CHEBYSHEV CONSTANTS, TRANSFINITE DIAMETER, AND COMPUTATION ON COMPLEX ALGEBRAIC CURVES

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ABSTRACT. New notions of directional Chebyshev constant and transfinite diameter have recently been studied on certain algebraic curves in $\mathbb{C}^2$ [9]. The theory is extended here to curves in $\mathbb{C}^N$ for arbitrary N. The results are analogous but require more methods from computational algebraic geometry.

1. INTRODUCTION

The goal of this paper is to study a notion of transfinite diameter on algebraic curves in $\mathbb{C}^N$ ($N > 1$). This will be a natural generalization of the Fekete-Leja transfinite diameter of a compact set in $\mathbb{C}^N$. The importance of the latter has increased in recent years as its geometric and analytic aspects have become better understood (see e.g. [10, 2, 4]).

We briefly recall the definition of the Fekete-Leja transfinite diameter. Let $\{z^{\alpha_j}\}_{j=1}^{\infty}$ be the monomials in N variables listed according to a *graded order* (i.e., $|\alpha_n| \leq |\alpha_k|$ whenever $n < k$). Here we are using standard multi-index notation: if $\alpha_j = (\alpha_{j1}, \ldots, \alpha_{jN})$, then $z^{\alpha_j} = z_1^{\alpha_{j1}} z_2^{\alpha_{j2}} \cdots z_N^{\alpha_{jN}}$ and $|\alpha_j| = \alpha_{j1} + \cdots + \alpha_{jN}$ denotes the total degree. Given a positive integer M and points $\{\zeta_1, \ldots, \zeta_M\} \subset \mathbb{C}^N$, the $M \times M$ determinant

$$(1.1) \qquad \text{Van}(\zeta_1, \ldots, \zeta_M) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z^{\alpha_2}(\zeta_1) & z^{\alpha_2}(\zeta_2) & \cdots & z^{\alpha_2}(\zeta_M) \\ \vdots & \vdots & \ddots & \vdots \\ z^{\alpha_M}(\zeta_1) & z^{\alpha_M}(\zeta_2) & \cdots & z^{\alpha_M}(\zeta_M) \end{pmatrix}$$

is called a *Vandermonde determinant* of order M. Note that $z^{\alpha_1} = 1$ since $\alpha_1 = (0, 0, \ldots, 0)$.

Given a compact set $K \subset \mathbb{C}^N$, the n-th order diameter $d_n(K)$ is defined as follows. Let m_n be the number of monomials of degree at most n in N variables, and let $l_n = \sum_{j=1}^{m_n} |\alpha_j|$ be the sum of the degrees. Then

$$(1.2) \qquad d_n(K) = \sup\{|\text{Van}(\zeta_1, \ldots, \zeta_{m_n})|^{\frac{1}{l_n}} : \{\zeta_1, \ldots, \zeta_{m_n}\} \subset K\}.$$

The Fekete-Leja transfinite diameter of K is then given by

$$(1.3) \qquad d(K) := \lim_{n \to \infty} d_n(K).$$

The existence of the limit on the right-hand side of (1.2) was verified by Fekete [6] when $N = 1$ and by Zaharjuta [12] in general. Recent studies of the transfinite diameter and related notions are presented in [4] and the survey [13].

Now consider an algebraic variety $V \subset \mathbb{C}^N$. The polynomials restricted to V form the *coordinate ring* $\mathbb{C}[V]$; $p = q$ in $\mathbb{C}[V]$ means that $p(z) = q(z)$ for all $z \in V$. The monomials in N variables span $\mathbb{C}[V]$ (as a complex vector space), and linear dependencies among the monomials, induced by restricting to V, are given by the ideal

$$\mathbf{I}(V) = \{p \in \mathbb{C}[z] : p(a) = 0 \text{ for all } a \in V\}.$$

One can systematically reduce the set of monomials to a basis of $\mathbb{C}[V]$ as follows. Let $\{z^{\alpha_j}\}_{j=1}^{\infty}$ denote the monomials indexed according to a graded ordering. One then goes through the collection, removing linearly dependent monomials as they arise. For example, remove $z^{\alpha_{m+1}}$ if it is linearly dependent with respect to $\{z^{\alpha_j}\}_{j=1}^{m}$.

Let now $\mathcal{B} = \{e_j\}_{j=1}^{\infty}$ denote the reduced set of monomials, which is a basis for $\mathbb{C}[V]$ by definition. For a positive integer M and points $\{\zeta_1, \ldots, \zeta_M\} \subset V$, define the $M \times M$ Vandermonde determinant $\mathrm{Van}_{\mathcal{B}}(\zeta_1, \ldots, \zeta_M)$ to be given by (1.1) with each occurence of z^{α_j} replaced by e_j.

If we let now $m_n = m_n(V)$ be the number of monomials in $\mathcal{B}$ of degree at most n, and $l_n = l_n(V) := \sum_{j=1}^{m_n} \deg(e_j)$, then equation (1.2) defines the n-th order diameter of a compact set $K \subset V$ with $\mathrm{Van}_{\mathcal{B}}(\cdot)$ replacing $\mathrm{Van}(\cdot)$. Finally, define the *transfinite diameter* by

$$(1.4) \qquad\qquad d(K) = \limsup_{n \to \infty} d_n(K).$$

The main theorem (Theorem 5.4) says that when V is an algebraic curve (satisfying some additional technical properties), the lim sup in (1.4) may be replaced by a limit as in (1.3). When $N = 2$, this was done in [9]. The point of this paper is that the methods there generalize in a natural way to arbitrary N. The main idea, following [12], is to relate the transfinite diameter to various *Chebyshev constants*, whose limiting properties can be proved directly.

The paper is organized as follows. Section 2 is devoted to recalling background material in computational algebraic geometry and in describing the notation that will be used in the rest of the paper. As indicated above, systematic computation requires an ordering on monomials. In several variables there are several ways to order monomials that respect degree (such orderings are called *graded orderings*); we will work exclusively with the *grevlex ordering* (see Section 2.2).

In Section 3, we relate computation on an algebraic curve $V \subset \mathbb{C}^N$ to its geometry. To study this relationship it is convenient to view V projectively, i.e., consider its projective closure $V_{\mathbb{P}} \subset \mathbb{CP}^N = \mathbb{C}^N \cup H_{\infty}$ (where H_{∞} denotes the hyperplane at infinity) Under mild restrictions on points of $H_{\infty} \cap V_{\mathbb{P}}$, algebraic computation in $\mathbb{C}[V]$ has some nice properties. This section builds on preliminary investigations in $\mathbb{C}^3$ carried out in [1].

In Section 4, we study Chebyshev constants. Following an idea in [3], we define Chebyshev constants associated to homogeneous polynomials. This includes the directional Chebyshev constants of [9] as special cases. We then derive geometric properties of these Chebyshev constants.

In Section 5, we prove Theorem 5.4. The theorem relates the notions of transfinite diameter and directional Chebyshev constant, and shows that the transfinite diameter is given by a well-defined limiting process. Further properties of transfinite diameter are also shown.

The main results of Sections 4 and 5 are the same as those of [9], with some arguments simplified. In particular, most properties of the transfinite diameter given here are not proved directly but follow immediately from corresponding properties of directional Chebyshev constants, which are studied here in more detail. The directional Chebyshev constant is the more primitive notion and its properties are easier to derive. We remark that in a more general setting, the part of Theorem 5.4 dealing with existence of the limit can be proved using Bernstein-Markov measures rather than Chebyshev constants [2].

We close the paper by illustrating the relationship between directional Chebyshev constants associated to $K \subset V$ and Robin constants associated to the Siciak-Zaharjuta extremal function of K, which are important in pluripotential theory. This will be studied further in a future paper.

2. Preliminaries

This section reviews essential background and notation we will need, with proofs omitted. We refer to [5], whose notation we follow closely.

2.1. **Dimension and non-singularity.** Write $z = z_1, \ldots, z_N$ for the standard variables or coordinates on $\mathbb{C}^N$, and write $\mathbb{C}[z] = \mathbb{C}[z_1, \ldots, z_N]$ for the ring of polynomials over $\mathbb{C}$ in these variables. We use standard multi-index notation: if $\alpha = (\alpha_1, \ldots, \alpha_N)$ is a multi-index then $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_N}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_N$.

Given a (non-empty) algebraic variety $V = \{z \in \mathbb{C}^N : P_1(z) = \cdots = P_k(z) = 0\}$, where $P_1, \ldots, P_k$ are polynomials in $\mathbb{C}[z]$, let

$$\mathbf{I}(V) := \{p \in \mathbb{C}[z] : p(z) = 0 \text{ for all } z \in V\}$$

be the ideal of V. The polynomials restricted to V can be identified with elements of the factor ring $\mathbb{C}[z]/\mathbf{I}(V) =: \mathbb{C}[V]$, called the *coordinate ring of V*.

Define the degree on V of a polynomial p by

$$\deg_V(p) = \min\{\deg(q) : q(z) = p(z) \text{ for all } z \in V\},$$

where deg denotes the usual degree in $\mathbb{C}[z]$, i.e., $\deg(c_\alpha z^\alpha) := |\alpha|$ $(c_\alpha \in \mathbb{C} \setminus \{0\})$, and for any polynomial p, $\deg(p) := \max\{|\alpha| : c_\alpha z^\alpha \text{ is a term of } p\}$.

Next, for a non-negative integer s write

$$\mathbb{C}[z]_{\leq s} := \{p \in \mathbb{C}[z] : \deg(p) \leq s\} \text{ and } \mathbb{C}[V]_{\leq s} := \{p \in \mathbb{C}[V] : \deg_V(p) \leq s\}$$

for the polynomials of degree at most s. As a vector space over $\mathbb{C}$ we have $\dim(\mathbb{C}[z]_{\leq s}) = \binom{N+s}{s}$ as can be seen by counting the monomials of degree $\leq s$ in z, and $\dim(\mathbb{C}[V]_{\leq s}) \leq \dim(\mathbb{C}[z]_{\leq s})$. It is well-known that for large s, $\dim(\mathbb{C}[V]_{\leq s})$ is a polynomial in s, $H(s)$ (called the *Hilbert polynomial of V*). It is also a well-known fact that $\deg(H) = 1$ (i.e., $H(s) = as + b$, $a \in \mathbb{N}$, $b \in \mathbb{Z}$) if and only if at all but a finite number of exceptional points, V is a *complex manifold of dimension 1*. For such a non-exceptional point $p \in V$ there is a local one-to-one holomorphic map $\varphi : D \to V$ (where $D = \{|t| < 1\} \subset \mathbb{C}$) with $\varphi(0) = p$. V is then said to be an *algebraic curve*, and p is a *non-singular point*. We will work exclusively with algebraic curves in this paper.

We recall a useful characterization of non-singular points. Given a collection of polynomials $F = \{f_1, \ldots, f_s\}$ and a point p, define the $s \times N$ matrix of partial derivatives

$$J_p(F) = J_p(f_1, \ldots, f_s) := \begin{pmatrix} \partial f_1/\partial z_1(p) & \cdots & \partial f_1/\partial z_N(p) \\ \vdots & \ddots & \vdots \\ \partial f_s/\partial z_1(p) & \cdots & \partial f_s/\partial z_N(p) \end{pmatrix}.$$

Proposition 2.1. *Let V be an algebraic curve in $\mathbb{C}^N$, and $\mathbf{I}(V) = \langle f_1, \ldots, f_s \rangle$. Then $s \geq N - 1$ and $p \in V$ is non-singular if and only if $J_p(f_1, \ldots, f_s)$ has rank $N - 1$.* $\qquad\square$

Hence the normal vectors $\{\nabla f_1(p), \ldots, \nabla f_s(p)\}$ span a (complex) hyperplane H where $\nabla f_i(p) = (\partial f_i/\partial z_1(p), \ldots, \partial f_i/\partial z_N(p))$. The tangent space to V at p is then the orthogonal complement.

2.2. Groebner bases and computation.

The main tool to carry out computation in $\mathbb{C}[V]$ is a generalized division algorithm using Groebner bases. This requires an ordering on the monomials. In one variable, the natural ordering is by degree: $1, z, z^2, z^3, \ldots$. For $n > 1$, there are several natural generalizations, and we recall one called *grevlex ordering*.

Let α, β be multi-indices in N variables. Writing $\prec$ for grevlex, it is defined by setting $z^\alpha \prec z^\beta$ whenever:

- $|\alpha| < |\beta|$; or
- $|\alpha| = |\beta|$ and there exists $i \in \{1, \ldots, n\}$ such that
  $\alpha_i > \beta_i$ and $\alpha_j = \beta_j$ for any positive integer $j < i$.

For example, the first few monomials in $\mathbb{C}[z_1, z_2]$ listed according to $\prec$ are

$$1, z_1, z_2, z_1^2, z_1 z_2, z_2^2, z_1^3, z_1^2 z_2, z_1 z_2^2, z_2^3, \ldots$$

We will work exclusively with grevlex in what follows.

We can now order terms of a polynomial $p(z) = \sum_\alpha a_\alpha z^\alpha$ unambiguously and define the *leading term* $\mathrm{LT}(p) = a_\beta z^\beta$ to be the term for which $a_\beta \neq 0$ and for all α such that $a_\alpha \neq 0$, we have $z^\alpha \prec z^\beta$.

Consider the monomials as elements of $\mathbb{C}[V]$, where V is a curve. We go through the monomials in increasing order (according to grevlex), throwing out linearly dependent monomials as they arise. Let $H(s) = as + b$ be the Hilbert polynomial of V. When s is a sufficiently large positive integer, this says that our reduction process will keep a monomials of degree s and throw out the rest.

Let $\mathcal{B} = \{z^{\alpha_j}\}_{j=1}^\infty$ denote our reduced collection of monomials, which forms a basis of $\mathbb{C}[V]$. Computation in terms of $\mathcal{B}$ is done systematically using a Groebner basis of $\mathbf{I}(V)$, whose definition we now recall. First, given an ideal $I \subset \mathbb{C}[z]$, let $\mathrm{LT}(I) = \{\mathrm{LT}(p) : p \in I\}$, and denote the ideal generated from $\mathrm{LT}(I)$ by $\langle \mathrm{LT}(I) \rangle$.

Definition 2.2. A collection of polynomials $\{g_1, \ldots, g_k\}$ is called a *Groebner basis of I* if $\langle g_1, \ldots, g_k \rangle = I$ and $\langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_k) \rangle = \langle \mathrm{LT}(I) \rangle$.

More precisely, this defines a Groebner basis for the grevlex ordering. (Other orderings may give different leading terms $\mathrm{LT}(g_i)$ for which the Groebner basis condition fails.)

We have the following result on computation in $\mathbb{C}[V]$ (c.f., [5, 5§3]).

Proposition 2.3 (Algebraic computation in $\mathbb{C}[V]$)**.**

> *(1) A Groebner basis for $I = \mathbf{I}(V)$ always exists.*
> *(2) $\mathcal{B} = \{z^\alpha : z^\alpha \notin \langle \mathrm{LT}(I) \rangle\}$.*
> *(3) For any $p \in \mathbb{C}[z]$ there is a unique $r \in \mathbb{C}[z]$ such that*

$$(2.1) \qquad p(z) = \sum_{j=1}^{k} q_j(z) g_j(z) \ + \ r(z)$$

> *where $q_1, \ldots, q_k \in \mathbb{C}[z]$ and all terms of r are in $\mathcal{B}$.* $\qquad\square$

The representation (2.1) is usually computed using a generalized division algorithm, in which $\{g_1, \ldots, g_k\}$ are the divisors, $\{q_1, \ldots, q_k\}$ the quotients, and r the remainder ([5, 2§3]). Although r is unique, the quotients q_i may not be.

Note that for all $z \in V$, $p(z) = r(z)$; so $p(z) = r(z)$ in $\mathbb{C}[V]$. We call r the *normal form of p*. For convenience, we will sometimes write $\rho_V(p) = r$. The above proposition says that $p \mapsto \rho_V(p)$ is a well-defined operation on polynomials, and corresponds to choosing the unique representative of the class of polynomials equal to p on V that can be expressed as a linear combination of elements of $\mathcal{B}$.

3. Computation on curves

3.1. **Multiplication.** From Proposition 2.3, given $f, g \in \mathbb{C}[z]$ it is easy to see that $f = g$ on V if and only if $\rho_V(f) = \rho_V(g)$. Rather than considering $\mathbb{C}[V]$ as a factor ring, one can take the alternative view of $\mathbb{C}[V]$ as the collection of normal forms. This is the linear subspace of $\mathbb{C}[z]$ spanned by $\mathcal{B}$. From this point of view, $\rho_V : \mathbb{C}[z] \to \mathbb{C}[V]$ is a linear map whose kernel is $\mathbf{I}(V)$. Multiplication descends to a bilinear map $* : \mathbb{C}[V] \times \mathbb{C}[V] \to \mathbb{C}[V]$ given by

$$(p, q) \mapsto \rho_V(pq) =: p * q.$$

We will stick to this point of view in what follows: i.e., $\mathbb{C}[V]$ is the space spanned by $\mathcal{B}$, with a multiplication given by $*$. The total degree can also be read off easily: if p is in normal form, then $\deg_V(p) = \deg(p)$ where the latter denotes the usual total degree in $\mathbb{C}[z]$. This works because grevlex is a graded ordering.

Chebyshev constants that we will study in the next section are defined by fixing properties of leading homogeneous parts of polynomials. Given $p = \sum_{|\alpha| \le d} a_\alpha z^\alpha \in \mathbb{C}[V]$, write $\widehat{p} = \sum_{|\alpha| = d} a_\alpha z^\alpha$ for the leading homogeneous part of p.

For polynomials p and q we also want to consider $\widehat{p * q}$, the leading homogeneous part of the product. Given a non-negative integer d, write $\mathbb{C}[V]_{=n}$ for the homogeneous polynomials of degree n in $\mathbb{C}[V]$. Fixing p with $\deg(p) = n$, we want $q \mapsto \widehat{p * q}$ to be a linear map $\mathbb{C}[V]_{=n} \to \mathbb{C}[V]_{=n+\deg p}$. But this is not always the case as cancellations may occur.

Example 3.1. *Let $V = \{z_2^2 - z_1^2 - z_1 - 1 = 0\}$. Take $p = z_1 + z_2$ and $q = z_1 - z_2$. Then $\widehat{p * q} = -z_1$ which is not of degree 2.*

To account for cancellation, we therefore define

$$p \,\widehat{*}\, q = \begin{cases} \widehat{p * q} & \text{if } \deg p + \deg q = \deg(pq) \\ 0 & \text{otherwise} \end{cases}.$$

Note that we let zero be an element of $\mathbb{C}[V]_{=n}$ for each n so that it becomes a vector space. It is easy to see the following.

Lemma 3.2. *Let p be a fixed homogeneous polynomial. Then $q \mapsto p \widehat{*} q$ is a linear transformation $\mathbb{C}[V]_{=n} \mapsto \mathbb{C}[V]_{=n+\deg p}$.* □

When V is a curve, there is a positive integer d, such that for sufficiently large degree n, $\mathbb{C}[V]_{=n}$ is of dimension d (here $H(n) = dn + c$ ($d \in \mathbb{Z}_+, c \in \mathbb{Z}$) is the Hilbert polynomial), and $\mathbb{C}[V]_{=n}$ has basis $\{z^\alpha \notin \langle \mathrm{LT}(I) \rangle : |\alpha| = n\}$. In what follows, we will take n to be sufficiently large that the homogeneous polynomials of a given degree form a space of dimension d.

The grevlex order gives an unambiguous representation of $\mathbb{C}[V]_{=n}$ by $\mathbb{C}^d$; if the basis of $\mathbb{C}[V]_{=n}$ listed in (increasing) grevlex order is $\{z^{\alpha_1}, \dots, z^{\alpha_d}\}$, then match z^{α_j} with the standard j-th coordinate in $\mathbb{C}^d$. Using this, we form vector and matrix representations of polynomials. For $p \in \mathbb{C}[V]_{=n}$ given by $p(z) = a_{\alpha_1} z^{\alpha_1} + \cdots + a_{\alpha_d} z^{\alpha_d}$, set

$$(3.1) \qquad\qquad [p] := (a_{\alpha_1}, \dots, a_{\alpha_d}) \in \mathbb{C}^d.$$

Similarly, denote by $[[p]]$ the representation of p as a $d \times d$ matrix, i.e., as representing the linear map $q \mapsto p \widehat{*} q$. That is, $[[p]]$ is the matrix defined by the equation

$$[[p]][q] = [p \widehat{*} q].$$

3.2. Projective space.
We will be interested in curves whose coordinate rings have additional nice properties for computation. Computational properties of $\mathbb{C}[V]$ are closely related to geometric properties of V.

To study this relationship, we will consider V as a curve in projective space $\mathbb{CP}^N = \mathbb{C}^N \cup H_\infty$, under the usual embedding $(z_1, \dots, z_N) \mapsto [1 : z_1 : \cdots : z_N]$ where the latter are homogeneous coordinates, i.e.,

$$[z_0 : z_1 : \cdots : z_N] = [y_0 : y_1 : \cdots : y_N] \text{ if and only if } z_j y_k = z_k y_j \; \forall j, k = 0, \dots, N.$$

Dehomogenization at z_0 (when $z_0 \neq 0$) recovers affine coordinates via

$$[z_0 : z_1 : \cdots : z_N] = [1 : z_1/z_0 : \cdots : z_N/z_0] \mapsto (z_1/z_0, \dots, z_N/z_0),$$

at points of $\mathbb{C}^N = \mathbb{CP}^N \setminus H_\infty$. Also, we will move relatively freely between standard coordinates $(z_1, \dots, z_n)$ and homogeneous coordinates $[z_0 : z_1 : \cdots : z_N]$. Dehomogenization at z_j (for other j) is defined similarly; this is useful to study points at infinity.

Recall that a *projective variety* is a set of the form

$$\{[z_0 : z_1 : \cdots : z_N] \in \mathbb{CP}^n : q_1(z_0, \dots, z_N) = q_2(z_0, \dots, z_N) = \cdots = q_s(z_0, \dots, z_N) = 0\}$$

where $\{q_j\}_{j=1}^s$ are homogeneous polynomials in $\mathbb{C}[z_0, \dots, z_N]$.

The *homogenization (in z_0) of $p \in \mathbb{C}[z]$* is the unique homogeneous polynomial $p^h \in \mathbb{C}[z_0, z_1, \dots, z_N] = \mathbb{C}[z_0, z]$ for which

$$\deg(p^h) = \deg(p) \text{ and } p(z_1, z_2, \dots, z_N) = p^h(1, z_1, z_2, \dots, z_N).$$

Also, given an ideal $I \subset \mathbb{C}[z_1, \dots, z_N]$, its *homogenization* $I^h \subset \mathbb{C}[z_0, z]$ is given by

$$(3.2) \qquad\qquad I^h = \langle p^h(z_0, z) : p(z) \in I \rangle.$$

The *dehomogenization (at z_0)* of a homogeneous polynomial $h(z_0, z)$ is $h(1, z)$.

For our curve $V \subset \mathbb{C}^N \subset \mathbb{CP}^N$, define $V_\mathbb{P} := \bigcap \mathcal{W}$, where

$$\mathcal{W} = \{W \supset V : W \text{ is a projective variety in } \mathbb{CP}^N\}.$$

We list some well-known properties of $V_\mathbb{P}$ (c.f., [5, 8§4]).

Proposition 3.3. *Let $I = \mathbf{I}(V)$.*

(1) $V_{\mathbb{P}}$ is the smallest projective variety containing V. In particular, $V_{\mathbb{P}} \setminus V$ is a finite subset of H_{∞}.

(2) $V_{\mathbb{P}} = \{[z_0 : \cdots : z_N] : P(z_0, \ldots, z_N) = 0 \text{ for all } P \in I^h\}$.

(3) If G is a Groebner basis for I, then $G^h = \{g^h : g \in G\}$ is a Groebner basis of I^h for the grevlex ordering on $\mathbb{C}[z_0, z_1, \ldots, z_N]$. $\square$

We call $V_{\mathbb{P}}$ the *projective closure of V in $\mathbb{CP}^N$*.

Remark 3.4. Part (3) of the proposition says that $\langle I^h \rangle = \langle G^h \rangle$. It is essential that G is a Groebner basis for a graded ordering. If not, then $I = \langle G \rangle$ only gives $I^h \supset \langle G^h \rangle$.

For the projective curve $V_{\mathbb{P}}$, define
(3.3)
$$\mathbf{I}_h(V_{\mathbb{P}}) := \{p \in \mathbb{C}[z_0, \ldots, z_N] : p(z_0, \ldots, z_N) = 0 \text{ whenever } [z_0 : z_1 : \cdots : z_N] \in V_{\mathbb{P}}\}.$$

Remark 3.5. Write $p = \sum_{j=0}^{\deg p} p_j$ where for each j, p_j is homogeneous of degree j. It is easy to see that $p \in \mathbf{I}_h(V_{\mathbb{P}})$ if and only if $p_j \in \mathbf{I}_h(V_{\mathbb{P}})$ for each j. Hence we need only consider homogeneous polynomials in (3.3).

Proposition 3.6. *Let $V \subset \mathbb{C}^N$ be an algebraic curve with projective closure $V_{\mathbb{P}} \subset \mathbb{CP}^N$. Then $\mathbf{I}_h(V_{\mathbb{P}})$ is the homogenization of $\mathbf{I}(V)$.* $\square$

Lemma 3.7. *If $\langle G_1, \ldots, G_s \rangle = \mathbf{I}_h(V_{\mathbb{P}})$ where $G_1, \ldots, G_s \in \mathbb{C}[z_0, \ldots, z_N]$ are homogeneous polynomials, then $\langle g_1, \ldots, g_s \rangle = \mathbf{I}(V)$ where $g_k(z_1, \ldots, z_N) = G_k(1, z_1, \ldots, z_N)$ for all $k = 1, \ldots, s$.* $\square$

We can study points of $V_{\mathbb{P}} \cap \{[z_0 : \cdots : z_N] : z_j \neq 0\}$ by dehomogenizing the polynomials $G_1, \ldots, G_s$ at z_j ($j \in \{1, \ldots, N\}$). This will be useful in the next section when studying $V_{\mathbb{P}}$ near H_{∞}. If we start with a Groebner basis G of $\mathbf{I}(V)$ in standard coordinates (for grevlex), then by Propositions 3.3 and 3.6, homogenization of the polynomials of G followed by their dehomogenization at z_j gives a Groebner basis for V in local coordinates on $\{z_j \neq 0\}$.

Henceforth, we may refer to a Groebner basis G as being associated to either V or $V_{\mathbb{P}}$, implicitly homogenizing and dehomogenizing the elements of G as the context demands. The associated ideal (e.g. $\mathbf{I}(V)$, $\mathbf{I}_h(V_{\mathbb{P}})$) will be clear from the context.

3.3. **Algebraic and Geometric properties.** Let V be a curve and let $I = \mathbf{I}(V)$ be its ideal in standard affine coordinates. Computation in $\mathbb{C}[V]$ simplifies when $\langle \text{LT}(I) \rangle$ contains monomials of the form $z_k^{a_k}$ for each $k = 2, \ldots, N$. The following proposition gives a condition under which this occurs.

Proposition 3.8. *Suppose the curve V satisfies the following condition on the coordinates of its points at infinity:*

(3.4) $\qquad [0 : z_1 : z_2 : \cdots : z_N] \in V_{\mathbb{P}} \implies z_j \neq 0 \text{ for all } j = 1, \ldots, N.$

Then

(1) $z_1^a \notin \langle \text{LT}(I) \rangle$ for any positive integer a, and for each $k = 2, \ldots, N$ there is a positive integer a_k such that $z_k^{a_k} \in \langle \text{LT}(I) \rangle$.

(2) $[[z_1]]$ is the $d \times d$ identity matrix, I_d (where d is as in (3.1)).

Proof. First, suppose $z_1^a \in \langle \mathrm{LT}(I) \rangle$ for some a. Let $r(z) = \rho_V(z_1^a)$. Then in $\mathbb{C}[z_1, \ldots, z_N]$, we have $\mathrm{LT}(r) \prec z_1^a$, which implies $\deg(r) < a$. The equation $z_1^a = r(z)$ holds for any $z \in V$. Projectively, this means

$$z_1^a = z_0^b r^h(z) \text{ whenever } [z_0 : z_1 : \cdots : z_N] \in V_{\mathbb{P}};$$

where $b = a - \deg(r)$ and $r^h \in \mathbb{C}[z_0, z_1, \ldots, z_N]$ is the homogenization of r. Hence $z_0 = 0$ implies $z_1 = 0$. This contradicts our hypothesis. So $z_1^a \notin \langle \mathrm{LT}(I) \rangle$ for any positive integer a.

Now suppose $z_2^a \notin \langle \mathrm{LT}(I) \rangle$ for any positive integer a. As before, let d be the dimension of $\mathbb{C}[V]_{=a}$ for a sufficiently large. If $a > d$, the number of monomials in z_1, z_2 of total degree a is at least a; hence this collection is so large that at least one monomial $z_1^m z_2^n$ cannot be a monomial in $\mathbb{C}[V]_{=a}$. This is equivalent to $z_1^m z_2^n \in \langle \mathrm{LT}(I) \rangle$. Consider the smallest such monomial with respect to grevlex, and let $r(z) = \rho_V(z_1^m z_2^n)$; then $\deg(r) < d$, and hence projectively,

$$z_1^m z_2^n = z_0^b r^h(z) \text{ for } [z_0 : \cdots : z_N] \in V_{\mathbb{P}}$$

so that $z_0 = 0$ and $z_1 \neq 0$ implies $z_2 = 0$, contradicting our hypothesis. Hence for some $a \leq d$ we have $z_2^a \in \langle \mathrm{LT}(I) \rangle$.

The same argument as above can be repeated inductively for $z_3, \ldots, z_N$. In this case, if z_1 and z_j can have arbitrarily large powers in $\mathbb{C}[V]$ but the powers of $z_2, \ldots, z_{j-1}$ remain bounded, then for a sufficiently large integer a we can deduce the existence of a monomial $z^\alpha := z_1^{a_1} z_2^{a_2} \cdots z_j^{a_j} \in \langle \mathrm{LT}(I) \rangle$ with $|\alpha| = a$ such that $\deg \rho_V(z^\alpha) < a$, and by the same reasoning as above, $z_0 = 0$ and $z_2, \ldots, z_{j-1} \neq 0$ must imply $z_j = 0$, a contradiction. This proves (1).

To prove the second part, note that for each monomial $z^b = z_1^{b_1} z_2^{b_2} \cdots z_N^{b_N} \in \mathbb{C}[V]$, we have $b_k \leq a_k$ whenever $k = 2, \ldots, N$. Assume that the a_k's are the minimum such integers, i.e., if $b_k > a_k$ for some $k = 2, \ldots, N$ then $z^b \in \langle \mathrm{LT}(I) \rangle$; and set $a_1 = \infty$ for convenience. Thus the condition $b_k \leq a_k$ for all k characterizes the monomials $z^b \in \mathbb{C}[V]$.

It follows that on monomials the map $z^b \mapsto z_1 z^b$ is a bijection from $\mathbb{C}[V]_{|b|}$ to $\mathbb{C}[V]_{|b|+1}$, since in $\mathbb{C}[z]$, the fact that $b_k > a_k$ or $b_k \leq a_k$ remains the same for the image under this map. Since the grevlex ordering is also unaffected, the representation of $[[z_1]]$ must be the identity, giving (2). $\qquad \square$

The first part of the above proposition has a partial converse.

Proposition 3.9. *Suppose* $\langle \mathrm{LT}(I) \rangle$ *satisfies Proposition 3.8(1). Then* $z_1 \neq 0$, *i.e.,*

$$V_{\mathbb{P}} \cap \{ [0 : 0 : z_2 : \cdots : z_N] : z_j \in \mathbb{C} \} = \emptyset.$$

Proof. Suppose $z_2^{a_2} \in \langle \mathrm{LT}(I) \rangle$. Then it must be generated by a leading term of a Groebner basis polynomial; in fact, if a_2 is the minimum such positive integer then there is a Groebner basis polynomial of the form

$$g(z) = z_2^{a_2} + z_1 q_1(z) + q_0(z)$$

where $\deg q_0, \deg q_1 < a_2$. The polynomial $z_1 q_1$ consists of the rest of the terms in the leading homogeneous part of g (which according to grevlex comprise powers of z_1 and z_2 only), and q_0 is the lower degree terms. Projectively, we have for all $[z_0 : z_1 : \cdots : z_N] \in V_{\mathbb{P}}$ that

$$0 = g^h(z_0, \ldots, z_N) = z_2^{a_2} + z_1 z_0^{c_1} q_1^h(z_0, \ldots, z_N) + z_0^{c_2} q_0^h(z_0, \ldots, z_N)$$

where we write $c_1 = a_2 - 1 - \deg q_1$, $c_2 = a_2 - \deg q_2$. When $z_0 = z_1 = 0$ the above equation reduces to $z_2^{a_2} = 0$, hence $z_2 = 0$. So

$$V_{\mathbb{P}} \cap \{[0:0:z_2:\cdots:z_N]\} = V_{\mathbb{P}} \cap \{[0:0:0:z_3\cdots:z_N]\}.$$

Using our hypothesis again, we can find by the same process a Groebner basis polynomial whose homogenization has the form $z_3^{a_3} + z_2 q_2 + z_1 q_1 + z_0 q_0$ from which we can show that if $z_0 = z_1 = z_2 = 0$ then $z_3 = 0$ for all points on V.

This forms the basis of an inductive argument that results in the statement that

$$z \in V_{\mathbb{P}} \cap \{[0:0:z_2:\cdots:z_N]\} \implies z = [0:\cdots:0].$$

Since no such z in $\mathbb{CP}^N$ exists, $V_{\mathbb{P}} \cap \{[0:0:z_2:\cdots:z_N]\} = \emptyset$. $\qquad\square$

Proposition 3.10. *Suppose (3.4) holds. Then for $j = 2,\ldots,N$, the following are equivalent:*

(1) λ is an eigenvalue of $[[z_j]]$.
(2) There is $[0:1:\lambda_2\cdots:\lambda_N] \in V_{\mathbb{P}} \cap H_\infty$ such that $\lambda = \lambda_j$.

Hence the matrices $[[z_j]]$ are non-singular for all j.

Proof. $\underline{(1) \Rightarrow (2)}$ Suppose λ is an eigenvalue of $[[z_j]]$. Then there exists a non-zero homogeneous polynomial $v \in \mathbb{C}[V]$ with the property that $[[z_j]][v] = \lambda[v]$. Since $[[z_1]]$ is the identity matrix I_d (Proposition 3.8(2)), we have $([[z_j]] - \lambda[[z_1]])[v] = 0$, and so

$$(z_j - \lambda z_1)v(z_1,\ldots,z_N) = z_0 r(z_0,z_1,\ldots,z_N) \text{ for } [z_0:\cdots:z_N] \in V_{\mathbb{P}}$$

for some homogeneous polynomial r. Thus $(\lambda_j - \lambda)v(1,\lambda_2\ldots,\lambda_N) = 0$ whenever $[0:1:\lambda_2:\cdots:\lambda_N] \in V_{\mathbb{P}}$. If $\lambda_j = \lambda$ at one of these points, we are done.

Otherwise, we aim to derive a contradiction. Suppose $[0:1:\lambda_2:\cdots:\lambda_N] \in V_{\mathbb{P}}$ implies $\lambda_j \neq \lambda$. Then $v(1,\lambda_2,\ldots,\lambda_N) = 0$ always holds. By homogeneity, this means $v(z_1,z_2,\ldots,z_N) = 0$ whenever $[0:z_1:\cdots:z_N] \in V_{\mathbb{P}}$, and so

$$v(z_1,\ldots,z_N) = z_0 r(z_0,z_1\ldots,z_N) \text{ for all } [z_0:z_1:\cdots:z_N] \in V_{\mathbb{P}}$$

for some homogeneous polynomial $r(z_0,\ldots,z_N)$. Viewed affinely in $\mathbb{C}[z_1,\ldots,z_N]$, this implies that $\mathrm{LT}(v(z_1,\ldots,z_N) - r(1,z_1,\ldots,z_N)) = \mathrm{LT}(v(z_1,\ldots,z_N))$. But we have

$$v(z_1,\ldots,z_N) - r(1,z_1,\ldots,z_N) = \sum_{g_i \in G} q_i g_i \text{ in } \mathbb{C}[z_1,\ldots,z_N],$$

where we sum on the right-hand side over a Groebner basis G for $I = \mathbf{I}(V)$. Equating coefficients in this equation yields $\mathrm{LT}(v) \in \langle \mathrm{LT}(G) \rangle = \langle \mathrm{LT}(I) \rangle$, but this contradicts the fact that v is a normal form. Therefore, the statement that $\lambda_j \neq \lambda$ whenever $[0:1:\lambda_2:\cdots:\lambda_N] \in V_{\mathbb{P}}$ is false.

$\underline{(2) \Rightarrow (1)}$ Suppose $[0:1:\lambda_2:\cdots:\lambda_N] \in V_{\mathbb{P}}$. Let P be the characteristic polynomial of $A = [[z_j]]$, and P^h its homogenization (in one more variable), such that $P(\lambda) = P^h(1,\lambda)$. By the Cayley-Hamilton theorem of linear algebra a matrix satisfies its characteristic equation, so $0 = P(\mathrm{I}_d,[[z_j]]) = P^h([[z_1]],[[z_j]])$. Translated back to computation on $V_{\mathbb{P}}$, this says that

$$P^h(z_1,z_j) = z_0 r(z_0,z_1,\ldots,z_N) \text{ whenever } [z_0:z_1:\cdots:z_N] \in V_{\mathbb{P}},$$

for some polynomial r. Plugging in $[0:1:\lambda_2:\cdots:\lambda_N]$ to the above equation gives $0 = P^h(1,\lambda_j) = P(\lambda_j)$. Hence λ_j is an eigenvalue of $[[z_j]]$.

Note that since (3.4) holds, clearly $[[z_j]]$ is non-singular since all of its eigenvalues are non-zero. □

Recall that a curve $V_\mathbb{P}$ intersects a hyperplane H *transversally* at a point p if any tangent line to $V_\mathbb{P}$ at p does not lie in H. The following is a straightforward consequence of Proposition 2.1.

Lemma 3.11. *Suppose $V_\mathbb{P} \subset \mathbb{CP}^N$ is a curve and $\langle g_1, \ldots, g_s \rangle = \mathbf{I}_h(V_\mathbb{P})$, where g_k is a homogeneous polynomial for each $k = 1, \ldots, s$. Then*

   (1) For any non-singular point a, the matrix $J_a(g_1, \ldots, g_s)$ has rank $N - 1$ in any local coordinate.[*]
   (2) If $V_\mathbb{P}$ intersects the hyperplane
   $$H = \{[z_0 : \cdots : z_N] : h(z_0, \ldots, z_N) = A_0 z_0 + \cdots + A_N z_N = 0\}$$
   transversally at p, then in any local coordinate at p, $J_p(g_1, \ldots, g_s, h)$ has rank N. □

We have the following.

Proposition 3.12. *Suppose $V \subset \mathbb{C}^N$ is an algebraic curve, and $V_\mathbb{P} \cap H_\infty$ is a set of d points*
$$\{\lambda_i = [0 : 1 : \lambda_{i,2} : \cdots : \lambda_{i,N}]\}_{i=1}^d,$$
each of which is a non-singular point of $V_\mathbb{P}$ that intersects H_∞ transversally. Suppose for each $i = 1, \ldots, d$ and $j = 2, \ldots, N$, $\lambda_{i,j} \neq 0$. Further, suppose for each $j \in \{2, \ldots, N\}$, no two distinct points of $V_\mathbb{P} \cap H_\infty$ have the same j-th coordinate (i.e. $i \neq i'$ implies $\lambda_{i,j} \neq \lambda_{i',j}$).

Then $\dim \mathbb{C}[V]_{=n} = d$ and $[[z_j]]$ is a $d \times d$ matrix with eigenvalues $\{\lambda_{i,j}\}_{i=1}^d$. Hence the eigenvalues are all of multiplicity one.

Proof. By Proposition 3.10, we have exactly d distinct eigenvalues of $[[z_j]]$, which are given by the $\lambda_{i,j}$, $i = 1, \ldots, d$. Hence $\dim \mathbb{C}[V]_{=n} \geq d$. Strict inequality means that there is an eigenvalue of multiplicity greater than one. Then an exercise in calculus shows that if $\mathbf{I}_h(V_\mathbb{P}) = \langle g_1, \ldots, g_s \rangle$ for some homogeneous polynomials $g_1, \ldots, g_s$, then (dehomogenizing in local coordinates with $z_1 = 1$) the matrix $J_{\lambda_i}(g_1, \ldots, g_s, z_0)$ has rank $\leq N - 1$, contradicting the previous lemma. Therefore all eigenvalues are of multiplicity one, so $\dim \mathbb{C}[V]_{=n} = d$ and $[[z_j]]$ is a $d \times d$ matrix. □

Remark 3.13. Proposition 3.12 says that the intersection number of V with a hyperplane is the same as the leading coefficient of its (linear) Hilbert polynomial. This can be extracted as a special case of a general formula in algebraic geometry (cf. [7, Thm. 7.7]). This number d is called the *degree of V*, denoted $\deg(V)$.

Lemma 3.14. *Suppose V satisfies the hypotheses of Proposition 3.12. Let $V_j \subset \mathbb{CP}^2$ be the curve given by projecting $V_\mathbb{P}$ to the coordinates $[z_0 : z_1 : z_j]$. Then there is a homogeneous polynomial $P_j \in \mathbb{C}[z_0, z_1, z_j]$ such that $\deg P_j = d$, $\mathbf{I}(V_j) = \langle P_j \rangle$, and $\lambda \mapsto P_j(0, 1, \lambda)$ is the characteristic polynomial of $[[z_j]]$.*

Proof. Since V_j is a curve in $\mathbb{CP}^2$, we have $V_j = \{P_j = 0\}$ and $\mathbf{I}(V_j) = \langle P_j \rangle$ for some polynomial $P_j \in \mathbb{C}[z_0, z_1, z_j]$. That $\deg(P_j) = d$ follows from our hypotheses on V: $V_j \cap \{z_0 = 0\}$ contains precisely d points, and projection does not increase

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[*]e.g. dehomogenize $g_1, \ldots, g_s$ at z_j if $a \in \{[z_0 : \cdots : z_N] : z_j \neq 0\}$.

the degree of a curve. So $\deg(P_j) \leq d$. It is easy to see equality by applying the one-variable Factor Theorem to P_j (setting $z_0 = 0$ and $z_1 = 1$).

Let $Q_j(z_1, z_j)$ be the homogenization in two variables of the characteristic polynomial of $[[z_j]]$, i.e., $\lambda \mapsto Q_j(1, \lambda)$ is the characteristic polynomial of $[[z_j]]$. Factoring out $\lambda - \lambda_j$ from the characteristic polynomial and translating the characteristic equation to computation on $V_\mathbb{P}$ (using the Cayley-Hamilton theorem as in the proof of Proposition 3.12), we have

$$Q_j(z_1, z_j) = (z_j - \lambda_j z_1)v_j(z_1, z_j) = 0 \quad \text{on } V_\mathbb{P} \cap H_\infty$$

for some homogeneous polynomial $v_j(z_1, z_j)$ of degree $d-1$. Since the above equation is independent of all coordinates other than z_0, z_1, z_j we can consider it on the projection $V_j \subset \mathbb{CP}^2$. Then $(z_j - \lambda_j z_1)v_j(z_1, z_j) = 0$ on $V_j \cap \{z_0 = 0\}$, which says that

$$(3.5) \qquad Q_j(z_1, z_j) = (z_j - \lambda_j z_1)v_j(z_1, z_j) = z_0 r_j(z_0, z_1, z_j) \quad \text{if } [z_0 : z_1 : z_j] \in V_j,$$

where r_j is a homogeneous polynomial of degree $d - 1$. Now (3.5) says that $Q_j(z_1, z_j) - z_0 r_j(z_0, z_1, z_j) \in \langle P_j \rangle$. Since $\deg(Q_j) = \deg(P_j) = d$, $Q_j(z_1, z_j)$ must be a constant multiple of $P_j(0, z_1, z_j)$. Renormalizing P_j yields the result. □

Given $j \in \{2, \ldots, N\}$, suppose λ_j is an eigenvalue of $[[z_j]]$. Then there is an associated eigenvector, which we translate into a homogeneous polynomial as follows. Let n be the smallest positive integer such that $\mathbb{C}[V]_{=n}$ has dimension d. Set $v_{\lambda_j}(z) := \sum_{k=1}^{d} a_k z^{\alpha_k}$ where $(a_1, \ldots, a_d)$ is an eigenvector of λ_j and $\{z^{\alpha_1}, \ldots, z^{\alpha_d}\}$ is the basis for $\mathbb{C}[V]_{=n}$. (So $[[z_j]][v_{\lambda_j}] = \lambda_j[v_{\lambda_j}]$.)

Lemma 3.15. *Suppose V satisfies the hypotheses of Proposition 3.12, and v_{λ_j} is a polynomial, as defined above, associated to some eigenvalue λ_j of $[[z_j]]$. For $w = (1, w_2, \ldots, w_N)$, suppose $[0 : w] = [0 : 1 : w_2 : \cdots : w_N] \in V_\mathbb{P} \cap H_\infty$.*
Then $v_{\lambda_j}(w) = 0$ iff $w_j \neq \lambda_j$.

Proof. From $[[z_j]][v_{\lambda_j}] = \lambda_j[v_{\lambda_j}]$ we obtain for $z = [1 : z_1 : \cdots : z_N] \in V_\mathbb{P}$ that $z_j v_{\lambda_j}(z) = \lambda_j z_1 v_{\lambda_j}(z) + r(z)$ for some polynomial r with $\deg(r) \leq \deg(v_{\lambda_j})$, and hence

$$(z_j - \lambda_j z_1)v_{\lambda_j}(z) = z_0^a r^h(z_0, z_1, \ldots, z_N) \quad \text{whenever } [z_0 : z_1 : \cdots : z_N] \in V_\mathbb{P}$$

for some positive integer a. Plugging in $[0 : w] \in V_\mathbb{P} \cap H_\infty$, we obtain $(w_j - \lambda_j)v_{\lambda_j}(w) = 0$. Hence $w_j \neq \lambda_j$ implies $v_{\lambda_j}(w) = 0$.

We now show that $w_j = \lambda_j$ implies $v_{\lambda_j}(w) \neq 0$. As before, let V_j be the algebraic curve in $\mathbb{CP}^2$ given by projecting $V_\mathbb{P}$ to the coordinates $[z_0 : z_1 : z_j]$, with $V_j = \{P_j = 0\}$, $\deg(P_j) = d$. Let v_j and r_j be as in (3.5). Since v_j is formed by factoring out $(z_j - \lambda_j z_1)$ where λ_j is a simple eigenvalue of $[[z_j]]$, we have $v_j(1, \lambda_j) \neq 0$. This will give us what we want after transferring our calculations back to V. To this end, set

$$v(z) = v(z_1, \ldots, z_n) := \rho_V(v_j(z_1, z_j)), \quad \text{and} \quad r(z) = r(z_1, \ldots, z_n) := \rho_V(r_j(1, z_1, z_j))$$

in $\mathbb{C}[z_1, \ldots, z_N]$. Equation (3.5) then translates to

$$(3.6) \qquad z_j v(z) = \lambda_j z_1 v(z) + r(z), \quad \text{for all } [1 : z_1 : \cdots : z_N] \in V_\mathbb{P}.$$

We verify that $\deg v = \deg v_j$. Clearly $\deg v \leq \deg v_j{}^*$ so that

$$z_0^c v^h(z_0, z_1, \ldots, z_N) - v(z_1, z_j) = 0 \ \text{ for all } [z_0 : z_1 : \cdots : z_N] \in V_{\mathbb{P}},$$

where $c = \deg v_j - \deg v$ and v^h is the homogenization of v in the variable z_0. But if $c > 0$, then evaluating at a point $[0 : 1 : \lambda_2 : \cdots : \lambda_N] \in V_{\mathbb{P}}$ would give $v_j(1, \lambda_j) = 0$, a contradiction.

Hence $\deg v = \deg v_j > \deg r_j \geq \deg r$ and

$$(3.7) \qquad 0 \neq v_j(1, \lambda_j) = v^h(0, 1, \lambda_2, \ldots, \lambda_N) = \widehat{v}(1, \lambda_2, \ldots, \lambda_N)$$

where v^h is as above and $\widehat{v}$ is the leading homogeneous part of v in the variables $z_1, z_2, \ldots, z_n$.

We now rewrite equation (3.6), grouping the lower degree terms of v with r to obtain

$$z_j \widehat{v}(z) = \lambda_j z_1 \widehat{v}(z) + \tilde{r}(z_1, \ldots, z_N) \text{ for all } [1 : z_1 : \cdots : z_N] \in V_{\mathbb{P}},$$

with $\deg \tilde{r} \leq \deg \widehat{v}$. The above equation says that $[\widehat{v}]$ is an eigenvector of $[[z_j]]$, therefore using the uniqueness of eigenvectors of multiplicity one up to scalar multiples, $\widehat{v} = c z_1^a v_{\lambda_j}$ for some non-zero constant c and non-negative integer a. Plugging in $[0 : w] = [0 : 1 : \lambda_2 : \cdots : \lambda_N] \in V_{\mathbb{P}}$ and using (3.7), we have

$$0 \neq v_j(1, \lambda_j) = \widehat{v}(w) = c v_{\lambda_j}(w),$$

so $v_{\lambda_j}(w) \neq 0$, as required. □

To get a unique polynomial associated to λ_j we can choose a convenient normalization. We will normalize as follows: put

$$(3.8) \qquad\qquad\qquad\qquad v_{\lambda_j}(\lambda) = 1$$

where $\lambda = [0 : 1 : \lambda_2 : \cdots : \lambda_N]$ is the unique point of $V_{\mathbb{P}} \cap H_\infty$ whose j-th coordinate is λ_j. Let us call this normalized polynomial v_{λ_j} the *eigenvector polynomial associated to λ_j*.

When a curve V is reducible, an eigenvector polynomial for one of its components is related to one for the entire curve.

Proposition 3.16. *Suppose $V = V_1 \cup V_2$ where V_1, V_2 are algebraic curves of degree d_1, d_2, with $d_1 + d_2 = d$. Let $\lambda = [0 : 1 : \lambda_1 : \cdots : \lambda_N] \in V_{1\mathbb{P}} \cap H_\infty$. For $j \in \{1, \ldots, N\}$, let $w_j \in \mathbb{C}[V_1]$, $v_j \in \mathbb{C}[V]$ be the eigenvector polynomials for λ_j on $V_{1\mathbb{P}}$ and $V_{\mathbb{P}}$ respectively. Then there is a homogeneous polynomial $\varphi \in \mathbb{C}[V]$ and non-negative integer a such that*

$$(3.9) \qquad\qquad\qquad z_1^a v_j(z) = w_j(z) \,\widehat{*}\, \varphi(z).$$

holds in $\mathbb{C}[V]$.

Remark 3.17. By construction, monomials in $\mathbb{C}[V_1]$ are also in $\mathbb{C}[V]$. Hence on the right-hand side of (3.9), w_j also makes sense as a polynomial in $\mathbb{C}[V]$.

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*Since grevlex is a graded order, the computation of v from v_j does not increase degree. For the same reason, $\deg(r_j) \geq \deg(r)$.

Proof. Let $\pi : \mathbb{CP}^N \to \mathbb{CP}^2$ be the projection to the coordinates $[z_0 : z_1 : z_j]$. Then $\pi(V_{\mathbb{P}}) = \pi(V_{1\mathbb{P}}) \cup \pi(V_{2\mathbb{P}})$ with $\pi(V_{2\mathbb{P}}) = \{[z_0 : z_1 : z_j] : P(z_0, z_1, z_j) = 0\}$ for some homogeneous polynomial P. Let $\varphi_0(z) := \rho_V(P(1, z_1, z_j)) \in \mathbb{C}[V]$, with leading homogeneous part $\widehat{\varphi}_0(z)$.

We show that $w_j \widehat{*} \widehat{\varphi}_0$ satisfies the eigenvector property in $\mathbb{C}[V]$ for λ_j. To show this, let $z \in V$. If $z \in V_1$, then we use computational properties of w_j in $\mathbb{C}[V_1]$ (in particular, the equivalent of (3.6) for w_j on V_1):

$$\begin{aligned}
z_j(w_j \widehat{*} \widehat{\varphi}_0)(z) &= z_j w_j(z)\varphi_0(z) + r_1(z) \\
&= (\lambda_j z_1 w_j(z) + r_2(z))\varphi_0(z) + r_1(z) \\
&= \lambda_j z_1 w_j(z) \widehat{*} \widehat{\varphi}_0(z) + r(z) = \lambda_j z_1 v(z) + r(z)
\end{aligned}$$

where $\deg(r_1) \leq \deg v$, $\deg(r_2) \leq \deg w_j$ and $\deg r \leq \deg v$.

On the other hand, if $z \in V_2$, then

$$\begin{aligned}
z_j(w_j \widehat{*} \widehat{\varphi}_0)(z) &= z_j w_j(z)\varphi_0(z) + r_3(z) \\
&= \lambda_j z_1 w_j(z)\varphi_0(z) + r_3(z)
\end{aligned}$$

where $\deg r_3 \leq \deg(w_j \widehat{*} \widehat{\varphi}_0)$. Note that the first term on the right-hand side of each line is zero since $P(z_0, z_1, z_j) = 0$ on $\pi(V_{2\mathbb{P}})$ implies $\varphi_0(z) = 0$ on V_2. We have simply used this fact to cast the expression into the form we want (i.e. replacing z_j with $\lambda_j z_1$).

In summary, this shows that $[w_j \widehat{*} \widehat{\varphi}_0]$ is an eigenvector for $[[z_j]]$ for λ_j (representing computation in $\mathbb{C}[V]$). By uniqueness of eigenvectors up to scalar multiples (since λ_j has multiplicity one), $C[w_j \widehat{*} \widehat{\varphi}_0] = [v_j]$ where v_j is the eigenvector polynomial in $\mathbb{C}[V]$ for λ_j and $C \in \mathbb{C}$ is a constant. So $z_1^a v_j(z) = w_j \widehat{*} C\widehat{\varphi}_0$. for some non-negative integer a. Setting $\varphi := C\widehat{\varphi}_0$ gives the result. $\qquad \square$

3.4. **An illustration.**
We illustrate the concepts of Section 3.3 with a concrete example. Consider the curve V in $\mathbb{C}^3$ given by

$$z_2^2 + z_3^2 - z_1^2 - 1 = z_3^2 + z_2 z_3 - 2z_2^2 + z_1 z_3 - z_1 z_2 + 1 = 0.$$

A Groebner basis for $\mathbf{I}(V)$ (for grevlex) is given by[†]

$$\begin{aligned}
G = \{&z_2 z_3 + z_1 z_3 - 3z_2^2 - z_1 z_2 + z_1^2 + 2, z_3^2 + z_2^2 - z_1^2 - 1, \\
&10z_2^3 - 2z_1 z_2^2 - 6z_1^2 z_2 + z_1^2 z_3 + z_1^3 - 7z_2 - 2z_3 + 3z_1\}.
\end{aligned}$$

This gives $\langle \text{LT}(G) \rangle = \langle z_2 z_3, z_3^2, 10z_2^3 \rangle$. From this we obtain that the monomial basis of $\mathbb{C}[V]_{=n}$ (for $n \geq 2$) is $\{z_1^n, z_1^{n-1} z_2, z_1^{n-1} z_3, z_1^{n-2} z_2^2\}$. Clearly $[[z_1]]$ is the 3×3 identity matrix with respect to this basis. More calculations yield

$$[[z_2]] = \begin{bmatrix} 0 & 0 & -1 & -1/10 \\ 1 & 0 & 1 & 6/10 \\ 0 & 0 & -1 & -1/10 \\ 0 & 1 & 3 & 2/10 \end{bmatrix}, \quad [[z_3]] = \begin{bmatrix} 0 & -1 & 1 & 7/10 \\ 0 & 1 & 0 & -2/10 \\ 1 & -1 & 0 & 7/10 \\ 0 & 3 & -1 & -14/10 \end{bmatrix}.$$

Observe that $[0 : 1 : \frac{1}{\sqrt{2}} : \frac{1}{\sqrt{2}}]$ is a point in $V_{\mathbb{P}}$ (using homogeneous coordinates $[z_0 : z_1 : z_2 : z_3]$), and that $\frac{1}{\sqrt{2}}$ is an eigenvalue of both $[[z_2]]$ and $[[z_3]]$.

[†]All calculations were done by a computer algebra system.

4. Chebyshev constants

We define notions of Chebyshev constant associated to a compact subset of an algebraic curve.

Let V be an algebraic curve of degree d whose points at infinity satisfy the hypotheses of Proposition 3.12. Recall that such an algebraic curve has the following properties:

(i) We have $(V_{\mathbb{P}} \cap H_{\infty}) \subset \{z_1 \neq 0\}$ in homogeneous coordinates $[z_0 : z_1 : \cdots : z_N]$.

(ii) The intersection points $V_{\mathbb{P}} \cap H_{\infty}$ are non-singular on $V_{\mathbb{P}}$ and all intersections are transverse.

(iii) If $\lambda_1 = [0 : 1 : \lambda_{12} : \cdots \lambda_{1N}]$ and $\lambda_2 = [0 : 1 : \lambda_{22} : \cdots : \lambda_{2N}]$ are points of $V_{\mathbb{P}} \cap H_{\infty}$, then $\lambda_{1j} \neq \lambda_{2j}$ for all $j = 2, \ldots, N$.

Let $\lambda = [0 : 1 : \lambda_2 : \cdots : \lambda_N] \in V_{\mathbb{P}} \cap H_{\infty}$ be a point at infinity. We have the following.

Lemma 4.1. *There is a unique polynomial* $\mathbf{v}_\lambda \in \mathbb{C}[V]$ *of minimal degree with the following properties:*

(1) $\mathbf{v}_\lambda(\lambda) = 1$.

(2) $\mathbf{v}_\lambda(w) = 0$ *if* $w \in (V_{\mathbb{P}} \cap H_{\infty}) \setminus \{\lambda\}$.

(3) For any polynomial $p \in \mathbb{C}[V]$,

$$(4.1) \qquad (p * \mathbf{v}_\lambda)(z) = z_1^{\deg(p)} \widehat{p}(\lambda) \mathbf{v}_\lambda(z) + r(z)$$

for some polynomial $r \in \mathbb{C}[V]$ *with* $\deg(r) < \deg(p) + \deg(\mathbf{v}_\lambda)$.

(4) If (1)–(3) hold with $\mathbf{v}_\lambda$ *replaced by some polynomial* w, *then* $w = z_1^a \mathbf{v}_\lambda$ *for some non-negative integer* a.

Proof. Set

$$\tilde{\mathbf{v}}_\lambda(z) := (v_2 \, \widehat{*} \, v_3 \cdots \widehat{*} \, v_N)(z),$$

where $v_j(z)$ is the eigenvector polynomial in $\mathbb{C}[V]$ corresponding to the eigenvalue λ_j of $[[z_j]]$, normalized so that $v_j(\lambda) = 1$.

Then by Lemma 3.15 and the normalization equation (3.8), $\tilde{\mathbf{v}}_\lambda(z)$ satisfies the first two properties. By linearity, it suffices to verify the third property when p is a monomial. This is a calculation that uses the fact that $\tilde{\mathbf{v}}_\lambda$ is formed from eigenvector polynomials. Explicitly, given $p(z) = z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_N^{\alpha_N}$ (so $\deg(p) = |\alpha|$), then for $z \in V$, we have by repeated application of (3.6) that

$$
\begin{aligned}
p(z)\tilde{\mathbf{v}}_\lambda(z) &= z_1^{\alpha_1} \Big(\prod_{j=2}^{N} z_j^{\alpha_j}\Big)\Big(\prod_{j=2}^{N} v_j(z)\Big) = z_1^{\alpha_1} \prod_{j=2}^{N} z_j^{\alpha_j} v_j(z) \\
&= z_1^{\alpha_1} \prod_{j=2}^{N} \Big((\lambda_j z_1)^{\alpha_j} v_j(z) + r_j(z)\Big) \quad (\text{with } \deg(r_j) < \deg(v_j) + \alpha_j) \\
&= \Big(z_1^{\alpha_1 + \cdots + \alpha_N} \prod_{j=2}^{N} \lambda_j^{\alpha_j} \prod_{j=2}^{N} v_j(z)\Big) + r(z) = z_1^{|\alpha|} \widehat{p}(\lambda)\tilde{\mathbf{v}}_\lambda + r(z),
\end{aligned}
$$

with $\deg(r) < |\alpha| + \deg(\tilde{\mathbf{v}}_\lambda)$. This proves property (3).

Suppose $w \in \mathbb{C}[V]$ is a homogeneous polynomial satisfying the first three properties; then

$$z_1^{\deg(w)} \tilde{\mathbf{v}}_\lambda(z) + r(z) = w(z) * \tilde{\mathbf{v}}_\lambda(z) = z_1^{\deg(\tilde{\mathbf{v}}_\lambda)} w(z) + \tilde{r}(z)$$

where $\deg(r), \deg(\tilde{r}) < \deg(w * \tilde{\mathbf{v}}_\lambda)$. Since the first and last polynomials are identical, equating coefficients gives

$$(4.2) \qquad z_1^{\deg(w)} \tilde{\mathbf{v}}_\lambda(z) = z_1^{\deg(\tilde{\mathbf{v}}_\lambda)} w(z).$$

The collection $\mathcal{W}$ of all homogeneous polynomials $w \in \mathbb{C}[V]$ satisfying the first three properties is thus a non-empty subset of

$$\{p \in \mathbb{C}[z_1, \ldots, z_N] : p(z) = z_1^a \tilde{\mathbf{v}}_\lambda \text{ for some } a \in \mathbb{Z}\},$$

which is well-ordered by (total) degree. By the well-ordering principle, we can take $\mathbf{v}_\lambda \in \mathcal{W}$ to be the element with minimal degree. It is unique and satisfies property (4). $\qquad \square$

Definition 4.2. We will call $\mathbf{v}_\lambda$ the *directional polynomial* for λ.

Example 4.3. Let V be the algebraic curve in $\mathbb{C}^3$ given by

$$z_2^2 + z_3^2 - z_1^2 - 1 = z_3^2 + z_2 z_3 - 2 z_2^2 + z_1 z_3 - z_1 z_2 + 1 = 0,$$

which was considered in Section 3.4 above. One can verify that $\lambda = [0 : 1 : -\frac{4}{5} : \frac{3}{5}]$ is a point on $V_\mathbb{P}$. An eigenvector of $[[z_2]]$ associated to the eigenvalue $-\frac{4}{5}$ is $\begin{bmatrix} 1 \\ -1 \\ 1 \\ -2 \end{bmatrix}$,

and this is also an eigenvector of $[[z_3]]$ for $\frac{3}{5}$. Hence the eigenvector polynomials for multiplication by z_2 and z_3 are both given by

$$v_2(z) = \tfrac{25}{28}(z_1^2 - z_1 z_2 + z_1 z_3 - 2 z_2^2) = v_3(z),$$

where we normalize so that at $z = (1, -\frac{4}{5}, \frac{3}{5})$ the polynomial evaluates to 1 (see (3.8)). It follows easily that the directional polynomial for $\lambda = [0 : 1 : -\frac{4}{5} : \frac{3}{5}]$ is

$$\mathbf{v}_\lambda(z) = \tfrac{25}{28}(z_1^2 - z_1 z_2 + z_1 z_3 - 2 z_2^2).$$

Another calculation gives

$$(\mathbf{v}_\lambda * \mathbf{v}_\lambda)(z) = \tfrac{25}{28} z_1^2(z_1^2 - z_1 z_2 + z_1 z_3 - 2 z_2^2) - \tfrac{75}{112} z_1^2 + \tfrac{100}{49} z_1 z_2 - \tfrac{25}{14} z_1 z_3 + \tfrac{1625}{392} z_2^2 - \tfrac{125}{98}$$

(cf. (4.3) below).

For a positive integer $s \geq \deg \mathbf{v}_\lambda := a$, define the homogeneous polynomial $\mathbf{v}_{\lambda,s}(z) := z_1^{s-a} \mathbf{v}_\lambda(z)$. With $\mathcal{W}$ as above, we have $\mathcal{W} = \{\mathbf{v}_{\lambda,s} : s \geq \deg(\mathbf{v}_\lambda)\}$. Equation (4.1) also has the following useful consequence:

$$(4.3) \qquad \text{For all } z \in V, \quad (\mathbf{v}_{\lambda,s}(z))^q = \mathbf{v}_{\lambda,sq}(z) + r(z) \quad (\deg(r) < sq).$$

Remark 4.4. In (4.3) above we mean $(\mathbf{v}_{\lambda,s}(z))^q = \mathbf{v}_{\lambda,s}(z) * \cdots * \mathbf{v}_{\lambda,s}(z)$ (q times). In what follows we will simplify things by writing pq, p^2, etc. for $p * q$, $p * p$, etc. This will present no problem as we will be restricting our attention to points of V. More generally, we will implicitly take normal forms of various expressions (i.e., apply $\rho_V(\cdot)$) so that we stay in $\mathbb{C}[V]$.

Definition 4.5. Let $K \subset V$ be a compact set, and $Q \in \mathbb{C}[V]$. Let $\mathcal{C}(Q)$ denote the collection of polynomials

$$\mathcal{C}(Q) := \{p \in \mathbb{C}[V] : \exists n \in \mathbb{N}, p(z) = Q(z)^n + r(z), \ \deg(r) < n \deg(Q)\}.$$

Define the n-th order Q-Chebyshev constant of K by

(4.4) $\tau(K, Q, n) := (\inf\{\|p\|_K : p \in \mathcal{C}(Q), \ \deg(p) \leq n \deg(Q)\})^{\frac{1}{n \deg(Q)}},$

and define the Q-Chebyshev constant of K by

$$\tau(K, Q) := \limsup_{n \to \infty} \tau(K, Q, n).$$

Proposition 4.6. *(1) $\tau(K, Q) = \tau(K, \widehat{Q})$.[*]*
(2) $\tau(K, \alpha Q) = |\alpha|^{\frac{1}{\deg Q}} \tau(K, Q)$ for all $\alpha \in \mathbb{C}$, $\alpha \neq 0$.
(3) We have $\tau(K, Q) = \lim_{n \to \infty} \tau(K, Q, n)$, i.e., the limit exists.

Proof. The first property follows almost immediately by definition, as $\mathcal{C}(Q) = \mathcal{C}(\widehat{Q})$. For the second property, a calculation (i.e., factoring out the correct power of α) shows that $q \in \mathcal{C}(\alpha Q)$ implies $q = \alpha^{\frac{\deg q}{\deg Q}} p$ for some $p \in \mathcal{C}(Q)$. The second property then follows easily.

For the last property, it suffices to show that $\tau(K, Q) \leq \liminf_{n \to \infty} \tau(K, Q, n)$. Let $\epsilon > 0$ and choose $n_0 \in \mathbb{N}$ such that $\tau(K, Q, n_0) \leq \liminf_{n \to \infty} \tau(K, Q, n) + \epsilon$. Given an integer $m > n_0 \deg(Q)$, write $m = n_0 \deg(Q) q + r$ with $0 \leq r < n_0 \deg(Q)$. Take a polynomial $p_0 \in \mathcal{C}(Q)$ with $\deg(p_0) = n_0 \deg(Q)$ such that

$$\|p_0\|_K \leq (\tau(K, Q, n_0) + \epsilon)^{n_0 \deg(Q)}.$$

We have $p := p_0^q Q^r$ is a polynomial in $\mathcal{C}(Q)$ with $\deg(p) = m$, so

$$\begin{aligned}
\tau(K, Q, m) \leq \|p_0^q Q^r\|_K^{1/m} &\leq \|Q\|_K^{r/m} \|p_0\|_K^{n_0 \deg(Q) q / m} \\
&\leq \|Q\|_K^{r/m} (\tau(K, Q, n_0) + \epsilon)^{n_0 \deg(Q) q / m},
\end{aligned}$$

and hence

$$\tau(K, Q, m) \leq \|Q\|_K^{r/m} (\liminf_{n \to \infty} \tau(K, Q, n) + 2\epsilon)^{n_0 \deg(Q) q / m}.$$

Since $\frac{r}{m} \in [0, \frac{n_0 \deg(Q)}{m})$ and $\frac{n_0 \deg(Q) q}{m} \in (\frac{m - n_0 \deg(Q)}{m}, 1]$, we have $\frac{r}{m} \to 0$ and $\frac{n_0 \deg(Q) q}{m} \to 1$ as $m \to \infty$. Taking the lim sup as $m \to \infty$ of both sides of the above inequality then gives

$$\tau(K, Q) \leq \liminf_{n \to \infty} \tau(K, Q, n) + 2\epsilon.$$

Letting $\epsilon \to 0$ yields the result. $\square$

Definition 4.7. Let $K \subset V$ be a compact set, and $\lambda \in V_{\mathbb{P}} \cap H_\infty$. For a positive integer $s \geq \deg \mathbf{v}_\lambda$, we define the *$s$-th order directional Chebyshev constant for the direction λ* by

(4.5) $\tau_s(K, \lambda) := (\inf\{\|p\|_K : p(z) = \mathbf{v}_{\lambda, s}(z) + q(z), \deg(q) < s\})^{1/s},$

and define the *directional Chebyshev constant for the direction λ* by

(4.6) $\tau(K, \lambda) := \limsup_{s \to \infty} \tau_s(K, \lambda).$

Proposition 4.8. *We have $\tau(K, \lambda) = \lim_{s \to \infty} \tau_s(K, \lambda)$, i.e., the limit of the right-hand side exists, and $\tau(K, \lambda) = \tau(K, \mathbf{v}_\lambda)$.*

---

[*]Recall $\widehat{Q}$ is the leading homogeneous part of Q: $\deg(Q - \widehat{Q}) < \deg Q$.

Proof. Let $s > \deg(\mathbf{v}_\lambda) =: a$ be a large positive integer. Write $s = na + r$ where $n, r \in \mathbb{N}$ and $0 < r < a$. Let $p_s(z) = \mathbf{v}_{\lambda,s} + q(z)$ $(\deg(q) < s)$ be such that $\|p_s\|_K = \tau_s(K, \lambda)^s$. Then

$$z_1^{a-r} p_s = \mathbf{v}_{\lambda,(n+1)a} + z_1^{a-r} q = (\mathbf{v}_\lambda(z))^{n+1} + \tilde{r}(z), \quad \text{with } \deg(\tilde{r}) < (n+1)a,$$

where we use equation (4.3) to get $\tilde{r}(z)$. Hence

$$(4.7) \qquad \|z_1\|_K^{a-r} \tau_s(K,\lambda)^s \geq \tau(K, \mathbf{v}_\lambda, n+1)^{(n+1)a}.$$

On the other hand, taking $q_n \in \mathcal{C}(\mathbf{v}_\lambda)$ with $\deg(q_n) = na$ such that $\|q_n\|_K = \tau(K, \mathbf{v}_\lambda, n)^{na}$, we have, using equation (4.3) again, that $z_1^r q_n = \mathbf{v}_{\lambda,s} + q(z)$ where $\deg(q) < s$. A similar argument as above yields

$$(4.8) \qquad \|z_1\|_K^r \tau(K, \mathbf{v}_\lambda, n)^{na} \geq \tau_s(K, \lambda)^s.$$

We now take s-th roots in (4.7) and (4.8) and let $s \to \infty$. It is easy to see that $\frac{na}{s}, \frac{(n+1)a}{s} \to 1$ and $\frac{r}{s}, \frac{a-r}{s} \to 0$. The proposition follows. $\qquad \square$

Remark 4.9. More generally, the same proof gives $\tau(K, \lambda) = \tau(K, z_1^a \mathbf{v}_\lambda)$ for any fixed non-negative integer a.

Notation 4.10. In what follows, to distinguish Chebyshev constants on different curves, we will put the curve in subscripts. Write $\tau_{V,s}(K, Q)$ and $\tau_{V,s}(K, \lambda)$ to denote s-th order Chebyshev constants for K on the curve V (where $\tau_{V,s}(Q)$ is the same as $\tau(K, Q, s)$ defined in (4.4), and $\tau_{V,s}(K, \lambda)$ is as in (4.5)), and write $\tau_V(K, Q)$ and $\tau_V(K, \lambda)$ for the respective Chebyshev constants.

The next result shows how Chebyshev constants transform under linear changes of coordinates. First, note that an invertible linear transformation $T = (T_1, \ldots, T_N) : \mathbb{C}^N \to \mathbb{C}^N$ (so $T_k(z) = a_{k1} z_1 + \cdots + a_{kN} z_N$, $k = 1, \ldots, N$) extends to an automorphism of $\mathbb{CP}^N = \mathbb{C}^N \cup H_\infty$, which we also denote by T, with the property that $T(H_\infty) = H_\infty$. In homogeneous coordinates,

$$T([1 : z_1 : \cdots : z_N]) = [1 : T_1(z_1, \ldots, z_N) : \cdots : T_N(z_1, \ldots, z_N)],$$

and for points at H_∞ of the form $[0 : 1 : w_2 : \cdots : w_N] = [0 : w]$, we have

$$(4.9) \qquad T([0 : w]) = [0 : T_1(w) : \cdots : T_N(w)] = [0 : 1 : \tfrac{T_2(w)}{T_1(w)} : \cdots : \tfrac{T_N(w)}{T_1(w)}]$$

as long as $T_1(w) \neq 0$.

Proposition 4.11. *Let $T = (T_1, \ldots, T_N) : \mathbb{C}^N \to \mathbb{C}^N$ be an invertible linear transformation and suppose $T_1(w) = T_1(w_1, \ldots, w_N) \neq 0$ whenever $[0 : w_1 : \cdots : w_N] \in H_\infty \cap V_{\mathbb{P}}$. Suppose in addition that both curves V and $T(V)$ satisfy properties (i)–(iii) at the beginning of this section. Then*

(1) For any compact set $K \subset V$ and polynomial $Q \in \mathbb{C}[T(V)]$,

$$\tau_{T(V)}(T(K), Q) = \tau_V(K, Q \circ T).^{\dagger}$$

(2) For any compact set $K \subset V$ and direction η of V,

$$\frac{1}{|T_1(\eta)|} \tau_{T(V)}(T(K), T(\eta)) = \tau_V(K, \eta),$$

where $T(\eta)$ is as in equation (4.9), using the extension of T across H_∞.

---

†We understand $Q \circ T \in \mathbb{C}[V]$; more precisely, take $\rho_V(Q \circ T)$. (See Remark 4.4.)

Proof. For any polynomial $Q \in \mathbb{C}[T(V)]$, a calculation shows that $\deg(Q) = \deg(Q \circ T)$. Another calculation shows that $\mathcal{C}(Q \circ T) = \mathcal{C}(Q) \circ T$,[‡] and if $q \in \mathcal{C}(Q)$, then $\|q\|_{T(K)} = \|q \circ T\|_K$. The first part now follows easily from Definition 4.5.

For part (2), let $\eta \in H_\infty \cap V_\mathbb{P}$, and let $\lambda = T(\eta)$. In what follows, $\mathbf{v}_\lambda$ will denote the directional polynomial for λ in $\mathbb{C}[T(V)]$ while $\mathbf{w}_\eta$ will denote the directional polynomial for η in $\mathbb{C}[V]$. Write $b = \deg(\mathbf{v}_\lambda)$ and define

$$w(z) := \left(\tfrac{1}{T_1(\eta)}\right)^b \mathbf{v}_\lambda \circ T(z),$$

considered as a polynomial in $\mathbb{C}[V]$ by taking the normal form (which we also denote by w). We have $\deg(w) = b$, and evaluating at η,

$$w(\eta) = \left(\tfrac{1}{T_1(\eta)}\right)^d \mathbf{v}_\lambda \circ T(\eta) = \mathbf{v}_\lambda(1, \tfrac{T_2(\eta)}{T_1(\eta)}, \dots, \tfrac{T_N(\eta)}{T_1(\eta)}) = \mathbf{v}_\lambda(1, \lambda_2, \dots, \lambda_N) = 1.$$

Similar calculations show that $w(\tilde{\eta}) = 0$ for any direction $\tilde{\eta} \neq \eta$ of V, and that $p(z)w(z) = \widehat{p}(\eta)z_1^{\deg p}w(z) + r(z)$ where $\deg(r) < \deg(w) + \deg(p)$. Hence w satisfies properties (1)–(3) of Lemma 4.1, which implies that $w = z_1^a \mathbf{w}_\eta$ in $\mathbb{C}[V]$ for some non-negative integer $a < b$. We have

$$\tfrac{1}{|T_1(\eta)|}\tau_{T(V)}(T(K), \mathbf{v}_\lambda) = \tau_{T(V)}(T(K), \tfrac{\mathbf{v}_\lambda}{T_1(\eta)^b}) = \tau_V(K, \tfrac{\mathbf{v}_\lambda}{T_1(\eta)^b} \circ T) = \tau_V(K, z_1^a \mathbf{w}_\eta),$$

where we use Proposition 4.6(2) to get the first equality, and part (1) above to get the second. Finally, applying Proposition 4.8 and Remark 4.9 yields the result. $\square$

Remark 4.12. Chebyshev constants are invariant under translation of V. Suppose $z \mapsto z + c =: \tilde{z}$ is a translation by some vector c and $\tilde{V} = V + c$. Then straightforward calculations show that $V_\mathbb{P} \cap H_\infty = \tilde{V}_\mathbb{P} \cap H_\infty$. Also, it is easy to check that for any large positive integer n, the homogeneous normal forms $\mathbb{C}[V]_{=n}$ and $\mathbb{C}[\tilde{V}]_{=n}$ are spanned by the same reduced collection of monomials of degree n, and for any polynomial p, the diagram

$$
\begin{array}{ccc}
\mathbb{C}[V]_{=n} & \longrightarrow & \mathbb{C}[\tilde{V}]_{=n} \\
\\
\downarrow \widehat{\ast}\,\rho_V(p) & & \downarrow \widehat{\ast}\,\rho_V(p) \\
\\
\mathbb{C}[V]_{=n+\deg(p)} & \longrightarrow & \mathbb{C}[\tilde{V}]_{=n+\deg(p)}
\end{array}
$$

commutes, where the horizontal arrows are given by $q(z) \mapsto \rho_{\tilde{V}}(\widehat{q(\tilde{z} - c)})$, i.e., make the required change of coordinates and take the leading homogeneous part (see Section 3 for the notation).[*] Hence all of the algebraic computations used to define and compute directional polynomials and Chebyshev constants on V and $\tilde{V}$ are identical. As a consequence, the previous proposition is also true if T is replaced by an affine transformation.

The next proposition shows that the study of Chebyshev constants can be restricted to irreducible curves.

[‡]i.e., $q \in \mathcal{C}(Q)$ if and only if $q \circ T \in \mathcal{C}(Q \circ T)$.

[*]It is easy to see that for large n we just obtain the same polynomial.

Proposition 4.13. *Suppose $V = V_1 \cup V_2$ is a union of algebraic curves, and $K \subset V$ is compact. Let $K_1 := K \cap V_1$. Then for $\lambda \in V_{1\mathbb{P}} \cap H_\infty$ ($\lambda = [0 : 1 : \lambda_2 : \cdots : \lambda_N]$), we have the following equalities:*

$$(4.10) \qquad \tau_{V_1}(K_1, \lambda) \;=\; \tau_V(K_1, \lambda) \;=\; \tau_V(K, \lambda).$$

Proof. We first prove $\tau_V(K_1, \lambda) = \tau_V(K, \lambda)$. The inequality $\tau_V(K_1, \lambda) \leq \tau_V(K, \lambda)$ follows easily from the definition, since $K_1 \subset K$ and hence $\|p\|_{K_1} \leq \|p\|_K$.

We need to prove that $\tau_V(K_1, \lambda) \geq \tau_V(K, \lambda)$. First, we fix a polynomial $g \in \mathbf{I}(V_2)$ such that $\|g\|_{K_1} > 0$ and $\widehat{g}(\lambda) \neq 0$. This is possible because $\lambda \notin V_2$ and K_1 contains points not in V_2.

For each positive integer s, let $q_s = \mathbf{v}_{\lambda,s} + \cdots$ be a Chebyshev polynomial of degree s with $\tau_{V,s}(K_1, \lambda)^s = \|q_s\|_{K_1}$. Consider the polynomial $q_s g$. Then since $K = K_1 \cup (K \cap V_2)$, we have

$$(4.11) \qquad \|q_s g\|_K \;=\; \|q_s g\|_{K_1} \leq \tau_{V,s}(K_1, \lambda)^s \|g\|_{K_1}.$$

On the other hand, writing $a = \deg(g)$, we have

$$g(z)q_s(z) = \widehat{g}(\lambda)\mathbf{v}_{\lambda,s+a}(z) + \cdots \;=\; \widehat{g}(\lambda)p(z)$$

for some polynomial $p(z) = \mathbf{v}_{\lambda,s+a}(z) + \cdots$. Hence

$$(4.12) \qquad \|q_s g\|_K \;\geq\; |\widehat{g}(\lambda)|\tau_{V,s+a}(K, \lambda)^{s+a}.$$

Putting together (4.11) and (4.12) gives $|\widehat{g}(\lambda)|\tau_{V,s+a}(K, \lambda)^{s+a} \leq \tau_{V,s}(K_1, \lambda)^s \|g\|_{K_1}$; taking s-th roots and letting $s \to \infty$, we obtain $\tau_V(K, \lambda) \leq \tau_V(K_1, \lambda)$. Altogether, $\tau_V(K_1, \lambda) = \tau_V(K, \lambda)$.

We now prove $\tau_{V_1}(K_1, \lambda) = \tau_V(K_1, \lambda)$. Take a large positive integer s and let p_s be a Chebyshev polynomial of degree s for K_1 on V. Then for all $z \in V$,

$$(4.13) \qquad p_s(z) \;=\; \mathbf{v}_{\lambda,s}(z) \;+\; r_1(z) \;=\; z_1^{s-a}v_1(z)v_2(z)\cdots v_N(z) \;+\; r_1(z)$$

where $a = \deg(v_1 v_2 \cdots v_N)$ and $\deg(r_1) < s$.

By Proposition 3.16 there are homogeneous polynomials $\varphi_1, \ldots, \varphi_N$ such that

$$(4.14) \qquad z_1^{a_j}v_j(z) = w_j(z)\varphi_j(z) \quad \text{for all } z \in V, \; j \in \{1, \ldots, N\}.$$

Here the a_j's are non-negative integers, and w_j denotes the eigenvector polynomial associated to the eigenvalue λ_j of the matrix $[[z_j]]$ that represents multiplication by z_j on the curve V_1.

Let $\varphi := \varphi_1 \varphi_2 \cdots \varphi_N$ and $b := \deg \varphi - \sum_i a_i$. (Note that b is positive since $\deg \varphi_i \geq a_i$ for all i.) Let $\mathbf{w}_\lambda$ be the directional polynomial on V_1 for λ. We have $w_1 w_2 \cdots w_N = \mathbf{w}_{\lambda,c}$ for some non-negative integer c. Using (4.13), we have for all $z \in V_1$ that

$$p_s(z) \;=\; z_1^{s-a}\mathbf{w}_{\lambda,c}(z)\varphi(z) \;+\; r_1(z) \;=\; \varphi(\lambda)z_1^{s-a+b}\mathbf{w}_{\lambda,c}(z) \;+\; r_2(z),$$

and the fact that $\mathbf{v}_\lambda(\lambda) \neq 0$ means that $\varphi(\lambda) \neq 0$. Comparing degrees also gives $c = a - b$. We now normalize p_s by setting $q_s := \frac{p_s}{\varphi(\lambda)}$ to obtain a competitor for a Chebyshev polynomial of degree s for K_1 in the direction λ on V_1. Hence

$$\tau_{V,s}(K_1, \lambda)^s \;=\; \|p_s\|_{K_1} \;=\; |\varphi(\lambda)|\|q_s\|_{K_1} \;\geq\; |\varphi(\lambda)|\tau_{V_1,s}(K_1, \lambda)^s.$$

Taking s-th roots and letting $s \to \infty$, we have $\tau_V(K_1, \lambda) \geq \tau_{V_1}(K_1, \lambda)$.

For the reverse inequality, let q_s be a Chebyshev polynomial for K_1 of degree s with $\|q_s\|_{K_1} = \tau_{V_1,s}(K_1,\lambda)^s$, and let $p_{s+b} := q_s\varphi$. In what follows we assume that $\|\varphi\|_{K_1} > 0$. We have

$$p_{s+b}(z) \ = \ z_1^{s-(a-b)}\mathbf{w}_\lambda(z)\varphi(z) \ + \ r_3(z) \ = \ z_1^{s+b-a}\mathbf{v}_\lambda(z) \ + \ r_3(z),$$

so that

$$\tau_{V,s+b}(K_1,\lambda)^{s+b} \ \leq \ \|p_{s+b}\|_{K_1} \ \leq \ \|q_s\|_{K_1}\|\varphi\|_{K_1} \ = \ \tau_{V_1,s}(K_1,\lambda)^s\|\varphi\|_{K_1}.$$

Taking s-th roots and letting $s \to \infty$ gives $\tau_V(K_1,\lambda) \leq \tau_{V_1}(K_1,\lambda)$, as desired. If it happens that $\|\varphi\|_{K_1} = 0$, then $\tau_V(K_1,\lambda) \leq \tau_{V_1}(K_1,\lambda)$ may be shown by replacing φ with $\varphi + 1$ in the preceding argument.

Hence $\tau_{V_1}(K_1,\lambda) = \tau_V(K_1,\lambda)$. $\qquad\square$

Remark 4.14. The directional Chebyshev constants $\tau(K,\lambda)$ were first defined in [9] for curves in $\mathbb{C}^2$. Propositions 4.8 and 4.13 above generalize, respectively, Theorem 4.5 and Proposition 4.13 in [9].

The following characterization of directional Chebyshev constants will be useful when studying the transfinite diameter in the next section. For a compact set $K \subset V$ and a direction $\lambda \in H_\infty \cap V_\mathbb{P}$, define
(4.15)
$$t_s(K,\lambda) := \inf\left\{\|p\|_K : p(z) = \mathbf{v}_{\lambda,s}(z) + \sum_{\mu \neq \lambda} a_\mu \mathbf{v}_{\mu,s}(z) + q(z),\ a_\mu \in \mathbb{C},\ \deg q < s\right\}^{\frac{1}{s}}.$$

Note that the polynomials of Definition 4.7 have $a_\mu = 0$. We now verify that the same constant is obtained in the limit.

Lemma 4.15. *With $t_s(K,\lambda)$ as defined above,* $\lim\limits_{s\to\infty} t_s(K,\lambda) \ = \ \tau(K,\lambda)$.

Proof. Clearly $t_s(K,\lambda) \leq \tau_s(K,\lambda)$ for all s, and so $\limsup_{s\to\infty} t_s(K,\lambda) \leq \tau(K,\lambda)$.

Next, let $a = \deg(\mathbf{v}_\lambda)$. Given a large positive integer $s > a$, take a polynomial $p_s = \mathbf{v}_{\lambda,s} + \sum_{\mu\neq\lambda} a_\mu\mathbf{v}_{\mu,s} \ + \ q_1(z)$ with the property that $\|p_s\|_K = t_s(K,\lambda)^s$. Then

$$\begin{aligned}
p_s(z)\mathbf{v}_\lambda(z) \ &= \ \mathbf{v}_\lambda(\lambda)\mathbf{v}_{\lambda,s+a}(z) + \sum_{\mu\neq\lambda} a_\mu\mathbf{v}_\mu(\lambda)\mathbf{v}_{\lambda,s}(z) \ + \ q_2(z) \\
&= \ \mathbf{v}_{\lambda,s+a}(z) + q_2(z),
\end{aligned}$$

where $\deg(q_2) < s + a$. Here we use equation (4.1) for the first equality and parts (1) and (2) of Lemma 4.1 for the second. Hence $\|p_s\mathbf{v}_\lambda\| \geq \tau_{s+a}(K,\lambda)$, and

$$t_s(K,\lambda)^s\|\mathbf{v}_\lambda\|_K \ \geq \ \|p_s\mathbf{v}_\lambda\| \ \geq \ \tau_{s+a}(K,\lambda)^{s+a}.$$

Taking s-th roots and letting $s \to \infty$, we have $\liminf_{s\to\infty} \tau_s(K,\lambda) \ \geq \tau(K,\lambda)$. This concludes the proof. $\qquad\square$

5. Transfinite diameter

In this section, we study the transfinite diameter of a compact subset of an algebraic curve in $\mathbb{C}^N$. To make use of previous results, we restrict for the moment to an algebraic curve that satisfies properties (i)–(iii) listed at the beginning of the previous section.

Let $V_{\mathbb{P}} \cap H_\infty = \{\lambda_1, \ldots, \lambda_d\}$, where we write $\lambda_j = [0 : 1 : \lambda_{j2} : \cdots : \lambda_{jN}]$ for each $j \in \{1, \ldots, d\}$. Fix a positive integer $a \geq \max_{j=1,\ldots,d} \deg(\mathbf{v}_{\lambda_j})$. Consider the following collection $\mathcal{C}$ of polynomials:

$z^\alpha \in \mathcal{C}$ for all $z^\alpha \in \mathcal{B}$ with $|\alpha| < a$;[†] and

$\mathbf{v}_{\lambda_j,s} \in \mathcal{C}$ for all $j \in \{1, \ldots, d\}$ and all $s \in \mathbb{N}$ with $s \geq a$.

We put an ordering $\prec$ on $\mathcal{C}$ as follows. First by degree, i.e., $\deg(p) < \deg(q)$ implies $p \prec q$. For elements of the form z^α with $|\alpha| < a$, we use any graded ordering (e.g. grevlex). For higher degree elements of the form $\mathbf{v}_{\lambda_j,s}$, we induce an ordering on $\mathcal{C}$ by ordering the directions at infinity, e.g.

$$\mathbf{v}_{\lambda_j,s} \prec \mathbf{v}_{\lambda_k,s} \quad \text{if} \quad j < k.$$

For a positive integer n, let $\mathbb{C}[V]_{\leq n} = \{p \in \mathbb{C}[V] : \deg(p) \leq n\}$.[‡] Let m_n denote the dimension of this vector space, and define $l_n := \sum_{k=1}^n k(m_k - m_{k-1})$.

Lemma 5.1. *For each positive integer n, the polynomials $\mathcal{C}_n := \{p \in \mathcal{C} : \deg(p) \leq n\}$ form a basis for $\mathbb{C}[V]_{\leq n}$, and hence $\mathcal{C}$ is a basis for $\mathbb{C}[V]$.*

Proof. For $n < a$ this is trivial as $\mathcal{B}$ is the monomial basis. For $n \geq a$, note that $\mathbb{C}[V]_{=n}$ has dimension d (by Proposition 3.12). We verify that the set $\{\mathbf{v}_{\lambda_j,n}\}$ is linearly independent in $\mathbb{C}[V]_{=n}$. For any linear combination $p = \sum_{k \neq j} c_k \mathbf{v}_{\lambda_k,n}$, we have $p(1, \lambda_j) = 0$ but $\mathbf{v}_{\lambda_j}(1, \lambda_j) \neq 0$. Hence $\{\mathbf{v}_{\lambda_j,n}\}_{j=1}^d$ spans $\mathbb{C}[V]_{=n}$. Assuming $\mathcal{C}_{n-1}$ is a basis for $\mathbb{C}[V]_{\leq n-1}$, clearly $\mathcal{C}_n = \mathcal{C}_{n-1} \cup \{\mathbf{v}_{\lambda_j,s}\}_{j=1}^d$ spans $\mathbb{C}[V]_{\leq n}$, as d linearly independent elements are added, and the dimension increases by d.

The lemma now follows by induction. $\qquad\square$

Remark 5.2. As an immediate consequence, l_n is given by the sum of the degrees of all polynomials in $\mathcal{C}$ of degree $\leq n$.

Write $\mathcal{C}$ as a sequence $\{e_j\}_{j=1}^\infty$ by listing the polynomials according to the ordering $\prec$ defined above (i.e., $e_1 = 1$, $e_j \prec e_k$ iff $j < k$). Next, for a positive integer n, consider a collection of points $\{\zeta_1, \ldots, \zeta_n\} \subset V$. Define the Vandermonde determinant

$$\mathrm{Van}_{\mathcal{C}}(\zeta_1, \ldots, \zeta_n) := \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ e_2(\zeta_1) & e_2(\zeta_2) & \cdots & e_2(\zeta_n) \\ \vdots & \vdots & \ddots & \vdots \\ e_n(\zeta_1) & e_n(\zeta_2) & \cdots & e_n(\zeta_n) \end{pmatrix}.$$

Definition 5.3. For a positive integer n, put

$$V_n := \sup\{|\mathrm{Van}_{\mathcal{C}}(\zeta_1, \ldots, \zeta_{m_n})| : \{\zeta_1, \ldots, \zeta_{m_n}\} \subset K\}.$$

The *transfinite diameter of K, $d(K)$,* is defined by

(5.1) $$d_V(K) = d(K) := \limsup_{n \to \infty} (V_n)^{1/l_n}.$$

The main theorem relates the transfinite diameter to the directional Chebyshev constants, and generalizes [9, Thm. 5.7] to the $\mathbb{C}^N$ setting.

[†]$\mathcal{B}$ is as in Section 2.2.

[‡]Note that in our notation, $(\mathbb{C}[V]_{\leq n} \setminus \mathbb{C}[V]_{\leq n-1}) \supsetneq \mathbb{C}[V]_{=n}$, as the latter set contains only homogeneous polynomials.

Theorem 5.4. *Let $K \subset V$ be a compact set. Then the limit $\lim\limits_{n\to\infty} (V_n)^{1/l_n} = d(K)$ exists and*

$$(5.2) \qquad d(K) = \left(\prod_{j=1}^{d} \tau(K, \lambda_j) \right)^{1/d}.$$

We first establish some bounds relating Chebyshev constants, specifically the constants of finite order given in Definition 4.7 and equation (4.15). We will need some more notation. For $n > a$ and $j = 1, \ldots, d$, set

$$V_{n,j} = \sup \left\{ \mathrm{Van}_{\mathcal{C}}(\zeta_1, \ldots, \zeta_{m_n+j}) : \{\zeta_1, \ldots, \zeta_{m_n+j}\} \subset K \right\}.$$

Note that $V_{n,d} = V_{n+1}$. For convenience, we also put $V_{n,0} = V_n$.

Lemma 5.5. *For $n > a$ and $j = 1, \ldots, d$, we have*

$$t_{n+1}(K, \lambda_j)^{n+1} \leq \frac{V_{n,j}}{V_{n,j-1}} \leq (m_n + j)\tau_{n+1}(K, \lambda_j)^{n+1}.$$

Proof. Let $\{\zeta_1, \ldots, \zeta_{m_n+j-1}\}$ be a collection of $m_n + j - 1$ points such that $V_{n,j-1} = \mathrm{Van}_{\mathcal{C}}(\zeta_1, \ldots, \zeta_{m_n+j-1})$. Define the polynomial

$$p(z) = \frac{\mathrm{Van}_{\mathcal{C}}(\zeta_1, \ldots, \zeta_{m_n+j-1}, z)}{\mathrm{Van}_{\mathcal{C}}(\zeta_1, \ldots, \zeta_{m_n+j-1})}.$$

Then expanding the Vandermonde determinant down the last column, we have

$$p(z) = \frac{\mathbf{v}_{\lambda_j,n+1}(z)\mathrm{Van}_{\mathcal{C}}(\zeta_1, \ldots, \zeta_{m_n+j-1}) + q(z)}{\mathrm{Van}_{\mathcal{C}}(\zeta_1, \ldots, \zeta_{m_n+j-1})} = \mathbf{v}_{\lambda_j,n+1}(z) + r(z)$$

where $\deg(q) = \deg(r) \leq n + 1$. Hence $t_{n+1}(K, \lambda_j)^{n+1} \leq \|p\|_K \leq \frac{V_{n,j}}{V_{n,j-1}}$. This proves the lower inequality.

Next, let now $\{\zeta_1, \ldots, \zeta_{m_n+j}\}$ be a collection of $m_n + j$ points such that $V_{n,j} = \mathrm{Van}_{\mathcal{C}}(\zeta_1, \ldots, \zeta_{m_n+j})$. Now let $p(z) = \mathbf{v}_{\lambda_j,n+1}(z) + q(z)$ (with $\deg(q) \leq n$) be a polynomial such that $\|p\|_K = \tau_{n+1}(K, \lambda)^{n+1}$. Then

$$\det \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ \mathbf{v}_{\lambda_j,n+1}(\zeta_1) & \cdots & \mathbf{v}_{\lambda_j,n+1}(\zeta_{m_n+j}) \end{bmatrix} = \det \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ p(\zeta_1) & \cdots & p(\zeta_{m_n+j}) \end{bmatrix},$$

since replacing $\mathbf{v}_{\lambda_j,n+1}$ with $p = \mathbf{v}_{\lambda_j,n+1} + q$ in the last row is the same as adding to this row a linear combination of previous rows (given by the coefficients of q). Expanding the determinant along the last row and taking absolute values yields

$$V_{n,j} \leq \sum_{s=1}^{m_n+j} |\mathrm{Van}_{\mathcal{C}}(\zeta_1, \ldots, \widehat{\zeta}_s, \ldots, \zeta_{m_n+j})| \cdot |p(\zeta_s)|$$

$$\leq (m_n + j)V_{n,j-1}\|p\|_K = (m_n + j)V_{n,j-1}(\tau_{n+1}(K, \lambda))^{n+1},$$

where $\widehat{\zeta}_s$ indicates that ζ_s is omitted. This proves the upper inequality. $\qquad \square$

Corollary 5.6. *For $n > a$, we have*

$$\prod_{j=1}^{d} t_{n+1}(K, \lambda_j)^{n+1} \leq \frac{V_{n+1}}{V_n} \leq \frac{m_{n+1}!}{m_n!} \prod_{j=1}^{d} \tau_{n+1}(K, \lambda_j)^{n+1}.$$

Proof. We have

$$\frac{V_{n+1}}{V_n} = \frac{V_{n,d}}{V_{n,d-1}} \frac{V_{n,d-1}}{V_{n,d-2}} \cdots \frac{V_{n,1}}{V_{n,0}}.$$

Now apply, to each quotient on the right-hand side, the upper and lower bounds in the previous result. $\qquad\square$

Proof of Theorem 5.4. Let $\epsilon > 0$. By Proposition 4.8 and Lemma 4.15, we have for each j that

$$\lim_{n\to\infty} t_n(K, \lambda_j) = \tau(K, \lambda_j) \text{ and } \lim_{n\to\infty} \tau_n(K, \lambda_j) = \tau(K, \lambda_j).$$

Hence there exists an integer $n_0 > a$ sufficiently large such that for all $j = 1, \ldots, d$, we have

(5.3) $t_{n_0}(K, \lambda_j) \geq \tau(K, \lambda_j) - \epsilon$ and $\tau_{n_0}(K, \lambda_j) \leq \tau(K, \lambda_j) + \epsilon$ for all $n \geq n_0$.

For $n > n_0$, write

$$V_n = \frac{V_n}{V_{n-1}} \frac{V_{n-1}}{V_{n-2}} \cdots \frac{V_{n_0+1}}{V_{n_0}} V_{n_0};$$

applying the previous corollary to the product on the right-hand side of the above equation, we obtain

$$V_{n_0} \prod_{s=n_0+1}^{n} \prod_{j=1}^{d} t_s(K, \lambda_j)^s \leq V_n \leq V_{n_0} \frac{m_n!}{m_{n_0}!} \prod_{s=n_0+1}^{n} \prod_{j=1}^{d} \tau_s(K, \lambda_j)^s,$$

and by (5.3), this becomes

$$V_{n_0} \prod_{j=1}^{d} (\tau(K, \lambda_j) - \epsilon)^{A_n} \leq V_n \leq V_{n_0} \frac{m_n!}{m_{n_0}!} \prod_{j=1}^{d} (\tau(K, \lambda_j) + \epsilon)^{A_n}$$

where $A_n = \sum_{s=n_0+1}^{n} s = \frac{(n-n_0)(n+n_0+1)}{2}$. Taking l_n-th roots, we have for all $n > n_0$ that

$$V_{n_0}^{1/l_n} \prod_{j=1}^{d} (\tau(K, \lambda_j) - \epsilon)^{A_n/l_n} \leq V_n^{1/l_n} \leq V_{n_0}^{1/l_n} \left(\frac{m_n!}{m_{n_0}!}\right)^{1/l_n} \prod_{j=1}^{d} (\tau(K, \lambda_j) + \epsilon)^{A_n/l_n}.$$

We want to take the limit as $n \to \infty$. Note that $m_n - m_{n-1} = d$ for $n > n_0$; hence $m_n = dn + c$ for some integer c. Also, we have

$$l_n = l_{n_0} + \sum_{s=n_0+1}^{n} sd = l_{n_0} + \frac{d(n-n_0)(n+n_0+1)}{2},$$

which implies that $\lim_{n\to\infty} \frac{A_n}{l_n} = \frac{1}{d}$. Clearly $V_{n_0}^{1/l_n}, (m_{n_0}!)^{1/l_n} \to 1$ as $n \to \infty$. Since l_n is of order n^2 and m_n is of order n, there is a constant $b > 0$ such that $b \cdot nm_n \leq l_n$ for all $n > n_0$, and so

$$1 \leq (m_n!)^{1/l_n} \leq m_n^{m_n/l_n} \leq (dn+c)^{\frac{1}{bn}} \longrightarrow 1 \quad \text{as } n \to \infty.$$

Hence

$$\left(\prod_{j=1}^{d} (\tau(K, \lambda_j) - \epsilon)\right)^{\frac{1}{d}} \leq \liminf_{n\to\infty} V_n^{\frac{1}{l_n}} \leq \limsup_{n\to\infty} V_n^{\frac{1}{l_n}} \leq \left(\prod_{j=1}^{d} (\tau(K, \lambda_j) + \epsilon)\right)^{\frac{1}{d}}.$$

Finally, let $\epsilon \to 0$. Then (5.2) follows, which completes the proof. $\qquad\square$

From an algebraic point of view, it is natural to define transfinite diameter using monomials (as in the Introduction).

Let $K \subset V \subset \mathbb{C}^N \subset \mathbb{CP}^N$ be a compact set, and let $I = \mathbf{I}(V)$ be the ideal of V. Consider the monomials $\{z^\alpha : z^\alpha \notin \langle \mathrm{LT}(I) \rangle\}$ (which form a basis for $\mathbb{C}[V]$) listed according to grevlex order as a sequence $\{z^{\alpha_j}\}_{j=1}^\infty$. Define

$$\mathrm{Van}(\zeta_1, \ldots, \zeta_n) := \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ z^{\alpha_2}(\zeta_1) & z^{\alpha_2}(\zeta_2) & \cdots & z^{\alpha_2}(\zeta_n) \\ \vdots & \vdots & \ddots & \vdots \\ z^{\alpha_n}(\zeta_1) & z^{\alpha_n}(\zeta_2) & \cdots & z^{\alpha_n}(\zeta_n) \end{pmatrix}.$$

Corollary 5.7. *For a positive integer n, put*

$$\tilde{V}_n := \sup\{|\mathrm{Van}(\zeta_1, \ldots, \zeta_{m_n})| : \{\zeta_1, \ldots, \zeta_{m_n}\} \subset K\}.$$

Then $\lim_{n \to \infty} \tilde{V}_n^{1/l_n} = d(K)$.

The corollary says that $d(K)$ may be given by Definition 5.3 with $\mathrm{Van}_{\mathcal{C}}$ replaced by Van. That the two limits are equal can be seen as follows. For large n, a basis of $\mathbb{C}[V]_{=n}$ is given alternatively by $\{z^{\alpha_j}\}_{j=m_{n-1}+1}^{m_n}$ and $\{\mathbf{v}_{\lambda_k,n}\}_{k=1}^d$. One can therefore use row operations to transform the rows in positions $m_{n-1} + 1, \ldots, m_n$ of the matrix for $\mathrm{Van}_{\mathcal{C}}$ (there are d of these) into the corresponding rows of the matrix for Van. Note that the bases of $\mathbb{C}[V]_{=n}$ are the same for each n up to a power of z_1. Precisely, for a large positive integer n_0 and $k \in \{1, \ldots, d\}$, we have $\mathbf{v}_{\lambda_k,n}(z) = z_1^{n-n_0} \mathbf{v}_{\lambda_k,n_0}(z)$ and $z^{\alpha_{j+dn}} = z_1^{n-n_0} z^{\alpha_{j+dn_0}}$ for all positive integers j and $n > n_0$ (cf. Proposition 3.8). Hence the *same* row operations work for each block of rows corresponding to a fixed degree.

It follows from properties of determinants that

(5.4) $$\mathrm{Van}_{\mathcal{C}}(\zeta_1, \ldots, \zeta_{m_n}) = CR^{n-n_0} \mathrm{Van}(\zeta_1, \ldots, \zeta_{m_n})$$

where $C = C(n_0)$ is a fixed constant and R is the factor obtained each time we use the row operations on the rows of $\mathrm{Van}_{\mathcal{C}}$ corresponding to $\{\mathbf{v}_{\lambda_k,s}\}_{k=1}^d$ to get the rows of Van corresponding to the monomials $\{z^{\alpha_j}\}_{j=m_{s-1}+1}^{m_s}$ for each $s = n_0 + 1, \ldots, n$. Clearly $R \neq 0$.

The important point is that l_n is quadratic in n and therefore $(CR^{n-n_0})^{1/l_n} \to 1$ as $n \to \infty$. Hence the l_n-th roots of the Vandermonde determinants in (5.4) are almost equal for large n.

A precise proof along the above lines was given as [9, Cor. 5.14]; although only curves in $\mathbb{C}^2$ (or $\mathbb{CP}^2$) were considered there, the argument is general.

The following properties of transfinite diameter follow immediately from properties of directional Chebyshev constants (Propositions 4.11 and 4.13).

Corollary 5.8. *(1) Let $\lambda_1, \ldots, \lambda_d$ be the directions of V, and $T = (T_1, \ldots, T_N) : \mathbb{C}^N \to \mathbb{C}^N$ be a linear transformation such that $T_1(\lambda_j) \neq 0$ for all $j = 1, \ldots, d$. Suppose V and $T(V)$ satisfy properties (i)–(iii) at the beginning of Section 4. Then for any compact set $K \subset V$,*

(5.5) $$d_V(K) \prod_{j=1}^d T_1(\lambda_j) = d_{T(V)}(T(K)).$$

(2) *Let $V = V_1 \cup V_2$ where V_1, V_2 are curves of degrees d_1 and d_2 respectively, and satisfy properties (i)–(iii). Then writing $d = d_1 + d_2$, we have*

$$d_V(K) = d_{V_1}(K \cap V_1)^{\frac{d_1}{d}} d_{V_2}(K \cap V_2)^{\frac{d_2}{d}}. \qquad \square$$

By the first part of the above corollary, $d_V(K) = 0$ if and only if $d_{T(V)}(T(K)) = 0$. Also, ratios of transfinite diameters are invariant under linear changes of coordinates (as long as all quantities are defined and the ratio makes sense) since the extra factors on the left-hand side of (5.5) are independent of the set. Given compact sets $K_1, K_2 \subset V$, with $d_V(K_2) > 0$, we have

$$(5.6) \qquad \frac{d_V(K_1)}{d_V(K_2)} = \frac{d_{T(V)}(T(K_1))}{d_{T(V)}(T(K_2))} =: d_V(K_1, K_2).$$

Definition 5.9. *Given $K_1, K_2 \subset V$ with $d_V(K_2) > 0$, define the transfinite diameter of K_1 relative to K_2 to be $d_V(K_1, K_2)$.*

One can therefore normalize transfinite diameter by computing it relative to some fixed set (i.e. fix K_2 in (5.6)), to obtain an intrinsic notion independent of coordinates.

Example 5.10. For the complex line $V = \{z \in \mathbb{C}^N : z_1 = z_2 = \cdots = z_N\}$, let $d_V(\cdot)$ be the transfinite diameter on V (of Theorem 5.4 or Corollary 5.7). Set $\delta(K) := d_V(K, D)$ where $D = \{z \in V : |z_1|^2 + |z_2|^2 + \cdots + |z_N| \leq 1\}$ is the "unit disk" in V. Then $\delta(K)$ coincides with the classical transfinite diameter of K in the plane (defined in terms of the restriction to V of the usual metric in $\mathbb{C}^N$).

Fixing a normalization, we can extend the notion of relative transfinite diameter to any algebraic curve V with the property that $V_{\mathbb{P}} \cap H_\infty$ is a transverse intersection of non-singular points. It can be explicitly computed by changing, if necessary, to "good" coordinates, i.e., such that the image of V under this change of coordinates satisfies properties (i)–(iii) at the beginning of Section 4.

Example 5.11. Consider the curve V in $\mathbb{C}^2$ given by the equation $z_1 z_2 = 1$. Then V is a transverse intersection of non-singular points, but its coordinates are "bad" since H_∞ contains the point $[0 : 0 : 1]$. Letting $T(z_1, z_2) = (z_1 + z_2, z_1 - z_2)$, we have that $T(V)$ is the curve given by $z_1^2 - z_2^2 = 4$, in which our theory applies and we can compute $d_{T(V)}(K)$ for a compact $K \subset T(V)$. Let $D = \{(e^{i\theta}, e^{-i\theta}) : \theta \in \mathbb{R}\} \subset V$; then $T(D) = \{(2\cos\theta, 2i\sin\theta) : \theta \in \mathbb{R}\}$. We define $d_V(K, D) := d_{T(V)}(T(K), T(D))$.

In general, suppose V has "bad" coordinates. Let $T, S : \mathbb{C}^N \to \mathbb{C}^N$ be invertible linear maps that provide "good" coordinates for V. Fix a compact set D. Then using equation (5.6) applied to $T(V)$ and $S \circ T^{-1}$, we can see that for any compact set $K \subset V$, we have

$$d_{T(V)}(T(K), T(D)) = d_{S(V)}(S(K), S(D))$$

as long as e.g. $d_{T(V)}(T(D)) \neq 0$. Hence $d_V(K, D)$ is a well-defined quantity since its value is independent of which "good" coordinates are chosen for the computation.

Remark 5.12. A linear change of coordinates that provides "good" coordinates is given by a *Noether normalization* (cf. [8, Thm. 3.4.1]). The condition that $V_{\mathbb{P}} \cap H_\infty$ is a transverse intersection is still essential. For example, our theory does not apply

to a Noether normalization of $V \subset \mathbb{C}^2$ given by $z_2 = z_1^2$, since $[0 : 0 : 1]$ is not a transverse intersection of $V_\mathbb{P}$ with H_∞. In order to manage such a case, one would have to deal with multiple eigenvalues of multiplication matrices.

6. CONCLUDING REMARKS: PLURIPOTENTIAL THEORY

Let $\mathcal{L}$ be the class of global plurisubharmonic (psh) functions on $\mathbb{C}^N$ of logarithmic growth, i.e.,

$$\mathcal{L} = \{u \text{ psh on } \mathbb{C}^N : \exists C \in \mathbb{R} \text{ such that } u(z) \leq \log^+ |z| + C, \ \forall z \in \mathbb{C}^N\}.$$

Given a compact subset K of $\mathbb{C}^N$, define the *Siciak-Zaharjuta extremal function associated to K* to be

$$V_K(z) := \sup\{u(z) : u \in \mathcal{L}, u \leq 0 \text{ on } K\}.$$

Some authors define the Siciak-Zaharjuta extremal function to be the upper semicontinuous regularization $V_K^*(z) := \limsup_{t \to z} V_K(t)$. For convenience we will call V_K the *unregularized* extremal function and V_K^* the *regularized* extremal function.

Sadullaev showed in [11] that the unregularized extremal function V_K provides a pluripotential theoretic criterion for an analytic set to be algebraic.

Theorem 6.1. *Let $A \subset \mathbb{C}^N$ be an analytic set. Then A is contained in an algebraic set if and only if there exists a compact subset $K \subset A$ such that V_K is locally bounded on A. In that case, V_K has logarithmic growth on A.* $\square$

Suppose an algebraic curve V satisfies properties (i)–(iii) at the beginning of Section 4, with directions $\lambda_j = [0 : 1 : \lambda_{j2} : \cdots : \lambda_{jN}] \in V_\mathbb{P}$, $j = 1, \ldots, d$. When V_K is of logarithmic growth on V, we may define for each j the *j-th directional Robin constant* of K by

$$\rho_K(\lambda_j) := \limsup_{\substack{|z| \to \infty, z \in V \\ [1:z] \to \lambda_j}} V_K(z) - \log |z_1|,$$

where we write $[1 : z] = [1 : z_1 : \cdots : z_N] = [\frac{1}{z_1} : 1 : \frac{z_2}{z_1} : \cdots : \frac{z_N}{z_1}]$. We claim that for each $j = 1, \ldots, d$,

$$e^{-\rho_K(\lambda_j)} = \tau(K, \lambda_j).$$

When V is a complex line (i.e. $d = 1$) it is straightforward to show that the quantities in the above equation may be computed in terms of the variable z_1 only, reducing it to a classical relation between Robin and Chebyshev constants in the plane. The generalization to curves of higher degree will be proved in another paper.

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