing $c$ showing the 1:2 resonance tongue bounded by the blue curves of saddle-node bifurcations (SN) of periodic orbits. Also shown is the red curve of torus bifurcations (T); here $b = 1$ and $\kappa = 11$.

unstable (red) solutions, respectively, for the same parameter set $(c, \tau_n) = (1, 0.5)$. These solutions are shown as a projection onto the $(h(t), h(t - \tau_n), h(t - \frac{1}{2}\tau_n))$-space in panel (c). Note that viewing this projection from different angles (not shown) reveals that trajectories do not intersect. Panel (d) shows a one-parameter bifurcation diagram of these solutions when they are continued in $\tau_n$ for $c = 1$. The gray line at $\tau_n = 0.5$ intersects the solutions shown in panels (a)–(c). Small blue-filled circles represent saddle-node bifurcations of periodic orbits at the boundary of the resonance tongue.

Comparing the two solutions in each panel (a)–(b), one can see that the symmetry

$$h_2(t) = -h_1(t + \frac{1}{2})$$

(2.2)

gives two distinct solutions that are symmetric counterparts of each other. In general, this symmetry is an inherent property of Eq. (2.1), resulting from both the periodic nature of the forcing term and the fact that the delay term is an odd function. The symmetry 2.2 does not depend on the parameter values; however, for $p:q$ locked solutions with odd $p$ and $q$ integers, $h_2 \equiv h_1$. In this case, there is only one distinct solution with the symmetry $h_1(t) = -h_1(t + \frac{1}{2})$. This explains why only some of the resonance tongues (i.e. those with even $p$ or $q$) appear striped in the maximum maps.

The symmetry (2.2) appears in the phase space projection in panel (c) as a rotational invariance of 180 degrees. The two symmetric counterpart solutions can also be seen in the bifurcation diagram in panel (d). Continuing the solutions shown in panels (a)–(b) of Fig. 2.7
2.2. The bifurcation set in the \((c, \tau_n)\)-plane

![Figure 2.7](image)

Figure 2.7: Two stable (blue) and two unstable (red) periodic orbits within the 1:2 resonance tongue at \((c, \tau_n) = (1, 0.5)\) are shown in panels (a) and (b), respectively, as time series and in panel (c) as a projection onto the \((h(t), h(t - \tau_n), h(t - \frac{1}{2} \tau_n))\)-space. Panel (d) is the one-parameter bifurcation diagram in \(\tau_n\) for \(c = 1\), where the blue and red curves correspond to stable and unstable solutions, respectively. Saddle-node bifurcations of the periodic orbits are indicated by blue-filled circles; intersection points with the gray line at \(\tau_n = 0.5\) yield the solutions observed in panels (a)–(c). Here \(b = 1\) and \(\kappa = 11\).

across the resonance tongue for varying \(\tau_n\) reveals that either side of the tongue is bound by, not just one, but two symmetric saddle-node of periodic orbits bifurcations. This was not visible in Fig. 2.4 because both sets of saddle-node bifurcation curves that bound either side of the resonance tongue lie on top of each other as they relate to symmetrically related periodic solutions.
2.2.3 Criticality of torus bifurcation

For some parameter values in Fig. 2.4(b) there are discrepancies between the curve of torus bifurcations and the sharp interface seen in the maximum map. This can be seen more clearly in Fig. 2.8, which is an enlargement of part of Fig. 2.4. The curves of saddle-node bifurcations of periodic orbits are not shown here; nonetheless, the resonance tongues are easy to recognise. Figure 2.8(a) shows the maximum map for increasing $c$, where the curve $T$ does not seem to affect the solutions being followed. Instead, there are other sharp interfaces that will be discussed in section 2.3. Figure 2.8(b) shows a region where the curve $T$ agrees only partially with the sharp interface. For $\tau_n \lesssim 1.5$ or $\tau_n \gtrsim 1.6$ the curves agree, where the maximum of the solutions change rapidly at curve $T$ (from dark blue to red) as $c$ is decreased. However, for $1.5 \lesssim \tau \lesssim 1.6$, there is a gradual change (to light blue) after the torus bifurcation curve as $c$ is decreased, before a rapid change in maximum at $c$ values beyond the torus bifurcation curve.

The reason for the discrepancies between the curve of torus bifurcations and the sharp interface in panel (b) is that the torus bifurcation changes criticality along the curve. For $\tau_n \lesssim 1.5$ or $\tau_n \gtrsim 1.6$ in panel (b), the (dark blue) solution, which is dominated by the seasonal forcing, simply becomes unstable at the torus bifurcation curve as $c$ is decreased and the next solution jumps to a larger (red) maximum. The solution with a larger (red) maximum is one that lies on a different, larger torus that co-exists for these parameters, that is all parameters for which the maximum appears red in panel (a). This implies that the torus bifurcation at these values of $\tau_n$ is subcritical (resulting in an unstable invariant torus of saddle-type to the right of the torus bifurcation curve). As $c$ is decreased for $1.5 \lesssim \tau \lesssim 1.6$, the (dark blue) solution also becomes unstable at the curve of torus bifurcations. However, for these $\tau_n$
values, a stable torus emerges with a stable periodic or quasi-periodic solution. This implies that the torus bifurcation for those values of \( \tau_n \) is supercritical (resulting in a stable invariant torus). The torus grows in maximum (light blue) and, at some value of \( c \) past the torus bifurcation curve, it loses stability along a sharp interface, where the maximum jumps to another solution with a larger (red) maximum; the nature of this sharp interface in terms of bifurcations of the torus is discussed in section 2.3.

With the knowledge gained from the bifurcation analysis in section 2.2, the examples shown in Fig. 2.1 can be understood in terms of their position on the \((c, \tau_n)\)-plane relative to the bifurcation curves. The solutions in Figs. 2.1(c) and (e) belong to the resonance tongues of \( T = 5 \) and \( T = 3 \), respectively; see Fig. 2.4(a). The solution in Fig. 2.1(b) is dominated by the seasonal forcing and has a \( c \) value larger than the torus bifurcation curve; see Fig. 2.4(b). The solution in Fig. 2.1(d) is a high-period or quasi-periodic solution for a value of \( c \) slightly below that of a supercritical torus bifurcation; see Fig. 2.4(b).

The bifurcation curves shown in Fig. 2.4 explain most, but not all, of the results obtained by numerical integration in sections 2.1–2.2. For example, one might ask the question: why does the stable invariant torus seen to emerge from the curve of torus bifurcations in Fig. 2.8(b) disappear at certain combinations of parameters? What causes the sharp interfaces seen in Fig. 2.8(a)? These questions are discussed in the next section.

### 2.3 Bifurcations of tori

We now consider the sharp interfaces seen in the maximum maps of Figs. 2.4 and 2.8 that still remain unexplained. For example, as \( c \) is increased in Fig. 2.8(a) for each value of \( \tau_n \), the tori being followed have a relatively large maximum values (appearing red on the maximum map). However, these tori seem to suddenly lose their stability and disappear, after which the next stable solution has a considerably smaller maximum (appearing blue on the maximum map). As \( c \) is being decreased in Fig. 2.8(b) for \( \tau_n \) values where the torus bifurcation is supercritical, small (light blue on the maximum map) stable tori emerge from the torus bifurcation curve. They then soon lose their stability and disappear, after which the next stable solution has a larger (red) maximum value.

Figure 2.9(a) is a one-parameter bifurcation diagram for the parameter range \( c \in [2.9, 3.2] \) and \( \tau_n = 0.94 \) — which crosses a region in the \((c, \tau_n)\)-plane where such unexplained sharp interfaces can be seen in Fig. 2.4. For larger values of \( c \), the solutions in Fig. 2.9(a) are periodic and dominated by the seasonal forcing and are annotated 1:1. These periodic solutions can be continued with DDE-Biftool through the torus bifurcation (T). The stable torus can be followed by numerical integration while decreasing \( c \) in small steps, until there is a rapid transition in \( \max((h(t))) \) at \( c \approx 2.95 \). Notice the kink at \( c \approx 2.97 \), where the torus passes
Figure 2.9: Panel (a) is a one-parameter bifurcation diagram in $c$ for $\tau_n = 0.94$, where blue and red lines indicate stable and unstable solutions, respectively. The branches of stable tori were found by parameter scanning with numerical simulation, and they end at the points denoted SNT. The black arrows indicate the direction of change of $c$ and a hysteresis loop. The red circles represent unstable locked periodic solutions. Panels (b) and (c) are one-parameter bifurcation diagrams in $c$ for $\tau_n = 0.94$ of the periodic orbits in the 5:17 and 7:24 resonance tongues, respectively. Here $b = 1$ and $\kappa = 11$.

through the 3:10 resonance tongue and its rotation number is 3/10. Enlarging the blue curve would reveal further smaller kinks representing thinner resonance tongues. For smaller $c$ and larger max($h(t)$) values, there is a 2:7 resonance tongue whose periodic orbits can be continued with DDE-Biftool; it terminates in a saddle-node bifurcation of periodic orbits (SN). While increasing $c$, the solutions after the 2:7 resonance tongue can be followed by numerical integration and we find an upper stable torus until $c \approx 3.01$.

Notice how the upper and lower blue curves representing solutions on tori bend downwards and upwards, respectively, and become vertical before their rapid transitions. This is very reminiscent of a saddle-node bifurcation of periodic orbits. This comparison suggests that there is a fold or saddle-node bifurcation of tori, denoted SNT in Fig. 2.9(a). We discuss the intricacies of this phenomenon in more detail below. On the level of Fig. 2.9(a), the suggestion is that there is a branch of unstable tori between the two points labelled SNT. It is not possible with existing techniques to readily find and follow unstable tori in a DDE
by continuation or simulation. We can, however, use DDE-Biftool to locate locked periodic orbits along the unstable branch.

The small red circles in Fig. 2.9(a) represent periodic solutions in narrow resonance tongues. To find these unstable locked solutions, we make use of the fact that resonance tongues are ordered in the Farey sequence: the largest resonance tongue that exists between a $p:q$ and a $r:s$ resonance tongue is a $p+r:q+s$ resonance (for example, see [48]). Therefore, we know that between the 2:7 and 3:10 resonance tongues seen in Fig. 2.9(a) the torus must pass through a 5:17 resonance tongue. To find a periodic solution in this resonance tongue, we construct an initial guess and then use DDE-Biftool to correct it. To achieve an approximation of a 5:17 solution, we take a 17 year section from the time series of a nearby periodic solution, in this case the 3:10 periodic orbit. Based on an estimate of where the 5:17 solution would lie on the plot in Fig. 2.9(a), we scale the time series such that $\max(h(t)) \approx 0.85$ and let $c \approx 2.98$. DDE-Biftool is then able to correct this constructed initial guess to the true 5:17 periodic solution. Other unstable locked periodic orbits were found similarly. Shown as red circles in Fig. 2.9(a) are the associated very narrow resonance locations of the frequency ratios indicated, which appear to lie on a curve between the two points labelled SNT.

Figures 2.9(b) and (c) show the unstable 5:17 and 7:24 periodic solutions, respectively, continued for changing $c$. In each case, this gives us a slice of the resonance tongue to which the periodic solution belongs. Note that the respective $c$-ranges are very small. Also notice the double set of saddle-node bifurcations of periodic orbits in panel (c), as is expected for an even period (see section 2.2.2). As indicated by their red colour, all points in these slices of resonance tongues are unstable solutions. We find that these solutions always have at least one unstable Floquet multiplier, which implies that the torus along this part of the branch is indeed of saddle-type.

### 2.3.1 Resonance tongues and Chenciner bubbles

Figure 2.9(a) presents a convincing bifurcation diagram where branches of stable and saddle tori exist and come very close to each other near the points labelled SNT. Nevertheless, it is important to realise that the tori lose smoothness and cease to exist once they become sufficiently close to each other near SNT [5, 15]. In other words, the precise bifurcation diagram is not so simple and involves the break-up of tori. A good approach for investigating the sharp interface boundary formed by these bifurcations of tori is to consider resonance tongues of locked tori in the nearby $(c, \tau_n)$-plane.

We begin by continuing all the resonances identified in Fig. 2.9 in the $(c, \tau_n)$-plane. Figure 2.10 shows maximum maps with $c$ increasing in panel (a) and decreasing in panel (b), upon which we overlay curves of saddle-node bifurcations of periodic orbits (SN) that form the boundaries of these resonance tongues; they are labelled $p:q$ at the points where they bifurcate from the curve T. Near the curve T, where the resonance tongues become extremely
narrow, the continuation of the saddle-node bifurcations of periodic orbits eventually becomes impractical. To represent the extremely narrow segments of the tongues rooted on the curve T, we therefore compute and plot a curve of a single periodic orbit in each \( p:q \) resonance tongue. Most resonance tongues appear as single curves because they are very thin. Although this is not visible, except for the 5:17 tongue, one boundary is drawn in a lighter blue. The points where the resonance tongues intersect the line shown at \( \tau_n = 0.94 \) coincide with the red circles and the 3:10 kink seen in Fig. 2.9(a).

By calculating the Floquet multipliers of the \( p:q \) periodic solutions with DDE-Biftool, we establish that, as they bifurcate from curve T in Fig. 2.10, the invariant tori are stable, meaning that all of the resonance tongues contain a set of stable and unstable solutions. Yet, it was shown in Fig. 2.9 that these resonance tongues, except the 3:10 tongue, contain only unstable periodic solutions. To explain how this happens, let us consider, for example, the 7:24 resonance tongue. Following the 7:24 resonance tongue from the curve T, it contains a set of stable and unstable solutions. At \( c \approx 2.92 \) the boundary curves of this resonance tongue have local minima in \( c \). Since these curves are so close together, this can be interpreted as a fold of the resonance tongue. There is another fold of the 7:24 resonance tongue at \( c \approx 3.01 \), where it has a local maximum in \( c \). Calculations reveal that in-between the two folds with respect to \( c \) all solutions lie on a torus of saddle-type and have at least one unstable Floquet multiplier. This is why the 7:24 resonance tongue contains only unstable solutions as it passes \( \tau_n = 0.94 \) (cf. Fig. 2.9(c)). On the other hand, past the local maximum there is again a set of stable and unstable periodic solutions in the resonance tongue. The other resonance tongues in Fig. 2.10 have the same folding and stability properties. At \( \tau_n = 0.94 \) only the 3:10 resonance tongue has not undergone any fold and, hence, is seen to be on the stable branch in Fig. 2.9(a). Overall, the folding of resonance tongues explains why certain locked solutions seen in Fig. 2.9(a) lie on either a stable or saddle torus at \( \tau_n = 0.94 \).

As can be seen in Fig. 2.10, the folds of resonance tongues coincide with the saddle-node bifurcations of tori, represented by the sharp interface in \( \max(h(t)) \). However, there is actually no smooth curve of saddle-node bifurcations of tori. Near their folds the resonance tongues form so-called Chenciner bubbles: the invariant torus loses normal hyperbolicity and breaks up as it enters the region of Chenciner bubbles in the transition from a stable torus to a torus of saddle-type. In Chenciner bubbles the dynamics are generally very complicated [8, 99]. In Fig. 2.10(b) the sharp interface in maximum values might appear as a smooth curve; however, looking closer would reveal further smaller Chenciner bubbles. In this case the resonance tongues are simply thinner, so the Chenciner bubbles are smaller and not visible on the scale of Fig. 2.10.

Since we found that the resonance tongues bifurcate from the torus bifurcation curve T, we can identify many more in an extended region of the \( (c, \tau_n) \)-plane. Figure 2.11 shows (dark and light blue) curves of saddle-node bifurcations of the \( p:q \) periodic orbits for \( p < q \leq 30 \),
Figure 2.10: Maximum maps with increasing $c$ (a) and decreasing $c$ (b). The red and blue curves are torus bifurcations (T) and saddle-node bifurcations of $p:q$ periodic orbits (SN), respectively. One boundary of each resonance tongue is drawn in a lighter blue. The green line at $\tau_n = 0.94$ intersects the solutions seen in Fig. 2.9. Here $b = 1$ and $\kappa = 11$. 
Figure 2.11: Maximum maps with increasing $c$ (a) and decreasing $c$ (b). The red and blue curves are torus bifurcations (T) and saddle-node bifurcations of $p:q$ periodic orbits, respectively; shown are all $p:q$ resonance tongues with $p < q \leq 30$ bifurcating from a segment of the torus bifurcation curve. The upper/lower boundary of each resonance tongue is drawn in dark/light blue. Here $b = 1$ and $\kappa = 11$. 
that bifurcate from a segment of the curve $T$ (for $0.92 \lesssim \tau \lesssim 1$). Again, the resonance tongues fold near the two sharp transitions of the maximum maps for increasing and decreasing $c$, respectively. More specifically, the envelopes of these folds form the two boundaries. Notice also that in Fig. 2.11 the two (dark and light blue) boundary curves of the shown resonance tongues start to separate considerably near the second fold from $T$ (their local maxima in $c$). Indeed, the two large indentations in the boundary of the maximum map for increasing $c$ are formed by light blue boundary curves of the low resonances 2:7 and 3:11; compare with Fig. 2.10(a). Not only do the light and dark blue boundary curves of each tongue separate but these two sets of boundary curves converge to different limits in Fig. 2.11. This leads to parameter regions where many resonance tongues overlap. We find that, once this overlapping occurs, the resonance tongues contain cascades of period-doubling bifurcations and chaotic behaviour may occur.

Figure 2.12: Stable solutions of Eq. (2.1), shown as time series in panels (a1) and (b1) and as projections onto the $(h(t-\tau_n), h(t))$-plane in panels (a2) and (b2); here $\tau_n = 0.91, c = 2.6, b = 1$ and $\kappa = 11$.

Figure 2.12 shows two simultaneously stable solutions for $\tau_n = 0.91$ and $c = 2.6$, obtained by numerical integration of Eq. (2.1) with the Euler method. As seen in Fig. 2.11, this is a point in the $(c, \tau_n)$-plane where many resonance tongues are overlapping.

Row (a) of Fig. 2.12 displays a periodic solution that belongs to the 1:3 resonance tongue. It has a large maximum every three years (see panel (a1)) and corresponds to a closed loop in projection onto the $(h(t-\tau_n), h(t))$-plane in panel (a2). This periodic solution is similar to the one shown in Fig. 2.1(e1)–(e2), which also belongs to the 1:3 resonance tongue. An interpretation of this solution is that a strong El Niño event appears every three years.

Row (b) of Fig. 2.12 shows a different solution of Eq. (2.1) for the same parameter values. The time series in panel (b1) seems irregular, with the largest maxima occurring every three
to seven years. In projection onto the \((h(t - \tau_n), h(t))\)-plane, 200 years of trajectory traces an attracting object. The solution looks as if it might be periodic with period \(T = 37\) in panel (b1). Although it is not shown in the previous figures, there exists an 11:37 resonance tongue between the 3:10 and 8:27 tongues. However, this resonance tongue, like those nearby, has already undergone a cascade of period-doubling bifurcations at \(\tau_n = 0.91\) and \(c = 2.6\). Upon close inspection of panel (b1), one can see that the two local maxima at \(t \approx 12\) differ very slightly from the two local maxima at \(t \approx 49\). In fact, this trajectory appears to be chaotic: it is actually very sensitive to initial conditions, as has been checked with numerical simulations. A chaotic solution, as in Fig. 2.12(b), reflects the observed irregularity of the time intervals between successive large El Niño events, which occurs every three to seven years.

### 2.4 Conclusions regarding the GZT model

In this chapter we investigated the interaction of the negative time-delayed feedback mechanism and seasonal forcing in a simplified ENSO model. The bifurcation analysis of the governing DDE with the continuation software DDE-Biftool allowed us to explain certain features seen in numerical simulations in previous works \([40, 110]\). More specifically, we presented maximum maps, calculated by scanning the \((c, \tau_n)\)-plane up and down in the parameter \(c\), on which we overlaid the relevant bifurcation curves. We discussed resonance tongues and their role in multistability, including bistabilities within \(p:q\) resonance tongues with even \(p\) or \(q\). Our analysis found that the relevant parameter plane is organised by an infinite number of resonance tongues rooted on curves of torus bifurcations. We also focussed on sharp interfaces in the maximum maps that could not be explained by continuing bifurcations of periodic orbits. We presented evidence that they are due to the phenomenon of Chenciner bubbles associated with the folding of resonance tongues. Following the boundary curves of these resonance tongues also revealed parameter regions where they overlap and more complicated behaviour ensues.

It was previously suggested in \([110]\) that Fig. 2.5(a2) provides evidence of chaotic dynamics in the one-parameter \(\tau_n\)-interval of \([0.5, 0.59]\) for \(c = 2\). Our bifurcation analysis revealed that the behaviour seen in this interval actually represents quasi-periodic or high-period locked dynamics resulting from a torus bifurcation. We find that the associated resonance tongues in the \((c, \tau_n)\)-plane effectively do not overlap. On the other hand, chaos could exist in small domains of the parameter space where resonance tongues do overlap significantly. We illustrated that this is indeed the case with an example of a chaotic solution in Fig. 2.12, which is from an area of a high-degree of multistability due to clearly overlapping resonance tongues in the \((c, \tau_n)\)-plane.

Given that the scalar DDE (2.1) contains just two terms, the model shows surprisingly
2.4. Conclusions regarding the GZT model

The wealth of bifurcations, even within small parameter ranges, highlights the relevance of parameter sensitivity in climate modelling. In particular, it is interesting in the context of climate tipping. Some climate tipping events correspond to certain bifurcations, where the response of a climate system to a slight variation in parameter is a qualitative or drastic change of observed behaviour [6]. Saddle-node bifurcations have been identified as potential mechanisms for particular climate tipping events; for example, see [1]. As far as we are aware, the quasi-periodic saddle-node bifurcation of tori (characterised by complicated dynamics in the associated Chenciner bubbles) discussed in section 2.3 has not yet been considered in the context of tipping. The irreversibility of this form of tipping event is illustrated by the hysteresis loop in Fig. 2.9(a). The topic of folding tori and Chenciner bubbles, and their possible interpretation as tipping points will be addressed in chapter 4.


3

Positive and negative delayed feedback with asymmetric coupling and seasonal forcing

In this chapter, we focus on the conceptual ENSO model with delay terms that was introduced by Tziperman, Stone, Cane and Jarosh in [98]. To keep this chapter self-contained we briefly recall the DDE model from section 1.1.2, which takes the form:

\[ \dot{h}(t) = aA(h(t - \tau_p)) - bA(h(t - \tau_n)) + c \cos(2\pi t), \]

where

\[ A(h) = \begin{cases} \begin{align} d_u \tanh\left( \frac{h}{d_u} \right) & \text{if } h \geq 0, \\ d_l \tanh\left( \frac{h}{d_l} \right) & \text{if } h < 0 \end{align} \end{cases} \]

is the ocean-atmosphere coupling function as in section 1.1.2. We refer to Eq. (3.1) as the TSCJ model. It describes the change of the thermocline depth \( h \) at the eastern boundary of the Pacific Ocean (more specifically, its deviation from the annual mean), which depends on time measured in years. The first and second terms of Eq. (3.1) represent positive and negative delayed feedback mechanisms, respectively, that exist due to a coupling of processes in the ocean and atmosphere, as expressed by the function \( A(h) \). The third term of Eq. (3.1) reflects the periodic forcing effect of the annual cycle of the seasons with a period of one year. More details of the TSCJ model, including the physical background and meanings of the parameters, are provided in chapter 1.

The TSCJ model is significant because in [98] Tziperman et al. demonstrated that the irregularity that is characteristic of ENSO could be reproduced as chaotic behaviour of this simple DDE. This is in contrast to another school of thought that suggests that such irregularity is driven by noise, in particular, by local weather in the form of small-scale, high-frequency stochastic forcing; for example, see [69]. One aim in this chapter is to understand the mechanisms responsible for the chaotic behaviour observed in [98] and to establish how typical such
behaviour is in the TSCJ model.

A simplified version of the TSCJ model, the GZT model from chapter 2, was introduced by Ghil, Zaliapin and Thompson in [40] and then studied further in [110]. The GZT model is the TSCJ model without the positive feedback mechanism and without the asymmetry in the ocean-atmosphere coupling function; which is:

\[
\dot{\hat{h}}(t) = -\hat{b}A(\hat{h}(t - \tau_n)) + \hat{c}\cos(2\pi t)
= -\hat{b}\tanh[\hat{\kappa}\hat{h}(t - \tau_n)] + \hat{c}\cos(2\pi t),
\]

(3.2)

where \(\hat{h}, \hat{\kappa}, \hat{b}\) and \(\hat{c}\) are rescaled parameters defined below, which have the same physical meanings as \(h, \kappa, b\) and \(c\) in Eq. (2.1) of chapter 2, respectively.

A second aim of this chapter is to understand the differences in dynamical behaviour between the GZT and the TSCJ models. In other words, we want to understand how important the positive feedback and the asymmetry in the coupling are in the creation of more realistic solutions across large parameter regimes.

From the bifurcation analysis of the GZT model with DDE-Biftool in chapter 2, we know that resonance tongues are a prominent feature in its bifurcation diagrams and are responsible for a high degree of multistability. More specifically, areas in the \((\hat{c}, \tau_n)\)-plane where many resonance tongues ‘bunch up’ and overlap give rise to small parameter regions of high multistability where chaotic solutions can exist. This is interesting because, while it was argued in [98] that chaotic behaviour in the TSCJ model reproduces the irregularity in the ENSO system, the same is true for the simplified GZT model. However, the parameter regions where chaotic behaviour can be found in the GZT model are so small as to not be realistic. Indeed, any chaotic behaviour of practical relevance should be present over a large range of parameters (bearing in mind that the parameters of such a conceptual model do not necessarily correspond to measurable quantities in the real world).

The TSCJ model, on the other hand, does exhibit chaotic behaviour over a larger range of parameters. To understand why this is so, we investigate here the transition from the simplified GZT model, as studied in [40, 110] and chapter 2 of this thesis, to the more realistic TSCJ model, as studied in [98]. During this transition we gradually introduce the two additional model features — the positive feedback mechanism and the asymmetry in the coupling function — in order to identify how and for which corresponding parameters the qualitative behaviour in the system changes. In this way, we determine how important each additional model feature is for creating more realistic dynamics. To see if there are any differences in how the transition to the TSCJ model takes place, we consider both paths through parameter space. Figure 3.1 shows the two distinct paths in parameter space that we will follow from the GZT model to the TSCJ model.

In Fig. 3.1 the first step is to rescale the dependent variable \(h\) and parameters \(c, d\) and
Figure 3.1: Schematic of the transition from the GZT model (3.2) to the TSCJ model (3.1). After rescaling the variable $h$ and parameters $b$, $c$, $d_u$, $d_l$, and $\kappa$, the two paths, I and II, introduce the asymmetry of the coupling function ($d_l$) and positive feedback ($a$) in different orders. Each step is indicated by a figure in later sections.

Table 3.1: During the transition from the GZT model (3.2) to the TSCJ model (3.1) the parameters from chapter 2 (row 1) are changed to those from [98] (row 3).

The parameters we need to change in order to introduce the additional model features are $a$ and $d_l$. Path I first introduces the asymmetry into the coupling function by increasing $d_l$ from $-2$ to $-0.4$, and then introduces the positive feedback by increasing $a$ from $0$ to $2.02$ (cf. Table 3.1). Alternatively, path II reverses the order: first $a$ is increased, then $d_l$ is increased. In this way, the parameters change from those corresponding to the GZT model to those of the TSCJ model.

In this chapter, we begin our investigation with the GZT model as it was studied in chapter 2 and transform it into the TSCJ model studied in [98], where we calculate maximum
maps and bifurcation sets in the \((c, \tau_n)\)-plane in order to follow the changing dynamical landscape. This leads to a rise in complexity as resonances increasingly overlap and routes to chaos develop via the emergence of period-doubling cascades. At different stages throughout the transformation, we present example solutions to discuss how the additional model features lead to time series that are more reminiscent of observed data. Increasing the asymmetry in the coupling function is shown to be more effective than switching on the positive feedback for creating more realistic time series across large parameter ranges. We then conduct a bifurcation analysis of the TSCJ model in the \((c, \kappa)\)-plane, in order to understand observations made in [98], including the existence of a chaotic solution for certain parameters. Our analysis reveals that this chaotic solution is, as suggested in [98], due to the overlapping of resonances. Moreover, we show that this chaotic behaviour is the result of cascades of period-doubling bifurcations that emerge inside overlapping resonance tongues. In the past, various ENSO models have often been associated with a specific route to chaos. Yet, we demonstrate that in the TSCJ model multiple types of routes to the same chaotic solution are possible; these depend on the specific path taken through parameter space. Furthermore, we investigate the sensitivity with respect to the delay time parameters. We conduct a bifurcation analysis in the \((\tau_p, \tau_n)\)-plane across a range of realistic values from [13] and reveal how much the model behaviour changes if the two delay times vary. We find that within the range of realistic values many different solutions exist and that both the existence and stability of these solutions are very sensitive to changes in \(\tau_n\).

The chapter is organized as follows. In section 3.1 we consider the transformation from the GZT model to the full TSCJ model. Initial rescaling of parameters \(b, c, d_u, d_l\) and \(\kappa\) and the variable \(h\) is performed in section 3.1.1. First, the asymmetry in the coupling is introduced into the model in section 3.1.2. In section 3.1.3 the individual effect of the positive delayed feedback, without asymmetry in the coupling, is shown. To complete the transformation along path II, the asymmetry is introduced to the coupling, with the positive delayed feedback already present, in section 3.1.4. The transformation along path I is then completed by introducing the positive delayed feedback, with asymmetric coupling already present, in section 3.1.5. In section 3.1.6 we present the bifurcation set in the \((c, \tau_n)\)-plane for the full TSCJ model and then in section 3.1.7 discuss the importance of the asymmetric coupling in creating solutions with more realistic features. The bifurcation analysis of the full TSCJ model is presented in section 3.2. We review the transition to chaos that was observed in [98] in section 3.2.1. Period-doubling and intermittency routes to chaos are identified in sections 3.2.2 and 3.2.3, respectively. The bifurcation set across a large range of the \((c, \kappa)\)-plane is presented in section 3.2.4. In section 3.2.5 a sensitivity analysis is conducted for the delay times. Finally, section 3.3 contains concluding remarks.
3.1 Transition from the GZT model to the TSCJ model

There are two features in the TSCJ model that are not included in the GZT model: asymmetry in the ocean-atmosphere coupling function and the positive delayed feedback. In this section we begin with the GZT model and gradually introduce the two additional model features, one after the other. During the transition we calculate maximum maps with bifurcation sets in the \((c, \tau_n)\)-plane to identify changes in the dynamical landscape of the model.

We follow the schematic in Fig. 3.1. The GZT model as a special case of the TSCJ model (with \(|d_u| = |d_l|\) and \(a = 0\)) is presented in section 3.1.1. The first legs of paths I and II show the effects of each additional model feature individually; they are presented in sections 3.1.2 and 3.1.3. The transition to the TSCJ model via the second leg of path II is then completed in section 3.1.4, followed by the second leg of path I in section 3.1.5. The \((c, \tau_n)\)-plane at the end of the transition to the full TSCJ model is discussed in section 3.1.6. Section 3.1.7 illustrates that the asymmetric coupling function is particularly important for creating chaotic solutions over a large range of parameters.

3.1.1 The GZT model as a special case of the TSCJ model

We now consider the GZT model, or rather the TSCJ model with \(a = 0\) and \(|d_u| = |d_l|\). Figure 3.2 shows a maximum map in the \((c, \tau_n)\)-plane for the parameter values given in the second row of Table 3.1. Notice that the ranges of \(c\) and \(\tau_n\) have been chosen to include \(c = 2.6377\) and \(\tau_n = 0.4792\) as used in [98].

Figure 3.2: Maximum map for increasing \(c\) of the TSCJ model (3.1) with \(a = 0\), \(b = 3.03\), \(d_u = 2\), \(d_l = -2\) and \(\kappa = 3.63\). Also shown are (blue) curves SN of saddle-node bifurcations of periodic orbits and a (red) curve T of torus bifurcations. Labels \(p:q\) denote the resonance type.
The maximum map displays the maxima of sufficiently long numerical solutions as a function of $c$ and $\tau_n$; here, maxima are simply taken as the largest values of the time series. To ensure that the shown maximum is accurate, the length of the time series considered must be sufficiently long (typically many hundreds of years). Maximum maps have been employed extensively in [40, 110] as a way of identifying qualitative changes in the solutions: a sudden jump in the maximum of the trajectory is generally the result of a bifurcation. Comparing maximum maps with different scanning directions (for example, for both increasing and decreasing $c$) to reveal parameter regions where bistabilities exist is not considered in this chapter but is detailed in section 2.1.3 of chapter 2. The maximum map in Fig. 3.2 is calculated, again with the Euler method, for a range of fixed $\tau_n$ values while increasing $c$, as indicated by the black arrow. For each increment of $c$, the initial history is taken from the solution of the previous simulation (i.e. that of a slightly lower value of $c$). Transients are disregarded before considering a maximum value; in other words, the trajectory is first simulated for a sufficient time so that it can approach and reach a stable attractor.

Overlaid in Fig. 3.2 is a curve $T$ of torus bifurcations (red) and curves SN of saddle-node bifurcations of periodic orbits (blue). As expected and was checked, the same solutions and bifurcations exist for the GZT model with the parameter values given in the first row of Table 3.1. The corresponding bifurcation set and maximum map (not shown) appear identical to Fig. 3.2 with $c$ and $h$ rescaled; compare with the lower left part of Fig. 2.4(a) near the 1:2 resonance tongue.

As a starting point for the transition to the full TSCJ model, we now discuss the general properties of Fig. 3.2. A main feature is a 1:2 resonance tongue that is bounded by curves SN of saddle-node bifurcations of periodic orbits. It connects the zero-forcing line $c = 0$ and the torus bifurcation curve $T$ in the lower-right corner of the $(c, \tau_n)$-plane. In the upper-right corner there is also part of the boundary of the 1:3 resonance tongue. The bifurcation analysis of the GZT model in chapter 2, together with general theory [59], already provides an interpretation of Fig. 3.2. Firstly, the solutions situated to the right of the curve $T$ are those solutions with sufficiently large $c$, such that they are dominated by the seasonal forcing and have a period equal to one year. These period-one solutions have relatively small maxima, hence they appear blue in the maximum map. Secondly, along the zero-forcing line $c = 0$ are also stable oscillations that are driven by the negative delayed feedback. In between the curve of torus bifurcations and zero-forcing line are solutions driven by two frequencies both the negative delayed feedback and the seasonal forcing, so the solutions lie on a torus. As $c$ is increased from the zero-forcing line, the maximum map becomes orange and red, giving an indication of the increasing size of the tori. The tori then decrease in size before disappearing at the curve of torus bifurcations. Notice that traces of other resonance tongues can be seen in the maximum map, although the associated saddle-node bifurcation curves are not shown. There are, in fact, an infinite number of resonance tongues, each anchored on the curve $T$ at
3.1. Transition from the GZT model to the TSCJ model

Figure 3.3: Stable solutions found by numerical integration of the TSCJ model (3.1) with $\tau_n = 0.53$, and $c = 14$ (a), $c = 9$ (b) and $c = 4$ (c), displayed as a time series (left column), as a stroboscopic trace in the $(h(t), h(t - \tau_p), h(t - \tau_n))$-space (middle column), and as a power spectrum on a logarithmic scale in arbitrary units [a.u.] (right column). Other parameters are $a = 0$, $b = 3.03$, $d_u = 2$, $d_l = -2$, $\kappa = 3.63$ and $\tau_p = 0.0958$.

one end and on the zero-forcing line $c = 0$ at the other end.

Inside the 1:2 resonance tongue in Fig. 3.2 are stripes in the $c$-direction. In chapter 2 it was shown that they are due to a symmetry property of the governing DDE (for $|d_u| = |d_l|$). In $p$:$q$ resonance tongues of even $p$ or $q$ there are two symmetrically related, yet distinct, stable solutions. Therefore, as $c$ is increased or decreased for fixed $\tau_n$, the trajectory approaches one of two solutions depending on the initial condition as it enters the resonance tongue. In maximum maps of changing $c$ this bistability manifests itself as horizontal stripes inside the corresponding resonance tongues. If $\tau_n$ was varied for fixed $c$, the stripes would be vertical.

Figure 3.3 shows examples of typical solutions associated with the $(c, \tau_n)$-plane in Fig. 3.2(b). Each solution is displayed as a time series, a stroboscopic trace in the $(h(t), h(t - \tau_p), h(t - \tau_n))$-space and a logarithmic power spectrum obtained by taking the Fourier transform of the time series over 500 years (after transients died down). The stroboscopic traces are constructed by considering the solution, which is a function segment, after each forcing period, and then
plotting an appropriate projection of the associated first point, also called the headpoint of the function segment. Because the forcing period is one, \( h(t) \) is plotted at every \( t \in \mathbb{N} \). This stroboscopic trace is a type of Poincaré trace, which is defined formally for DDEs and discussed in [56]. These three representations of a solution are useful for identifying characterizing solution features.

Row (a) of Fig. 3.3 displays the solution for \( c = 14 \) and \( \tau_n = 0.53 \), which is to the right of the curve T in Fig. 3.2, so that the resulting solution is dominated by the seasonal forcing. The time series in panel (a1) shows that this periodic solution is of period one and almost sinusoidal. The stroboscopic trace in panel (a2) consists of a single point. Panel (a3) shows a single dominant peak at the forcing frequency of one. Figure 3.3 (b) is for \( c = 9 \) and \( \tau_n = 0.53 \), which is a point inside the 1:2 resonance tongue in Fig. 3.2. Indeed, the time series in panel (b1) reveals a solution of period two, the stroboscopic trace in panel (b2) consists of two points, and the dominant frequency in the spectrum in panel (b3) is now \( 0.5 \). Row (c) of Fig. 3.3, calculated for \( c = 4 \) and \( \tau_n = 0.53 \), reveals a solution that is quasi-periodic (or of very high period). The time series in panel (c1) shows a characteristic modulation of the heights of local maxima. The stroboscopic trace in panel (c2) no longer consists of isolated points; rather, the trace of the headpoints forms a closed curve in \( (h(t), h(t - \tau_p), h(t - \tau_n)) \)-space. Moreover, the power spectrum in panel (c3) has a dominant peak at a frequency that is incommensurate with the frequency 1 of the seasonal forcing. All this is clear evidence that the solution is indeed quasi-periodic (or of very high period).

We now discuss how these solutions can be interpreted in the context of the ENSO system. An interpretation of the solution in Fig. 3.3 (a) is that there is no resonance between the internal ENSO mechanisms and the seasonal forcing. There is an annual variation of the thermocline depth with the seasons, but no year is different from any other. In the case of row (b), there is a considerably larger peak in the thermocline depth and, therefore, in the sea-surface temperature in the eastern equatorial Pacific, every two years. The magnitude of this rise in sea-surface temperature is always the same. An interpretation of the solution in Fig. 3.3 (c) is that an El Niño event occurs every couple of years with varying strength. There is no phase-locking to the annual cycle, so that the El Niño events do not necessarily happen at the same time of the year. None of these solutions are good representation of observed El Niño events.

### 3.1.2 Path I: increasing the asymmetry in the coupling function

Figure 3.4 shows the effect of introducing the asymmetry into the coupling function of the TSCJ model. Specifically, \( d_l \) is increased from \(-2\) to its nominal value of \(-0.4\) with \( a = 0 \) fixed, which constitutes the first leg of path I in Fig. 3.1. Panels (a1)–(a5) of Fig. 3.4 are maximum maps in the \((c, \tau_n)\)-plane, each overlaid with a bifurcation set, for \( d_l = -2, -1.6, -1.2, -0.8 \) and \(-0.4\), respectively. Curves of saddle-node bifurcations of periodic orbits...
3.1. Transition from the GZT model to the TSCJ model

Figure 3.4: Effect of increasing $d_l$ for $a = 0$ along path I. Shown are maximum maps in the $(c, \tau_n)$-plane for increasing $c$ and for $d_l = -2$ (a1), $-1.6$ (a2), $-1.2$ (a3), $-0.8$ (a4) and $-0.4$ (a5) with curves of saddle-node bifurcations of periodic orbits (blue), torus bifurcations (red) and period-doubling bifurcations (black). Panel (b) is the maximum map in the $(c, d_l)$-plane with bifurcation curves, for increasing $d_l$ and fixed $\tau_n = 0.4792$. Panel (c) is a 3D representation of the maximum maps in panels (a1)–(a5). The grey lines in each panel indicate where the $(c, d_l)$-plane in panel (b) intersects the $(c, \tau_n)$-planes in panels (a1)–(a5) and (c). Colour bars are omitted, but are chosen in each panel to best display qualitative changes (red = large $h$, blue = small $h$). Other parameters are $a = 0$, $b = 3.03$, $d_u = 2$, $\kappa = 3.63$ and $\tau_p = 0.0958$. 
reveal the boundaries of resonance tongues; shown are the $1:q$ resonance tongues that exist in the visible range of parameters. Grey horizontal lines mark where $\tau_n = 0.4792$, which is the value used in [98], and the black arrows indicate that $c$ is increased in the calculations. Panel (b) is the associated maximum map in the $(c, d_l)$-plane for increasing $d_l$ with fixed $\tau_n = 0.4792$, overlaid with a bifurcation set. Specifically, this $(c, d_l)$-plane passes through the grey lines in panels (a1)–(a5). To illustrate the transition with increasing $d_l$ more clearly and emphasise the multidimensional approach to the transition, Fig. 3.4 (c) is a three-dimensional representation in $(c, \tau_n, d_l)$-space of the $(c, \tau_n)$-planes in panels (a1)–(a5).

Figure 3.4 (a1) is the maximum map for $d_u = 2$ and $d_l = -2$, as in Fig. 3.2, where the coupling function is still symmetric. The growth of the $1:2$ resonance tongue is reflected in panels (a1)–(a5). Moreover, with increasing $d_l$, the root point of the $1:2$ resonance tongue on the zero-forcing line, where the saddle-node bifurcation curves bounding the resonance tongue meet at $c = 0$, moves downwards so that the resonance tongue begins at lower values of $\tau_n$. In panel (a4) the $1:2$ resonance tongue covers most of the visible $(c, \tau_n)$-plane. Moreover, the torus bifurcation curve no longer exists in the displayed range of the $(c, \tau_n)$-plane. Hence, for the parameters shown here, the solutions that are dominated by the seasonal forcing only bifurcate to create period-two solutions. This is in contrast to panels (a1)–(a2), where the solutions dominated by the seasonal forcing may lose stability at a torus bifurcation and bifurcate to create tori of various frequencies. In Fig. 3.4 (a5) many resonance tongues overlap and cascades of period-doubling bifurcations have appeared inside them. A period-doubling cascade of the period-one solutions dominated by the seasonal forcing has also appeared.

The above mentioned changes in dynamics, specifically for $\tau_n = 0.4792$, are illustrated in the $(c, d_l)$-plane of Fig. 3.4 (b). Here, we see that the torus bifurcation terminates at a period-doubling curve at $d_l \approx -1.8$. The root of the $1:2$ resonance tongue is located at $d_l \approx -1.3$ in panel (b), meaning that it passes through $\tau_n = 0.4792$ in the $(c, \tau_n)$-plane at $d_l \approx -1.3$. The roots of other resonance tongues exist for larger values of $d_l$, meaning that they too pass through $\tau_n = 0.4792$, as $d_l$ increases. For $d_l > -0.6$ period-doubling curves appear inside the resonance tongues. Although not all period-doubling curves are shown in panel (b), we can see that, as $d_l$ increases, the first period-doubling curve grows in size within a resonance tongue, then a second period-doubling curve appears inside the first and grows in size. This process continues and eventually leads to a whole cascade of period-doubling curves inside the resonance tongue. How this process unfolds is further illustrated in the three-dimensional representation in panel (c).

The conclusion from Fig. 3.4 is that, as $d_l$ increases, the resonance tongues reposition themselves in the $(c, \tau_n)$-plane so that they overlap increasingly. The resulting multistability becomes more complicated as the attractors destabilize each other and the dynamics eventually becomes chaotic. This provides the system with period-doubling routes to chaos.

The full effect of introducing the asymmetry into the coupling function of the TSCJ model
3.1. Transition from the GZT model to the TSCJ model

Figure 3.5: Stable solutions found by numerical integration of the TSCJ model (3.1) in the $(c, \tau_n)$-plane for $d_l = -0.4$ and $a = 0$ (cf. Fig. 3.4(a5)). The solutions for $c = 0.2$ and $\tau_n = 0.4792$ (a), $c = 3$ and $\tau_n = 0.4792$ (b), and $c = 1.5$ and $\tau_n = 0.55$ (c) are displayed as a time series (left column), as a stroboscopic trace in the $(h(t), h(t - \tau_p), h(t - \tau_n))$-space (middle column), and as a power spectrum on a logarithmic scale in arbitrary units [a.u.] (right column). Other parameters are $a = 0$, $b = 3.03$, $d_u = 2$, $d_l = -0.4$, $\kappa = 3.63$ and $\tau_p = 0.0958$.

is represented by the bifurcation set in panel (a5), calculated for the nominal parameter value of $d_l = -0.4$. Figure 3.5 shows examples of solution types found in the $(c, \tau_n)$-plane of Fig. 3.4(a5). Row (a) of Fig. 3.5 for $c = 0.2$ and $\tau_n = 0.4792$ is evidence for a quasi-periodic solution. The value of $c$ is small enough that this solution exists in a small area of the $(c, \tau_n)$-plane of Fig. 3.4(a5) that is not occupied by resonance tongues. The time series in panel (a1) of Fig. 3.5 appears to have many different local maxima. The points of the stroboscopic trace form a closed loop and the power spectrum has distinct peaks. Row (b) is a periodic solution for $c = 3$ and $\tau_n = 0.4792$, so that the solution is inside a resonance tongue in Fig. 3.4(a5). Panel (b1) of Fig. 3.5 shows a solution of period three. The stroboscopic trace in panel (b2) has three distinct points and the power spectrum in panel (b3) has a dominant peak at $1/3$. For the parameters $c = 1.5$ and $\tau_n = 0.55$ a chaotic solution is displayed in row (c). It is
situated beyond a cascade of period-doubling bifurcations in Fig. 3.4(a5). The time series in panel (c1) of Fig. 3.5 is clearly irregular. Panel (c3) shows a broad power spectrum, which contains contributions from all frequencies and is typical for chaotic behaviour in a time series.

An interpretation of the quasi-periodic solution in row (a) is that a peak in the thermocline depth occurs approximately every 2.5 years with small variations in strength. However, there are no peaks significantly larger than the others that could be interpreted as El Niño events. The periodic solution in row (b) corresponds to El Niño events of identical intensity reoccurring every three years, followed by cold La Niña events. The solution in row (c) is considerably more realistic than the previous examples. Firstly, there is frequency locking to the annual cycle so that all the major peaks, say local maxima of \( h > 0.5 \), take place at the same and correct time of the year. Secondly, the time series is irregular with major peaks occurring occasionally between smaller local maxima. On the other hand, an unrealistic aspect of this solution is that the major peaks occur about every 10 years, in contrast to observational data, showing that El Niño events occur every 3–7 years.

### 3.1.3 Path II: increasing the positive delayed feedback

Figure 3.6 shows the effect of introducing the positive delayed feedback, while the coupling function remains symmetric. In other words, \( a \) is increased from zero to its nominal value of 2.02 with fixed \( d_l = -2 \), which is the first leg of path II in Fig. 3.1. Figure 3.6 has the same layout as Fig. 3.4. Panels (a1)–(a5) of Fig. 3.6 are maximum maps in the \((c, \tau_n)\)-plane with associated bifurcation sets for \( a = 0, 0.5, 1, 1.5 \) and 2.02, respectively. Panel (a1) is again the same initial maximum map and bifurcation set as in Fig. 3.4(a1). With each increase of \( a \) the roots of the resonance tongues move downwards to lower values of \( \tau_n \); this is reflected in Figs. 3.6(a1)–(a5). Notice that the stripes in the 1:2 resonance tongue persist, because the before-mentioned symmetry in the governing DDE is preserved. The 1:3 resonance tongue appears very large in panel (a3) and in panel (a4) clearly occupies most of the visible \((c, \tau_n)\)-plane. In panel (a5) the 1:3, 1:4 and 1:5 resonance tongues overlap, inducing a period-doubling cascade inside the 1:4 resonance tongue.

Panel (b) also reflects the downwards movement of the resonance tongues. Here we see the 1:2, 1:3 and 1:4 resonance tongues, indicating that they successively pass through \( \tau_n = 0.4792 \) as they move downwards in the \((c, \tau_n)\)-plane for increasing \( a \). The torus bifurcation curve is shown to move steadily to the right. In panel (c) the maximum maps remain largely red as \( a \) increases; these represent the maxima of the larger solutions that lie on a torus. Panel (c) also displays well how the blue region, representing the smaller period-one solutions, diminishes as the positive feedback becomes stronger.

Generally, Figs. 3.4 and 3.6 reveal that introducing either the positive feedback or the asymmetry into the coupling function leads to the same effect: namely, the resonance tongues
Figure 3.6: Effect of increasing $a$ for $d_l = -2$ along path II. Shown are maximum maps in the $(c, \tau_n)$-plane for increasing $c$ and for $a = 0$ (a1), 0.5 (a2), 1.0 (a3), 1.5 (a4) and 2.02 (a5) with curves of saddle-node bifurcations of periodic orbits (blue), torus bifurcations (red) and period-doubling bifurcations (black). Panel (b) is the maximum map in the $(c, a)$-plane with bifurcation curves, for increasing $a$ and fixed $\tau_n = 0.4792$. Panel (c) is a 3D representation of the maximum maps in panels (a1)–(a5). The grey lines in each panel indicate where the $(c, a)$-plane in panel (b) intersects the $(c, \tau_n)$-planes in panels (a1)–(a5) and (c). Colour bars are omitted, but are chosen in each panel to best display qualitative changes (red = large $h$, blue = small $h$). Other parameters are $b = 3.03$, $d_u = 2$, $d_l = -2$, $\kappa = 3.63$ and $\tau_p = 0.0958$. 

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Figure 3.7: Stable solutions found by numerical integration of the TSCJ model (3.1) in the $(c, \tau_n)$-plane for $a = 2.02$ and $d_l = -2$ (cf. Fig. 3.4(a5)). The solutions for $\tau_n = 0.4792$ and $c = 9$ (a), $c = 5$ (b) and $c = 1$ (c) are displayed as a time series (left column), as a stroboscopic trace in the $(h(t), h(t - \tau_p), h(t - \tau_n))$-space (middle column), and as a power spectrum on a logarithmic scale in arbitrary units [a.u.] (right column). Other parameters are $a = 2.02$, $b = 3.03$, $d_u = 2$, $d_l = -2$, $\kappa = 3.63$ and $\tau_p = 0.0958$.

move downwards in the $(c, \tau_n)$-plane and increasingly overlap, leading to the appearance of period-doubling cascades. However, in Fig. 3.4(b) the torus bifurcation curve terminates at a period-doubling curve and the 1:2 resonance tongue covers much of the visible plane. This is not the case in Fig. 3.6(b). Therefore, the increasing asymmetry in the coupling function alone results in the 1:2 resonance tongue becoming very dominant and the period-one solutions undergoing a cascade of period-doubling bifurcations.

The full effect of the positive delayed feedback in the TSCJ model is illustrated in panel (a5) and Fig. 3.7 shows again examples of solution types found in the $(c, \tau_n)$-plane with $a$ equal to its nominal value of 2.02. Row (a) of Fig. 3.7 is calculated for $c = 9$ and $\tau_n = 0.4792$, which are parameters inside the large 1:3 resonance tongue in Fig. 3.6(a5). This example is clearly a periodic solution of period three: in particular, the stroboscopic trace in panel (a2) of Fig. 3.7 consists of three points and the power spectrum in panel (a3) has...
3.1. Transition from the GZT model to the TSCJ model

a dominant peak at three years. The parameters for row (b), \( c = 5 \) and \( \tau_n = 0.4792 \), are positioned inside the cascade of period-doubling bifurcations of the 1:4 resonance tongue in Fig. 3.6(a5). The time series in panel (b1) of Fig. 3.7 is irregular and the power spectrum in panel (b3) is broad, showing that the solution is chaotic. Row (c), calculated for \( c = 1 \) and \( \tau_n = 0.4792 \), shows a quasi-periodic solution. Although the stroboscopic trace in panel (c2) looks similar to that in panel (b2), the distinct peaks in the power spectrum in panel (c3) confirm that this solution is not chaotic.

Interpreting the periodic solution in row (a) of Fig. 3.7 gives a scenario where El Niño events of identical intensity occur every three years. In the time series (b1) the modulation of the peak amplitudes is small, meaning that there are no distinctly strong peaks that could be interpreted as El Niño events. Therefore, although the solution in row (b) is chaotic so that it reproduces the unpredictable nature of peak magnitudes, it is not realistic that these magnitudes are all so similar. The quasi-periodic solution in row (c) corresponds to an increase in the thermocline depth of varying magnitude approximately every 4.5 years. However, there are again no peaks significantly larger than the others that resemble El Niño events.

3.1.4 Path II: increasing the asymmetry in the coupling function

We now continue along path II by introducing the asymmetry into the coupling function with the positive feedback already present. In Fig. 3.8, \( d_l \) is increased from \(-2\) to its nominal value of \(-0.4\) with fixed \( a = 2.02 \), and completes the transformation to the TSCJ model. Panels (a1)–(a5) of Fig. 3.8 are maximum maps in the \((c,\tau_n)\)-plane with associated bifurcation sets for \( d_l = -2, -1.6, -1.2, -0.8 \) and \(-0.4\), respectively; panel (a1) of Fig. 3.8, with \( a = 2.02 \) and \( d_l = -2 \), is as from Fig. 3.6(a5). As \( d_l \) is increased, the 1:4 resonance tongue grows in size and the resonance tongues move downwards in the \((c,\tau_n)\)-plane; see panels (a1)–(a5). In panels (a4)–(a5) the resonance tongues overlap to a large extent and period-doubling cascades emerge inside the resonance tongues. The torus bifurcation curve no longer exists in the visible range of the \((c,\tau_n)\)-plane; instead, there is a cascade of period-doubling bifurcation curves of the period-one solutions dominated by the seasonal forcing. These period-doubling bifurcation curves move to the left of the \((c,\tau_n)\)-plane as \( d_l \) is increased, so that more of the plane is occupied by period-one solutions.

Figure 3.8(b) shows the \((c,d_l)\)-plane associated with panels (a1)–(a5) for fixed \( \tau_n = 0.4792 \). Panel (b) illustrates that, at \( d_l \approx -1.3 \), the torus bifurcation curve terminates at a period-doubling curve. As \( d_l \) increases, more resonance tongues begin to overlap; in particular, for \( d_l \gtrsim -1 \) there are many curves of saddle-node bifurcations that intersect and some curves appear to converge towards each other. It is especially here, where so much overlapping occurs, that period-doubling cascades appear within the resonance tongues. The maximum maps in panel (c) show that the maxima of the attractors generally becomes much smaller
Figure 3.8: Effect of increasing $d_l$ for $a = 2.02$ along path II. Shown are maximum maps in the $(c, \tau_n)$-plane for increasing $c$ and for $d_l = -2$ (a1), $-1.6$ (a2), $-1.2$ (a3), $-0.8$ (a4) and $-0.4$ (a5) with curves of saddle-node bifurcations of periodic orbits (blue), torus bifurcations (red) and period-doubling bifurcations (black). Panel (b) is the maximum map in the $(c, d_l)$-plane with bifurcation curves, for increasing $d_l$ and fixed $\tau_n = 0.4792$. Panel (c) is a 3D representation of the maximum maps in panels (a1)–(a5). The grey lines in each panel indicate where the $(c, d_l)$-plane in panel (b) intersects the $(c, \tau_n)$-planes in panels (a1)–(a5) and (c). Colour bars are omitted, but are chosen in each panel to best display qualitative changes (red = large $h$, blue = small $h$). Other parameters are $a = 2.02$, $b = 3.03$, $d_u = 2$, $\kappa = 3.63$ and $\tau_p = 0.0958$. 
as the asymmetry in the coupling function is increased. This is the same effect of increasing \( d_l \) without the positive feedback.

Figure 3.8 verifies that increasing the asymmetry when the positive feedback is already present results in an intensified overlapping of resonance tongues and creation of chaotic solutions via period-doubling cascades; overall, with asymmetric coupling, chaotic behaviour can be found over a larger region of the \((c, \tau_n)\)-plane.

### 3.1.5 Path I: increasing the positive delayed feedback

We now return to path I to complete the transition to the full TSCJ model. Figure 3.9 shows the effect of introducing the positive delayed feedback with the coupling function already asymmetric, by increasing \( a \) from zero to its nominal value of 2.02 with fixed \( d_l = -0.4 \). Panels (a1)–(a5) of Fig. 3.9 are maximum maps in the \((c, \tau_n)\)-plane, each overlaid with a bifurcation set for \( a = 0, 0.5, 1.0, 1.5 \) and 2.02, respectively. Panel (a1), with \( d_l = -0.4 \) and \( a = 0 \), is identical to Fig. 3.4(a5). As Figs. 3.9(a1)–(a5) show, the resonance tongues move downwards in the \((c, \tau_n)\)-plane and the period-one solutions dominated by seasonal forcing occupy an increasingly large area on the right-hand side of the visible plane.

Panel (b) Fig. 3.9 of shows the associated \((c,a)\)-plane for \( \tau_n = 0.4792 \). The roots of many resonance tongues can be seen on the zero-forcing line, which reflects that as \( a \) is increased, they all move downwards in the \((c, \tau_n)\)-plane to pass through \( \tau_n = 0.4792 \). Moreover, the period-doubling curves that exist at relatively large \( c \) values at \( a = 0 \) shift towards lower \( c \) values as \( a \) is increased. In fact, in panel (b) it appears that many curves, both of period-doubling and saddle-node bifurcations, converge towards each other as \( a \) approaches its nominal value 2.02. As expected, this leads to many overlapping resonance tongues with period-doubling cascades inside. In panel (c) the maximum map for \( a = 2.02 \) shows that the difference in maxima between the period-one solutions for large \( c \) values and the solution for smaller \( c \) values has become more extreme.

### 3.1.6 The \((c, \tau_n)\)-plane of the TSCJ model

Figure 3.10 is an enlarged version of Figs. 3.8(a5) and 3.9(a5). It represents the final state of the \((c, \tau_n)\)-plane once the parameter values equal those used in [98] (cf. the third row of Table 3.1); it includes the effects of both additional model features of the asymmetric coupling function and the positive delayed feedback. The resonance tongues overlap to a high degree and period-doubling cascades have appeared inside these tongues. There is also a period-doubling cascade of the 1:1 solutions, meaning that the solutions dominated by seasonal forcing also become chaotic for lower values of \( c \). Overall, due to the period-doubling cascades, chaotic dynamics can be found throughout a large region of the parameter plane.

Figure 3.11 shows examples of solutions representative of Fig. 3.10, where the chosen
Figure 3.9: Effect of increasing $a$ for $d_l = -0.4$ along path I. Shown are maximum maps in the $(c, \tau_n)$-plane for increasing $c$ and for $a = 0$ (a1), 0.5 (a2), 1.0 (a3), 1.5 (a4) and 2.02 (a5) with curves of saddle-node bifurcations of periodic orbits (blue), torus bifurcations (red) and period-doubling bifurcations (black). Panel (b) is the maximum map in the $(c, a)$-plane with bifurcation curves, for increasing $a$ and fixed $\tau_n = 0.4792$. Panel (c) is a 3D representation of the maximum maps in panels (a1)–(a5). The grey lines in each panel indicate where the $(c, a)$-plane in panel (b) intersects the $(c, \tau_n)$-planes in panels (a1)–(a5) and (c). Colour bars are omitted, but are chosen in each panel to best display qualitative changes (red = large $h$, blue = small $h$). Other parameters are $b = 3.03$, $d_u = 2$, $d_l = -0.4$, $\kappa = 3.63$ and $\tau_p = 0.0958$. 
3.1. Transition from the GZT model to the TSCJ model

Figure 3.10: Maximum map of the TSCJ model (3.1) for increasing $c$ in the $(c, \tau_n)$-plane and for $a = 2.02$ and $d_l = -0.4$ with curves of saddle-node bifurcations of periodic orbits (blue) and period-doubling bifurcations (black). Green crosses mark the parameters of the example solutions in Fig. 3.11. Other parameters are $b = 3.03$, $d_u = 2$, $\kappa = 3.63$ and $\tau_p = 0.0958$. Parameters are marked with green crosses. Row (a) of Fig. 3.11 is a chaotic solution for $c = 2.75$ and $\tau_n = 0.4792$, which are parameter values inside the period-doubling cascade of the 1:7 resonance tongue of Fig. 3.10. In the time series in panel (a1) of Fig. 3.11 a major peak occurs every seven years, although the amplitude of the peaks is irregular. The influence of the 1:7 resonance tongue is evident here. The power spectrum in panel (a3) is broad, confirming that the solution is chaotic. Row (b) is for $c = 4$ and $\tau = 0.4792$. These parameter values are located in Fig. 3.10 within the 1:7 resonance tongue between the first and second period-doubling bifurcation curves. It is therefore periodic with a period of 14 years, as seen in the time series in panel (b1). This period is represented in panel (b2) in the stroboscopic trace by the 14 points, as well as the power spectrum with a dominant peak at 7 years and a smaller one at 14 years. The example shown in row (c) is calculated for $c = 4$ and $\tau_n = 0.4271$. This solution is chaotic, as is particularly evident in the time series in panel (c1) and the broad power spectrum in panel (c3). The magnitude of the major peaks in the time series (c1) occur irregularly, similar to (a1). Nonetheless, the chaotic solution in row (c) is very different from that in row (a). While in the time series (a1) the major peaks occur exactly every seven years, in (c1) they occur irregularly. There are even intermittent periods, in particular near $t \approx 90$, where no major peaks occur. This is also reflected in the
power spectrum in panel (c3), which has less structure compared to (a3). In other words, the solution in row (c) displays both a spatial and temporal sense of irregularity, observed in the magnitude of the major peaks and how often those major peaks occur, respectively.

An interpretation of the solution in Fig. 3.11 (a) in the context of the ENSO system is that a strong El Niño event of unpredictable magnitude would take place every seven years, after a gradual build-up, followed every time by a strong La Niña. This solution is not realistic because El Niño events do not occur exactly every 7 years; in other words, one could argue that the influence of the 1:7 resonance tongue on the solution is too strong. The example in row (b) is similar to row (a), except that there are only two slightly different El Niño strengths. This example also shows that amongst the chaos, there are still regions in the \((c, \tau_n)\)-plane where the periodic solutions are stable. The solution in row (c) is more realistic because the time between El Niño events varies. It is also realistic in that the El Niño events...
3.1. Transition from the GZT model to the TSCJ model

Figure 3.12: Bifurcation sets of the TSCJ model (3.1) in the $(c, \tau_n)$-plane for $a = 0$ and $d_l = -2$ (a), $a = 2.02$ and $d_l = -2$ (b), $a = 0$ and $d_l = -0.4$ (c), and $a = 2.02$ and $d_l = -0.4$ (d). The blue curves are saddle-node bifurcations of $p:q$ periodic orbits for all $p < q \leq 12$. The upper/lower boundary of each resonance tongue is drawn in dark/light blue. The red and black curves are torus bifurcations (T) and period-doubling (PD) of the 1:1 solutions, respectively. Other parameters are $b = 3.03$, $d_u = 2$, $\kappa = 3.63$ and $\tau_p = 0.0958$.

Figure 3.12: Bifurcation sets of the TSCJ model (3.1) in the $(c, \tau_n)$-plane for $a = 0$ and $d_l = -2$ (a), $a = 2.02$ and $d_l = -2$ (b), $a = 0$ and $d_l = -0.4$ (c), and $a = 2.02$ and $d_l = -0.4$ (d). The blue curves are saddle-node bifurcations of $p:q$ periodic orbits for all $p < q \leq 12$. The upper/lower boundary of each resonance tongue is drawn in dark/light blue. The red and black curves are torus bifurcations (T) and period-doubling (PD) of the 1:1 solutions, respectively. Other parameters are $b = 3.03$, $d_u = 2$, $\kappa = 3.63$ and $\tau_p = 0.0958$.

occur about every 4–5 years. Notice that the time series also reproduces decadal variability. It is the most convincing time series from the examples shown and demonstrates that the mechanisms described by the TSCJ model are adequate for reproducing important features of ENSO data; in particular, noise is not necessary to reproduce the irregularity of the frequency and amplitude of El Niño events.

3.1.7 The importance of an asymmetric coupling function

We have established that in the $(c, \tau_n)$-plane of Fig. 3.10 chaotic behaviour, leading to qualitatively realistic solutions, exists over a large range of parameters. Now we investigate whether this is due more to the positive delayed feedback or to the asymmetric coupling function. Figure 3.12 shows bifurcation sets in the $(c, \tau_n)$-plane for the TSCJ model with neither positive
feedback or asymmetric coupling (a), only positive feedback (b), only asymmetric coupling (c), and both positive feedback and asymmetric coupling (d). Specifically, panels (a)–(d) are calculated for $a = 0$ and $d_l = -2$, $a = 2.02$ and $d_l = -2$, $a = 0$ and $d_l = -0.4$, and $a = 2.02$ and $d_l = -0.4$, respectively. In order to highlight the movement of the resonance tongues in response to the additional model features, the presented range of parameters across the $(c, \tau_n)$-plane is larger compared to earlier figures, and the bifurcation sets now include all $p:q$ resonance tongues for $p < q \leq 12$.

In each panel of Fig. 3.12 the resonance tongues are rooted on the zero-forcing line on one side and on the curve $T$ on the other side. In panels (a) and (b), with a symmetric coupling function, the resonance tongues are ordered in neat rows with only some overlapping along the edges of the resonance tongues. On the other hand, in panels (c) and (d), with an asymmetric coupling function, the resonance tongues overlap each other to a much larger extent. As seen in earlier bifurcation sets, it is inside regions where the resonance tongues overlap that period-doubling cascades occur to create chaotic regions in the parameter plane. Therefore, Fig. 3.12 implies that the asymmetric coupling function contributes more to this chaos-inducing process than the addition of the positive delayed feedback.

In [66] the authors studied a more sophisticated coupled ocean-atmosphere model, which describes the processes of a linear shallow water equatorial ocean driven by a wind stress that depends on the thermocline depth at the eastern boundary. They suggested that adding asymmetry to the coupling function encourages aperiodicity. We find this to be accurate in the TSCJ model over a large range of parameters. However, it should be noted that there are special cases (for example, in Fig. 3.8(b)) in which increasing $d_l$ can bring the solution from a quasi-periodic or a chaotic state into a region of frequency locking making it periodic. Nonetheless, in general, this section illustrated the importance of the asymmetry in obtaining not only aperiodic, but chaotic solutions in the model over a large range of parameters.

With the transition from the GZT model to the TSCJ model complete, we conclude that for the range of parameters considered, the two additional model features do have a very significant effect on observed solution types. The additional model features bring chaotic behaviour into the system over a large range of parameters, with the asymmetric coupling function appearing to play a more important role than the positive delayed feedback. Although the GZT model can produce rich and complicated dynamics, the TSCJ model should be considered a more accurate model that is capable of creating solutions with more realistic features; in particular, it reproduces better the irregular and unpredictable nature of observable ENSO data.
3.2 Bifurcation analysis of the full TSCJ model

In the previous section we have presented bifurcation sets in parameters $a$, $c$, $d_t$ and $\tau_n$. To offer a comprehensive picture of the possible dynamics of the TSCJ model, we now present bifurcation sets that include parameters $\kappa$ and $\tau_p$. Note that the influence of the remaining parameters, $b$ and $d_u$, can be determined by simply rescaling them with respect to the other parameters; cf. Eq. (3.3). First, we focus on $\kappa$ as a bifurcation parameter in order to understand the observations made in [98]. In section 3.2.1 we review the main result from [98], which is a transition from a periodic solution to a chaotic solution that is observed in simulations while varying $\kappa$. We detail how this transition occurs via a cascade of period-doubling bifurcations in section 3.2.2. In section 3.2.3 we demonstrate that the same chaotic solution can be obtained by an intermittency route. We discuss how robust these transitions to chaos are and explore further effects of changing $\kappa$ by considering the $(c,\kappa)$-plane in section 3.2.4. Finally, in section 3.2.5 we present a bifurcation set in the $(\tau_p,\tau_n)$-plane to see how sensitive solutions to the TSCJ model are to changes in the delay times within a realistic range of values.

3.2.1 A transition to chaos in $\kappa$

In the investigation by Tziperman et al. in [98], the TSCJ model is given by Eq. (3.1) with the parameters from row 3 in Table 3.1. Upon increasing $\kappa$ from 0.9 to 2.0, Tziperman et al. observed periodic, quasi-periodic and finally chaotic behaviour. This first demonstrated that the irregularity seen in ENSO data could be modelled (even by a single, relatively simple delay differential equation) as chaotic behaviour intrinsic to the system, rather than requiring the addition of external noise.

Figure 3.13 displays solutions for varying $\kappa$ with the specific parameters used in [98]. Row (a) is for $\kappa = 0.9$ and shows a periodic time series in panel (a1). The stroboscopic trace and power spectrum in panels (a2)–(a3), respectively, confirm that the period is one. Row (b) is for $\kappa = 1.2$ and shows a quasi-periodic solution. In the stroboscopic trace in panel (b2) points form a closed loop and there are distinct peaks in panel (b3). Row (c) is evidence for a periodic solution when $\kappa = 1.5$. The stroboscopic trace in panel (c2) contains four points and the dominant frequency in panel (c3) is four years. If $\kappa$ is increased to 2.0, as is the case in row (d), the solution becomes chaotic. The time series in panel (d1) is irregular and the power spectrum in panel (d3) is broad.

The solutions shown in Fig. 3.13 embody a transition to chaos. Based on these solutions, the authors of [98] attribute the chaotic behaviour at $\kappa = 2.0$ to the coexistence of mode-locked solutions, in other words, to overlapping resonances. Those authors suggested that different mode-locked solutions coexist once the strength of nonlinearity in the model, which is proportional to $\kappa$, is sufficiently large. We now investigate this claim by means of bifurcation...
Figure 3.13: Stable solutions found by numerical integration of the TSCJ model (3.1) for \( \kappa = 0.9 \) (a), \( \kappa = 1.2 \) (b), \( \kappa = 1.5 \) (c) and \( \kappa = 2.0 \) (d) displayed as a time series (left column), as a stroboscopic trace in the \((h(t), h(t-\tau_p), h(t-\tau_n))\)-space (middle column), and as a power spectrum on a logarithmic scale in arbitrary units [a.u.] (right column). Other parameters are \( a = 2.02, b = 3.03, c = 2.6377, d_u = 2, d_l = -0.4, \tau_p = 0.0958 \) and \( \tau_n = 0.4792 \).

analysis to identify the boundaries of resonances in the parameter space.

### 3.2.2 Period-doubling route to chaos in \( \kappa \)

Figure 3.14 is a one-parameter bifurcation diagram in \( \kappa \) for fixed \( c = 2.6377 \). The vertical axis displays the largest value from the time series. Stable (blue) solutions are calculated by numerical integration of Eq. (3.1) for increasing \( \kappa \). As for the maximum maps shown
3.2. Bifurcation analysis of the full TSCJ model

Figure 3.14: A one-parameter bifurcation diagram of the TSCJ model (3.1) for $\kappa$ with stable (blue) solutions calculated by numerical integration, and unstable (red) solutions found using DDE-Biftool. Blue, red and black dots represent saddle-node bifurcations of periodic orbits, torus bifurcations and period-doubling bifurcations, respectively. Labels $p:q$ denote the resonance type. Grey lines mark the $\kappa$ values considered in Fig. 3.13. Other parameters are $(a, b, c, d, \tau_p, \tau_n) = (2.02, 3.03, 2.6377, 2, -0.4, 0.0958, 0.4972)$.

earlier, transients are excluded. The unstable (red) solutions are found by continuation with DDE-Biftool. The blue, red and black dots represent saddle-node bifurcations of periodic orbits, torus bifurcations and period-doubling bifurcations, respectively. The grey vertical lines indicate the $\kappa$ values of solutions (a)–(d) in Fig. 3.13.

The bifurcation diagram in Fig. 3.14 agrees very well with the observations made in [98], that is, the solutions displayed in Fig. 3.13. While increasing $\kappa$ in Fig. 3.14 we can follow how the solution changes. The 1:1 periodic solution loses stability at a torus bifurcation at $\kappa \approx 1.11$. A stable torus bifurcates, upon which lies solution (b). At $\kappa \approx 1.45$ the solution enters the 1:4 resonance tongue. The locked 1:4 periodic orbit (c) is passed before the period-doubling bifurcations occur, marked by black dots, after which the observed solution is chaotic. The 1:5 resonance tongue is entered at $\kappa \approx 1.75$ and the solution becomes periodic again. Finally, more period-doubling bifurcations are encountered inside the 1:5 resonance tongue before reaching the chaotic solution (d) at $\kappa = 2.0$. Afterwards, the solution becomes periodic again at $\kappa \approx 2.05$, where the 1:6 resonance tongue is entered.

Notice that solution (d) is located where both the 1:4 and 1:5 resonances coexist, confirming the claim made in [98] that the chaotic solution is due to overlapping resonances. It was also suggested in [98] that the chaotic solution (d) is obtained, while increasing $\kappa$,
via the quasi-periodic route to chaos and represents the system jumping irregularly between different possible resonances. Figure 3.14 illustrates that, at $\kappa = 2.0$, the 1:4 and 1:5 periodic solutions are unstable and there is no bistability, so that the system does not “jump” between resonances; rather, the chaotic behaviour seen in Fig. 3.13 is the result of a cascade of period-doubling bifurcations inside the overlapping resonance tongues.

### 3.2.3 Intermittency route to chaos in $\kappa$

Possible routes to chaos that account for irregular behaviour in ENSO models have been discussed at length in the past; for example, see the literature reviews in [79, 83, 102]. In various ENSO models a route to chaos has been observed and identified as either the period-doubling route [18, 21, 66], the quasi-periodic route [50, 96, 97] or the intermittency route [103]. We now demonstrate that in the TSCJ model different types of routes to chaos coexist and depend only on how the parameters are changed.

Aside from the period-doubling route to chaos seen when increasing $\kappa$ to 2.0 in section 3.2.2, an alternative route to the same chaotic attractor can be observed when $\kappa$ is decreased to 2.0. Figure 3.15 shows time series of Eq. (3.1) for fixed $c = 2.6377$ at $\kappa = 2.1$ (a), $\kappa = 2.045$ (b) and $\kappa = 2.0$ (c). Panel (a) is periodic with a period of 6; panel (b) is chaotic but with long windows of periodic-like behaviour; panel (c), on the other hand, is chaotic without any near-periodic intervals. Figure 3.15 is evidence of a so-called intermittent transition [74], whereby chaos suddenly appears at a saddle-node bifurcation. In this case, the chaos appears after crossing the lower boundary of the 1:6 resonance tongue, shown at $\kappa \approx 2.05$ in Fig. 3.14. As $\kappa$ is decreased, after exiting the 1:6 resonance tongue, the windows of periodic behaviour become shorter until they completely disappear. Co-existing routes to chaos may also be present in other ENSO models, since this phenomenon has been found in other dynamical systems in the past; for example, see [105].

### 3.2.4 The $(c, \kappa)$-plane of the TSCJ model

We now discuss the robustness of the observed chaotic solution and its transitions from periodic solutions. Figure 3.16 shows a bifurcation set in the $(c, \kappa)$-plane with curves of saddle-node bifurcations of periodic orbits (blue), torus bifurcations (red) and period-doubling bifurcations (black). The labels $p:q$ denote regions of locked periodic solutions. The colour scheme displays the maximal Lyapunov exponent; for clarity, colour is only used when it is positive and the corresponding solution is chaotic. The numerical computation of the maximal Lyapunov exponent was performed with the algorithm for DDEs described in [37]. In Fig. 3.16 the maximal Lyapunov exponent was calculated from simulations running for fixed $c$ and gradually increasing $\kappa$, so some multistabilities may not be represented. Also shown in Fig. 3.16 are the parameter points giving the solutions (a)–(d) referred to in [98] and displayed
Figure 3.15: Time series of stable solutions found by numerical integration of the TSCJ model (3.1) for $\kappa = 2.1$ (a), $\kappa = 2.045$ (b) and $\kappa = 2.0$ (c). Other parameters are $a = 2.02$, $b = 3.03$, $c = 2.6377$, $d_u = 2$, $d_l = -0.4$, $\tau_p = 0.0958$ and $\tau_n = 0.4972$.

In Fig. 3.13, In Fig. 3.16 a curve of torus bifurcations exists for low values of $\kappa$. Resonance tongues are rooted at the zero-forcing line $c = 0$ and contain cascades of period-doubling bifurcations. Some period-doubling cascades and associated positive maximal Lyapunov exponent appear in regions where the displayed resonance tongues are not overlapping. This is simply because smaller, higher-order resonance tongues exist in-between those shown and will overlap.

In Fig. 3.16, the parameter point of solution (a) in Fig. 3.13 is located in the parameter region dominated by the seasonal forcing; hence, it is a solution of period $T = 1$. When increasing $\kappa$ for fixed $c = 2.6377$ the solution loses its stability at a torus bifurcation, beyond which both periodic and quasi-periodic solutions exist. The parameter point of solution (b) in Fig. 3.13 is not inside a shown resonance tongue; hence it was observed in [98] to be a
Figure 3.16: Bifurcation set for the TSCJ model (3.1) in the \((c, \kappa)\)-plane. Saddle-node bifurcations of periodic orbits (blue), torus bifurcations (red) and period-doubling bifurcations (black) are shown with the \(p:q\) resonances of periodic solutions indicated. Points a to d show the locations in the \((c, \kappa)\)-plane of the parameters corresponding to solutions studied in [98] and shown in Fig. 3.13. The colour scheme indicated the strength of the maximal Lyapunov exponent when it is positive. Other parameters are \(a = 2.02, b = 3.03, d_u = 2, d_l = -0.4, \tau_p = 0.0958\) and \(\tau_n = 0.4972\).

quasi-periodic solution. The parameter point corresponding to solution (c) in Fig. 3.13 is located within the 1:4 resonance tongue; this also agrees with [98]. A more interesting case is solution (d) in Fig. 3.13, which is located in a parameter region of overlapping resonance tongues in Fig. 3.16. Furthermore, the bifurcation curves crossed while increasing \(\kappa\) for fixed \(c = 2.6377\) also exist for other values of \(c\). More specifically, according to Fig. 3.16, the same changes of solution type would have been observed not only for fixed \(c = 2.6377\), but for any fixed \(1.5 \lesssim c \leq 3.2\) and possibly for larger \(c\), albeit for slightly different values of \(\kappa\). Therefore, the transition to chaos observed in [98] and illustrated in Fig. 3.13 is a prominent feature of the TSCJ model, leading to chaotic behaviour across a substantial range of parameters. This is confirmed by the maximal Lyapunov exponent, which is positive over large regions of the parameter plane.

In order to identify further features of the TSCJ model and obtain a more complete picture of the possible dynamics in the \((c, \kappa)\)-plane, we now present a bifurcation set over a larger range of parameters in Fig. 3.17. The (blue) saddle-node bifurcation curves represent examples of resonance tongues that are rooted on the (red) torus bifurcation curve; the (blue) curves representing the resonance tongues rooted on the zero-forcing line at \(c = 0\)
3.2. Bifurcation analysis of the full TSCJ model

Figure 3.17: Bifurcation set for the TSCJ model (3.1) over a larger region of the \((c, \kappa)\)-plane. Saddle-node bifurcations of periodic orbits (blue), torus bifurcations (red) and period-doubling bifurcations (black) are shown with the \(p:q\) resonances of periodic solutions indicated. For clarity, the resonance tongues already shown in Fig. 3.16 (rooted along the \(c = 0\) line) are shown fainter. The colour scheme indicates the strength of positive Lyapunov exponent. Other parameters are \(a = 2.02, b = 3.03, d_u = 2, d_l = -0.4, \tau_p = 0.0958\) and \(\tau_n = 0.4972\).

are shown faint (cf. Fig. 3.16). The torus bifurcation curve in Fig. 3.17 terminates at a (black) period-doubling curve, where the 1:1 periodic solutions (dominated by the seasonal forcing) lose stability; beyond this point there is a cascade of period-doubling curves of the 1:1 periodic solution. Again, the maximal Lyapunov exponent is displayed by a colour scheme. As discussed in section 3.3, Fig. 3.17 is reminiscent of the bifurcation set of a similar ENSO model shown in Fig. 6 of [57].

In Fig. 3.17 we find chaotic behaviour across large regions of the \((c, \kappa)\)-plane, as indicated by the maximal Lyapunov exponent being positive. The torus bifurcation curve is another source of resonance tongues that overlap with each other, as well as with the resonance tongues rooted on the zero-forcing line. The bifurcation set in Fig. 3.17 shows that the number of overlapping resonance tongues increases with the value of \(\kappa\), confirming the suggestion made in [98] that the nonlinearity in the model must be sufficiently large in order for this to happen. Importantly, Fig. 3.17 shows that for larger values of \(c\) a period-doubling cascade of the 1:1 periodic solutions occurs, demonstrating that chaotic solutions can be obtained by increasing \(\kappa\) via a period-doubling route to chaos even without the presence of overlapping resonance tongues.
Figure 3.18: Maximum maps in the \((\tau_p, \tau_n)\)-plane for increasing \(\tau_p\) of the TSCJ model (3.1). Blue and black curves represent saddle-node of periodic orbits and period-doubling bifurcations, respectively. Note that although only the first set of period-doubling bifurcations are shown in each resonance tongue, they do all contain period-doubling cascades. Other parameters are \(a = 2.02\), \(b = 3.03\), \(c = 2.6377\), \(d_u = 2\), \(d_l = -0.4\), \(\kappa = 3.63\).

### 3.2.5 Delay sensitivity

We now address the issue of parameter uncertainty in the TSCJ model and determine how sensitive stability and existence of solutions are to variations in parameters. We focus on the delay parameters in Eq. (3.1) and investigate the quantitative and qualitative effects they have on the system within a range of realistic values. These are motivated by calculations of oceanic wave velocities based on measurements of sea surface anomalies by the TOPEX/POSEIDON satellite [13].

The authors of [13] calculated mean phase velocities and error estimates of one standard deviation as \(v_K = 2.7 \pm 0.9 \text{ ms}^{-1}\) and \(v_R = -1.0 \pm 0.2 \text{ ms}^{-1}\), where \(v_K\) is the velocity for eastward propagating Kelvin waves and \(v_R\) is the velocity of westward propagating first meridional mode Rossby waves. Here, we only consider the first meridional mode because, as part of the negative feedback mechanism, it is the dominant mode in Pacific western boundary reflections [26]. The width of the Pacific Ocean near the equator is approximately 15,500 ± 1,600 km, so these wave velocities correspond to mean delays with uncertainties; expressed as \(\tau_p \in [0.0612, 0.1506]\) and \(\tau_n \in [0.3061, 0.6401]\).

Figure 3.18 shows a maximum map and bifurcation set for the TSCJ model in the \((\tau_p, \tau_n)\)-plane. Note that the ranges of values of \(\tau_p\) and \(\tau_n\) contain the uncertainty bounds given
above. Other parameters are given by the nominal values used in [98], i.e. $a = 2.02$, $b = 3.03$, $c = 2.6377$, $d_a = 2$ and $d_l = -0.4$. The parameter $\kappa$, which is free in [98], is set to 3.63, which corresponds to the time series in Fig. 3.11(c1). This time series is considered to be more convincing compared to Fig. 3.13(d1) with $\kappa = 2.0$, because the El Niño events are more pronounced. Figure 3.18 shows parts of resonance tongues bound by saddle-node bifurcation curves with period-doubling curves inside. For clarity, only the first period-doubling bifurcation in each resonance tongue is shown; however, in each case a full cascade does exist. Also, for high $\tau_p$ and low $\tau_n$ values, there is a period-doubling cascade of the 1:1 solutions dominated by seasonal forcing.

The bifurcation curves in Fig. 3.18 extend predominantly into the $\tau_p$-direction, meaning that changes in $\tau_p$ are less likely to affect the observed attractor. Qualitatively, the parameter $\tau_p$ has no effect, except where bifurcation curves intersect. When intersections do occur in Fig. 3.18, which is only on a few occasions, the angles are so small that any qualitative changes take place very gradually. On the other hand, when varying $\tau_n$ within its uncertainty range of one standard deviation, the observed attractor may cross many bifurcation curves. This affects the number and types of coexisting resonances and, therefore, can affect the type or quantitative properties of the solution. Solutions calculated for different values of $\tau_n$ (not shown) result in time series that correspond to very different ENSO scenarios. Figure 3.18 shows that any solution loses its stability with relatively small changes in $\tau_n$; in other words, the observed dynamics is much more sensitive to changes in the delay $\tau_n$ of the negative feedback than it is to changes in the delay $\tau_p$ of the positive feedback.

### 3.3 Conclusions regarding the TSCJ model

This chapter considered the TSCJ model for the ENSO system from [98], with a positive and negative feedback and seasonal forcing. It is significant because, despite its relatively simple form, it was demonstrated in [98] to produce chaotic behaviour that resembles the irregular and unpredictable nature characteristic of the ENSO system. However, this observation of chaotic behaviour was only for a single point in parameter space and the dynamical processes involved in this observation were not entirely clear.

Here, we addressed how prominent the chaotic behaviour is in the TSCJ model and also where and how it arises when parameters are varied. More specifically, we presented an overall bifurcation analysis of the TSCJ model as a function of its parameters. This allowed us to determine the possible dynamics of the model and detail the qualitative changes and multistabilities that can give rise to complicated behaviour, including transitions to chaos.

The starting point of our work was the simpler GZT model [40], without positive feedback and with a symmetric ocean-atmosphere coupling function. We gradually introduced the additional model features of the positive feedback and asymmetry, one by one, to demonstrate
how each of them influences the dynamics. We found that the asymmetry in the coupling function is most important when it comes to creating more realistic solutions across large regions of parameters. We then explained the transition to chaos observed in [98] for increasing ocean-atmosphere coupling strength $\kappa$ as the result of emerging period-doubling cascades within overlapping resonance tongues. This results in large regions of chaotic dynamics in parameter space, as was checked by computations of the maximal Lyapunov exponent. Furthermore, for decreasing $\kappa$ the same chaotic solution can be reached via an intermittency route to chaos. Finally, the analysis of the delay times, $\tau_p$ and $\tau_n$, showed that the observed behaviour (with all other parameters at their nominal values) is not sensitive to changes in $\tau_p$, but very sensitive to changes of the delay $\tau_n$ of the negative feedback.

The results presented in this chapter have shown that, while the GZT model is capable of reproducing some characteristic features of the ENSO system, the TSCJ model is able to create solutions with more realistic irregular behaviour and pronounced peaks, significantly larger than other regular peaks, in the thermocline depth that can be interpreted as distinct El Niño events. In the future it will be interesting to see the effects of less stringent modelling assumptions on how realistic model behaviour can be for plausible parameter regimes. For example, the sensitivity analysis for the delay times suggests that much of the ENSO variability could be captured by considering non-constant delays. This will be an especially intriguing and challenging topic for future work, as is discussed in some more detail in chapter 5.
Chenciner bubbles in an ENSO model

Climate tipping, which describes transitions between climate states, has received much attention in recent years. A tipping point itself is defined in [62] as a critical point at which small perturbations may switch the future state of the system to a qualitatively different state. Climatic tipping points have been hugely influential for the global climate in the past, for example, about 10,700 years ago when the Arctic warmed by 7°C within 50 years [30], and may be equally influential in the context of present day climate change.

ENSO is one of several potential climate tipping elements identified in [62]. For instance, it was suggested in [95] that an increase in heat uptake by the Pacific Ocean could lead to a deeper thermocline in the eastern equatorial Pacific, resulting in an ENSO variability of greater amplitude and/or frequency compared to current conditions. An alternative scenario described in [17] suggests that adequate warming of the western equatorial Pacific could lead to permanent La Niña conditions.

When considering a climate system as a dynamical system, tipping points can be thought of as certain bifurcations. This is useful because lessons from bifurcation theory can help identify precursors to an imminent tipping event from a time series of observable data. Such precursors can be interpreted as valuable early warning signs [93]. A common approach to predict a tipping event uses the fact that a quasi-static steady-state becomes increasingly less stable as it approaches a bifurcation. One interprets the observed dynamics in a time series as a branch of stable steady-states being perturbed by noise. When perturbed by noise, the system will relax back towards the steady-state; however, as the steady-state approaches an instability, this relaxation time becomes longer. The so-called local (linear) decay rate (LDR) quantifies the slowing down of transients as the instability is approached. Held and Kleinen [45] demonstrated the use of auto-regressive modelling to determine LDR information from noise fluctuations in data from a climate model of intermediate complexity. Livina and Lenton [63] and Dakos et al. [29] developed this method further in order to apply it to recorded data, and they detected ancient tipping events, including the above-mentioned rapid warming of the Arctic and the desertification of North Africa. For a useful review relating climate tipping
to bifurcation theory see [93].

In [94] generic codimension-one bifurcations are classified into the categories “safe”, “explosive” and “dangerous”. “Safe” bifurcations are those that exhibit a gradual change to a nearby attractor. A sudden change in the size of the attractor occurs in an “explosive” bifurcation, but the trajectories in phase space remain near the old attractor. “Dangerous” bifurcations are those relevant to climate tipping, because a small perturbation will cause a jump to some unknown and generally far away attractor. Hysteresis may be present. In particular, in the context of climate the tipping event is irreversible by another small local perturbation.

An example of a “dangerous” bifurcation is the fold bifurcation, also known as the saddle-node bifurcation. Here, two co-existing steady-states of different stabilities meet, merge and mutual annihilate each other as a single parameter is changed. This bifurcation is typically encountered in one-parameter bifurcation diagrams. There are other “dangerous” bifurcations, such as the subcritical Hopf bifurcation. In this chapter, however, we focus on the fold bifurcation.

The simplest form of the fold bifurcation occurs when the steady-states involved are equilibria. A fold bifurcation of equilibria can take place in a phase space of dimension \( N \geq 1 \) and has been observed in many climate models. An example of a fold of equilibria appearing in the context of climate tipping is found in [52], where an energy balance equation is used to calculate equilibrium solutions for the mean global temperature. One saddle and two stable equilibria are found, which bifurcate at two fold bifurcations when varying the control parameter, which is the mean solar energy that falls onto one unit area of the Earth’s atmosphere per second.

Figure 4.1 shows a schematic illustration of a fold bifurcation of equilibria. As some parameter \( \rho \) is slowly increased or decreased, a (blue) stable node and a (green) saddle collide and annihilate each other. There are actually two fold bifurcations in Fig. 4.1 that form a hysteresis loop, as indicated by the black arrows.

In the next simplest case of fold bifurcation, the steady-states involved are periodic orbits, which requires a phase space of dimension \( N \geq 2 \). This case is highly relevant in climate science, because solar variation from diurnal, annual or Milankovich cycles is common in climate systems. Periodic forcing means that the topologically simplest solutions of the system are periodic orbits. Folds of periodic orbits have been found, for example, in an energy balance model for sea ice [34]. In this model the energy per unit area of ice and water is seasonally forced, resulting in two fold bifurcations of periodic orbits when varying a parameter that represents perturbations in surface heat flux. In [107], the authors discuss precursors that can be derived for tipping points of systems that are periodically forced. For example, they show that changes in the amplitude response to the periodic forcing and phase lag between the forcing and resulting oscillations can provide evidence of an imminent
bifurcation. Figure 4.1 provides also the schematic bifurcation diagram of a fold of periodic orbits, when the y-axis represents a scalar such as the amplitude of the periodic orbit or an averaged position over the time of one period.

The next step in complexity of fold bifurcations is the case that the steady-state solutions are on smooth invariant tori, on which the dynamics may be either locked (periodic) or unlocked (quasi-periodic). This requires that the phase space has dimension $N \geq 3$ and the system features at least two internal or external frequencies. In fact, oscillations involving multiple frequencies across a wide range are common in global climate systems [38], making dynamics on tori a likely occurrence. The aperiodic nature of ENSO is well known. ENSO is often associated with quasi-periodic behaviour, for example [31, 73]. We have seen in chapter 2 that quasi-periodic behaviour is commonly observed in the GZT model from section 1.1.2. Nonetheless, folding of invariant tori have not yet been considered as a tipping mechanism.

Compared to fold bifurcations of equilibria or periodic orbits, fold bifurcations of invariant tori are technically more challenging. In fact, the term “bifurcation” is used loosely here, because a fold bifurcation of two smooth invariant tori does not actually exist as such. Namely, the two smooth invariant tori involved cannot merge. As the two tori approach the fold locus, they lose normal hyperbolicity (smoothness) and then break up in this bifurcation scenario, which involves complicated dynamics [5, 14].

Figure 2.9 is a one-parameter bifurcation diagram with branches of invariant tori in the GZT model. It is similar to Fig. 4.1; seemingly featuring two fold bifurcations of tori and a hysteresis loop. However, a closer analysis of the solution curve near the fold points in Fig. 2.9 would reveal that they are in fact not smooth, because the tori on which the solutions lie must break up.
It turns out that folding tori are studied best as a phenomenon in a two-dimensional parameter space. As we have seen in section 2.3, it involves folds of curves of saddle-node bifurcations of periodic orbits, which bound resonance tongues, where the dynamics on tori is locked. The starting point of our investigation here is Fig. 4.2, which shows maximum maps of the GZT model in the \((c, \tau_n)\)-plane for increasing \(c\) in panel (a) and for decreasing \(c\) in panel (b). As in the previous chapters, these have been calculated with the Euler method by gradually varying the parameter \(c\) at a small step size and disregarding transients. Each panel is overlaid with a bifurcation set calculated with DDE-Biftool, where blue and red curves represent saddle-node bifurcations of periodic orbits and torus bifurcations, respectively. The blue bifurcation curves of periodic orbits represent boundaries of resonance tongues with one boundary of each resonance tongue shown in lighter blue. This figure is a version of Figs. 2.10 and 2.11 that includes the boundaries of the 2:7 resonance tongue, which we will study in detail in this chapter, and clearly shows how the resonance tongues bend over. Note also that within some resonance tongues, stripes are evident as a result of symmetry-related bistabilities, as discussed in section 2.2.2.

There is a clearly visible sharp interface in each maximum map of Fig. 4.2, where the maximum of the observed solution changes rapidly from large/small to small/large as \(c\) is increased/decreased. The sharp interfaces represent boundaries, where either a locked or unlocked stable solution is lost, as \(c\) is varied. Note that in between these two sharp interfaces bistability is present. Of course, as mentioned above, a fold of tori does not exist as a single bifurcation. Instead, the transition from a stable torus to a torus of saddle-type is described by complicated dynamics involving several bifurcations of both locked and unlocked solutions [8, 99]. That there is no smooth curve of folds of tori can be seen in Fig. 4.2(a), for example, where the sharp interface is disrupted by the folding 2:7 resonance tongue. Although the shapes of interfaces in Fig. 4.2 often appear smooth, this is only because of the scale of the maximum maps. Enlarging any segment of a sharp interface representing a fold of tori would eventually reveal more detailed features.

It is also evident in Fig. 4.2 that the blue curves bounding the resonance tongues have local minima and maxima in \(c\), such that the resonance tongues bend or fold in \(c\). These bending resonance tongues coincide with the sharp interfaces of the maximum maps. Specifically, the envelope of bends of resonance tongues form boundaries where tori fold. This is because some of the solutions that undergo a fold of tori are locked solutions on the torus \((p:q)\) periodic orbits or torus knots) and these are organised into resonance tongues. Analysing the solutions inside the resonance tongues with DDE-Biftool reveals that, before the first fold and after the second fold of tori, the resonance tongues contain a set of stable and saddle periodic solutions on invariant tori. In other words, the tori are stable. Naturally, this statement is not valid near the boundary where the tori no longer exist. In between the two folds of resonance tongues, all periodic orbits have at least one unstable Floquet multiplier, meaning
Figure 4.2: Maximum maps with increasing $c$ (a) and decreasing $c$ (b). The blue and red curves are saddle-node bifurcations of $p:q$ periodic orbits (SN) and torus bifurcations (T), respectively. One boundary of each resonance tongue is drawn in a lighter blue. Here $b = 1$ and $\kappa = 11$. 
that they lie on a torus of saddle-type. Inside each resonance tongue, where it bends, a so-called Chenciner bubble forms [8]: within such a bubble an invariant torus undergoes a transition from stable to saddle-type via a loss of normal hyperbolicity and subsequent break up. Therefore, analysing such Chenciner bubbles offers a good approach to understanding the phenomenon of folding tori.

The focus of this chapter is to unravel the complicated dynamics inside the Chenciner bubbles of the GZT model. This is the first time that Chenciner bubbles have been analysed in a DDE. We focus on the 2:7 resonance tongue, which is the largest visible in Fig. 4.2 and will form the largest Chenciner bubble. We use numerical techniques including continuation software to detail the bifurcation structure of the 2:7 Chenciner bubble.

We find that the bifurcation structure we uncover in the Chenciner bubble agrees well with the theoretical structure suggested in [8] for an ordinary differential equation model. As part of this structure, we detect curves of torus bifurcations, neutral saddles, homoclinic bifurcations of periodic orbits, heteroclinic bifurcations of periodic orbits and fold bifurcations of tori in the GZT DDE model. The only difference we find between the detected bifurcation structure and the theoretical one is that the torus bifurcation inside the Chenciner bubble we study is subcritical, instead of supercritical. We give detailed examples of how we calculate stroboscopic traces, a type of Poincaré trace, for the DDE model in order to detect the bifurcations that involve tori. Based on the additional bifurcations that take place within Chenciner bubbles, we discuss their potential role as a climate tipping mechanism.

The chapter is organised as follows. First, we review some of the theory behind the bifurcation structure suggested in [8] in section 4.1. We then present in section 4.2 the structure of a Chenciner bubble of the GZT model. In section 4.2.1 we determine the criticality of the torus bifurcation. We present detailed examples of transitions through certain bifurcations in the Chenciner bubble of the GZT DDE model in sections 4.2.2 and 4.2.3. Finally, in section 4.3 we discuss how our results relate to climate tipping and point to some interesting areas of future work.

4.1 Theory of Chenciner bubbles

The theoretical results in [8] provide a description of what should occur inside a Chenciner bubble. The authors begin with a vector field in three dimensions, which is the minimum number of dimensions to allow for dynamics on an invariant torus. With a winding number of $p/q$ on the torus, there exist two $p:q$ periodic orbits (one saddle and one stable or unstable). The authors assume an arbitrary $p:q$ locking on the torus and that the associated resonance tongue bends in an appropriate parameter plane. It is necessary for the methods used in [8] that $q > 4$, so that the $p:q$ resonance is weak [42, 59]. A classic approach to studying a vector field in three dimensions is to take a Poincaré section to study the dynamics in a two-
dimensional subspace of the phase space transverse to the torus. All important information about the dynamics is retained when analysing the associated Poincaré map; tori in the vector field appear as closed invariant curves in the Poincaré section, periodic orbits appear as periodic points, etc. In particular, a $p:q$ periodic orbit corresponds to $q$-periodic points in the map. For weak resonances the symmetry of the periodic orbits can be divided out, so that any $q$-periodic orbit corresponds to just a single fixed point. By the method of averaging, one then finds a two-dimensional vector field whose time-one map is an approximation to the symmetry-reduced Poincaré map \cite{7,42,64,82}; this step is also referred to as obtaining a normal form approximation. It reduces the problem to the analysis of a vector field in two dimensions, instead of three. More specifically, because the symmetry is divided out of the Poincaré section, the phase space is a cylinder and the phase portraits only contain one saddle and one node. The authors of \cite{8} then apply topological arguments to obtain a minimal bifurcation diagram of this two-dimensional vector field on the cylinder for the parameter values near and inside the Chenciner bubble where the resonance tongue bends.

Figure 4.3 is reproduced from \cite{8}; it shows the simplest robust bifurcation set for the two-dimensional approximation of the Poincaré map in two parameters. Representative phase portraits are shown in open regions and along the bifurcation curves. The phase portraits in Fig. 4.3 are drawn such that the left and right equilibria need to be identified in order to obtain the dynamics of the vector field on the cylinder. Hence, the saddles on the left and right sides are in fact one and the same point. In the centre of the phase portraits are nodes, which may be stable or unstable. The thicker ovals are contractible periodic orbits (meaning that the orbit can be contracted to a single point on the cylinder) and the thicker curves that travel left or right across some phase portraits represent periodic orbits around the cylinder.

The bifurcation set includes curves of saddle-node bifurcations of equilibria (sne), where a saddle and node are created. The curves sne represent the boundaries of the resonance tongue folding with respect to the horizontal direction. They are bridged by several other bifurcation curves. The Hopf bifurcation curve intersects the two curves sne at the two points denoted B. Along the curve labelled Hopf, the node at the centre of the phase portrait changes stability and gives birth to a periodic orbit. The curve of neutral saddles (ns) also intersects the two curves sne at the points B. The curve ns does not represent a bifurcation, but it still plays a role in how the bifurcations are organised. Between the curves Hopf and ns are curves of contractible homoclinic bifurcations (chc). Along the two curves chc, the periodic orbit that forms at the curve Hopf disappears in a homoclinic orbit at the saddle. The two curves chc also intersect the curves sne at the points marked B. Two curves of rotational homoclinic bifurcations (rhc) cross the resonance tongue and intersect the curves sne at points Z. Along the two curves rhc, there is a homoclinic connection at the saddle that extends around the cylindrical phase space. The two curves rhc intersect each other, as well as the two curves chc, at point N. From the homoclinic orbits along the two curves
rhc bifurcate periodic orbits. However, these periodic orbits soon disappear at curves of saddle-node bifurcations of periodic orbits (snp). Each curve snp terminates on a curve rhc at a point labelled K. One of the curves snp also intersects a curve rhc at point X, so that three periodic orbits co-exist inside the triangle formed by NKX. All these curves form a minimal example of a bifurcation structure, which describes the changes that must occur as a resonance tongue folds. The bifurcation set in Fig. 4.3 is consistent, meaning that one can move between the phase portraits at any two points on the visible parameter plane by the shown bifurcation mechanisms. It is also robust in the sense that it is structurally stable in the space of two-dimensional vector field on a cylinder.

We briefly elaborate on the so-called saddle quantity, because it will be helpful later when discussing the Chenciner bubble of the GZT model. The saddle quantity is defined as $\nu_1 + \nu_2$, where $\nu_{1,2}$ are the eigenvalues of the Jacobian matrix of the two-dimensional vector field at the saddle point. It is equal to zero along the curve ns with both eigenvalues $\nu_{1,2}$ on the real axis. Because the sum of the eigenvalues at the stable equilibrium is also equal to zero along the curve Hopf, the curve ns is a natural continuation of the curve Hopf and they both form a closed curve in Fig. 4.3. The saddle quantity is also important for the curves rhc and chc. A homoclinic bifurcation is supercritical if the saddle quantity is positive and subcritical if it is negative, so the curve ns helps to clarify the criticality of the homoclinic bifurcation curves. The curves rhc and chc to the left of the curve ns are supercritical, while the curves rhc to the right are subcritical. Hence, in Fig. 4.3 the curves rhc change criticality at points K. It is a result of this change in criticality of the curves rhc that points K, which are codimension-2 bifurcations, must involve the emergence of curves snp of saddle-node bifurcations of periodic orbits.

The suggested bifurcation diagram in Fig. 4.3 has been obtained by topological considerations in [8] and serves as a useful guide for what we can expect to find in the GZT model. Of course, Fig. 4.3 represents only a minimal bifurcation structure. It cannot be ruled out that the curves are arranged differently in the parameter plane and/or further bifurcation curves exist. For example, in [8] an extended bifurcation structure is depicted for the case that the curves Hopf and ns intersect once to form a figure-eight shape.

The bifurcation set and phase portraits in Fig. 4.3 can be interpreted in terms of the original vector field in three-dimensions. To begin with, a node at the centre of phase portraits corresponds to a stable or unstable $p:q$ periodic orbit in the three-dimensional system, while the saddle point corresponds to a saddle $p:q$ periodic orbit. The stable and unstable periodic orbits on the cylinder correspond to tori that are stable and of saddle-type, respectively. Clearly, the curves sne in Fig. 4.3 correspond to the curves of saddle-node bifurcations of periodic orbits that are boundaries of the resonance tongues, such as those shown in Fig. 4.2. The Hopf bifurcation of the two-dimensional vector field, where a periodic orbit bifurcates from an equilibrium, corresponds to a torus (Neimark-Sacker) bifurcation in the three-dimensional
4.1. Theory of Chenciner bubbles

Figure 4.3: Theoretical minimal bifurcation structure inside a Chenciner bubble. Shown are curves of rotational homoclinic bifurcations (rhc), curves of saddle-node bifurcations of periodic orbits (snp), curves of contractible homoclinic bifurcations (chc), curves of saddle-node bifurcations of equilibria (sne), a curve of neutral saddles (ns) and a curve of Hopf bifurcations. The points Z, B, N, X and K mark intersections of bifurcation curves. Representative phase portraits are also shown. Reproduced with permission from C. Baesens and R.S. MacKay, *Resonances for weak coupling of the unfolding of a saddle-node periodic orbit with an oscillator*, Nonlinearity, 20(5):1283, 2007. ©IOP Publishing & London Mathematical Society. All rights reserved.

system, whereby a torus bifurcates from a periodic orbit.

Interpreting the curves rhc, chc and snp is more difficult. These curves correspond to “bifurcations” in the three-dimensional vector field that do not actually exist because they involve tori that must break up before the bifurcation locus is reached. Consider, first, the curve chc in its two-dimensional vector field. As a bifurcation parameter reaches its critical value, a periodic orbit on the two-dimensional cylinder disappears at a homoclinic connection, where the stable and unstable manifolds of the saddle meet. By perturbing the time-one map of the averaged vector field, one acquires the Poincaré map of the original three-dimensional system, and the homoclinic connection becomes a homoclinic tangle [42]. As the same bifurcation parameter is varied in the three-dimensional system, one encounters the following scenario. First, a torus breaks up in a homoclinic tangency, where the stable and
unstable manifolds of the saddle periodic orbit intersect only tangentially. Second, one passes through a parameter range with homoclinic tangle, where an infinite number of transversal intersections between the stable and unstable manifolds occur. In this parameter range, chaotic behaviour ensues. Finally, another homoclinic tangency is met, after which the stable and unstable manifolds have disentangled. In the process the torus, even as a continuous object, is gone. As such, there is no smooth curve of homoclinic bifurcations in the parameter plane for the three-dimensional case. Instead, the homoclinic bifurcation curve is fattened into a strip bound on either side by the first and last homoclinic tangencies. The first and last homoclinic tangencies are exponentially close to each other near where they meet at points B [8].

In the next section, when analysing the GZT model, we can detect the occurrence of a homoclinic tangle because the asymptotic behaviour of the unstable manifold involved in the homoclinic tangle changes from before and to after the two homoclinic tangencies. More specifically, the invariant object towards which the unstable manifold tends must change. As an example, compare Figs. 4.6(b3) and (d3), where there is a switch in the asymptotic behaviour of one set of (red) unstable manifolds before and after a homoclinic tangle. As it turns out, in the GZT model, we cannot distinguish between the first and last homoclinic tangencies. Therefore, in the next section a strip of homoclinic tangle is represented well by a curve representing “homoclinic bifurcations of tori”.

The same arguments apply for the curves rhc, which also corresponds to a region bound by homoclinic tangencies in the original three-dimensional vector field, near where the bifurcating torus breaks up. However, without the symmetry divided out of the \( p:q \) periodic orbits in the three-dimensional vector field, the homoclinic tangencies are actually heteroclinic tangencies. Therefore, the curves rhc correspond to strips of heteroclinic tangle bounded by first and last heteroclinic tangencies. The two heteroclinic tangencies become exponentially close to each other near where they meet at points Z. Again, in the next section, we find the heteroclinic tangencies in the GZT model to be indistinguishable, and we can represent the strip of heteroclinic tangle by a curve.

The existence of curves snp translates to “fold bifurcations of tori” in the three-dimensional vector field. As explained above, these fold “bifurcations” do not actually exist because the tori involved must break up. Thus, in our analysis of Chenciner bubbles, which we conduct in order to study folds of tori, we encounter further folds of tori. Note the irony. Theoretically, perturbations of the time-one map along the curve snp fattens the curve into a string of Chenciner bubbles each belonging to a resonance tongues of weak resonance [8]. Depending on the frequencies involved, some of these Chenciner bubbles will be exponentially small. In our analysis of a large Chenciner bubble of the GZT model in the next section, we find all other nearby Chenciner bubbles to be exceptionally small in relation to the parameter plane we consider. Therefore, we approximate the folding tori that correspond to the curves snp
4.2 Chenciner bubbles in the GZT model

We now take a closer look at one of the resonance tongues near the folds of tori in the GZT model. We choose to focus on the 2:7 resonance tongue, shown in Fig. 4.2. As mentioned above, it is necessary that we consider a weak resonance \( q > 4 \). Furthermore, the 2:7 resonance tongue is a low-order resonance tongue with the smallest period compared to other nearby resonance tongues. This helps the exposition as the respective region of the resonance tongue in the parameter plane is largest and the small period minimises the computational expense of running numerical calculations. The part of the 2:7 resonance tongue that folds and forms the Chenciner bubble should have a bifurcation structure that is at least as complicated as that suggested in [8]. First, we present the bifurcation structure that we could detect. We also provide details about how we detect the bifurcation curves that cannot be continued as curves of periodic orbits with DDE-Biftool. We then summarise the comparison between our bifurcation structure to the hypothesised structure.

Figure 4.4 shows the bifurcation structure in the 2:7 resonance tongue near where it folds and forms a Chenciner bubble. The folds occur with respect to \( c \) along the horizontal axis, rather than the vertical axis as in Fig. 4.3. The dark blue, red and black curves in Fig. 4.4 are saddle-node bifurcations of periodic orbits (SN), torus (T) bifurcations, and neutral saddles (NP), respectively. The curves T and NP form a closed curve, which is anchored on the two curves SN at points B. The light blue, light green and dark green curves are fold or saddle-node “bifurcations” of tori (SNT), homoclinic “bifurcations” (HoC) and heteroclinic “bifurcations” (HeC), respectively. The two curves HeC span across the resonance tongue, meeting the curves SN at points Z. In the parameter range shown in Fig. 4.4, only the upper points Z are shown. At points K the curves HeC intersect the curve NP. The curves SNT also terminate at these points. Both curves HeC intersect the curves HoC at point N. In contrast to Fig. 4.3, the curves HeC and SNT converge very quickly as they approach point N. As a result, the point X, where one curve SNT intersects a curve HeC, is indistinguishable from point N.

The curves SN, T and NP are calculated by continuation of periodic orbits with DDE-Biftool. The criticality of curve T and the stability of the torus that bifurcates there is addressed in section 4.2.1. The curve NP is defined by \( \mu_1 \mu_2 = 1 \), where \( \mu_{1,2} \) are the two leading Floquet multipliers of the 2:7 saddle periodic orbits. The curve NP is therefore

with a curve found by observing the switching behaviour of the unstable manifolds of the nearby saddle periodic orbit (rather than attempting to compute further folding resonance tongues). For an example, compare already Figs. 4.7(c3) and (d3); before the fold of tori a set of (red) unstable manifolds approach a stable torus, while after the fold of tori the unstable manifolds approach some remote attractor.
Figure 4.4: Bifurcation structure inside the Chenciner bubble at the folding of the 2:7 resonance tongue. Shown are dark green curves (HeC) of heteroclinic “bifurcations”, light blue curves (SNT) of folding tori, light green curves (HoC) of homoclinic “bifurcations”, dark blue curves (SN) of saddle-node bifurcations of 2:7 periodic orbits, a black curve (NP) of neutral saddles and a red curve (T) of torus bifurcations. Points labelled by letters mark the same intersection points as in Fig. 4.3. Here $b = 1$ and $\kappa = 11$.

A natural continuation of the curve T, where for the curve NP the two leading Floquet multipliers lie on the real line instead of on the complex unit circle. As discussed above, the curves HoC, HeC and SNT are actually not smooth bifurcation curves. They cannot be continued with DDE-Biftool, and have been inferred from dedicated simulations (for detailed examples, see sections 4.2.2–4.2.3). It is impractical to represent all the phase portraits that occur in open regions and along bifurcation curves. Instead we demonstrate the effects of these bifurcations in sections 4.2.2–4.2.3.
4.2. Chenciner bubbles in the GZT model

4.2.1 Criticality of the torus bifurcation

We now address the criticality of the curve T shown in Fig. 4.4. In [8] it is assumed that the Hopf bifurcation is supercritical and that a family of stable limit cycles exists between the curve $\text{Hopf}$ and the homoclinic bifurcation ($\text{chc}$) curve. However, after a 2:7 stable periodic solution loses its stability at curve T for increasing $\tau$, simulations of the GZT model do not reveal a nearby stable torus. In other words, there is no evidence that the torus bifurcation along curve T is supercritical in the GZT DDE. Yet, because Fig. 4.3 represents only a minimal bifurcation structure, it cannot be ruled out that the torus bifurcation is in fact supercritical and that the stable torus very quickly loses its stability in some different bifurcation. The key to proving conclusively that curve T is indeed subcritical is to study the resonance tongues that emerge from curve T. Just as resonance tongues are rooted along the curve of torus bifurcation in Fig. 4.2, the same holds for the curve T in Fig. 4.4. The periodic orbits in these resonance tongues can be found, regardless of their stability, with DDE-Biftool.

Figure 4.5 is a detailed view of the bifurcation structure shown in Fig. 4.4. Other curves that are not directly related to the criticality of the curve T have not been included in Fig. 4.5. The curves SN to the far left and right of Fig. 4.5 are the boundaries of the 2:7 resonance tongue. The remaining curves SN represent the 1:11, 1:12 and 1:13 resonance tongues that are rooted on the curve T and contain periodic orbits of period 77, 84 and 91 (with respect to the one year forcing cycle), respectively. Because the resonance tongues are extremely narrow, as determined numerically, the continuation of the boundaries as saddle-node bifurcations of periodic orbits is impractical. Therefore, we compute and plot curves of single periodic orbits in each $p:q$ resonance tongue.

The 1:11, 1:12 and 1:13 resonance tongues represent the locked trajectories on the tori
that form from the torus bifurcations along curve T. These resonance tongues are located on the side of curve T, for which the 2:7 periodic orbits are stable. Further, calculating the stability of the periodic orbits in the 1:11, 1:12 and 1:13 resonance tongues reveals that all the solutions possess at least one unstable Floquet multiplier. Therefore, these locked periodic solutions exist on a torus of saddle-type that winds over the stable 2:7 periodic orbits, confirming that the curve T represents subcritical torus bifurcations.

Compared to the theoretical bifurcation structure from [8] in Fig. 4.3, the curve HoC is on the other side of the curve NP. This means that the curve HoC is subcritical instead of supercritical, allowing the tori of saddle-type that are created at curve T to terminate at curve HoC.

4.2.2 Transition through the curves T and HoC

Even though we do not divide out symmetry or reduce the dimensionality of the GZT model, we can still produce evidence, by performing a numerical study in an appropriate way, that the curves T and HoC in Fig. 4.4 correspond directly to the curves $\text{Hopf}$ and $\text{chc}$ in Fig. 4.3.

Figure 4.6 shows examples of solutions on either sides of and in between the curves T and HoC. Overall, we present periodic orbits and tori in different ways. Column 1 shows stable, unstable and saddle periodic solutions as blue, red and green trajectories, respectively, in projection onto the $(h(t), h(t - \tau_n))$-plane. Note that the two different shades of colours reflect the existence of two symmetrically-related periodic solutions, as mentioned above and discussed in detail in section 2.2.2. Black trajectories represent quasi-periodic or high-period solutions. The panels in column 2 are stroboscopic traces in the $(h(t), h(t - \tau_n))$-plane. The construction of Poincaré traces for a scalar DDE is discussed in [16], albeit with state-dependent delays. For Poincaré traces, the Poincaré section is defined spatially; for example, by $h(t) = 0$ and $h'(t) < 0$. For stroboscopic traces, the stroboscopic section is defined temporally, as discussed formally in [56]. Here, we construct the stroboscopic trace by taking the solution after each forcing period. For the associated function segment we plot the first point of the segment, called the headpoint, in projection onto the $(h(t), h(t - \tau_n))$-plane. The forcing period in the GZT model is one, so in column 2, $h(t)$ is plotted as a point whenever $t \in \mathbb{N}$. Blue triangles, red squares and green crosses represent stable, unstable and saddle periodic points, respectively. The trace of closed curves formed by black points in panel (b2) represents a quasi-periodic or high-period solution. The grey rectangles in column 2 show the areas that are enlarged in column 3. Additionally, in column 3 two different shades of red points trace the unstable manifolds of the saddle periodic points that correspond to the two symmetrically-related saddle periodic orbits. These are calculated by running simulations of various small perturbations in the unstable eigendirections of the saddle periodic points; the latter can be calculated with DDE-Biftool. Column 4 shows representative phase portraits of the two-dimensional vector field on a cylinder drawn in the same style as those in Fig. 4.3.
4.2. Chenciner bubbles in the GZT model

Figure 4.6: Solutions that pass through the curves T and HoC shown in Fig. 4.4. Rows (a)–(d) are calculated for $c = 2.9850$ and $\tau_n = 0.9540$ (a), $\tau_n = 0.9533$ (b), $\tau_n = 0.9531$ (c) and $\tau_n = 0.9530$ (d), respectively. Column 1 displays (blue/green/red) stable/saddle/unstable periodic orbits and a (black) unstable torus (approximated by a period-84 solution) projected onto the $(h(t), h(t - \tau_n))$-plane; column 2 is a stroboscopic trace in the $(h(t), h(t - \tau_n))$-plane with (blue/green/red) stable/saddle/unstable fixed points and a (black) unstable limit cycle; column 3 is an enlargement of the stroboscopic trace, as indicated by the grey box in column 2, with (red) unstable manifolds of the saddle points; column 4 is a representative phase portrait. The two different shades of each colour corresponds to the two different symmetrically-related solutions. Here $b = 1$, $c = 2.985$ and $\kappa = 11$. 
The points in the centre are stable equilibria, while those either side are saddles. Thicker curves represent periodic orbits.

Row (a) of Fig. 4.6 is calculated for \( c = 2.9850 \) and \( \tau_n = 0.9540 \), which is above curve T in Fig. 4.4. As shown in Fig. 4.6(a1), there exists one set of symmetrically-related unstable periodic orbits and one set of saddle periodic orbits. These 2:7 periodic orbits are each associated with 7 periodic points in the stroboscopic trace in panel (a2). The enlargement in panel (a3) shows that both sets of unstable manifold branches lead to some remote attractor. The phase portrait in panel (a4) is topologically the same as the corresponding phase portrait to the right of the curve Hopf and below the curves rhc in Fig. 4.3.

After \( \tau_n \) is decreased to 0.9533 for fixed \( c = 2.9850 \), the curve T has been crossed. The solutions are shown in row (b) of Fig. 4.6. The (red) unstable periodic orbits from row (a) are now (blue) stable in row (b). Because the curve T is subcritical and unstable quasi-periodic solutions can neither be simulated nor found by continuation, we approximate the torus by a high-period locked trajectory. The black trajectory in panel (b1) is actually an unstable period-84 solution that winds around the stable periodic orbits; this is particularly clear in panels (b2)–(b3). The black points trace out closed curves around the blue stable periodic points; in other words, the torus that forms at curve T encircles the stable periodic orbits. In panel (b4) the closed curve around the central stable equilibrium is an unstable periodic orbit. This phase portraits is different from any in Fig. 4.3, because the periodic orbit is unstable, owing to the subcriticality of the curve T.

Row (c) of Fig. 4.6 is calculated for \( c = 2.9850 \) and \( \tau_n = 0.9531 \). We show the stable and saddle periodic orbits in panel (c1) and the stable and saddle periodic points in panels (c2)–(c3). Notice that in panel (c3), the unstable manifold branches belonging to one of the saddle periodic orbits spiral into the stable periodic points, while those belonging to the other (symmetrically-related) saddle periodic orbit travel around the stable periodic points before approaching some remote attractor. Because of the symmetry in the GZT model, the sets of unstable manifolds belonging to both saddle periodic orbits should pass through the homoclinic tangle simultaneously. The fact that they do not is evidence that, firstly, the first and last homoclinic tangencies are indistinguishably close to each other and, secondly, small numerical error is sufficient to result in two simulated unstable manifold sets that are on opposite sides of the homoclinic tangle. Therefore, the solution in row (c) must be exceptionally close to, and within numerical accuracy of, the homoclinic tangle. For an averaged two-dimensional vector field, the homoclinic tangle becomes a single homoclinic connection, as illustrated in panel (c4).

Row (d) is calculated for \( c = 2.9850 \) and \( \tau_n = 0.9530 \), now on the other side of the curve HoC. The only difference between row (d) and row (c) is that now all small perturbations of the saddle periodic orbit in the same eigendirection lead to the stable periodic orbit, as depicted with their associated periodic points in panel (d3) and schematically in panel (d4).
4.2. Chenciner bubbles in the GZT model

To summarise, one set of unstable manifold branches have switched from all approaching some remote attractor before the homoclinic “bifurcation” (cf. panel (b3)) to all approaching the stable periodic orbit after the “bifurcation”; such as panel (d3).

Figure 4.6 proves that crossing the curves T and HoC in Fig. 4.4 produces qualitative changes of the dynamics that correspond to crossing the curves $\text{Hopf}$ and $\text{chc}$ in Fig. 4.3, taking into account the subcriticality of curve T.

4.2.3 Transition through the curves HeC and SNT

Figure 4.7 shows examples of solutions when crossing the curves HeC and SNT; its layout is identical to Fig. 4.6. Row (a) of Fig. 4.7 is calculated for $\tau_n = 0.9475$ and $c = 2.9850$, to the left of both curves HeC and SNT. As shown in panel (a1), there exists one pair of symmetrically-related stable periodic orbits and one pair of saddle periodic orbits. In panel (a2) there are seven periodic points in the stroboscopic trace, representing each 2:7 periodic orbit. In panel (a3) both branches of unstable manifolds from the saddle periodic points spiral into their neighbouring stable fixed points. The fact that the unstable manifolds are spiralling shows that the torus on which the 2:7 periodic orbits exist has already lost normal hyperbolicity, which is the first step of the torus break-up. The organisation of the unstable manifolds is again represented in panel (a4), which is qualitatively equivalent to the corresponding phase portrait above the left $\text{rch}$ curve in Fig. 4.3.

After increasing $c$ to 2.9915 for fixed $\tau_n = 0.9475$, the solutions shown in row (b) of Fig. 4.7 are now just on the other side of the curve HeC. In addition to the two stable and two saddle periodic orbits, there exists a stable torus shown by the black trajectory in panel (b1). It appears as a black closed curve in panels (b2)–(b3). In panel (b3) it is particularly clear that this torus gives rise to heteroclinic tangles between neighbouring saddle periodic points. In an averaged vector field, the heteroclinic tangle becomes a heteroclinic connection, as depicted schematically in panel (b4), where a periodic orbit overlaps the unstable manifolds of the saddle at the point of bifurcation.

Row (c) is calculated for $\tau_n = 0.9475$ and $c = 2.9940$, which is now on the other side of the curve HeC in Fig. 4.4. Row (c) of Fig. 4.7 is similar to row (b), except that the torus has moved further away from the saddle periodic orbits, shown in panel (c1). In panels (c2)–(c3) the black closed curves has clearly disconnected itself from the saddle periodic points. In panel (c3) one set of unstable manifold branches approaches the stable periodic points, whereas the other set of unstable manifold branches approaches the stable torus. This corresponds to the schematic illustration in panel (c4).

Row (d) of Fig. 4.7 is calculated for $\tau_n = 0.9475$ and $c = 2.9990$, now having crossed both curves HeC and SNT. As a result the stable torus has now disappeared after meeting a torus of saddle-type at the SNT curve. This torus of saddle-type is not shown in previous panels because it can neither be simulated nor calculated by continuation. Panel (d3) and
Figure 4.7: Solutions that pass through the curves HeC and SNT shown in Fig. 4.4. Rows (a)–(d) are calculated for $\tau_n = 0.9475$ and $c = 2.9850$ (a), $c = 2.9915$ (b), $c = 2.9940$ (c) and $c = 2.9990$ (d), respectively. Column 1 displays (blue/green) stable/saddle periodic orbits and (black) stable tori projected onto the $(h(t), h(t - \tau_n))$-plane; column 2 is a stroboscopic trace in the $(h(t), h(t - \tau_n))$-plane with (blue/green) stable/saddle fixed points and (black) stable limit cycles; column 3 is an enlargement of the stroboscopic trace, as indicated by the grey box in column 2, with (red) unstable manifolds of the saddle points; column 4 is a representative phase portrait. The two different shades of each colour corresponds to the two different symmetrically-related solutions. Here $b = 1$ and $\kappa = 11.$
its schematic counter-part (d4) agree with the transition in Fig. 4.3: one set of unstable manifold branches approaches the stable periodic points, while the other approaches some remote attractor. Notice also that from row (a) to (d) the stable and saddle periodic orbits are moving increasingly close to each other, in anticipation of the nearby saddle-node bifurcation of periodic orbits at the boundary of the 2:7 resonance tongue.

4.3 Discussion and interpretation for tipping

This chapter took a detailed look at the “saddle-node bifurcations of tori” found in the GZT model in chapter 2. This phenomenon actually involves parameter regions, known as Chenciner bubbles, where the tori break up and the dynamics becomes complicated. We focussed on understanding the dynamics inside the Chenciner bubbles of the GZT model, because the additional complexity of the dynamics may have important implications for climate tipping events in systems that are driven and feature multiple frequencies and exhibit bifurcations involving quasi-periodic dynamics. As far as we are aware, this is the first time that a Chenciner bubble has been detected and analysed in a DDE model.

We computed curves of bifurcations in the 2:7 Chenciner bubble of the GZT model in Fig. 4.4. The resulting bifurcation set compares very well with the structure suggested in [8] in Fig. 4.3. The curves SN, T and NP were found by continuation with DDE-Biftool. We also detailed how we determined the criticality of the curve T in Fig. 4.4. This involved computationally very expensive calculations of resonance tongues rooted on the curve T by continuation. These resonance tongues contain periodic orbits of period-77, -84 and -91, which represent locked trajectories on tori of saddle-type that exist between the subcritical curves T and HoC. The detection of the other bifurcation loci was comparatively difficult, because these “bifurcations” involve tori that must break up and cannot simply be continued as smooth curves, and was outlined in sections 4.2.2–4.2.3. Transitions through the curves T, HoC, HeC and SNT were demonstrated by calculating and analysing the unstable manifolds of the co-existing saddle periodic orbits by simulation. Changes in the asymptotic behaviour of these unstable manifolds correspond to the break up of tori at the curves HoC, HeC or SNT. Observing these changes in the unstable manifolds reveals the paths of the curves HoC, HeC or SNT in the \((c, \tau_n)\)-plane.

The bifurcation structures in Figs. 4.3 and 4.4 compare very well. This is evidence that the Poincaré map of the GZT DDE is “close” to a two-dimensional vector field approximation. For example, the first and last homoclinic tangencies were found to be indistinguishably close to each other in the 2:7 Chenciner bubble. A minor difference between Figs. 4.3 and 4.4 is that the curves \(\text{Hopf}\) and \(\text{chc}\) are supercritical and there exists stable periodic orbits between them; whereas, the corresponding curves T and HoC are subcritical and the tori that exists between them are of saddle-type.
Consider now a tipping event described by crossing a locus of folding tori; for example, see Fig. 2.9. Depending on the frequencies involved in the dynamics near the tipping point, it is possible that there exists a Chenciner bubble sufficiently large to be detectable. An obvious implication would be that the additional bifurcation structure, compared to the case of a simple fold bifurcation of equilibria or periodic orbits, may provide precursors to the imminent tipping event. Of course, how well a Chenciner bubble can be detected depends on various factors. There are many different parameter paths through a Chenciner bubble that a quasi-static solution could take, and this affects the specific dynamics that are encountered. Relative time scales also play a vital role; for example, that of the drifting parameter compared to that of the system transients.

To illustrate the role of relative time scales, Fig. 4.8 shows simulated time series of the GZT model with the parameter $\tau_n = 0.953$ fixed and $c$ drifting linearly from 2.96 to 3.01, across the 2:7 Chenciner bubble shown in Fig. 4.4, at two different rates. The time series in Fig. 4.8 drift across the Chenciner bubble in 2000 and 100 years in panels (a) and (b), respectively. The vertical grey lines mark where the drifting $c$ parameter passes through a value of $c$ corresponding to curves HeC (green), T (red) and SN (blue) in Fig. 4.4.

In Fig. 4.8(a) we can associate behaviour in the time series with the quasi-static solution passing through different bifurcations. Before and after passing through the point HeC, there is a clear change in frequency from quasi-periodic to periodic and a subtle change in amplitude. Before and after the point SN, there is a clear change in amplitude. These observations are due to the relatively slow drift rate of $c$, so that the trajectory remains on one type of attractor long enough that it can be observed in the time series, before the next bifurcation is encountered. We know from the investigation in this chapter that at $t = 0$, the solution begins on a torus, so that the time series appears quasi-periodic. At $t \approx 520$ the torus breaks up in a heteroclinic tangle corresponding to the curve HeC in Fig. 4.4. After the point HeC in Fig. 4.8(a) the trajectory is attracted to the stable 2:7 periodic solution and settles to this solution over the next few hundred years. At $t \approx 1180$ the point T is crossed and the 2:7 periodic solution loses stability in a torus bifurcation. Nonetheless, there is no obvious change in the time series after the point T. This is because the 2:7 periodic solution is so weakly unstable that transient motion away from the 2:7 periodic solution is very slow. After $t \approx 1460$ the amplitude of the time series decreases to about half its original size. This results from the solution passing through the saddle-node bifurcation of periodic orbits corresponding to the point SN, so that any 2:7 periodic solution ceases to exist. The trajectory must therefore approach a remote attractor. One can argue that, in this scenario, the loss of the torus through the heteroclinic tangle acts as a precursor to the more dramatic saddle-node bifurcation of periodic orbits. In Fig. 4.8(b) there is also a clear change in amplitude from $t = 0$ to $t = 100$. However, the subtle changes in the behaviour of the time series, such as a change in frequency and amplitude at the point HeC, are not so
4.3. Discussion and interpretation for tipping

Figure 4.8: Times series of the GZT model with the parameter $c$ drifting linearly across the 2:7 Chenciner bubble, from 2.96 to 3.01 over 2000 years (a) and 100 years (b), for fixed $\tau_n = 0.953$. Grey vertical lines represent where $c$ crosses certain curves shown in Fig. 4.4, as labelled. Here $b = 1$ and $\kappa = 11$.

Whether the parameter drifting rate in panel (a) or (b) is more relevant for comparisons to changes in seasonal forcing strength represented by the drifting parameter $c$ of the ENSO system is a question that requires further consideration. In either case, a general understanding of the phenomenon of quasi-periodic tipping is of great interest in the context of both long-term (palaeoclimatology) climate change or short-term (anthropogenic; that is, due to humans) climate change.

There is still much work to be done. For example, the presence of noise could introduce further effects, including switching between the bistabilities within Chenciner bubbles. Solutions that are near bifurcations are often either weakly stable or weakly unstable, making them particularly susceptible to the effects of noise.

Can quasi-periodic tipping points be identified in models of at least intermediate complexity or even in real-world data? This is a critical and intriguing question. There already exist possible candidates for quasi-periodic tipping events. The North Atlantic thermohaline circulation (THC) is known to be capable of hysteresis [78]. In [45] the local decay rate (LDR) method, mentioned above in the introduction to this chapter, is used to predict a tipping
event induced by an unspecified bifurcation in a THC model of intermediate complexity. Due to the many degrees of freedom in the model, the detected bifurcation, assuming it is a fold bifurcation belonging to its known hysteresis mechanism, may in fact be folding tori. The LDR method is also used in [29] to identify an ancient greenhouse to icehouse tipping event, described as “fold-like” in [93], in a time series of data taken from tropical Pacific sediment cores. Given the many degrees of freedom in real-world climate systems, this tipping event may be folding tori as well.

It would be necessary to analyse data from sophisticated models or historical climate observations capable of quasi-periodic behaviour to see what kind of resonances are present. Only certain resonances will result in Chenciner bubbles that are large enough to be detected. It would then be very interesting to consider candidate cases of folding tori and try to find proof of qualitative changes in their data that may belong to the complicated behaviour associated with Chenciner bubbles. So far, existing methods for detecting the influence of bifurcations in time series, such as the LDR method, have focussed on predicting local bifurcations, such as fold bifurcations of equilibria. However, in future work we will also be interested in, for example, heteroclinic tangencies, which will not show up in LDR results because they are global bifurcations and require a different detection approach.
5

Discussion and outlook

In this thesis, we applied a dynamical systems approach to the analysis of conceptual models of the ENSO system. More specifically, the focus was on the delayed action oscillator paradigm of the ENSO system to describe the dynamics of the thermocline depth \( h(t) \), which is closely related to the sea-surface temperature, at the eastern boundary of the equatorial Pacific. The dynamics is driven by delayed feedback mechanisms that are created by coupling processes between oceanic waves in the Pacific Ocean and the trade winds in the atmosphere. The model introduced by Tziperman et al. in [98], referred to here as the TSCJ model, includes both a positive and negative delayed feedback, described by an asymmetric and nonlinear coupling function, and seasonal forcing, described by sinusoidal motion. We also considered a simplified version, which we refer to as the GZT model, first studied by Ghil et al. in [40]. It includes the negative feedback and a symmetric nonlinear coupling function, together with seasonal forcing. These models take the form of scalar delay differential equations (DDEs), meaning that one is dealing with an infinite-dimensional phase space of initial conditions in the form of function segments. The analysis of these and similar models have in the past been primarily carried out by conventional simulation of the equations. However, by simulation alone it is difficult to achieve an overview of the type of dynamics these models are capable of displaying. Multistabilities, transients and the continua of possible initial conditions can obscure certain dynamical features from being observed by simulation.

In order to clearly decipher possible solutions of the model equations for given parameters, we used concepts and methods from bifurcation theory and investigated with continuation software how a set of possible solutions will change when parameters are varied. This bifurcation analysis allowed us to organise the parameter space into regions of different solutions types that are separated by various bifurcations. This gave insight into how the interactions of the negative feedback and seasonal forcing in the GZT model are capable of producing elaborate dynamics that resemble certain ENSO features. By transitioning from the GZT model to the more general TSCJ model, we studied the effects of additional model features (an asymmetric coupling function and a positive delayed feedback) and the role they play in
creating more realistic solutions with characteristic features of observed data. We detected different types of routes to chaos and explained results found in previous publication concerning the existence of chaotic behaviour, which is critical for creating realistically irregular time series. Notably, this dynamical systems approach enabled us to consider large ranges of parameters, which is particularly important for conceptual models because some parameters relate to the real-world system only in a loosely defined way. Moreover, some parameters can only be estimated with certain precision and, hence, our bifurcation analysis also serves as a sensitivity analysis. We also uncovered new phenomenon. In particular, we laid the groundwork for the study of quasi-periodic climate tipping. Specifically, we identified and analysed so-called Chenciner bubbles along quasi-periodic fold bifurcations in the GZT ENSO model, where invariant tori break up before disappearing.

**Interdisciplinary relevance**

Overall, the work presented here shows that qualitatively realistic behaviour is possible in simple models that describe how delayed feedback in the ENSO system interacts with the seasons. We have gained an understanding of how this comes to be by studying how certain model features can affect the dynamics and, ultimately, lead to very complicated behaviour. This thesis advocates the important role of conceptual models for, not necessarily forecasting, but better understanding underlying mechanisms in climate systems. Much can be learned from these simple models. Furthermore, the results may be, on the one hand, interesting from a dynamics point of view; for example, the inherent bistability inside certain $p:q$ resonance tongues of the GZT model. On the other hand, some results will be more relevant from a climate science point of view; for example, the significant role of the asymmetry in the coupling function in the TSCJ model — which is due to the off-equatorial position of the intertropical convergence zone (ITCZ), where winds from the northern and southern hemispheres converge — draws a connection between the ITCZ and aperiodic thermocline behaviour. Although the interdisciplinary nature of this topic, at the interface between nonlinear dynamical systems and climate science, makes it exciting, it also comes with a difficult challenge: the more general issue of how to feed knowledge back into climate research. It is hoped that the work in this thesis will assist in bridging the gap between the two fields. In this way, it may serve as a basis for further studies that apply the methods used here to understanding more sophisticated models, studied by climate researchers.

Despite the relatively simple form of the DDE ENSO models studied here, their infinite-dimensional phase space allows for surprisingly rich and complicated dynamics. Feedback mechanisms that involve delays may be present in many forms, also in other systems, not just in climate. For example, DDEs have been used to describe the dynamics of epidemics [60], networks of neurons [84], coupled chemical oscillators [12] and laser systems [55, 61]. Delayed feedback is also successfully and widely used as a form of control [76]. The specific
combination of both delayed effects and periodic forcing has also been the subject of investigation in different fields; for example, in ecology [54], gene networks [104] and laser dynamics [4]. Therefore, the results in this thesis, regarding general analysis of dynamical systems with delay and periodic forcing, may be of relevance in a broader sense than simply within the context of conceptual climate modelling.

The bifurcation analysis was conducted here with the state-of-the-art continuation software, specifically, the Matlab package DDE-Biftool, which is designed for the study of DDEs. A more recent capability of DDE-Biftool is the continuation of codimension-one bifurcations of periodic orbits, which was vital for the completion of this thesis. Moreover, DDE-Biftool’s robust correction algorithm played a pivotal role in completing a bifurcation diagram of folding tori that involved both locked and unlocked solutions, by allowing us to find unstable periodic solution belonging to certain $p:q$ resonance tongues. The calculation of unstable manifolds is a particularly good example of how continuation and simulation can be combined. The initial histories given by the saddle periodic orbits and the initial perturbations in the direction of the unstable eigenmodes are both found with DDE-Biftool, while the one-dimensional unstable manifolds are then extended or grown by using simulation. The bifurcation study presented in this thesis may be of interest more generally, because it showcases how state-of-the-art continuation methods (in particular, those for periodic solutions) can be utilised for the bifurcation analysis of a DDE. Compared to an investigation solely reliant on numerical simulations, the approach of continuing bifurcation curves of various types offers a very efficient and systematic method for gaining insight into the possible dynamics over a large region in the parameter space. Recent investigations conducted in the same spirit are, for example, [75, 88].

**Future work**

The TSCJ model (and by default the GZT model) is, of course, a much simplified representation of suggested mechanisms behind ENSO observations. It would, therefore, be interesting to see how our findings relate to the behaviour of more complex models. For example, as mentioned above, the important role of the asymmetry in the coupling function highlights the importance of accurately describing the position of the ITCZ. Do the solutions of ENSO models that describe wind patterns across the Pacific Ocean become more regular if the ITCZ is not situated away from the equator? In the TSCJ model we observed an intermittency route to chaos. In fact, intermittencies have been observed in certain time series produced by the more sophisticated Cane–Zebiak forecasting model [111]. It would be intriguing to see whether these actually belong to a route to chaos and, if so, whether the co-existence of different routes to chaos could also be observed in the parameter space of such a more sophisticated model.

Another important issue for future research is to study further extensions of the TSCJ model.
with less stringent modelling assumptions and the inclusion of additional features. One specific modelling assumption that requires further consideration is the nature of the seasonal forcing. In the TSCJ model the seasonal forcing is assumed to be additive; in other words, the forcing appears as an individual term added onto the model equation. A variation of the TSCJ model with seasonal forcing in the form of multiplicative (or parametric) forcing of \( \kappa \) was introduced in [97]. A bifurcation analysis of this model can be found in [57]; in particular, many aspects of the bifurcation diagram in [57] are reminiscent of those presented in this thesis. However, the resonance tongues in [57] seem to be organized in the parameter plane differently in that they approach each other rather than overlap. This difference between additive and multiplicative forcing warrants further analysis.

It is a strong assumption that the delay times associated with the feedback mechanisms are constant in the TSCJ model. In order to push future investigations into a more realistic setting and build on the knowledge gained through analysis of the TSCJ model, it will be necessary to allow the delay times to be non-constant and investigate how this influences the resulting dynamics. In the ENSO system, the oceanic wave velocities are, in fact, distributed around mean velocities [13] and the spatial lengths travelled by them depend also on the geography of the Pacific basin near the equator. Hence, one approach would be to determine and consider the distribution of delays. Moreover, the delay times also depend on the state, that is, on the thermocline depth \( h(t) \). To begin with, the position of the western Pacific warm-pool, which itself is influenced by changes in \( h(t) \), affects where the oceanic waves are created and, therefore, affects the associated delay times. Furthermore, \( h(t) \) describes deviations from a background state, which is a long-term mean of \( h \) and assumed to be constant, but realistically this mean should depend on \( h(t) \) itself. Exactly how best to incorporate state-dependency into the model will require further consideration; nonetheless, an implicit expression for a state-dependent delay in an ENSO model has already been suggested in [28]. It was demonstrated in [47] that a wealth of dynamical diversity may be created when state-dependency is introduced into a scalar DDE. While the theory of state-dependent DDEs is still under development [33, 43, 46, 85, 100, 108], numerical continuation for these types of DDEs can be performed with the package DDE-Biftool. Therefore, a study of a state-dependent conceptual model for ENSO is feasible and would provide an interesting, yet challenging, case study of the effect of state dependency.

Another modelling assumption is that the TSCJ model is deterministic and includes no noise. From a modelling perspective, it is useful to include noise because there are many processes, such as localised high-frequency weather events, that might affect the dynamics, but are not typically modelled. Therefore, even though the TSCJ model does not require noise to produce irregular behaviour, it is still important to understand how noise affects the dynamics. How best to represent noise in a conceptual ENSO model is not immediately clear. Noise can have various distributions, and will most likely not be Gaussian white noise
due to the various nonlinear processes in the ENSO system. There are different ways of introducing noise into a model, in particular, as additive or multiplicative noise. In the case of multiplicative noise, the question arises as to what are suitable distributions for the different noise terms, and to what extent they affect the resulting behaviour. One of these many modelling options was demonstrated in [91] with the inclusion of an additive white noise term to the TSCJ model, which for some parameters was shown to affect the type of observed dynamics. Noise is known to cause switching between stable states of the corresponding deterministic system and it is important to first understand the system without noise. Therefore, the study presented here forms a useful basis of knowledge for future work in this direction.

Finally, results in this thesis also lay the foundation for exciting future work into quasi-periodic climate tipping. Climate tipping is presently very topical, particularly due to worldwide concerns about anthropogenic climate change. Yet, some possible scenarios may have so far been overlooked, because largely only tipping events described by bifurcations of equilibria or periodic orbits have been considered. This is despite the fact that climate systems are generally complex and commonly include (forcing of) various frequencies [38], which in turn results in quasi-periodic dynamics. A better understanding of the complicated dynamics near quasi-periodic bifurcations could lead to practical developments in identifying characteristic dynamics that prelude tipping events in climate models. We hope that our analysis of the dynamics invoked by Chenciner bubbles in an ENSO model may serve as a first step.
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