Half-arc-transitive graphs of arbitrary even valency greater than 2

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Abstract

A *half-arc-transitive* graph is a regular graph that is both vertex- and edge-transitive, but is not arc-transitive. If such a graph has finite valency, then its valency is even, and greater than 2. In 1970, Bouwer proved that there exists a half-arc-transitive graph of every even valency greater than 2, by giving a construction for a family of graphs now known as $B(k,m,n)$, defined for every triple (k,m,n) of integers greater than 1 with $2^m \equiv 1 \mod n$. In each case, $B(k,m,n)$ is a $2k$-valent vertex- and edge-transitive graph of order mn^{k-1}, and Bouwer showed that $B(k,6,9)$ is half-arc-transitive for all $k > 1$.

For almost 45 years the question of exactly which of Bouwer's graphs are half-arc-transitive and which are arc-transitive has remained open, despite many attempts to answer it. In this paper, we use a cycle-counting argument to prove that almost all of the graphs constructed by Bouwer are half-arc-transitive. In fact, we prove that $B(k,m,n)$ is arc-transitive only when $n = 3$, or $(k,n) = (2,5)$, or $(k,m,n) = (2,3,7)$ or $(2,6,7)$ or $(2,6,21)$. In particular, $B(k,m,n)$ is half-arc-transitive whenever $m > 6$ and $n > 5$. This gives an easy way to prove that there are infinitely many half-arc-transitive graphs of each even valency $2k > 2$.

Keywords: graph, half-arc transitive, edge-transitive, vertex-transitive, arc-transitive, automorphisms, cycles

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1 Introduction

In the 1960s, W.T. Tutte [10] proved that if a connected regular graph of odd valency is both vertex-transitive and edge-transitive, then it is also arc-transitive. At the same time, Tutte observed that it was not known whether the same was true for even valency. Shortly afterwards, I.Z. Bouwer [2] constructed a family of vertex- and edge-transitive graphs of any given even valency $2k > 2$, that are not arc-transitive.

Any graph that is vertex- and edge-transitive but not arc-transitive is now known as a *half-arc-transitive* graph. Every such graph has even valency, and since connected graphs of valency 2 are cycles, which are arc-transitive, the valency must be at least 4.

Quite a lot is now known about half-arc-transitive graphs, especially in the 4-valent case — see [3, 4, 7, 8] for example. Also a lot of attention has been paid recently to half-arc-transitive group actions on edge-transitive graphs — see [5] for example. In contrast, however, relatively little is known about half-arc-transitive graphs of higher valency. Bouwer's construction produced a vertex- and edge-transitive graph $B(k, m, n)$ of order mn^{k-1} and valency $2k$ for every triple (k, m, n) of integers greater than 1 such that $2^m \equiv 1 \bmod n$, and Bouwer proved in [2] that $B(k, 6, 9)$ is half-arc-transitive for every $k > 1$. Bouwer also showed that the latter is not true for every triple (k, m, n); for example, $B(2, 3, 7)$, $B(2, 6, 7)$ and $B(2, 4, 5)$ are arc-transitive.

For the last 45 years, the question of exactly which of Bouwer's graphs are half-arc-transitive and which are arc-transitive has remained open, despite a number of attempts to answer it. Three decades after Bouwer's paper, C.H. Li and H.-S. Sim [6] developed a quite different construction for a family of half-arc-transitive graphs, using Cayley graphs for metacyclic p-groups, and in doing this, they proved the existence of infinitely many half-arc-transitive graphs of each even valency $2k > 2$. Their approach, however, required a considerable amount of group-theoretic analysis.

In this paper, we use a cycle-counting argument to prove that almost all of the graphs constructed by Bouwer in [2] are half-arc-transitive, and thereby give an easier proof of the fact that there exist infinitely many half-arc-transitive graphs of each even valency $2k > 2$. Specifically, we prove the following:

Theorem. *The graph $B(k,m,n)$ is arc-transitive if and only if $n = 3$, or $(k,n) = (2,5)$, or $(k,m,n) = (2,3,7)$ or $(2,6,7)$ or $(2,6,21)$. In particular, $B(k,m,n)$ is half-arc-transitive whenever $m > 6$ and $n > 5$.*

We do this by considering the 6-cycles containing a given 2-arc. For any positive integer s, an *s-arc* in a simple graph X is a sequence $(v_0, v_1, \ldots, v_s)$ of $s+1$ vertices in X such that every two consecutive vertices v_{i-1} and v_i in the sequence are adjacent, and every three consecutive vertices v_{i-1}, v_i and v_{i+1} are distinct. We will denote an s-arc by $v_0 \sim v_1 \sim \ldots \sim v_s$. Also we take a cycle to be a closed path, with specified orientation (and sometimes with a given 'base vertex'), and an s-cycle to be a cycle of length s.

In Section 3 we prove that $B(k,m,n)$ is half-arc-transitive whenever $m > 6$ and $n > 7$, and then we adapt this for the other half-arc-transitive cases in Section 5. In between, we prove arc-transitivity in the cases given in the above theorem in Section 4. But first we give some additional background about the Bouwer graphs in the following section.

2 Further background

First we give the definition of the Bouwer graph $B(k,m,n)$, for every triple (k,m,n) of integers greater than 1 such that $2^m \equiv 1 \bmod n$.

The vertices of $B(k,m,n)$ are the k-tuples $(a, b_2, b_3, \ldots, b_k)$ with $a \in \mathbb{Z}_m$ and $b_j \in \mathbb{Z}_n$ for $2 \leq j \leq k$. We will sometimes write a given vertex as $(a, \mathbf{b})$, where $\mathbf{b} = (b_2, b_3, \ldots, b_k)$. Any two such vertices are adjacent if they can be written as $(a, \mathbf{b})$ and $(a+1, \mathbf{c})$ where either $\mathbf{c} = \mathbf{b}$, or $\mathbf{c} = (c_2, c_3, \ldots, c_k)$ differs from $\mathbf{b} = (b_2, b_3, \ldots, b_k)$ in exactly one position, say the $(j-1)$st position, where $c_j = b_j + 2^a$. We will often use X to denote $B(k,m,n)$.

Note that the condition $2^m \equiv 1 \bmod n$ ensures that 2 is a unit mod n, and hence that n is odd. Also note that the graph is simple, and that the definition of adjacency makes the projection $(a, \mathbf{b}) \mapsto a$ a graph homomorphism from $B(k,m,n)$ onto the cycle graph C_m, with obvious 'fibres'.

In what follows, we let $\mathbf{e}_j$ be the element of $(\mathbb{Z}_n)^{k-1}$ with jth term equal to 1 and all other terms equal to 0, for $1 \leq j < k$. With this notation, and with 2^{-1} denoting the multiplicative inverse of 2 in $\mathbb{Z}_n$, we see that the neighbours of a vertex $(a, \mathbf{b})$ are precisely the vertices $(a+1, \mathbf{b})$, $(a-1, \mathbf{b})$, $(a+1, \mathbf{b}+2^a\mathbf{e}_j)$ and $(a-1, \mathbf{b}-2^{a-1}\mathbf{e}_j)$ for $1 \le j < k$, and in particular, this shows that the graph $B(k, m, n)$ is regular of valency $2k$.

Next, we recall that for all (k, m, n), the graph $B(k, m, n)$ is both vertex- and edge-transitive; see [2, Proposition 1]. Also $B(k, m, n)$ is bipartite if and only if m is even. Moreover, it is easy to see that $B(k, m, n)$ has the following three automorphisms:

(i) θ, of order $k-1$, taking each vertex $(a, \mathbf{b}) = (a, b_2, b_3, \dots, b_{k-1}, b_k)$ to the vertex $(a, \mathbf{b}') = (a, b_3, b_4, \dots, b_k, b_2)$, obtained by shifting its last $k-1$ entries,

(ii) τ, of order m, taking each vertex $(a, \mathbf{b}) = (a, b_2, b_3, \dots, b_{k-1}, b_k)$ to the vertex $(a+1, \mathbf{b}'') = (a+1, 2b_2, 2b_3, \dots, 2b_{k-1}, 2b_k)$, obtained by increasing its first entry a by 1 and multiplying the others by 2, and

(iii) ψ, of order 2, taking each vertex $(a, \mathbf{b}) = (a, b_2, b_3, \dots, b_{k-1}, b_k)$ to the vertex $(a, \mathbf{b}''') = (a, 2^a - 1 - (b_2 + b_3 + \cdots + b_k), b_3, \dots, b_{k-1}, b_k)$, obtained by replacing its second entry b_2 by $2^a - 1 - (b_2 + b_3 + \cdots + b_k)$.

(Note: in the notation of [2], the automorphism ψ is $T_2 \circ S_2$.)

The automorphisms θ and ψ both fix the 'zero' vertex $(0, \mathbf{0}) = (0, 0, \dots, 0)$, and θ induces a permutation of its $2k$ neighbours that fixes each of the two vertices $(1, \mathbf{0}) = (1, 0, 0, \dots, 0)$ and $(-1, \mathbf{0}) = (-1, 0, 0, \dots, 0)$ and induces two $(k-1)$-cycles on the others, while ψ swaps $(1, \mathbf{0})$ with $(1, \mathbf{e}_1) = (1, 1, 0, \dots, 0)$, and swaps $(-1, \mathbf{0})$ with $(-1, -2^{-1}\mathbf{e}_1) = (-1, -2^{-1}, 0, \dots, 0)$, since $2^{-1} - 1 = 2^{-1}(1-2) = -2^{-1}$, and fixes all the others.

It follows that the subgroup generated by θ and ψ fixes the vertex $(0, \mathbf{0})$ and has two orbits of length k on its neighbours, with one orbit consisting of the vertices of the form $(1, \mathbf{b})$ where $\mathbf{b} = \mathbf{0}$ or $\mathbf{e}_j$ for some j, and the other consisting of those of the form $(-1, \mathbf{c})$ where $\mathbf{c} = \mathbf{0}$ or $-2^{-1}\mathbf{e}_j$ for some j. (In fact, $\langle \theta, \psi \rangle$ induces the symmetric group S_k on each of these two orbits.)

By edge-transitivity, the graph $B(k, m, n)$ is arc-transitive if and only if it admits an automorphism that interchanges the 'zero' vertex $(0, \mathbf{0})$ with one of its neighbours, in which case the above two orbits of $\langle \theta, \psi \rangle$ on the neighbours of $(0, \mathbf{0})$ are merged into one under the action of the stabiliser of $(0, \mathbf{0})$ in the full automorphism group. We will use the automorphism τ in the next section.

We will also use the following, which is valid in all cases, not just those with $m > 6$ and $n > 7$ considered in the next section.

Lemma 2.1. *Every* 3*-arc* $v_0 \sim v_1 \sim v_2 \sim v_3$ *in* $B(k,m,n)$ *with first vertex* $v_0 = (0,\mathbf{0})$ *is of one of the following forms, with* $r,s,t \in \{1,\ldots,k-1\}$ *in each case*:

(1) $(0,\mathbf{0})\sim(1,\mathbf{0})\sim(2,\mathbf{0})\sim(3,\mathbf{d})$, *where* $\mathbf{d}=\mathbf{0}$ *or* $4\mathbf{e}_r$,

(2) $(0,\mathbf{0})\sim(1,\mathbf{0})\sim(2,\mathbf{0})\sim(1,-2\mathbf{e}_r)$,

(3) $(0,\mathbf{0})\sim(1,\mathbf{0})\sim(2,2\mathbf{e}_r)\sim(3,2\mathbf{e}_r+\mathbf{d})$, *where* $\mathbf{d}=\mathbf{0}$ *or* $4\mathbf{e}_s$,

(4) $(0,\mathbf{0})\sim(1,\mathbf{0})\sim(2,2\mathbf{e}_r)\sim(1,2\mathbf{e}_r-\mathbf{d})$, *where* $\mathbf{d}=\mathbf{0}$ *or* $2\mathbf{e}_s$ *with* $s\neq r$,

(5) $(0,\mathbf{0})\sim(1,\mathbf{0})\sim(0,-\mathbf{e}_r)\sim(1,-\mathbf{e}_r+\mathbf{d})$, *where* $\mathbf{d}=\mathbf{0}$ *or* $\mathbf{e}_s$ *with* $s\neq r$,

(6) $(0,\mathbf{0})\sim(1,\mathbf{0})\sim(0,-\mathbf{e}_r)\sim(-1,-\mathbf{e}_r-\mathbf{d})$, *where* $\mathbf{d}=\mathbf{0}$ *or* $2^{-1}\mathbf{e}_s$,

(7) $(0,\mathbf{0})\sim(1,\mathbf{e}_r)\sim(2,\mathbf{e}_r)\sim(3,\mathbf{e}_r+\mathbf{d})$, *where* $\mathbf{d}=\mathbf{0}$ *or* $4\mathbf{e}_s$,

(8) $(0,\mathbf{0})\sim(1,\mathbf{e}_r)\sim(2,\mathbf{e}_r)\sim(1,\mathbf{e}_r-2\mathbf{e}_s)$,

(9) $(0,\mathbf{0})\sim(1,\mathbf{e}_r)\sim(2,\mathbf{e}_r+2\mathbf{e}_s)\sim(3,\mathbf{e}_r+2\mathbf{e}_s+\mathbf{d})$, *where* $\mathbf{d}=\mathbf{0}$ *or* $4\mathbf{e}_t$,

(10) $(0,\mathbf{0})\sim(1,\mathbf{e}_r)\sim(2,\mathbf{e}_r+2\mathbf{e}_s)\sim(1,\mathbf{e}_r+2\mathbf{e}_s-\mathbf{d})$,
where $\mathbf{d}=\mathbf{0}$ *or* $2\mathbf{e}_t$ *with* $t\neq s$,

(11) $(0,\mathbf{0})\sim(1,\mathbf{e}_r)\sim(0,\mathbf{e}_r)\sim(1,\mathbf{e}_r+\mathbf{e}_s)$,

(12) $(0,\mathbf{0})\sim(1,\mathbf{e}_r)\sim(0,\mathbf{e}_r)\sim(-1,\mathbf{e}_r-\mathbf{d})$, *where* $\mathbf{d}=\mathbf{0}$ *or* $2^{-1}\mathbf{e}_s$,

(13) $(0,\mathbf{0})\sim(1,\mathbf{e}_r)\sim(0,\mathbf{e}_r-\mathbf{e}_s)\sim(1,\mathbf{e}_r-\mathbf{e}_s+\mathbf{d})$, *where* $s\neq r$,
and $\mathbf{d}=\mathbf{0}$ *or* $\mathbf{e}_t$ *with* $t\neq s$,

(14) $(0,\mathbf{0})\sim(1,\mathbf{e}_r)\sim(0,\mathbf{e}_r-\mathbf{e}_s)\sim(-1,\mathbf{e}_r-\mathbf{e}_s-\mathbf{d})$, *where* $s\neq r$,
and $\mathbf{d}=\mathbf{0}$ *or* $2^{-1}\mathbf{e}_t$,

(15) $(0,\mathbf{0})\sim(-1,\mathbf{0})\sim(0,2^{-1}\mathbf{e}_r)\sim(1,2^{-1}\mathbf{e}_r+\mathbf{d})$, *where* $\mathbf{d}=\mathbf{0}$ *or* $\mathbf{e}_s$,

(16) $(0,\mathbf{0})\sim(-1,\mathbf{0})\sim(0,2^{-1}\mathbf{e}_r)\sim(-1,2^{-1}\mathbf{e}_r-\mathbf{d})$,
where $\mathbf{d}=\mathbf{0}$ *or* $2^{-1}\mathbf{e}_s$ *with* $s\neq r$,

(17) $(0,\mathbf{0})\sim(-1,\mathbf{0})\sim(-2,\mathbf{0})\sim(-1,2^{-2}\mathbf{e}_r)$,

(18) $(0,\mathbf{0})\sim(-1,\mathbf{0})\sim(-2,\mathbf{0})\sim(-3,-\mathbf{d})$, *where* $\mathbf{d}=\mathbf{0}$ *or* $2^{-3}\mathbf{e}_r$,

(19) $(0,\mathbf{0})\sim(-1,\mathbf{0})\sim(-2,-2^{-2}\mathbf{e}_r)\sim(-1,-2^{-2}\mathbf{e}_r+\mathbf{d})$,
where $\mathbf{d}=\mathbf{0}$ *or* $2^{-2}\mathbf{e}_s$ *with* $s\neq r$,

(20) $(0,\mathbf{0})\sim(-1,\mathbf{0})\sim(-2,-2^{-2}\mathbf{e}_r)\sim(-3,-2^{-2}\mathbf{e}_r-\mathbf{d})$,
where $\mathbf{d}=\mathbf{0}$ *or* $2^{-3}\mathbf{e}_s$,

(21) $(0,\mathbf{0})\sim(-1,-2^{-1}\mathbf{e}_r)\sim(0,-2^{-1}\mathbf{e}_r)\sim(1,-2^{-1}\mathbf{e}_r+\mathbf{d})$,
where $\mathbf{d}=\mathbf{0}$ *or* $\mathbf{e}_s$,

(22) $(0,\mathbf{0})\sim(-1,-2^{-1}\mathbf{e}_r)\sim(0,-2^{-1}\mathbf{e}_r)\sim(-1,-2^{-1}\mathbf{e}_r-2^{-1}\mathbf{e}_s)$,

(23) $(0,\mathbf{0})\sim(-1,-2^{-1}\mathbf{e}_r)\sim(0,-2^{-1}\mathbf{e}_r+2^{-1}\mathbf{e}_s)\sim(1,-2^{-1}\mathbf{e}_r+2^{-1}\mathbf{e}_s+\mathbf{d})$, *where* $s\neq r$, *and* $\mathbf{d}=\mathbf{0}$ *or* $\mathbf{e}_t$,

(24) $(0,\mathbf{0})\sim(-1,-2^{-1}\mathbf{e}_r)\sim(0,-2^{-1}\mathbf{e}_r+2^{-1}\mathbf{e}_s)\sim(-1,-2^{-1}\mathbf{e}_r+2^{-1}\mathbf{e}_s-\mathbf{d})$, *where* $s\neq r$, *and* $\mathbf{d}=\mathbf{0}$ *or* $2^{-1}\mathbf{e}_t$ *with* $t\neq s$,

(25) $(0,\mathbf{0})\sim(-1,-2^{-1}\mathbf{e}_r)\sim(-2,-2^{-1}\mathbf{e}_r)\sim(-1,-2^{-1}\mathbf{e}_r+2^{-2}\mathbf{e}_s)$,

(26) $(0,\mathbf{0})\sim(-1,-2^{-1}\mathbf{e}_r)\sim(-2,-2^{-1}\mathbf{e}_r)\sim(-3,-2^{-1}\mathbf{e}_r-\mathbf{d})$, *where* $\mathbf{d}=\mathbf{0}$ *or* $2^{-3}\mathbf{e}_s$,

(27) $(0,\mathbf{0})\sim(-1,-2^{-1}\mathbf{e}_r)\sim(-2,-2^{-1}\mathbf{e}_r-2^{-2}\mathbf{e}_s)\sim(-1,-2^{-1}\mathbf{e}_r-2^{-2}\mathbf{e}_s+\mathbf{d})$, *where* $\mathbf{d}=\mathbf{0}$ *or* $2^{-2}\mathbf{e}_t$ *with* $t\neq s$,

(28) $(0,\mathbf{0})\sim(-1,-2^{-1}\mathbf{e}_r)\sim(-2,-2^{-1}\mathbf{e}_r-2^{-2}\mathbf{e}_s)\sim(-3,-2^{-1}\mathbf{e}_r-2^{-2}\mathbf{e}_s-\mathbf{d})$, *where* $\mathbf{d}=\mathbf{0}$ *or* $2^{-3}\mathbf{e}_t$.

Proof. This follows directly from the definition of $B(k,m,n)$. $\square$

Note that the 28 cases given in Lemma 2.1 fall naturally into 14 pairs, with each pair determined by the form of the initial 2-arc $v_0\sim v_1\sim v_2$.

Also it is easy to see that the number of 3-arcs in each case is

$$\begin{cases} k & \text{in cases 1 and 18,}\\ k-1 & \text{in cases 2 and 17,}\\ k(k-1) & \text{in cases 3, 6, 7, 12, 15, 20, 21 and 26,}\\ (k-1)^2 & \text{in cases 4, 5, 8, 11, 16, 19, 22 and 25,}\\ k(k-1)^2 & \text{in cases 9 and 28,}\\ (k-1)^3 & \text{in cases 10 and 27,}\\ (k-1)^2(k-2) & \text{in cases 13 and 24,}\\ k(k-1)(k-2) & \text{in cases 14 and 23,} \end{cases}$$

and the total of all these numbers is $2k(2k-1)^2$, as expected.

3 The main approach

Let k be any integer greater than 1, and suppose that $m>6$ and $n>7$. We will prove that in every such case, the graph $X=B(k,m,n)$ is not arc-transitive, and is therefore half-arc-transitive.

We do this simply by considering the ways in which a given 2-arc or 3-arc lies in a cycle of length 6 in X. By vertex-transitivity, we can consider what happens locally around the vertex $(0, \mathbf{0})$.

Lemma 3.1. *The girth of X is* 6.

Proof. First, $X = B(k, m, n)$ is simple, by definition. Next, since $m > 6$, it is clear from the definition that there are no cycles of length 3 or 5 in X. Also there are no cycles of length 4 in X, since in the list of cases for a 3-arc $v_0 \sim v_1 \sim v_2 \sim v_3$ in X with first vertex $v_0 = (0, \mathbf{0})$ given by Lemma 2.1, the vertex v_1 is uniquely determined by v_2. On the other hand, there are certainly some cycles of length 6 in X, such as $(0, \mathbf{0}) \sim (1, \mathbf{0}) \sim (2, 2\mathbf{e}_k) \sim (1, 2\mathbf{e}_k) \sim (0, \mathbf{e}_k) \sim (1, \mathbf{e}_k) \sim (0, \mathbf{0})$. $\square$

Next, we can find all 6-cycles based at the vertex $v_0 = (0, \mathbf{0})$ in X.

Lemma 3.2. *Up to reversal, every* 6*-cycle based at the vertex $v_0 = (0, \mathbf{0})$ has exactly one of the forms below, with r, s, t all different when they appear*:

- $(0,\mathbf{0})\sim(1,\mathbf{0})\sim(2,2\mathbf{e}_r)\sim(1,2\mathbf{e}_r)\sim(0,\mathbf{e}_r)\sim(1,\mathbf{e}_r)\sim(0,\mathbf{0})$,
- $(0,\mathbf{0})\sim(1,\mathbf{0})\sim(0,-\mathbf{e}_r)\sim(1,-\mathbf{e}_r)\sim(2,\mathbf{e}_r)\sim(1,\mathbf{e}_r)\sim(0,\mathbf{0})$,
- $(0,\mathbf{0})\sim(1,\mathbf{0})\sim(0,-\mathbf{e}_r)\sim(1,\mathbf{e}_s-\mathbf{e}_r)\sim(0,\mathbf{e}_s-\mathbf{e}_r)\sim(1,\mathbf{e}_s)\sim(0,\mathbf{0})$,
- $(0,\mathbf{0})\sim(1,\mathbf{0})\sim(0,-\mathbf{e}_r)\sim(-1,-\mathbf{e}_r)\sim(0,-2^{-1}\mathbf{e}_r)\sim(-1,-2^{-1}\mathbf{e}_r)\sim(0,\mathbf{0})$,
- $(0,\mathbf{0})\sim(1,\mathbf{e}_r)\sim(2,2\mathbf{e}_s+\mathbf{e}_r)\sim(1,2\mathbf{e}_s-\mathbf{e}_r)\sim(0,\mathbf{e}_s-\mathbf{e}_r)\sim(1,\mathbf{e}_s)\sim(0,\mathbf{0})$,
- $(0,\mathbf{0})\sim(1,\mathbf{e}_r)\sim(0,\mathbf{e}_r)\sim(1,\mathbf{e}_s+\mathbf{e}_r)\sim(0,\mathbf{e}_s)\sim(1,\mathbf{e}_s)\sim(0,\mathbf{0})$,
- $(0,\mathbf{0})\sim(1,\mathbf{e}_r)\sim(0,\mathbf{e}_r)\sim(-1,2^{-1}\mathbf{e}_r)\sim(0,2^{-1}\mathbf{e}_r)\sim(-1,\mathbf{0})\sim(0,\mathbf{0})$,
- $(0,\mathbf{0})\sim(1,\mathbf{e}_r)\sim(0,\mathbf{e}_r-\mathbf{e}_s)\sim(1,\mathbf{e}_r-\mathbf{e}_s+\mathbf{e}_t)\sim(0,\mathbf{e}_t-\mathbf{e}_s)\sim(1,\mathbf{e}_t)\sim(0,\mathbf{0})$,
- $(0,\mathbf{0})\sim(1,\mathbf{e}_r)\sim(0,\mathbf{e}_r-\mathbf{e}_s)\sim(-1,2^{-1}\mathbf{e}_r-\mathbf{e}_s)\sim(0,2^{-1}\mathbf{e}_r-2^{-1}\mathbf{e}_s)$ $\sim(-1,-2^{-1}\mathbf{e}_s)\sim(0,\mathbf{0})$,
- $(0,\mathbf{0})\sim(-1,\mathbf{0})\sim(0,2^{-1}\mathbf{e}_r)\sim(1,2^{-1}\mathbf{e}_r)\sim(0,-2^{-1}\mathbf{e}_r)\sim(-1,-2^{-1}\mathbf{e}_r)$ $\sim(0,\mathbf{0})$,
- $(0,\mathbf{0})\sim(-1,\mathbf{0})\sim(0,2^{-1}\mathbf{e}_r)\sim(-1,2^{-1}\mathbf{e}_r-2^{-1}\mathbf{e}_s)\sim(0,2^{-1}\mathbf{e}_r-2^{-1}\mathbf{e}_s)$ $\sim(-1,-2^{-1}\mathbf{e}_s)\sim(0,\mathbf{0})$,
- $(0,\mathbf{0})\sim(-1,\mathbf{0})\sim(-2,-2^{-2}\mathbf{e}_r)\sim(-1,-2^{-2}\mathbf{e}_r)\sim(-2,-2^{-1}\mathbf{e}_r)\sim(-1,-2^{-1}\mathbf{e}_r)$ $\sim(0,\mathbf{0})$,
- $(0,\mathbf{0})\sim(-1,-2^{-1}\mathbf{e}_r)\sim(0,-2^{-1}\mathbf{e}_r)\sim(-1,-2^{-1}\mathbf{e}_r-2^{-1}\mathbf{e}_s)\sim(0,-2^{-1}\mathbf{e}_s)$ $\sim(-1,-2^{-1}\mathbf{e}_s)\sim(0,\mathbf{0})$,

- $(0,\mathbf{0})\sim(-1,-2^{-1}\mathbf{e}_r)\sim(0,-2^{-1}\mathbf{e}_r+2^{-1}\mathbf{e}_s)\sim(1,2^{-1}\mathbf{e}_r+2^{-1}\mathbf{e}_s)$
  $\sim(0,2^{-1}\mathbf{e}_r-2^{-1}\mathbf{e}_s)\sim(-1,-2^{-1}\mathbf{e}_s)\sim(0,\mathbf{0})$,
- $(0,\mathbf{0})\sim(-1,-2^{-1}\mathbf{e}_r)\sim(0,-2^{-1}\mathbf{e}_r+2^{-1}\mathbf{e}_s)\sim(-1,-2^{-1}\mathbf{e}_r+2^{-1}\mathbf{e}_s-2^{-1}\mathbf{e}_t)$
  $\sim(0,2^{-1}\mathbf{e}_s-2^{-1}\mathbf{e}_t)\sim(-1,-2^{-1}\mathbf{e}_t)\sim(0,\mathbf{0})$, *or*
- $(0,\mathbf{0})\sim(-1,-2^{-1}\mathbf{e}_r)\sim(-2,-2^{-1}\mathbf{e}_r-2^{-2}\mathbf{e}_s)\sim(-1,-2^{-2}\mathbf{e}_r-2^{-2}\mathbf{e}_s)$
  $\sim(-2,-2^{-2}\mathbf{e}_r-2^{-1}\mathbf{e}_s)\sim(-1,-2^{-1}\mathbf{e}_s)\sim(0,\mathbf{0})$.

Proof. This is left as an exercise for the reader. It may be helpful to note that a 6-cycle of the first form is obtainable as a 3-arc of type 4 with final vertex $2\mathbf{e}_r$ followed by the reverse of a 3-arc of type 11 with the same final vertex. The 6-cycles of the other 15 forms are similarly obtainable as the concatenation of a 3-arc of type i with the reverse of a 3-arc of type j, for $(i,j)=(5,8)$, $(5,13)$, $(6,22)$, $(10,13)$, $(11,11)$, $(12,16)$, $(13,13)$, $(14,24)$, $(15,21)$, $(16,24)$, $(19,25)$, $(22,22)$, $(23,23)$, $(24,24)$ and $(27,27)$, respectively. Uniqueness of each of the 16 forms follows from the assumptions about m and n. $\square$

Corollary 3.3. *The number of different* 6*-cycles in* X *that contain a given* 2*-arc* $v_0\sim v_1\sim v_2$ *with first vertex* $v_0=(0,\mathbf{0})$ *is always* 0, 1 *or* k*. More precisely, this number is*

0 *for the* 2*-arcs* $(0,\mathbf{0})\sim(1,\mathbf{0})\sim(2,\mathbf{0})$ *and* $(0,\mathbf{0})\sim(-1,\mathbf{0})\sim(-2,\mathbf{0})$,
and those of the form $(0,\mathbf{0})\sim(1,\mathbf{e}_r)\sim(2,3\mathbf{e}_r)$,
and those of the form $(0,\mathbf{0})\sim(-1,-2^{-1}\mathbf{e}_r)\sim(-2,-(2^{-1}+2^{-2})\mathbf{e}_r)$,

1 *for the* 2*-arcs of the form* $(0,\mathbf{0})\sim(1,\mathbf{0})\sim(2,2\mathbf{e}_r)$,
and those of the form $(0,\mathbf{0})\sim(1,\mathbf{e}_r)\sim(2,\mathbf{e}_r)$,
and those of the form $(0,\mathbf{0})\sim(1,\mathbf{e}_r)\sim(2,\mathbf{e}_r+2\mathbf{e}_s)$, *when* $k>2$,
and those of the form $(0,\mathbf{0})\sim(-1,\mathbf{0})\sim(-2,-2^{-2}\mathbf{e}_r)$,
and those of the form $(0,\mathbf{0})\sim(-1,-2^{-1}\mathbf{e}_r)\sim(-2,-2^{-1}\mathbf{e}_r)$,
and those of the form $(0,\mathbf{0})\sim(-1,-2^{-1}\mathbf{e}_r)\sim(-2,-2^{-1}\mathbf{e}_r-2^{-2}\mathbf{e}_s)$,
when $k>2$, *and*

k *for the* 2*-arcs of the form* $(0,\mathbf{0})\sim(1,\mathbf{0})\sim(0,-\mathbf{e}_r)$,
and those of the form $(0,\mathbf{0})\sim(1,\mathbf{e}_r)\sim(0,\mathbf{e}_r)$,
and those of the form $(0,\mathbf{0})\sim(1,\mathbf{e}_r)\sim(0,\mathbf{e}_r-\mathbf{e}_s)$, *when* $k>2$,
and those of the form $(0,\mathbf{0})\sim(-1,\mathbf{0})\sim(0,2^{-1}\mathbf{e}_r)$,
and those of the form $(0,\mathbf{0})\sim(-1,-2^{-1}\mathbf{e}_r)\sim(0,-2^{-1}\mathbf{e}_r)$,
and those of the form $(0,\mathbf{0})\sim(-1,-2^{-1}\mathbf{e}_r)\sim(0,-2^{-1}\mathbf{e}_r+2^{-1}\mathbf{e}_s)$,
when $k>2$.

In particular, every 2*-arc of the form* $(0,\mathbf{0})\sim(1,\mathbf{0})\sim(2,2\mathbf{e}_r)$ *lies in just one* 6*-cycle, namely* $(0,\mathbf{0})\sim(1,\mathbf{0})\sim(2,2\mathbf{e}_r)\sim(1,2\mathbf{e}_r)\sim(0,\mathbf{e}_r)\sim(1,\mathbf{e}_r)\sim(0,\mathbf{0})$, *while every* 2*-arc of the form* $(0,\mathbf{0})\sim(1,\mathbf{e}_r)\sim(0,\mathbf{e}_r)$ *lies in* k 6*-cycles.*

Proof. This follows easily from inspection of the list of 6-cycles given in Lemma 3.2, and their reverses. (The reader might also like to be reassured that these facts were checked and confirmed for all Bouwer graphs on up to 5000 vertices, by a computation using Magma [1].) □

At this stage, we could repeat the above calculations for 2-arcs and 3-arcs with first vertex $(1,\mathbf{0})=(1,0,0,\dots,0)$, but it is much easier to simply apply the automorphism τ defined in Section 2.

Hence in particular, the 2-arcs $v_0\sim v_1\sim v_2$ with first vertex $v_0=(1,\mathbf{0})$ that lie in a unique 6-cycle are those of the form $(1,\mathbf{0})\sim(2,\mathbf{0})\sim(3,4\mathbf{e}_r)$, or $(1,\mathbf{0})\sim(2,2\mathbf{e}_r)\sim(3,2\mathbf{e}_r)$, or $(1,\mathbf{0})\sim(2,2\mathbf{e}_r)\sim(3,2\mathbf{e}_r+4\mathbf{e}_s)$ when $k>2$, or $(1,\mathbf{0})\sim(0,\mathbf{0})\sim(-1,-2^{-1}\mathbf{e}_r)$, or $(1,\mathbf{0})\sim(0,-\mathbf{e}_r)\sim(-1,-\mathbf{e}_r)$, or $(1,\mathbf{0})\sim(0,-\mathbf{e}_r)\sim(-1,-\mathbf{e}_r-2^{-1}\mathbf{e}_s)$ when $k>2$.

Let v and w be the neighbouring vertices $(0,\mathbf{0})$ and $(1,\mathbf{0})$. We will show that there is no automorphism of X that reverses the arc (v,w).

Let $A=\{(2,2\mathbf{e}_r):1\le r<k\}$, which is the set of all vertices x in X that extend the arc (v,w) to a 2-arc (v,w,x) which lies in a unique 6-cycle, and similarly, let $B=\{(0,-\mathbf{e}_r):1\le r<k\}$, the set of all vertices that extend (v,w) to a 2-arc which lies in k different 6-cycles. Also let $C=\{(-1,-2^{-1}\mathbf{e}_r):1\le r<k\}$ and $D=\{(1,\mathbf{e}_r):1\le r<k\}$ be the analogous sets of vertices extending the arc (w,v), as illustrated in Figure 1.

Now suppose there exists an automorphism ξ of X that interchanges the two vertices of the edge $\{v,w\}$. Then by considering the numbers of 6-cycles that contain a given 2-arc, we see that ξ must interchange the sets A and C, and interchange the sets B and D.

Next, observe that if $x=(2,2\mathbf{e}_r)\in A$, then the unique 6-cycle containing the 2-arc $v\sim w\sim x$ is $v\sim w\sim x\sim y\sim z\sim u\sim v$, where $(y,z,u)=((1,2\mathbf{e}_r),(0,\mathbf{e}_r),(1,\mathbf{e}_r))$; in particular, the 6th vertex $u=(1,\mathbf{e}_r)$ lies in D. Similarly, if $x'=(-1,-2^{-1}\mathbf{e}_r)\in C$, then the unique 6-cycle containing the 2-arc $w\sim v\sim x'$ is $w\sim v\sim x'\sim y'\sim z'\sim u'\sim w$, where $(y',z',u')=((0,-2^{-1}\mathbf{e}_r),(-1,-\mathbf{e}_r),(0,-\mathbf{e}_r))$, and the 6th vertex $u'=(0,-\mathbf{e}_r)$ lies in B.

The arc-reversing automorphism ξ must take every 6-cycle of the first kind to a 6-cycle of the second kind, and hence must take each 2-arc of the

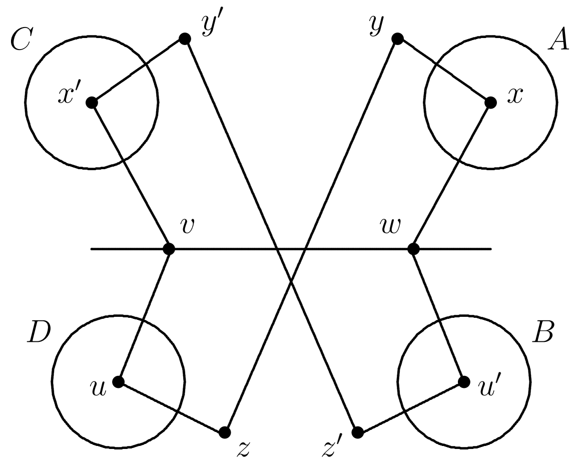

Figure 1: 6-cycles containing the edge from $v = (0, \mathbf{0})$ to $w = (1, \mathbf{0})$

form $v \sim u \sim z$ $(= (0, \mathbf{0}) \sim (1, \mathbf{e}_r) \sim (0, \mathbf{e}_r))$ to a 2-arc of the form $w \sim u' \sim z'$ $(= (1, \mathbf{0}) \sim (0, -\mathbf{e}_r) \sim (-1, -\mathbf{e}_r))$. By Corollary 3.3, however, each 2-arc of the form $(0, \mathbf{0}) \sim (1, \mathbf{e}_r) \sim (0, \mathbf{e}_r)$ lies in k different 6-cycles, while each 2-arc of the form $(1, \mathbf{0}) \sim (0, -\mathbf{e}_r) \sim (-1, -\mathbf{e}_r)$ is the τ-image of the 2-arc $(0, \mathbf{0}) \sim (-1, -2^{-1}\mathbf{e}_r) \sim (-2, -2^{-1}\mathbf{e}_r)$ and hence lies in only one 6-cycle. This is a contradiction, and shows that no such arc-reversing automorphism exists.

Hence the Bouwer graph $B(k, m, n)$ is half-arc-transitive whenever $m > 6$ and $n > 7$.

If $m \leq 6$, then the order m_2 of 2 as a unit mod n is at most 6, and so n divides $2^{m_2} - 1 = 3, 7, 15, 31$ or 63. In particular, if $m_2 = 2$, 3, 4 or 5 then $n \in \{3\}$, $\{7\}$, $\{5, 15\}$ or $\{31\}$ respectively, while if $m_2 = 6$, then n divides 63 but not 3 or 7, so $n \in \{9, 21, 63\}$. Conversely, if $n = 3, 5, 7, 9, 15, 21, 31$ or 63, then $m_2 = 2$, 4, 3, 6, 4, 6, 5 or 6, respectively, and of course in each case, m is a multiple of m_2.

We deal with these exceptional cases in the next two sections.

4 Arc-transitive cases

The following observations are very easy to verify.

When $n = 3$ (and m is even), the Bouwer graph $B(k, m, n)$ is always arc-transitive, for in this case $2 \equiv -1 \bmod n$, and it follows easily that there is an automorphism taking each vertex $(a, \mathbf{b})$ to $(1–a, -\mathbf{b})$, and this automorphism reverses the arc from $(0, \mathbf{0})$ to $(1, \mathbf{0})$.

Similarly, when $k = 2$ and $n = 5$ (and m is divisible by 4), the graph $B(k, m, n)$ is always arc-transitive, since in this case the fact that $2^{-1} = -2$ in $\mathbb{Z}_n$ implies that there exists an automorphism taking (a, b_2) to $(1-a, -b_2)$ for all $a \equiv 0$ or $1 \bmod 4$, and to $(1-a, 2 - b_2)$ for all $a \equiv 2$ or $3 \bmod 4$, and again this interchanges $(0, \mathbf{0})$ with $(1, \mathbf{0})$.

Next, $B(2, m, 7)$ is arc-transitive when $m = 3$ or 6, because in each of those two cases there is an automorphism that takes

$(a, 0)$ to $(1-a, 0)$,	$(a, 1)$ to $(a+1, 2)$,	$(a, 2)$ to $(a-1, 1)$,
$(a, 3)$ to $(–a, 6)$,	$(a, 4)$ to $(a+3, 4)$,	$(a, 5)$ to $(1-a, 5)$,
$(a, 6)$ to $(1-a, 3)$,		

for every $a \in \mathbb{Z}_m$.

Similarly, $B(2, 6, 21)$ is arc-transitive since it has an automorphism taking

$(a, 0)$ to $(1-a, 0)$,	$(a, 1)$ to $(a+1, 2)$,	$(a, 2)$ to $(a-1, 1)$,
$(a, 3)$ to $(5-a, 6)$,	$(a, 4)$ to $(a+3, 11)$,	$(a, 5)$ to $(5-a, 19)$,
$(a, 6)$ to $(5-a, 3)$,	$(a, 7)$ to $(1-a, 14)$,	$(a, 8)$ to $(a+1, 16)$,
$(a, 9)$ to $(a-1, 15)$,	$(a, 10)$ to $(5-a, 20)$,	$(a, 11)$ to $(a+3, 4)$,
$(a, 12)$ to $(5-a, 12)$,	$(a, 13)$ to $(5-a, 17)$,	$(a, 14)$ to $(1-a, 7)$,
$(a, 15)$ to $(a+1, 9)$,	$(a, 16)$ to $(a-1, 8)$,	$(a, 17)$ to $(5-a, 13)$,
$(a, 18)$ to $(a+3, 18)$,	$(a, 19)$ to $(5-a, 5)$,	$(a, 20)$ to $(5-a, 10)$,

for every $a \in \mathbb{Z}_m$.

5 Other half-arc-transitive cases

In this final section, we give sketch proofs of half-arc-transitivity in all the remaining cases. We first consider the three cases where m and the multiplicative order of 2 mod n are both equal to 6 (and $n = 9$, 21 or 63), and then deal with the cases where the multiplicative order of 2 mod n is less than 6 (and $n = 5$, 7, 15 or 31). In the cases $n = 15$ and $n = 31$, we can assume that $m < 6$ as well.

Note that one of these remaining cases is the one considered by Bouwer, namely where $(m, n) = (6, 9)$. As this is one of the exceptional cases, it is not representative of the generic case — which may help explain why previous attempts to generalise Bouwer's approach did not get very far.

To reduce unnecessary repetition, we will introduce some further notation that will be used in most of these cases. As in Section 3, we let v and w be the vertex $(0, \mathbf{0})$ and its neighbour $(1, \mathbf{0})$.

Next, for all $i \geq 0$, we define $V^{(i)}$ as the set of all vertices x in X for which $v \sim w \sim x$ is a 2-arc that lies in exactly i different 6-cycles, and $W^{(i)}$ as the analogous set of vertices x' in X for which $w \sim v \sim x'$ is a 2-arc that lies in exactly i different 6-cycles. (Hence, for example, the sets A, B, C and D used in Section 3 are $V^{(1)}$, $V^{(k)}$, $W^{(1)}$ and $W^{(k)}$, respectively.)

Also for $i \geq 0$ and $j \geq 0$, we define $T_v^{(i,j)}$ as the set of all 2-arcs $v \sim u \sim z$ that come from a 6-cycle of the form $v \sim w \sim x \sim y \sim z \sim u \sim v$ with $x \in V^{(i)}$ and $u \in W^{(i)}$, and lie in exactly j different 6-cycles altogether, and define $T_w^{(i,j)}$ as the analogous set of all 2-arcs $w \sim u' \sim z'$ that come from a 6-cycle of the form $w \sim v \sim x' \sim y' \sim z' \sim u' \sim w$ with $x' \in W^{(i)}$ and $u' \in V^{(i)}$, and lie in exactly j different 6-cycles altogether.

Note that if the graph under consideration is arc-transitive, then it has an automorphism ξ that reverses the arc (v, w), and then clearly ξ must interchange the two sets $V^{(i)}$ and $W^{(i)}$, for each i. Hence also ξ interchanges the two sets $T_v^{(i,j)}$ and $T_w^{(i,j)}$ for all i and j, and therefore $|T_v^{(i,j)}| = |T_w^{(i,j)}|$ for all i and j. Equivalently, if $|T_v^{(i,j)}| \neq |T_w^{(i,j)}|$ for some pair (i, j), then the graph cannot be arc-transitive, and therefore must be half-arc-transitive.

The approach taken in Section 3 was similar, but compared the 2-arcs $v \sim u \sim z$ that come from a 6-cycle of the form $v \sim w \sim x \sim y \sim z \sim u \sim v$ where $x \in V^{(1)}$ and $u \in W^{(k)}$, with the 2-arcs $w \sim u' \sim z'$ that come from a 6-cycle of the form $w \sim v \sim x' \sim y' \sim z' \sim u' \sim w$ where $x' \in W^{(1)}$ and $u' \in V^{(k)}$.

We proceed by considering the first three cases below, in which the girth of the Bouwer graph $B(k, m, n)$ is 6, but the numbers of 6-cycles containing a given arc or 2-arc are different from those found in Section 3.

5.1 The graphs $B(k, 6, 9)$

Suppose $m = 6$ and $n = 9$. This case was considered by Bouwer in [2], but it can also be dealt with in a similar way to the generic case in Section 3.

Here $2^3 = -1$ in $\mathbb{Z}_n$, and it follows that the 6-cycles are as found in Section 3, plus others having six distinct first coordinates (from $\mathbb{Z}_m = \mathbb{Z}_6$). Every 2-arc lies in either k or $k + 1$ cycles of length 6, and each arc lies in exactly k distinct 2-arcs of the first kind, and $k-1$ of the second kind. Next, the set $V^{(k)}$ consists of $(2,\mathbf{0})$ and $(0,-\mathbf{e}_r)$ for $1 \le r < k$, while $W^{(k)}$ consists of $(-1,\mathbf{0})$ and $(1,\mathbf{e}_s)$ for $1 \le s < k$.

Now consider the 2-arcs $v \sim u \sim z$ that come from a 6-cycle of the form $v \sim w \sim x \sim y \sim z \sim u \sim v$ with $x \in V^{(k)}$ and $u \in W^{(k)}$. Again, these can be found using the 3-arcs from Lemma 2.1. One of them comes from combining 3-arcs of the forms (1) and (18), and another $k-1$ come from those of the forms (5) and (8) with $s = r$, and another $(k-1)(k-2)$ from those of the forms (5) and (13) with $s \neq r$, giving a total of $k^2 - 2k + 2$. Among them, exactly $k-1$ lie in $k+1$ different 6-cycles altogether (namely the 2-arcs of the form $(0,\mathbf{0}) \sim (1,\mathbf{e}_s) \sim (2,\mathbf{e}_s)$), while the other $k^2 - 3k + 3$ lie on only k different 6-cycles. In particular, $|T_v^{(k,k+1)}| = k-1$.

On the other hand, among the 2-arcs $w \sim u' \sim z'$ coming from a 6-cycle of the form $w \sim v \sim x' \sim y' \sim z' \sim u' \sim w$ with $x' \in W^{(k)}$ and $u' \in V^{(k)}$, none lies in $k+1$ different 6-cycles. Hence $|T_w^{(k,k+1)}| = 0$. Thus $|T_v^{(k,k+1)}| \neq |T_w^{(k,k+1)}|$, and so the Bouwer graph $B(k,6,9)$ cannot be arc-transitive, and is therefore half-arc-transitive.

5.2 The graphs $B(k,6,21)$ for $k > 2$

Next, suppose $m = 6$ and $n = 21$, and $k > 2$. (The case $k = 2$ was dealt with in Section 4.) Here $2^6 = 1$ but $2^3 \neq -1$ in $\mathbb{Z}_n$, so there are not as many 'new' 6-cycles as for $B(k,6,9)$. In fact, the new 6-cycles include just one for each 2-arc $v_0 \sim v_1 \sim v_2$ when the first coordinates of the three vertices v_i are all different. Each 2-arc lies in one, two or k cycles of length 6, and each arc lies in exactly one 2-arc of the first kind, and $k-1$ distinct 2-arcs of each of the second and third kinds. The set $V^{(k)}$ consists of the $k-1$ vertices of the form $(0,-\mathbf{e}_r)$, while $W^{(k)}$ consists of the $k-1$ vertices of the form $(1,\mathbf{e}_s)$. (Note that this does not hold when $k = 2$.) Also $T_v^{(k,2)}$ consists of the 2-arcs of the form $(0,\mathbf{0}) \sim (1,\mathbf{e}_s) \sim (2,\mathbf{e}_s)$, so $|T_v^{(k,2)}| = k-1$, but on the other hand, $T_w^{(k,2)}$ is empty. Hence there can be no automorphism that reverses the arc (v,w), and so the graph is half-arc-transitive.

5.3 The graphs $B(k,6,63)$

Suppose $m = 6$ and $n = 63$. This case is similar to the previous one, but a little easier. Every 2-arc lies in either one or k different cycles of length

6, and each arc lies in exactly k 2-arcs of the first kind, and $k-1$ of the second kind. The sets $V^{(k)}$ and $W^{(k)}$ are precisely as in the previous case, but for all $k \geq 2$, and in this case $T_v^{(k,1)}$ consists of the 2-arcs of the form $(0, \mathbf{0}) \sim (1, \mathbf{e}_s) \sim (2, \mathbf{e}_s)$, so $|T_v^{(k,1)}| = k-1$, but on the other hand, $T_w^{(k,1)}$ is empty. Hence there can be no automorphism that reverses the arc (v, w), and so the graph is half-arc-transitive.

Now we turn to the cases where $n = 5$, 7, 15 or 31. In these cases, the order of 2 as a unit mod n is 4, 3, 4 or 5, respectively, and indeed when $n = 15$ or 31 we may suppose that $m = 4$ or $m = 5$, while the cases $n = 5$ and $n = 7$ are much more tricky.

5.4 The graphs $B(k, 4, 15)$

Suppose $n = 15$ and $m = 4$. Then the girth is 4, with $2k$ different 4-cycles passing through the vertex $(0, \mathbf{0})$. Apart from this difference, the approach taken in Section 3 for counting 6-cycles still works in the same way as before. (In fact there are no additional 6-cycles, largely because $m \neq 6$.) Hence the graph $B(k, 4, 15)$ is half-arc-transitive for all $k \geq 2$.

5.5 The graphs $B(k, 5, 31)$

Similarly, when $n = 31$ and $m = 5$, the girth is 5, and the same approach as taken in Section 3 using 6-cycles works, to show that the graph $B(k, 5, 31)$ is half-arc-transitive for all $k \geq 2$.

5.6 The graphs $B(k, m, 5)$ for $k > 2$

Suppose $n = 5$ and $k > 2$. (The case $k = 2$ was dealt with in Section 4.) Here we have $m \equiv 0 \bmod 4$, and the number of 6-cycles is much larger than in the generic case considered in Section 3 and the cases with $m = 6$ above (because n is so small, and $2^{-1} = -2$ in $\mathbb{Z}_n$), but a similar argument works.

When $m > 4$, the girth of $B(k, m, n)$ is 6, and every 2-arc lies in either $2k$, $2k+3$ or $4k-4$ cycles of length 6. Those of the first kind with initial vertex $(0, \mathbf{0})$ have the form $(0, \mathbf{0}) \sim (1, \mathbf{0}) \sim (2, 2\mathbf{e}_r)$, or $(0, \mathbf{0}) \sim (1, \mathbf{e}_r) \sim (2, \mathbf{e}_r)$, or $(0, \mathbf{0}) \sim (1, \mathbf{e}_r) \sim (2, \mathbf{e}_r + 2\mathbf{e}_s)$ with $s \neq r$, or $(0, \mathbf{0}) \sim (-1, \mathbf{0}) \sim (-2, \mathbf{e}_r)$, or $(0, \mathbf{0}) \sim (-1, 2\mathbf{e}_r) \sim (-2, 2\mathbf{e}_r)$, or $(0, \mathbf{0}) \sim (-1, 2\mathbf{e}_r) \sim (-2, 2\mathbf{e}_r + \mathbf{e}_s)$ with $s \neq r$, while those of the second kind with initial vertex $(0, \mathbf{0})$ have the form $(0, \mathbf{0}) \sim$

$(1, \mathbf{0}) \sim (0, -\mathbf{e}_r)$, or $(0, \mathbf{0}) \sim (1, \mathbf{e}_r) \sim (0, \mathbf{e}_r)$, or $(0, \mathbf{0}) \sim (1, \mathbf{e}_r) \sim (0, \mathbf{e}_r - \mathbf{e}_s)$ with $s \neq r$, or $(0, \mathbf{0}) \sim (-1, \mathbf{0}) \sim (0, -2\mathbf{e}_r)$, or $(0, \mathbf{0}) \sim (-1, 2\mathbf{e}_r) \sim (0, 2\mathbf{e}_r)$, or $(0, \mathbf{0}) \sim (-1, 2\mathbf{e}_r) \sim (0, 2\mathbf{e}_r - 2\mathbf{e}_s)$ with $s \neq r$, In particular, $V^{(2k+3)}$ consists of the $k-1$ vertices of the form $(0, -\mathbf{e}_r)$. A little further analysis shows that $W^{(2k+3)}$ consists of the $k-1$ vertices of the form $(1, \mathbf{e}_s)$, and then $T_v^{(2k+3,2k)}$ consists of the 2-arcs of the form $(0, \mathbf{0}) \sim (1, \mathbf{e}_s) \sim (2, \mathbf{e}_s)$, so $|T_v^{(2k+3,2k)}| = k–1$, but on the other hand, $T_w^{(2k+3,2k)}$ is empty.

When $m = 4$, the girth of $B(k, m, n)$ is 4, and the situation is similar, but slightly different. In this case, every 2-arc lies in either $2k + 2$, $2k + 5$ or $6k - 6$ cycles of length 6, and $|T_v^{(2k+5,2k+2)}| = k-1$ while $|T_w^{(2k+5,2k+2)}| = 0$.

Once again, it follows that no automorphism can reverse the arc (v, w), and so the graph is half-arc-transitive, for all $k > 2$ and all $m \equiv 0 \bmod 4$.

5.7 The graphs $B(k, m, 7)$ for $(k, m) \neq (2, 3)$ or $(2, 6)$

Suppose finally that $n = 7$, with $m \equiv 0 \bmod 3$, but $(k, m) \neq (2, 3)$ or $(2, 6)$. Here we treat four sub-cases separately: (a) $k = 2$ and $m > 6$; (b) $k > 2$ and $m = 3$; (c) $k > 2$ and $m = 6$; and (d) $k > 2$ and $m > 6$.

In case (a), where $k = 2$ and $m > 6$, every 2-arc lies in 1, 2 or 3 cycles of length 6. Also the set $V^{(3)}$ consists of the single vertex $(0, -1)$, while $W^{(3)}$ consists of the single vertex $(1, 1)$, and then $T_v^{(3,1)}$ consists of the single 2-arc $(0, 0) \sim (1, 1) \sim (2, 1)$, so $|T_v^{(3,1)}| = 1$, but on the other hand, $T_w^{(3,1)}$ is empty.

In case (b), where $k > 2$ and $m = 3$, every 2-arc lies in 0, 2 or k cycles of length 6. In this case the set $V^{(k)}$ consists of the $k-1$ vertices of the form $(0, -\mathbf{e}_r)$, while $W^{(k)}$ consists of the $k-1$ vertices of the form $(1, \mathbf{e}_s)$, and then $T_v^{(k,2)}$ consists of the 2-arcs of the form $(0, \mathbf{0}) \sim (1, \mathbf{e}_s) \sim (2, \mathbf{e}_s)$, so $|T_v^{(k,2)}| = k - 1$, but on the other hand, $T_w^{(k,2)}$ is empty.

In case (c), where $k > 2$ and $m = 6$, every 2-arc lies in 3, $k+1$ or $4k-3$ cycles of length 6. Here $V^{(k+1)}$ consists of the $k-1$ vertices of the form $(0, -\mathbf{e}_r)$, while $W^{(k+1)}$ consists of the $k-1$ vertices of the form $(1, \mathbf{e}_s)$, and then $T_v^{(k+1,3)}$ consists of the 2-arcs of the form $(0, \mathbf{0}) \sim (1, \mathbf{e}_s) \sim (2, \mathbf{e}_s)$, and so $|T_v^{(k+1,3)}| = k - 1$, but on the other hand, $T_w^{(k+1,3)}$ is empty.

In case (d), where $k > 2$ and $m > 6$, every 2-arc lies in 1, $k+1$ or $2k-2$ cycles of length 6. Next, if $k > 3$ then $V^{(k+1)}$ consists of the $k-1$ vertices of the form $(0, -\mathbf{e}_r)$, while $W^{(k+1)}$ consists of the k–1 vertices of the form $(1, \mathbf{e}_s)$, but if $k = 3$ then $k+1 = 2k-2$, and $V^{(k+1)}$ contains also $(2, \mathbf{0})$ while $W^{(k+1)}$ contains also $(-1, \mathbf{0})$. Whether $k = 3$ or $k > 3$, the set $T_v^{(k+1,1)}$ consists of the 2-arcs of the form $(0, \mathbf{0}) \sim (1, \mathbf{e}_s) \sim (2, \mathbf{e}_s)$, and so $|T_v^{(k+1,1)}| = k - 1$, but on the other hand, $T_w^{(k+1,1)}$ is empty.

Hence in all four of cases (a) to (d), no automorphism can reverse the arc (v, w), and therefore the graph is half-arc-transitive.

This completes the proof of our Theorem.

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