Trace Positive, Non-commutative Polynomials and the Truncated Moment Problem

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Abstract

The (multivariate) truncated moment problem is an important question in analysis with applications to mathematical physics, probability theory, etc. In the 1990's Curto & Fialkow began their expedition of solving the truncated moment problem. It has since been studied with a variety of motivations, due to its wide ranging applications, such as to multivariate integral computations (in physics, statistics, etc.), option pricing (in finance) and optimization (in mathematics).

The tracial moment problem is a non-commutative analogue of the classical moment problem. A sequence of real numbers indexed by words in non-commuting variables, invariant under cyclic permutations is called a *tracial sequence*. In this work we study conditions for when a tracial sequence is given by the tracial moments of some matrices, focusing on the bivariate problem in low degrees. We present sufficient conditions for Curto and Yoo's construction of an explicit representing measure μ on the classical quartic binary moment problem, to hold in the tracial analogue.

To each tracial sequence one can associate a multivariate Hankel matrix. If the sequence is given by moments, this matrix is positive semi-definite, but not vice-versa. In the bivariate quartic case this matrix is (7×7). It is known that positive definiteness of this matrix implies the existence of a representing measure. Here we also present a comprehensive analysis on the column relations in the Hankel matrix for lower rank cases in an attempt to find or disprove the existence of a representing measure.

Contents

Chapter 1

Introduction

Given a sequence of numbers (x_n) and arbitrary functions $M_n(x)$, the (generalized) moment problem entails finding a measure μ such that

$$x_n = \int_{-\infty}^{\infty} M_n(x)\, d\mu \quad \text{for all } n \in \mathbb{N}.$$

During 1894-1895 Stieltjes studied this problem for $M_n(x) = x^n$, and first introduced the term 'moment problem'. Since its introduction, the moment problem has gone through extensive studies by Stieltjes [23], Hamburger [15], Hausdorff [16], Riesz [20], and others which revealed deep connections between the moment problem and several areas of analysis.

An important variation of the moment problem is the truncated moment problem. Given a finite sequence of numbers $\mathbf{x} \equiv (x_n)_{n \le k}$, we search for a measure μ such that

$$x_n = \int_{-\infty}^{\infty} M_n(x)\, d\mu \quad \text{for } n = 1, \ldots, k.$$

The solution of the truncated moment problem can be obtained by analyzing the associated Hankel matrix, $\mathcal{H}(\mathbf{x})$. Given a set of moments $\boldsymbol{\mu} = \{\mu_0, \mu_1, \ldots, \mu_{2n}\}$, the associated Hankel matrix $\mathcal{H}(\boldsymbol{\mu})$ has the form

$$\mathcal{H}(\boldsymbol{\mu}) = \begin{pmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \cdots & \mu_{n+1} \\ \mu_2 & \vdots & \ddots & \cdots & \mu_{n+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \mu_{n+2} & \cdots & \mu_{2n} \end{pmatrix}.$$

In fact, for the (univariate) truncated moment problem to be realized, i.e., for a finite sequence $\mathbf{x}$ to be a truncated moment sequence, it is sufficient that $\mathcal{H}(\mathbf{x})$ be positive definite (refer to

[3], section 3.5.2). For a more detailed solution, the reader is referred to [1] and [22]. The truncated moment problem also has many applications, e.g. in optimization and in probability theory. This is largely due to its relation with the cone of non-negative polynomials, a relation which is retained with further generalizations.

In the 1990's, Curto and Fialkow began investigating one of these generalization [10]. Given a doubly indexed finite sequence of numbers, $\gamma \equiv \{\gamma_{00}, \gamma_{01}, \gamma_{10}, \ldots, \gamma_{2n,0}, \ldots, \gamma_{0,2n}\}$, with $\gamma_{00} > 0$, they searched for a positive Borel measure μ on $\mathbb{R}^2$ such that

$$\gamma_{ij} = \int x^i y^j \, d\mu.$$

Their approach was to use the multivariate Hankel matrix, given by $\mathcal{M}(n)_{\mathbf{i},\mathbf{j}} := \gamma_{\mathbf{i}+\mathbf{j}}$, where $\mathbf{i}, \mathbf{j} \in \mathbb{Z}_+^2$ and $|\mathbf{i}|, |\mathbf{j}| \leq n$. After an exhaustive study, they completely characterized the truncated quartic moment problem ($n = 2$), and gave necessary and sufficient conditions for when a representing measure exists (see [8, 9]). One of their crowning achievements was the discovery that if $\mathcal{M}(n)$ admits a rank preserving (*flat*) extension to $\mathcal{M}(n+1)$, then γ admits a representing measure. We present an overview of this in Chapter 2. Like the univariate truncated moment problem, the multivariate truncated moment problem has many deep connections and application in mathematics. For example, [24] highlights connections between the moment problem and operator theory, [14] looks at an application of the moment problem in geometric optics.

The tracial moment problem looks to generalize this in another direction. Let $\mathbb{SR}^{n\times n}$ be the space of symmetric real $n \times n$ matrices and let $\langle X, Y\rangle$ denote the *free monoid* generated by X and Y ($\langle X, Y\rangle$ consists of *words* in the non-commuting letters X and Y). For a word $w \in \langle X, Y\rangle$, w^* is its reverse, and $v \in \langle X, Y\rangle$ is cyclically equivalent to w ($v \sim w$), if and only if v is a cyclic permutation of w. The tracial moment problem is this; given a sequence of real numbers (y_w), indexed by words w of length at most k, that satisfies

$$y_w = y_v \quad \text{whenever } v \sim w, \qquad y_w = y_{w^*} \quad \text{for all } w,$$

when does there exist an $n \in \mathbb{N}$ and a positive probability measure μ on $(\mathbb{SR}^{n\times n})^2$ such that

$$y_w = \int \mathrm{Tr}(w(A, B)) \, d\mu(A, B).$$

Like the classical moment problem, the tracial moment problem is quite versatile. Its duality pairing with trace-positive polynomials, means the tracial moment problem has implications reaching far and wide. From operator algebras (Conne's embedding conjecture [17, 7]), to mathematical physics (the BMV conjecture [2, 18]) and even touching on the realms of quantum chemistry [19].

In this thesis, we study the quartic tracial moment problem using Curto and Fialkow's Hankel-matrix approach, and give sufficient conditions for a representing measure to exist.

Chapter 3 focuses on positive definite multivariate Hankel matrices. Decomposing the given Hankel matrix as $\mathcal{M} = M + U$, we attempt to find flat extensions for the simplified Hankel matrices M and U. While this attempt works for certain classes of matrices, it does not always yield a solution. We give conditions for when Curto and Yoo's method for the explicit construction of a representing measure in the truncated classical moment problem, extends to the truncated tracial moment problem.

In Chapter 4, we present a detailed analysis of the truncated tracial moment problem with positive semi-definite multivariate Hankel matrices $\mathcal{M}$. We analyze the column relations that exist in $\mathcal{M}$, depending on the rank of $\mathcal{M}$. We prove tracial analogues of fundamental results from the classical truncated moment problem, and implicitly present cases, which act as counter-examples for some of the fundamental results from the truncated moment problem. In all cases that arise, we give sufficient conditions for the existence of a representing measure $\mathcal{M}$ and provide examples that aid in better understanding the theory.

Chapter 2

Commutative Variables

In this chapter, we present Curto and Yoo's method for the explicit construction of a representing measure for positive definite multivariate Hankel matrices, expanding on some of the details in [9]. This elegant construction will serve to introduce the unfamiliar reader with the many well known results of the classical moment problem.

Given real numbers $\beta \equiv \beta^{(4)} := (\beta_{ij})$ with $0 \leq i+j \leq 4$ and $\beta_{00} > 0$, the *Quartic Real Moment Problem* for β entails finding conditions for the existence of a positive Borel measure μ, supported in $\mathbb{R}^2$, such that for $0 \leq i+j \leq 4$,

$$\beta_{ij} = \int s^i t^j \, d\mu.$$

Let $\mathcal{M}(2)$ be the multivariate Hankel matrix for $\beta^{(4)}$, given by $\mathcal{M}(2)_{\mathbf{i},\mathbf{j}} := \beta_{\mathbf{i}+\mathbf{j}}$, where $\mathbf{i}, \mathbf{j} \in \mathbb{Z}_+^2$ and $|\mathbf{i}|, |\mathbf{j}| \leq 2$; this 6×6 matrix is shown below. The rows and columns of $\mathcal{M}(2)$ are labeled by $\mathbb{1}, X, Y, X^2, XY$ and Y^2.

$$\mathcal{M}(2) \equiv \begin{pmatrix} \beta_{00} & \beta_{10} & \beta_{01} & \beta_{20} & \beta_{11} & \beta_{02} \\ \beta_{10} & \beta_{20} & \beta_{11} & \beta_{30} & \beta_{21} & \beta_{12} \\ \beta_{01} & \beta_{11} & \beta_{02} & \beta_{21} & \beta_{12} & \beta_{03} \\ \beta_{20} & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} \\ \beta_{11} & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} \\ \beta_{02} & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} \end{pmatrix}.$$

Assume now that $\mathcal{M}(2)$ is non-singular. Hilbert's theorem on sums of squares of bivariate quartics yields the existence of a finitely atomic representing measure as follows. Let $\mathcal{P}_4$ be the cone of non-negative polynomials of degree at most 4 in x and y, regarded as a subset of $\mathbb{R}^{15}$. The dual cone is $\mathcal{P}_4^* := \{\xi \in \mathbb{R}^{15} : \langle \xi, p \rangle \geq 0$ for all $p \in \mathcal{P}_4\}$. If $a, b \in \mathbb{R}$ and $\xi_{(a,b)} := (1, a, b, a^2, ab, b^2, a^3, a^2b, ab^2, b^3, a^4, a^3b, a^2b^2, ab^3, b^4) \in \mathbb{R}^{15}$, then $\langle \xi_{(a,b)}, p \rangle = p(a, b) \geq 0$, for all $p \in \mathcal{P}_4$. Thus $\xi_{(a,b)} \in \mathcal{P}_4^*$ for all $a, b \in \mathbb{R}$ and $\xi_{(a,b)}$ is also an extreme point. Consider now an arbitrary moment sequence $\beta^{(4)}$. Regarded as a point in $\mathbb{R}^{15}$, $\beta^{(4)}$ is an interior point in $\mathcal{P}_4^*$, since every $p \in \mathcal{P}_4$ is a sum of squares of polynomials. By Carathéodory's Theorem and the Krein-Milman Theorem, the Riesz functional $\Lambda_{\beta(4)}$ is a real convex combination of evaluations $\xi_{(a,b)}$: meaning that $\beta^{(4)}$ admits a finitely atomic representing measure with at most 15 atoms.

Proposition 2.0.1 ([8], Proposition 5.5)**.** *If $\mathcal{M}(2) \geq 0$ (positive semi-definite),* rank $\mathcal{M}(2) = 5$*, and if $XY = 0$ in the column space of $\mathcal{M}(2)$, then $\mathcal{M}(2)$ admits a representing measure μ with* card supp $\mu \leq 6$.

Proof. See Appendix B. □

When combined with previous work on truncated moment problems, Proposition 2.0.1 led to the following solution to the truncated moment problem on planar curves of degree ≤ 2. Given a moment matrix $\mathcal{M}(n)$ and a polynomial $p(x, y) = \sum p_{ij} x^i y^j$, we let $p(X, Y) := \sum p_{ij} X^i Y^j$. A column relation in $\mathcal{M}(n)$ is therefore always described as $p(X, Y) = 0$ for some polynomial p with deg $p \leq n$. We say that $\mathcal{M}(n)$ is recursively generated if for every p with $p(X, Y) = 0$ and every q such that deg $pq \leq n$ we have $(pq)(X, Y) = 0$. In what follows, v denotes the cardinality of the algebraic variety ($\mathcal{V}(\beta)$) associated with the sequence β (or the multivariate Hankel matrix $\mathcal{M}(n)$), which is defined as the intersection of the zero sets of all polynomials that describe a column relation in $\mathcal{M}(2)$.

Theorem 2.0.2 ([12], Theorem 1.2)**.** *Let $p \in \mathbb{R}[x, y]$, with* deg $p(x, y) \leq 2$*. Then $\beta^{(2n)}$ has a representing measure supported in the curve $p(x, y) = 0$ if and only if $\mathcal{M}(n)$ has a column dependence relation $p(X, Y) = 0$, $\mathcal{M}(n) \geq 0$, $\mathcal{M}(n)$ is recursively generated and* rank $\mathcal{M}(n) \leq$ card $\mathcal{V}(\beta)$.

Proof. See Appendix A. □

When $n = 2$, we want to find a rank preserving extension of the 6×6 moment matrix $\mathcal{M}(2)$ to a 10×10 moment matrix $\mathcal{M}(3)$ by adding B and C blocks, so

$$\mathcal{M}(3) = \begin{pmatrix} \mathcal{M}(2) & B(3) \\ B(3)^* & C(3) \end{pmatrix}$$

(in general we wish to extend $\mathcal{M}(n)$ to $\mathcal{M}(n + 1)$, by adding similar $B(n + 1)$ and $C(n + 1)$ blocks). To do this we use the following result of Smul'jan.

Theorem 2.0.3 (Smul'jan)**.** *For a positive semi-definite matrix A, an extension*

$$A' = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix},$$

is positive semi-definite if and only if *(i)* $B = AW$ *for some matrix* W*,*

(ii) $C \geq W^* AW$.

Moreover, if $C = W^* AW$ *for some* W*, then* $\text{rank } A' = \text{rank } A$.

Proof. Suppose that A' is positive semi-definite. Then $\mathbf{w}^* A' \mathbf{w} \geq 0$ for all vectors $\mathbf{w}$ (of the appropriate size). Now partition $\mathbf{w}$ as

$$\mathbf{w} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix},$$

so that

$$\begin{aligned} \mathbf{w}^* A' \mathbf{w} &= \mathbf{u}^* A \mathbf{u} + 2\mathbf{v}^* B^* \mathbf{u} + \mathbf{v}^* C \mathbf{v} \\ &= \mathbf{u}^* A \mathbf{u} + 2(B\mathbf{v})^* \mathbf{u} + \mathbf{v}^* C \mathbf{v}. \end{aligned}$$

For any fixed $\mathbf{v}$, this expression is a quadratic. It is well known that the optimal value δ of the general (non-convex) quadratic optimization problem

$$\text{Minimize } (1/2)\mathbf{x}^* P \mathbf{x} + \mathbf{q}^* \mathbf{x} + r$$

where P is a square symmetric matrix, can be expressed as

$$\delta = \begin{cases} -(1/2)\mathbf{q}^* P^\dagger \mathbf{q} + r & \text{if } P \geq 0 \text{ and } \mathbf{q} \in \text{ Ran}(A) \\ -\infty & \text{otherwise} \end{cases}$$

where $P^\dagger$ is the Moore-Penrose psuedo-inverse. Hence we have that $B\mathbf{v} \in \text{ Ran}(A)$ for every $\mathbf{v}$, and therefore we must have that $\text{Ran}(B) \subseteq \text{ Ran}(A)$, i.e., $B = AW$ for some W. Now if we had that $C < W^* AW$, then for $\mathbf{w}$ such that $\mathbf{v} \neq \mathbf{0}$

$$\begin{aligned} \mathbf{u}^* A \mathbf{u} + 2\mathbf{v}^* B^* \mathbf{u} + \mathbf{v}^* C \mathbf{v} &= \mathbf{u}^* A \mathbf{u} + 2\mathbf{v}^* W^* A \mathbf{u} + \mathbf{v}^* C \mathbf{v} \\ &< \mathbf{u}^* A \mathbf{u} + 2\mathbf{v}^* W^* A \mathbf{u} + \mathbf{v}^* W^* A W \mathbf{v} \\ &= (\mathbf{u} + W\mathbf{v})^* A (\mathbf{u} + W\mathbf{v}). \end{aligned}$$

We may choose $\mathbf{u}$ such that

$$\mathbf{u} + W\mathbf{v} \in \text{Null}(A).$$

Then with this $\mathbf{u}$ and $\mathbf{v}$ we have that

$$\mathbf{w} = \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix},$$

and

$$\mathbf{w}^* A' \mathbf{w} < 0.$$

This is a clear contradiction. Therefore if A' is positive semi-definite then the two conditions above must be satisfied. The rest of the proof is easy. For a more generalized operator version of this result, see [21]. □

Theorem 2.0.4. *Assume $\mathcal{M}(2)$ is positive definite and invertible. Then $\mathcal{M}(2)$ admits a representing measure with exactly 6 atoms; that is, $\mathcal{M}(2)$ admits a flat extension $\mathcal{M}(3)$.*

The proof of this theorem is constructive, in the sense that it is first shown that it's always possible to change the invertible $\mathcal{M}(2)$ to a related singular $\widetilde{\mathcal{M}(2)}$ for which Theorem 2.0.2 applies. Then using previously established results, it is concluded that there is a representing measure with at most 7 atoms. It is then shown that all invertible $\mathcal{M}(2)$'s actually have a *flat* extension, and hence a representing measure with exactly 6 atoms. We begin with a result that will allow us to convert a given moment problem into a simpler, equivalent, moment problem. This simplification will allow us to assume that the submatrix $\mathcal{M}(1)$ is the identity. For $a, b, c, d, e, f \in \mathbb{R}$, with $bf - ce \neq 0$, let $\Phi(x, y) \equiv (\Phi_1(x, y), \Phi_2(x, y)) := (a + bx + cy, d + ex + fy)$, $(x, y) \in \mathbb{R}^2$ (Notice that Φ is invertible). Given $\beta^{(2n)}$, define $\tilde{\beta}^{(2n)}$ by $\tilde{\beta}_{ij} := L_\beta(\Phi_1^i \Phi_2^j)$ $(0 \leq i + j \leq 2n)$, where L_β is the Riesz functional associated with β, which acts on $\mathbb{R}[x, y]_{2n}$, and is defined by

$$L_\beta(\sum a_{ij} x^i y^j) := \sum a_{ij} \beta_{i,j}.$$

It is straightforward to verify that $L_\beta(p \circ \Phi) = L_{\tilde{\beta}}(p)$ for every polynomial p of degree at most n. For a polynomial $p(x, y) = \sum a_{ij} x^i y^j \in \mathbb{R}[x, y]_{2n}$, let $\hat{p} = (a_{ij})$ be the coefficient vector of p with respect to the basis $\{x^i y^j\}_{0 \leq i+j \leq 2n}$ of $\mathbb{R}[x, y]_{2n}$ (ordered lexicographically).

Proposition 2.0.5. *Let $\mathcal{M}(n)$ and $\widetilde{\mathcal{M}(n)}$ be moment matrices associated with β and $\tilde{\beta}$, and let $J\hat{p} := \widehat{p \circ \Phi}$. Then the following hold,*

(i) $\widetilde{\mathcal{M}(n)} = J^* \mathcal{M}(n) J$.

(ii) *J is invertible.*

(iii) $\widetilde{\mathcal{M}(n)} \geq 0 \Leftrightarrow \mathcal{M}(n) \geq 0$.

(iv) rank $\widetilde{\mathcal{M}(n)} =$ rank $\mathcal{M}(n)$.

(v) *The formula $\tilde{\mu} = \mu \circ \Phi$ establishes a one-to-one correspondence between the sets of representing measures of β and $\tilde{\beta}$, which preserve measure class and cardinality of the support.*

(vi) $\mathcal{M}(n)$ *admits a flat extension if and only if* $\widetilde{\mathcal{M}(n)}$ *admits a flat extension.*

Proof. It is clear that (iii) and (iv) follow from (i) and (ii). For (i), notice that $\langle \mathcal{M}(n)\hat{p}, \hat{q}\rangle = L_\beta(pq)$ $(p, q \in \mathbb{R}[x, y]_{2n})$. Thus,

$$\begin{aligned}\langle J^*\mathcal{M}(n)J\hat{p}, \hat{q}\rangle &= \langle \mathcal{M}(n)J\hat{p}, J\hat{q}\rangle\,, \\ &= \left\langle \mathcal{M}(n)\widehat{p\circ\Phi}, \widehat{q\circ\Phi} \right\rangle, \\ &= L_\beta((p\circ\Phi)(q\circ\Phi)), \\ &= L_\beta((pq)\circ\Phi) = L_{\tilde{\beta}}(pq), \\ &= \left\langle \widetilde{\mathcal{M}(n)}\hat{p}, \hat{q}\right\rangle.\end{aligned}$$

For (ii),

$$J\hat{p} = 0 \Rightarrow \widehat{p\circ\Phi} = 0 \Rightarrow p\circ\Phi = 0.$$

Therefore,

$$p\circ\Phi = 0 \quad \text{for all } x, y \in \mathbb{R}.$$

Since Φ is invertible we have

$$p(\tilde{x}, \tilde{y}) = p(\Phi\circ\Phi^{-1}(x, y)) = 0 \text{ for all } \tilde{x}, \tilde{y} \in \mathbb{R}.$$

It readily follows that $p \equiv 0$, therefore $\hat{p} = 0$, which proves that J is invertible. To prove (v), assume that μ is a representing measure for β and observe that

$$\begin{aligned}\tilde{\beta}_{ij} = L_{\tilde{\beta}}(x^iy^j) &= L_\beta(\Phi_1^i\Phi_2^j) \\ &= \int \Phi_1^i(x, y)\Phi_2^j(x, y)\, d\mu(x, y) \\ &= \int x^iy^j\, d\tilde{\mu}(x, y),\end{aligned}$$

so $\tilde{\mu}$ is a representing measure for $\tilde{\beta}$. Conversely, it follows as above that if $\tilde{\mu}$ is a representing measure for $\tilde{\beta}$, then $\mu = \tilde{\mu}\circ\Omega$ (where $\Omega := \Phi^{-1}$) is a representing measure for β.

For (vi), suppose that $\mathcal{M}(n)$ admits a flat extension. Then $\mathcal{M}(n)$ admits a rank $\mathcal{M}(n)$ atomic measure, so (v) implies that $\widetilde{\mathcal{M}(n)}$ admits a rank $\mathcal{M}(n)$ atomic measure $\tilde{\mu}$. Now (iv) implies that $\tilde{\mu}$ is rank $\widetilde{\mathcal{M}(n)}$ atomic, and hence it follows that $\widetilde{\mathcal{M}(n)}$ admits a flat extension. The converse is entirely similar. □

We now put $\mathcal{M}(n)$ in "normalised form". Without loss of generality, we may assume that $\beta_{00} = 1$. Let d_i denote the leading principal minors of $\mathcal{M}(2)$. Now we take the degree one transformation Φ with $a := \frac{\beta_{01}\beta_{02}-\beta_{10}\beta_{11}}{\sqrt{d_2d_3}}$, $b := \frac{\beta_{11}-\beta_{01}\beta_{10}}{\sqrt{d_2d_3}}$, $c := -\sqrt{\frac{d_2}{d_3}}$, $d := -\frac{\beta 10}{\sqrt{d_2}}$,

$e := \frac{1}{\sqrt{d_2}}$, and $f := 0$. Note that $bf - ce \neq 0$. Using this transformation, and a straightforward calculation, we can prove that any positive definite moment matrix $\mathcal{M}(2)$ can be transformed into

$$\widetilde{\mathcal{M}(2)} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & \tilde{\beta}_{30} & \tilde{\beta}_{21} & \tilde{\beta}_{12} \\ 0 & 0 & 1 & \tilde{\beta}_{21} & \tilde{\beta}_{12} & \tilde{\beta}_{03} \\ 1 & \tilde{\beta}_{30} & \tilde{\beta}_{21} & \tilde{\beta}_{40} & \tilde{\beta}_{31} & \tilde{\beta}_{22} \\ 0 & \tilde{\beta}_{21} & \tilde{\beta}_{12} & \tilde{\beta}_{31} & \tilde{\beta}_{22} & \tilde{\beta}_{13} \\ 1 & \tilde{\beta}_{12} & \tilde{\beta}_{03} & \tilde{\beta}_{22} & \tilde{\beta}_{13} & \tilde{\beta}_{04} \end{pmatrix}$$

Thus, we can assume that $\mathcal{M}(1)$ is the identity. Assume now that $\mathcal{M}(2)$ is invertible and $\mathcal{M}(1)$ is the identity. For $u \in \mathbb{R}$, decompose $\mathcal{M}(2)$ as

$$\mathcal{M}(2) = \begin{pmatrix} 1-u & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & \beta_{30} & \beta_{21} & \beta_{12} \\ 0 & 0 & 1 & \beta_{21} & \beta_{12} & \beta_{03} \\ 1 & \beta_{30} & \beta_{21} & \beta_{40} & \beta_{31} & \beta_{22} \\ 0 & \beta_{21} & \beta_{12} & \beta_{31} & \beta_{22} & \beta_{13} \\ 1 & \beta_{12} & \beta_{03} & \beta_{22} & \beta_{13} & \beta_{04} \end{pmatrix} + \begin{pmatrix} u & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Denote the first summand by $\widehat{\mathcal{M}(2)}$ and the second by $\mathcal{P}$. It is clear that $\mathcal{P}$ is positive semi-definite, and has rank 1 if and only if $u > 0$, and in that case $\mathcal{P}$ is the moment matrix of the 1-atomic measure $u\delta_{(0,0)}$, where $\delta_{(0,0)}$ is the point mass at the origin.

The following result is an easy consequence of the multilinearity of the determinant (but for completeness we prove it here), and will be used in the proof of the subsequent theorem.

Lemma 2.0.6. *Let M be an $n \times n$ invertible matrix of real numbers, let E_{11} be the rank one matrix with $(1,1)$ entry equal to 1, and all others equal to zero, and let $u \in \mathbb{R}$. Then,* $\det(M - uE_{11}) = \det(M) - u\det(M_{\{2,3,\ldots,n\}})$, *where $M_{\{2,3,\ldots,n\}}$ is the compression of M to the rows and columns indexed by $\{2,3,\ldots,n\}$. In particular, if $u = \frac{1}{(M^{-1})_{11}}$, then* $\det(M - uE_{11}) = 0$.

Proof. Write the matrix M as

$$M = \begin{pmatrix} a & b & \ldots \\ d & \ddots & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Then

$$M - uE_{11} = \begin{pmatrix} a-u & b & \ldots \\ d & \ddots & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

A simple cofactor expansion gives that

$$\begin{aligned}\det(M - uE_{11}) =& (a-u)\det(M_{\{2,3,\dots,n\}}) + R \\ & (R \text{ is the remaining terms given by the expansion}) \\ =& (a\det(M_{\{2,3,\dots,n\}}) + R) - u\det(M_{\{2,3,\dots,n\}}) \\ =& \det(M) - u\det(M_{\{2,3,\dots,n\}}).\end{aligned}$$

Now for any invertible matrix A the inverse can be written as

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A),$$

where $\operatorname{adj}(A)$ is the adjugate matrix. Hence, $(M^{-1})_{11} = \frac{\det(M_{\{2,3,\dots,n\}})}{\det(M)}$. So if $u = \frac{1}{(M^{-1})_{11}}$, then clearly, $\det(M - uE_{11}) = 0$. □

Proposition 2.0.7. *Let $u := \frac{1}{(\mathcal{M}(2)^{-1})_{11}}$ and $\mathcal{M}(2)$, $\widehat{\mathcal{M}(2)}$ and $\mathcal{P}$ be as above. Then with this non-negative value of u we have*

(i) $\widehat{\mathcal{M}(2)} \geq 0$

(ii) rank $\widehat{\mathcal{M}(2)} = 5$ *and*

(iii) $\widehat{\mathcal{M}(2)}$ *is recursively generated.*

Moreover this is the only value of u for which $\widehat{\mathcal{M}(2)}$ satisfies $(i) - (iii)$.

Proof. (ii) Observe that $6 = \operatorname{rank} \mathcal{M}(2) \leq \operatorname{rank} \widehat{\mathcal{M}(2)} + \operatorname{rank} \mathcal{P} = \operatorname{rank} \widehat{\mathcal{M}(2)} + 1$. Since $\det(\widehat{\mathcal{M}(2)}) = 0$ (by Lemma 2.0.6), we have $\operatorname{rank} \widehat{\mathcal{M}(2)} = 5$.

(i) Using the Nested Determinants Test starting from the lower right hand corner of $\widehat{\mathcal{M}(2)}$, we know that $\widehat{\mathcal{M}(2)}$ is positive semi-definite since the principal minors of size 1, 2, 3, 4, and 5 are all positive for $\mathcal{M}(2)$, and the rank of $\widehat{\mathcal{M}(2)}$ is 5. This also implies that $1 - u \geq 0$. If $1 - u = 0$, then the positive semi-definiteness of $\widehat{\mathcal{M}(2)}$ would force all entries in the first row to be zero. Since this is evidently false, we conclude that $1 - u > 0$.

(iii) It is sufficient to show that the first three columns of $\widehat{\mathcal{M}(2)}$ are linearly independent, as then any linear polynomial $p(X, Y) = 0$ if and only if $p = 0$, which is enough to show recursive generation for $\mathcal{M}(2)$. Consider the third leading principal minor of $\widehat{\mathcal{M}(2)}$, which is $1 - u > 0$. Thus there is no linear dependence in this submatrix, and as a result the same holds in $\widehat{\mathcal{M}(2)}$.

Finally, the uniqueness of u as the only value satisfying (i)-(iii) is clear. □

For the proof of Theorem 2.0.4 we first observe that by combining Proposition 2.0.7 with Theorem 2.0.2, it suffices to consider the case when $\widehat{\mathcal{M}(2)}$ has a column relation corresponding to a pair of intersecting lines. For, in all other cases there exists a representing measure for $\widehat{\mathcal{M}(2)}$ with exactly five atoms (see Appendix B); when combined with the additional atom from $\mathcal{P}$, we have that $\mathcal{M}(2)$ admits a 6-atomic representing measure.

We thus focus on the case when $\widehat{\mathcal{M}(2)}$ is subordinate to a degenerate hyperbola. After applying an additional degree one transformation, we can assume that the relation $XY = 0$ is present in $\mathcal{M}(2)$, however we may no longer assume that $\mathcal{M}(1)$ is the identity matrix (since the degree one transformation that produces $XY = 0$ will, in general change the lower order moments). So $\widehat{\mathcal{M}(2)}$ is of the form

$$\widehat{\mathcal{M}(2)} = \begin{pmatrix} 1 & a & b & c & 0 & d \\ a & c & 0 & e & 0 & 0 \\ b & 0 & d & 0 & 0 & f \\ c & e & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & f & 0 & 0 & h \end{pmatrix}.$$

In this case the original moment matrix is written as

$$\mathcal{M}(2) = \widehat{\mathcal{M}(2)} + u \begin{pmatrix} 1 & p & q & p^2 & pq & q^2 \end{pmatrix}^T \begin{pmatrix} 1 & p & q & p^2 & pq & q^2 \end{pmatrix},$$

for some $u > 0$, and $pq \neq 0$. Without loss of generality, we may assume that $p = q = 1$ (using a degree one transformation). With this we have

$$\mathcal{M}(2) = \begin{pmatrix} 1+u & a+u & b+u & c+u & u & d+u \\ a+u & c+u & u & e+u & u & u \\ b+u & u & d+u & u & u & f+u \\ c+u & e+u & u & g+u & u & u \\ u & u & u & u & u & u \\ d+u & u & f+u & u & u & h+u \end{pmatrix}.$$

We will now show that that $\mathcal{M}(2)$ admits a flat extension, which will imply that it admits a rank $\mathcal{M}(2)$ atomic (that is, 6-atomic) representing measure. The $B(3)$ block of an extension $\mathcal{M}(3)$ can be generated by letting $\beta_{41} = \beta_{32} = \beta_{23} = \beta_{14} = u$, so that $B(3)$ then takes the form,

$$\begin{pmatrix} e+u & u & u & f+u \\ g+u & u & u & u \\ u & u & u & h+u \\ \beta_{50} & u & u & u \\ u & u & u & u \\ u & u & u & \beta_{05} \end{pmatrix}.$$

As usual let the matrix $W := \mathcal{M}(2)^{-1}B(3)$ and let $C(3) \equiv (C_{ij}) := W^*\mathcal{M}(2)W$. Note that if $C(3)$ is Hankel, then $\mathcal{M}(3)$ is a flat extension of $\mathcal{M}(2)$. Since $C(3)$ is symmetric, to ensure that $C(3)$ is Hankel we only need to solve the following system of equations:

$$\begin{cases} E_1 := C_{13} - C_{22} = 0, \\ E_2 := C_{14} - C_{23} = 0, \\ E_3 := C_{24} - C_{33} = 0. \end{cases} \tag{2.1}$$

This is a system of equations involving quadratic polynomials with 2 unknown variables (the new moments β_{50} and β_{05}). A straightforward calculation shows that $E_1 = E_3 = 0$, and that

$$\begin{aligned} E_2 = 0 \Leftrightarrow & (c^2 - ae)(d^2 - bf)\beta_{05}\beta_{50} + (c^2 - ae)(f^3 - 2dfh + bh^2 - d^2u + bfu)\beta_{50} \\ & + (d^2 - bf)(e^3 - 2ceg + ag^2 - c^2u + aeu)\beta_{05} \\ & + (e^3 - 2ceg + ag^2 - c^2u + aeu)(f^3 - 2dfh + bh^2 - d^2u + bfu) = 0 \\ \Leftrightarrow & \kappa\lambda\beta_{50}\beta_{05} + \kappa\mu\beta_{50} + \lambda\nu\beta_{05} + \nu\mu = 0, \end{aligned}$$

where

$$\begin{aligned} \kappa &= (c^2 - ae), \\ \lambda &= (d^2 - bf), \\ \mu &= (f^3 - 2dfh + bh^2 - d^2u + bfu), \quad \text{and} \\ \nu &= (e^3 - 2ceg + ag^2 - c^2u + aeu). \end{aligned}$$

If $\kappa, \lambda \neq 0$, then $\beta_{05} = \frac{-\mu\nu + \kappa\mu\beta_{50}}{\kappa\lambda\beta_{50} + \lambda\nu}$ (for $\beta_{50} \neq -\frac{\nu}{\kappa}$), which implies that $E_2 = 0$ has infinitely many solutions. When $\kappa = 0$ and $\lambda \neq 0$, we see that $E_2 = \lambda\nu\beta_{05} + \mu\nu$, from which it follows that a solution always exists (and it is unique when $\nu \neq 0$). Similarly, when $\lambda = 0$ and $\kappa \neq 0$ a solution exists (which is unique when $\mu \neq 0$). We are thus left with the case when $\kappa \equiv c^2 - ae$ and $\lambda \equiv d^2 - bf$ are equal to zero. Since c and d are on the diagonal of a positive matrix, they must be positive. Thus, all of a, b, e and f are non-zero and we can set $e := c^2/a$ and $f := d^2/b$. In this case the moment matrix is

$$\mathcal{M}(2) = \begin{pmatrix} 1+u & a+u & b+u & c+u & u & d+u \\ a+u & c+u & u & \frac{c^2}{a}+u & u & u \\ b+u & u & d+u & u & u & \frac{d^2}{b}+u \\ c+u & \frac{c^2}{a}+u & u & g+u & u & u \\ u & u & u & u & u & u \\ d+u & u & \frac{d^2}{b}+u & u & u & h+u \end{pmatrix}.$$

Let $k := \det(\mathcal{M}(2))/\det(\mathcal{M}(2)_{\{2,3,4,5,6\}})$. As in the proof of Proposition 2.0.7, we see that $k > 0$ and the first summand in the following decomposition of $\mathcal{M}(2)$ has rank 5 and is positive semidefinite (note that the (1,1) entry is $1+u-k$):

$$\mathcal{M}(2)=\begin{pmatrix} \frac{b^2c+a^2d+cdu}{cd} & a+u & b+u & c+u & u & d+u \\ a+u & c+u & u & \frac{c^2}{a}+u & u & u \\ b+u & u & d+u & u & u & \frac{d^2}{b}+u \\ c+u & \frac{c^2}{a}+u & u & g+u & u & u \\ u & u & u & u & u & u \\ d+u & u & \frac{d^2}{b}+u & u & u & h+u \end{pmatrix}+\begin{pmatrix} k & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The only relation in the first summand is

$$XY=\frac{cd}{-bc-ad+cd}\mathbb{1}-\frac{ad}{-bc-ad+cd}X-\frac{bc}{-bc-ad+cd}Y=:\xi\mathbb{1}-\eta X-\theta Y. \quad (2.2)$$

Unless $\eta\theta=-\xi$, the conic that represents this column relation is a non-degenerate hyperbola, and therefore the moment sequence associated to the moment matrix has a 5-atomic measure, by Theorem 2.0.2. In the case when the conic represents a pair of intersecting lines (i.e., $(x+\theta)(y+\eta)=0$), we must have $c=a$ or $d=b$.

Thus the remaining two specific cases to cover are $\mathcal{M}(2)$ with $c=a$ or $d=b$. Since $\mathcal{M}(2)$ is invertible, for any $B(3)$ block we will be able to find W such that $\mathcal{M}(2)W=B(3)$. We propose to use a B block with new moments $\beta_{32}=\beta_{23}=\beta_{14}=0$, and then to extend $\mathcal{M}(2)$ to $\mathcal{M}(3)$ using Smul'jan's Lemma, by defining $C(3):=W^*B(3)$. To establish that $C(3)$ is Hankel requires (2.1). We begin with a few observations.

Let d_i denote the leading principal minors of $\mathcal{M}(2)$ for $i=1,2,\dots,6$; since $\mathcal{M}(2)$ is invertible and positive, we know the minors are all positive. Then

$$d_5=-\frac{(b^2c+a^2d-cd)(-c^3+a^2g)u}{a^2} \quad \text{and} \quad d_6=\frac{d_6(-d^3+b^2h)}{b^2},$$

which implies

$$(b^2c+a^2d-cd)(-c^3+a^2g)<0 \text{ and } -d^3+b^2h>0. \quad (2.3)$$

Next, we use $Mathematica$ to solve for $E_1=0$ for β_{50} and $E_3=0$ for β_{05}, and we obtain

$$\beta_{50}=\frac{1}{a^2c(b^2c+a^2d-cd)(-d^3+b^2h)u}(\alpha_{11}\beta_{41}^2+\alpha_{12}\beta_{41}+\alpha_{13}),$$
$$\beta_{05}=\frac{1}{b^2d(b^2c+a^2d-cd)(c^3-a^2g}(\alpha_{21}\beta_{41}+\alpha_{22}),$$

where the α_{ij}'s are polynomials in a,b,c,d,g,h and u. Since $a,b\neq 0$, and $c,d>0$, we can use (2.3) to show that both β_{50} and β_{05} above are well defined. We now substitute these values in E_2 and check that E_2 is a quadratic polynomial in β_{41}; indeed, we can readily show that the leading coefficient of E_2 is non-zero if $c = a$ or $d = b$. Thus, if the discriminant Δ of this quadratic polynomial is nonnegative then (2.1) has at least one solution.

If $c = a$, then

$$\Delta = \frac{a^2u^2(a-g)^2(-d^3+b^2h)^2F_1(a,d,b,h)}{b^4d^2},$$

where

$$F_1(a,d,b,h) = (-1+a)^2b^2h^2 + 2b^2d(2b^2-3d+3ad)h - d^4(3b^2-4d+4ad)$$

is a concave upward quadratic polynomial in h. Notice that $\Delta \geq 0$ if and only if $F_1 \geq 0$, which means that the discriminant of F_1, $\Delta_1 := 16b^2d^2(b^2-d+ad)^3$, needs to be zero or negative. In this case, observe that

$$\begin{aligned}
c &= a > 0, \\
d_3 &= -ab^2 + ad - a^2d + au - a^2u - b^2u + du - adu > 0, \\
d_4 &= -d_3(a-g) > 0 \ (\Rightarrow a - g < 0), \\
d_5 &= a(b^2 - d + ad)(a-g)u > 0,
\end{aligned}$$

which leads to $b^2 - d + ad < 0$. Therefore $\Delta_1 < 0$ and $\Delta > 0$.

Similarly, if $d = b$ then

$$\Delta = \frac{(-c^3+a^2g)^2(b-h)^2u^2F_2(a,d,c,h)}{a^4},$$

where

$$\begin{aligned}
F_2(a,d,c,h) =& (-1+b)^2h^2c^2 + 2a(-1+b)(-2b^3+3b^2h+ah^2-bh^2)c \\
&+ a^2(-4ab^3+b^4+6ab^2h-2b^3h+a^2h^2-2abh^2+b^2h^2)
\end{aligned}$$

is a concave upward quadratic polynomial in c. The discriminant of F_2 is $\Delta_2 := 16a^2(-1+b)^2b^3(b-h)^3$; we observe that $d = b > 0$ and $d_6 = -d_5(b-h) > 0$, which leads to $b - h < 0$. Therefore, $\Delta_2 < 0$ and $\Delta > 0$, which completes the proof.

Chapter 3

Non-commutative Variables - Positive Definite Hankel Matrix

We begin our study of the tracial moment problem. Focusing on positive definite multivariate Hankel matrices, we consider when an explicit construction of a representing measure is possible. We also present examples which highlight areas of possible future research.

3.1 Constructing a Flat Extension

Let $\langle X, Y\rangle$ denote the free monoid generated by X and Y, i.e., $\langle X, Y\rangle$ consists of words in the non-commuting letters X and Y. Consider the free algebra $\mathbb{R}\langle X, Y\rangle$ of polynomials in the non-commuting variables X and Y, with coefficients in $\mathbb{R}$. Endow $\mathbb{R}\langle X, Y\rangle$ with the involution $p \mapsto p^*$ fixing $\mathbb{R} \cup \{X, Y\}$ pointwise. Hence for each word $w \in \langle X, Y\rangle$, w^* is its reverse. The length of the longest word in a polynomial $f \in \mathbb{R}\langle X, Y\rangle$ is the degree of f and is denoted deg f or $|f|$. We write $\mathbb{R}\langle X, Y\rangle_{\leq k}$ for the set of all polynomials of degree at most k and $\langle X, Y\rangle_{\leq k}$ for the set of words $w \in \langle X, Y\rangle$ of length at most k.

Definition 3.1.1. $f, g \in \mathbb{R}\langle X, Y\rangle$ are *cyclically equivalent* ($f \sim g$) if $f - g$ is a sum of commutators.

Lemma 3.1.2. *Two words $v, w \in \langle X, Y\rangle$ are cyclically equivalent if and only if one is a cyclic permutation of the other.*

Proof. Let $v, w \in \langle X, Y\rangle$, and suppose that v is a cyclic permutation of w, i.e., there are words $u_1, u_2 \in \langle X, Y\rangle$ such that

$$v = u_1u_2,$$
$$w = u_2u_1.$$

Then clearly we have

$$\begin{aligned} v - w &= u_1 u_2 - u_2 u_1, \\ &= [u_1, u_2]. \end{aligned}$$

The reverse implication is also easy. □

Definition 3.1.3. A sequence of real numbers (β_w) indexed by words $w \in \langle X, Y \rangle$ satisfying

$$\beta_w = \beta_v \text{ whenever } w \sim v \ , \quad \beta_w = \beta_{w^*} \text{ for all } w \text{ and } \quad \beta_\varnothing = 1$$

is called a (normalized) *tracial sequence*.

For example, given $n \in \mathbb{N}$ and a positive probability measure μ on $(\mathbb{SR}^{n \times n})^2$, the sequence given by

$$\beta_w := \int \text{Tr}(w(A, B)) \, d\mu(A, B) \tag{3.1}$$

is a tracial sequence since the traces of cyclically equivalent words are the same, and for all real matrices A, $\text{Tr}(A) = \text{Tr}(A^*)$.

We are interested in answering the truncated converse of this problem (the *truncated tracial moment problem*): For which sequences $\beta \equiv (\beta_w)_{\leq k}$, where deg $w \leq k$ for some $k \in \mathbb{N}$, do there exist $n \in \mathbb{N}$ and a positive probability measure μ on $(\mathbb{SR}^{n \times n})^2$ such that (3.1) holds? When this is true, we call β a *truncated tracial moment sequence*. The following theorem from [5] answers this question for quartic sequences with positive definite multivariate Hankel matrices.

Theorem 3.1.4. *Let $\beta \equiv (\beta_w)_{\leq 4}$ be a bivariate truncated tracial sequence with a positive definite multivariate Hankel matrix. Then β is a truncated tracial moment sequence.*

Inspired by the work of Curto and Yoo in [9], we will tackle Theorem 3.1.4 in a similar manner. We start by creating our multivariate Hankel matrix $\mathcal{M}(2)$ which has its columns and rows indexed by the words $\{\mathbb{1}, X, Y, X^2, XY, YX, Y^2\}$ (for convenience, the words in the subscripts have been replaced by their cyclic equivalence class representatives),

$$\mathcal{M}(2) = \begin{pmatrix} \beta_1 & \beta_x & \beta_y & \beta_{x^2} & \beta_{xy} & \beta_{yx} & \beta_{y^2} \\ \beta_x & \beta_{x^2} & \beta_{xy} & \beta_{x^3} & \beta_{x^2y} & \beta_{x^2y} & \beta_{xy^2} \\ \beta_y & \beta_{xy} & \beta_{y^2} & \beta_{x^2y} & \beta_{xy^2} & \beta_{xy^2} & \beta_{y^3} \\ \beta_{x^2} & \beta_{x^3} & \beta_{x^2y} & \beta_{x^4} & \beta_{x^3y} & \beta_{x^3y} & \beta_{x^2y^2} \\ \beta_{xy} & \beta_{x^2y} & \beta_{xy^2} & \beta_{x^3y} & \beta_{x^2y^2} & \beta_{xyxy} & \beta_{xy^3} \\ \beta_{xy} & \beta_{x^2y} & \beta_{xy^2} & \beta_{x^3y} & \beta_{xyxy} & \beta_{x^2y^2} & \beta_{xy^3} \\ \beta_{y^2} & \beta_{xy^2} & \beta_{y^3} & \beta_{x^2y^2} & \beta_{xy^3} & \beta_{xy^3} & \beta_{y^4} \end{pmatrix}.$$

First we prove an analogue of Proposition 2.0.5.

For $a, b, c, d, e, f \in \mathbb{R}$, with $bf - ce \neq 0$, let

$$\Phi(x,y) = (\Phi_1(x,y), \Phi_2(x,y)) := (a + bx + cy, d + ex + fy),\ (x,y) \in \mathbb{R}^2.$$

Given $\beta^{(2n)}$ we define $\tilde{\beta}^{(2n)}$ by

$$\tilde{\beta}_w := L_\beta(w(\Phi_1, \Phi_2)), \quad w \in \langle X, Y \rangle_{\leq 2n},$$

where L_β is the Riesz functional associated with β, which acts on $\mathbb{R}\langle X, Y \rangle_{\leq 2n}$ and is defined by

$$L_\beta(\sum_w a_w w) := \sum_w a_w \beta_w.$$

We have $L_\beta(p \circ \Phi) = L_{\tilde{\beta}}(p)$ for every polynomial p of degree at most n. For a polynomial $p \in \mathbb{R}\langle X, Y \rangle_{\leq 2n}$, let $\hat{p} = (a_w)_w$ be the coefficient vector of p with respect to the basis $\{w\}_{\deg(w) \leq 2n}$ of $\mathbb{R}\langle X, Y \rangle_{\leq 2n}$ ordered lexicographically.

Proposition 3.1.5. *Let $M(n)$ and $\widetilde{M(n)}$ be moment matrices associated with β and $\tilde{\beta}$ respectively, and let*

$$J_n^\Phi : \mathbb{R}\langle X, Y \rangle_{\leq 2n} \to \mathbb{R}\langle X, Y \rangle_{\leq 2n}, \quad J_n^\Phi \hat{p} := \widehat{p \circ \Phi}.$$

Then the following hold:

(i) $\widetilde{M(n)} = (J_n^\Phi)^* M(n) J_n^\Phi$.

(ii) J_n^Φ *is invertible.*

(iii) $\widetilde{M(n)} \geq 0 \Leftrightarrow M(n) \geq 0$.

(iv) $\operatorname{rank} \widetilde{M(n)} = \operatorname{rank} M(n)$.

(v) The formula $\tilde{\mu} = \mu \circ \Phi$ establishes a one-to-one correspondence between the sets of representing measures of β and $\tilde{\beta}$, which preserve measure class and cardinality of the support of the measure.

(vi) $M(n)$ admits a flat extension if and only if $\widetilde{M(n)}$ admits a flat extension.

Proof. Same proof as in commutative case except for (vi). To prove this notice that $\widetilde{M(n+1)} = (J_{n+1}^\Phi)^* M(n+1) J_{n+1}^\Phi$. □

Using Proposition 3.1.5 we can assume that $\mathcal{M}(2)$ has the form

$$\mathcal{M}(2) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & a & b & b & c \\ 0 & 0 & 1 & b & c & c & d \\ 1 & a & b & e & f & f & g \\ 0 & b & c & f & g & h & i \\ 0 & b & c & f & h & g & i \\ 1 & c & d & g & i & i & j \end{pmatrix}.$$

If $g = h$, then the columns of $\mathcal{M}(2)$ satisfy $XY = YX$, and it is clear that in this case, we may proceed with Curto and Yoo's method of proof to show that there is a representing measure for β with 5 atoms. However, if $g = h$, then rank $\mathcal{M}(2) \leq 6$ meaning $\mathcal{M}(2)$ cannot be positive definite. We will return to the positive semi-definite cases in the next chapter. When $g \neq h$ the columns of $\mathcal{M}(2)$ satisfy $XY \neq YX$, i.e., $\mathcal{M}(2)$ is a non-commutative Hankel matrix. For the remainder of the chapter we will assume that we are working with a non-commutative, positive definite Hankel matrix.

Loosely speaking, our initial approach is to make the problem commutative. For this, we first decompose our given matrix $\mathcal{M}(2)$ as follows,

$$\mathcal{M}(2) = \begin{pmatrix} 1-\varepsilon_1 & 0 & 0 & 1-\varepsilon_2 & 0 & 0 & 1-\varepsilon_3 \\ 0 & 1-\varepsilon_2 & 0 & a & b & b & c \\ 0 & 0 & 1-\varepsilon_3 & b & c & c & d \\ 1-\varepsilon_2 & a & b & e-\varepsilon_4 & f & f & g-u \\ 0 & b & c & f & g-u & h & i \\ 0 & b & c & f & h & g-u & i \\ 1-\varepsilon_3 & c & d & g-u & i & i & j-\varepsilon_5 \end{pmatrix}$$

$$+ \begin{pmatrix} \varepsilon_1 & 0 & 0 & \varepsilon_2 & 0 & 0 & \varepsilon_3 \\ 0 & \varepsilon_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon_3 & 0 & 0 & 0 & 0 \\ \varepsilon_2 & 0 & 0 & \varepsilon_4 & 0 & 0 & u \\ 0 & 0 & 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u & 0 \\ \varepsilon_3 & 0 & 0 & u & 0 & 0 & \varepsilon_5 \end{pmatrix}.$$

Denote the first summand by M and the second by U. We choose $u = g - h$, and ε_i such that M is still positive semidefinite. By the choice of u, we now have that the columns of M satisfy $XY = YX$. Hence, we may proceed with Curto and Yoo's method to find a 6-atomic representing measure for M.

To ease computation, we simplify things, by imposing conditions on U. Specifically, we require that the columns of U satisfy $X^2 = \mathbb{1}$, thus

$$U = \begin{pmatrix} \varepsilon_1 & 0 & 0 & \varepsilon_1 & 0 & 0 & u \\ 0 & \varepsilon_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 & 0 & 0 \\ \varepsilon_1 & 0 & 0 & \varepsilon_1 & 0 & 0 & u \\ 0 & 0 & 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u & 0 \\ u & 0 & 0 & u & 0 & 0 & \varepsilon_2 \end{pmatrix}.$$

The B block of a hypothetical extension has the form

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_1 & 0 & 0 & 0 & u & 0 & u & 0 \\ 0 & u & 0 & u & 0 & 0 & 0 & \varepsilon_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_1 & b_1 & b_2 \\ 0 & 0 & 0 & 0 & b_1 & b_1 & 0 & b_2 \\ 0 & 0 & b_1 & 0 & b_2 & b_2 & b_2 & b_3 \end{pmatrix},$$

for some $b_1, b_2, b_3 \in \mathbb{R}$. A calculation of the C block reveals it to be a very large matrix, which we have moved to Appendix C (see section C.1).

As the extension must have the form of a multivariate Hankel matrix, the C block must satisfy some equations, which we will refer to as the *Hankel System*.

$$\begin{aligned} C_{18} = C_{25} &= C_{47}, \\ C_{28} = C_{48} = C_{55} &= C_{77}, \\ C_{38} = C_{56} &= C_{67}, \\ C_{58} = C_{68} &= C_{78}, \\ C_{12} = C_{13} &= C_{14}, \\ C_{15} = C_{17} = C_{22} &= C_{44}, \\ C_{16} = C_{23} &= C_{34}, \\ C_{24} &= C_{33}, \\ C_{26} = C_{27} = C_{35} = C_{37} = C_{45} &= C_{46}, \\ C_{57} &= C_{66}. \end{aligned} \tag{3.2}$$

Solving the Hankel system (3.2) in *Mathematica* for $\varepsilon_1, \varepsilon_2, b_1, b_2$, and b_3 shows that we must have

$$\begin{aligned}\varepsilon_2 &= \frac{3u^2}{2\varepsilon_1},\\ b_1 &= \frac{u^{3/2}}{\sqrt{2\varepsilon_1}},\\ b_2 &= 0,\\ b_3 &= \frac{u^{5/2}}{2\sqrt{2}\varepsilon_1^{3/2}}.\end{aligned}$$

This now gives a flat extension of U. Hence, there is a 6-atomic representing measure for U, which can be obtained using the method given in [4]. If u and ε_1 satisfy certain conditions, then we have successfully found a 12-atomic representing measure for $\mathcal{M}(2)$ (6 atoms from M and 6 atoms from U).

3.2 The Bounds

We now present sufficient bounds on u that ensure a 12-atomic representing measure for $\mathcal{M}(2)$. Recall that

$$\begin{aligned}\mathcal{M}(2) =& M + U\\ =& \begin{pmatrix} 1-\varepsilon_1 & 0 & 0 & 1-\varepsilon_1 & 0 & 0 & 1-u\\ 0 & 1-\varepsilon_1 & 0 & a & b & b & c\\ 0 & 0 & 1-u & b & c & c & d\\ 1-\varepsilon_1 & a & b & e-\varepsilon_1 & f & f & g-u\\ 0 & b & c & f & g-u & h & i\\ 0 & b & c & f & h & g-u & i\\ 1-u & c & d & g-u & i & i & j-\varepsilon_2 \end{pmatrix}\\ &+ \begin{pmatrix} \varepsilon_1 & 0 & 0 & \varepsilon_1 & 0 & 0 & u\\ 0 & \varepsilon_1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & u & 0 & 0 & 0 & 0\\ \varepsilon_1 & 0 & 0 & \varepsilon_1 & 0 & 0 & u\\ 0 & 0 & 0 & 0 & u & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & u & 0\\ u & 0 & 0 & u & 0 & 0 & \varepsilon_2 \end{pmatrix}.\end{aligned}$$

With the decomposition of $\mathcal{M}(2)$, we require that M is positive semi-definite. So ε_2 must satisfy

$$\varepsilon_2 < j.$$

With regards to the solution for the C block given above, we must therefore have that

$$\frac{3u^2}{2j} < \varepsilon_1.$$

So we choose $\varepsilon_1 = 3u^2/j$, which means $\varepsilon_2 = j/2$. To obtain bounds on u, we use Sylvester's Criterion. The matrix M has the form

$$M = \begin{pmatrix} 1-\frac{3u^2}{j}-\alpha & 0 & 0 & 1-\frac{3u^2}{j}-\alpha & 0 & 0 & 1-u \\ 0 & 1-\frac{3u^2}{j}-\alpha & 0 & a & b & b & c \\ 0 & 0 & 1-u & b & c & c & d \\ 1-\frac{3u^2}{j}-\alpha & a & b & e-\frac{3u^2}{j}-\alpha & f & f & g-u \\ 0 & b & c & f & g-u & h & i \\ 0 & b & c & f & h & g-u & i \\ 1-u & c & d & g-u & i & i & \frac{j}{2} \end{pmatrix}.$$

Let $m_1(u), \ldots, m_7(u)$ be the principal minors of M, viewed as functions of u. It is clear that these functions are continuous. We know that $m_i(0) > 0$ for $1 \le i \le 6$. Let

$$K(\beta) = \min\{\nu \in \mathbb{R} : \nu > 0, \text{ and } m_1(\nu) \cdots m_7(\nu) = 0\}$$

i.e., the minimum of the positive roots of the minors (that arise from linearly independent columns). If

$$u \le K(\beta),$$

then the principal minors of M are non-negative and hence M is positive semi-definite. Therefore, $K(\beta)$ is a sufficient bound on u that ensures our previous construction works. Notice that for this method it isn't necessary to have the submatrix $\mathcal{M}(1)$ be the identity, but having $\mathcal{M}(1)$ as the identity makes the *Mathematica* calculations easier.

3.3 Illustrative Examples

Example 1. Consider the truncated tracial sequence $\beta = (1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 4, 0, 2, 1, 0, 4)$ with associated multivariate Hankel matrix

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 4 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 2 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 & 4 \end{pmatrix}.$$

With this example we have that

$$K(\beta) \approx 0.16,$$

however

$$u = \beta_{x^2y^2} - \beta_{xyxy} = 2 - 1 = 1 > K(\beta).$$

It is shown in [6] that β admits no representing measure. In Chapter 4 we will show that if β admits a representing measure μ, then $\mathcal{M}(n)$ must be recursively generated. Using this result, we can show that $\mathcal{M}$ has no measure. In the columns of $\mathcal{M}$ we have $X = Y$, however $XY \neq X^2$. Thus $\mathcal{M}$ is not recursively generated, and hence there is no representing measure.

Example 2. Consider the multivariate Hankel matrix

$$\mathcal{M}(2) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

With this example we have that

$$K(\beta) = \frac{1}{\sqrt{3}}$$

however we have

$$u = \beta_{x^2y^2} - \beta_{xyxy} = 1 + 1 = 2 > K(\beta).$$

It can be verified that the matrix

$$\mathcal{M}(3) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

is a flat extension of $\mathcal{M}(2)$. The reason for $\mathcal{M}(2)$ having a flat extension is that it is of rank 4, satisfying the relations

$$X^2 = \mathbb{1} = Y^2$$
$$XY + YX = 0$$

and will be discussed in the next chapter.

This example shows that unlike Curto and Yoo's algorithm, our method works only for positive definite multivariate Hankel matrices.

Chapter 4

Non-commutative Variables - Positive Semi-definite Hankel Matrices

In this chapter we study the structure of flat extensions $\mathcal{M}(3)$ of a multivariate Hankel matrix $\mathcal{M}(2)$ of the form

$$\mathcal{M}(3) = \begin{pmatrix} \mathcal{M}(2) & B(3) \\ B(3)^* & C(3) \end{pmatrix}.$$

To do this we analyze the column relations in the matrix $\mathcal{M}(2)$ depending on the rank of the matrix. We first present some preliminary results that will help in our analysis. Throughout this chapter we give several examples to demonstrate the theory.

4.1 Fundamental Results

Lemma 4.1.1. *Suppose μ is a positive representing measure for β. For $p \in \mathbb{R}\langle X, Y\rangle_{\leq n}$ let $\mathcal{Z}(p)$ be the zero set of p. Then*

$$\text{supp}(\mu) \subseteq \mathcal{Z}(p) \Leftrightarrow p(X,Y) = \mathbf{0}.$$

Proof. Suppose that $p(X,Y) = \mathbf{0}$, i.e.,

$$\sum_{|w|\leq n} a_w w(X,Y) = \mathbf{0} \quad \text{for words } w \in \langle X, Y\rangle. \tag{4.1}$$

Since μ is a positive Borel measure, to prove that supp$(\mu) \subseteq \mathcal{Z}(p)$ it suffices to show,

$$\int \text{Tr}(p(A,B)p^*(A,B))\, d\mu = 0.$$

Now,

$$p(A,B)p^*(A,B) = \sum_{|w|,|v|\le n} a_w a_v w(A,B)v^*(A,B).$$

therefore

$$\begin{aligned}\int \mathrm{Tr}(p(A,B)p^*(A,B))\, d\mu &= \sum_{|w|,|v|\le n} a_w a_v \int \mathrm{Tr}(w(A,B)v^*(A,B))\, d\mu \\ &= \sum_{|w|,|v|\le n} a_w a_v \beta_{wv^*}.\end{aligned}$$

For any fixed v with $|v| \le n$, the element in row v of (4.1) is

$$\sum_{|w|\le n} a_w \beta_{wv^*} = 0,$$

whence,

$$\int \mathrm{Tr}(p(A,B)p^*(A,B))\, d\mu = \sum_{|v|\le n} a_v \sum_{|w|\le n} a_w \beta_{wv^*} = 0,$$

which implies

$$\mathrm{supp}(\mu) \subseteq \mathcal{Z}(p).$$

For the converse, suppose that $p \equiv 0$ on $\mathrm{supp}(\mu)$, then for fixed $v \in \langle X, Y\rangle$ with $|v| \le n$, the row v element of $p(X,Y)$ is

$$\sum_{|w|\le n} a_w \beta_{wv} = \int \mathrm{Tr}(v(A,B)p(A,B))\, d\mu = 0.$$

Hence,

$$p(X,Y) = \mathbf{0}.$$

□

Lemma 4.1.2. *Suppose μ is a representing measure for β, and $\mathcal{M}(n) \ge 0$. Let $f \in \mathbb{R}\langle X, Y\rangle_{n-1}$ and suppose that*

$$f(X,Y) = \mathbf{0}.$$

Then,

$$\begin{aligned}(xf)(X,Y) &= \mathbf{0},\\ (fx)(X,Y) &= \mathbf{0},\\ (yf)(X,Y) &= \mathbf{0},\\ (fy)(X,Y) &= \mathbf{0}.\end{aligned}$$

Proof. If $f(X,Y) = \mathbf{0}$, i.e.,

$$\sum_{|w|\leq n-1} a_w w(X,Y) = \mathbf{0},$$

then we know from Lemma 4.1.1 that

$$\text{supp}(\mu) \subseteq \mathcal{Z}(f).$$

This means that for any point $(A,B) \in \text{supp}(\mu) \subseteq (\mathbb{SR}^{t\times t})^2$, $f(A,B) = \mathbf{0}_{t\times t}$. Indeed, we have

$$\int \text{Tr}(f(A,B)f^*(A,B))\, d\mu = \int \text{Tr}(f(A,B)(f(A,B))^T)\, d\mu = 0,$$

and we know that for any matrix N

$$\text{Tr}(N^T N) \geq 0.$$

Since μ is assumed to be a *positive* Borel measure, for any $R \geq 0$,

$$\int R\, d\mu = 0 \Leftrightarrow R = 0,$$

(the continuity required for this is implicit in f being a polynomial and the continuity of the Trace norm). Hence we must have

$$\text{Tr}(f(A,B)(f(A,B))^T) = 0 \Leftrightarrow f(A,B) = \mathbf{0}_{t\times t}.$$

Now for any $v \in \langle X,Y\rangle$ with $|v| \leq n$, the element in row v of $(xf)(X,Y)$ is

$$\begin{aligned}
\sum_{|w|\leq n-1} a_w \beta_{xw(x,y)v^*(x,y)} &= \sum_{|w|\leq n-1} a_w \beta_{(v^*(x,y)x)w(x,y)} \\
&= \sum_{|w|\leq n-1} a_w \int \text{Tr}((v^*(A,B)A)(w(A,B))\, d\mu \\
&= \int \text{Tr}(\sum_{|w|\leq n-1} a_w (v^*(A,B)A)(w(A,B)))\, d\mu \\
&= \int \text{Tr}((v^*(A,B)A)(\sum_{|w|\leq n-1} a_w w(A,B)))\, d\mu \\
&= \int \text{Tr}((v^*(A,B)A)(f(A,B)))\, d\mu \\
&= \int \text{Tr}((v^*(A,B)A)\mathbf{0}_{t\times t})\, d\mu \\
&= \int \text{Tr}(\mathbf{0}_{t\times t})\, d\mu \\
&=0.
\end{aligned}$$

Therefore, since the element of every row of $(xf)(X,Y)$ is zero, we must have,

$$(xf)(X,Y) = \mathbf{0}.$$

Similarly,

$$\begin{aligned}(fx)(X,Y) &= \mathbf{0}\\ (yf)(X,Y) &= \mathbf{0}\\ (fy)(X,Y) &= \mathbf{0}.\end{aligned}$$

□

Theorem 4.1.3 (Recursive Generation)**.** *Let $\mathcal{M}(n)$ be a multivariate Hankel matrix that admits a measure and $f, g, (fg) \in \mathbb{R}\langle X, Y\rangle_{\leq n}$. If*

$$f(X,Y) = \mathbf{0},$$

then

$$(fg)(X,Y) = \mathbf{0}.$$

Proof. Repeated application of Lemma 4.1.2. □

Column relations forced upon $\mathcal{M}(n)$ with an application of Theorem (4.1.3), will henceforth be referred to as *RG relations*.

Proposition 4.1.4. *Given a multivariate Hankel matrix $\mathcal{M}(n)$ and an extension M of $\mathcal{M}(n)$, if M is a flat extension, then M satisfies the symmetry property*

$$\langle p, q\rangle = \langle q^*, p^*\rangle,$$

where $\langle p, q\rangle$ is the inner product associated with M, denoting the element in row q column p, i.e., β_{qp}.

Proof. It suffices to prove the result for monomials of degree $n+1$. Let v and w be words of degree $n+1$, and $v(X,Y)$, $w(X,Y)$ the corresponding columns/rows of M. Since M is a flat extension, it follows that

$$w(X,Y) = \sum_{|u|\leq n} a_u u(X,Y),$$

and

$$v(X,Y) = \sum_{|t|\leq n} b_t t(X,Y).$$

Now by the properties of inner products we have,

$$\begin{aligned}
\langle v, w\rangle &= \sum_{|t|\leq n} b_t \langle t, w\rangle \\
&= \sum_{|t|\leq n} b_t \left\langle t, \sum_{|u|\leq n} a_u u \right\rangle \\
&= \sum_{|t|\leq n} b_t \sum_{|u|\leq n} a_u \langle t, u\rangle \\
&= \sum_{|t|\leq n} b_t \sum_{|u|\leq n} a_u \langle u^*, t^*\rangle \\
&= \sum_{|t|\leq n} b_t \left\langle \sum_{|u|\leq n} a_u u^*, t^* \right\rangle \\
&= \left\langle \sum_{|u|\leq n} a_u u^*, \sum_{|t|\leq n} b_t t^* \right\rangle \\
&= \langle w^*, v^*\rangle .
\end{aligned}$$

□

Lemma 4.1.5. *If $\mathcal{M}(2)$ is a non-commutative Hankel matrix admitting a measure, then the set of columns $\{\mathbb{1}, X, Y, XY\}$ of $\mathcal{M}(2)$ is linearly independent.*

Proof. First we prove that the set $\{\mathbb{1}, X, Y\}$ is linearly independent. Suppose that $a\mathbb{1} + bX + cY = 0$ for some $a, b, c \in \mathbb{R}$. Multiplying by X on the left gives $0 = aX + bX^2 + cXY$. Comparing rows XY,YX it follows that $c = 0$. Similarly on multiplying by Y one concludes that $b = 0$. Finally $0 = a\mathbb{1}$ implies that $a = 0$.

Now we prove that the set $\{\mathbb{1}, X, Y, XY\}$ is linearly independent. Suppose that $XY = a\mathbb{1} + bX + cY$ for some $a, b, c \in \mathbb{R}$. Since the rows XY,YX on the left hand side are different, while same on the right hand side, it is easy to see that this cannot be true for any $a, b, c \in \mathbb{R}$. □

We begin with an easy assessment of the low rank cases.

4.2 Ranks 1-3

Corollary 4.2.1. *If $\mathcal{M}(2)$ is a noncommutative Hankel matrix of rank at most 3, then it does not admit a measure.*

Proof. By Lemma 4.1.5, $\mathcal{M}(2)$ must have a rank at least 4 to admit a measure. □

4.3 Rank 4

We may assume that $\{\mathbb{1}, X, Y, XY\}$ is linearly independent. Therefore, we may assume that $\{\mathbb{1}, X, Y, XY\}$ is a basis for $\mathcal{C}_{\mathcal{M}(2)}$ (the column space of $\mathcal{M}(2)$). We now have, for some constants in $\mathbb{R}$,

$$\begin{aligned} X^2 &= A\mathbb{1} + BX + CY + DXY, \\ YX &= a\mathbb{1} + bX + cY + dXY, \\ Y^2 &= \alpha\mathbb{1} + \beta X + \gamma Y + \delta XY. \end{aligned}$$

Since X^2 (resp. Y^2) has the same entries in rows XY and YX, we must have $D = 0$ (resp. $\delta = 0$) . With similar considerations, it is also easy to show that $d = -1$.

Theorem 4.3.1. *Let $\mathcal{M}(2)$ be the Hankel matrix of rank 4 with linearly independent columns $\{\mathbb{1}, X, Y, XY\}$. The following statements are true:*

(i) *$\mathcal{M}(2)$ has a flat extension if and only if $\mathcal{M}(2)$ has a measure.*

(ii) *$\mathcal{M}(2)$ satisfies the relations*

$$X^2 = A_1\mathbb{1} + B_1X + C_1Y, \tag{4.2}$$
$$YX = A_2\mathbb{1} + B_2X + C_2Y - XY, \tag{4.3}$$
$$Y^2 = A_3\mathbb{1} + B_3X + C_3Y. \tag{4.4}$$

for some $A_1, \ldots, C_3 \in \mathbb{R}$.

(iii) *$\mathcal{M}(2)$ has a flat extension if and only if $C_1 = B_3 = 0$ and after the transformation*

$$X \mapsto \frac{X - \frac{B_1}{2}}{\sqrt{A_1 + \frac{B_1^2}{4}}}, \quad Y \mapsto \frac{Y - \frac{C_3}{2}}{\sqrt{A_3 + \frac{C_3^2}{4}}}, \tag{4.5}$$

the relations become

$$X^2 = \mathbb{1}, \tag{4.6}$$
$$YX + XY = a\mathbb{1}, \quad \textit{for some } a \in (-2, 2), \tag{4.7}$$
$$Y^2 = \mathbb{1}, \tag{4.8}$$

while $\mathcal{M}(2)$ is of the form

$$\mathcal{M}(2) = \begin{pmatrix} \beta_1 & 0 & 0 & \beta_1 & \frac{a}{2}\beta_1 & \frac{a}{2}\beta_1 & \beta_1 \\ 0 & \beta_1 & \frac{a}{2}\beta_1 & 0 & 0 & 0 & 0 \\ 0 & \frac{a}{2}\beta_1 & \beta_1 & 0 & 0 & 0 & 0 \\ \beta_1 & 0 & 0 & \beta_1 & \frac{a}{2}\beta_1 & \frac{a}{2}\beta_1 & \beta_1 \\ \frac{a}{2}\beta_1 & 0 & 0 & \frac{a}{2}\beta_1 & \beta_1 & (\frac{a^2}{2}-1)\beta_1 & \frac{a}{2}\beta_1 \\ \frac{a}{2}\beta_1 & 0 & 0 & \frac{a}{2}\beta_1 & (\frac{a^2}{2}-1)\beta_1 & \beta_1 & \frac{a}{2}\beta_1 \\ \beta_1 & 0 & 0 & \beta_1 & \frac{a}{2}\beta_1 & \frac{a}{2}\beta_1 & \beta_1 \end{pmatrix}. \tag{4.9}$$

(iv) After fixing $\beta_1 > 0$, $\mathcal{M}(2)$ satisfying relations (4.6), (4.7) and (4.8), is unique. The representing measure is 1-atomic, with the atom (up to unitary equivalence) given by the pair

$$A = \sqrt{\frac{\beta_1}{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad B = \sqrt{\frac{\beta_1}{2}} \begin{pmatrix} \frac{a}{2} & -\frac{1}{2}\sqrt{4-a^2} \\ -\frac{1}{2}\sqrt{4-a^2} & -\frac{a}{2} \end{pmatrix}, \tag{4.10}$$

i.e., for all words $w \in \langle X, Y \rangle_{\leq 4}$,

$$\beta_{w(x,y)} = \text{Tr}(w(A, B)).$$

Proof. First we prove (i). If $\mathcal{M}(2)$ admits a measure, then by *Recursive Generation*, $\mathcal{M}(3)$ satisfies RG relations, i.e.,

$$X^3 = X, \quad X^2Y = Y, \quad YX^2 = Y, \quad Y^3 = Y, \quad XY^2 = X, \quad Y^2X = X,$$
$$Y^2X + YXY = aY + bYX + cY^2, \quad YX^2 + XYX = aX + bX^2 + cYX.$$

for some $a, b, c \in \mathbb{R}$. Hence $\mathcal{M}(3)$ is a flat extension of $\mathcal{M}(2)$. Clearly, if $\mathcal{M}(2)$ has a flat extension, then it also has a representing measure. The proof of (ii) follows immediately from Lemma 4.1.5.

Now we prove (iii). Multiplying (4.2) by X on the left gives

$$X^3 = A_1X + B_1X^2 + C_1XY.$$

By symmetry we must have $C_1 = 0$ if $\mathcal{M}(2)$ has a measure. Similarly multiplying (4.4) by Y we get $B_3 = 0$ if $\mathcal{M}(2)$ has a measure. Now after the transformation (4.5) the relations become

$$X^2 = \mathbb{1}, \tag{4.11}$$
$$YX + XY = a\mathbb{1} + bX + cY, \tag{4.12}$$
$$Y^2 = \mathbb{1}. \tag{4.13}$$

Multiplying (4.11) by Y gives $X^2Y = YX^2 = Y$ and, while multiplying (4.13) by X gives $XY^2 = Y^2X = X$. Multiplying (4.12) by Y from left gives

$$X + YXY = Y^2X + YXY = aY + bYX + cY^2$$

which implies $b = 0$ if $\mathcal{M}(2)$ has a measure. Multiplying (4.12) by X from left gives

$$XYX + Y = XYX + X^2Y = aX + cXY$$

which implies $c = 0$ if $\mathcal{M}(2)$ has a measure. It is easy to see that $\mathcal{M}(2)$ is of the form (4.9) and by the positive definiteness of $\mathcal{M}(2)$ restricted to rows and columns $\{\mathbb{1}, X, Y, XY\}$, $a \in (-2, 2)$. The proof of (iv) and the claim about the representing measure (4.10) can be easily checked. □

4.4 Rank 5

For rank 4, the results we have obtained on the tracial moment problem for the existence of a representing measure have been analogous to the classical moment problem. In this section, we shall see a divergence between the tracial and the classical moment problems.

Lemma 4.4.1. *Let $\mathcal{M}(2)$ be the Hankel matrix of rank 5 with linearly independent columns $\{\mathbb{1}, X, Y, XY\}$. We have to consider the following three possibilities:*

1. *The columns $\{\mathbb{1}, X, Y, XY, YX\}$ are linearly independent and $X^2 \in \operatorname{span}\{\mathbb{1}, X, Y, XY\}$, $Y^2 \in \operatorname{span}\{\mathbb{1}, X, Y, YX\}$.*
2. *The columns $\{\mathbb{1}, X, Y, X^2, XY\}$ are linearly independent.*
3. *The columns $\{\mathbb{1}, X, Y, Y^2, YX\}$ are linearly independent.*

4.4.1 Case 1: The columns $\{\mathbb{1}, X, Y, XY, YX\}$ are linearly independent.

Since $X^2 \in \operatorname{span}\{\mathbb{1}, X, Y, XY\}$ it follows that

$$X^2 = A_1\mathbb{1} + B_1X + C_1Y \quad \text{for some } A_1, B_1, C_1 \in \mathbb{R}.$$

Similarly from $Y^2 \in \operatorname{span}\{\mathbb{1}, X, Y, XY\}$ it follows that

$$Y^2 = A_2\mathbb{1} + B_2X + C_2Y \quad \text{for some } A_2, B_2, C_2 \in \mathbb{R}.$$

If $\mathcal{M}(2)$ will satisfy RG relations, we must have $C_1 = 0$ and $B_2 = 0$. But then we have

$$\left(X - \frac{B_1}{2}\right)^2 = \left(A_1 + \frac{B_1^2}{4}\right)\mathbb{1},$$

$$\left(Y - \frac{C_2}{2}\right)^2 = \left(A_2 + \frac{C_1^2}{4}\right)\mathbb{1}.$$

After the transformation

$$X \mapsto \frac{X - \frac{B_1}{2}}{\sqrt{A_1 + \frac{B_1^2}{4}}}, \quad Y \mapsto \frac{Y - \frac{C_2}{2}}{\sqrt{A_2 + \frac{C_1^2}{4}}},$$

we get the relations

$$X^2 = \mathbb{1}, \quad Y^2 = \mathbb{1}.$$

Thus $\mathcal{M}(2)$ is of the form

$$\mathcal{M}(2) = \begin{pmatrix} 1 & \beta_x & \beta_y & 1 & \beta_{xy} & \beta_{xy} & 1 \\ \beta_x & 1 & \beta_{xy} & \beta_x & \beta_y & \beta_y & \beta_x \\ \beta_y & \beta_{xy} & 1 & \beta_y & \beta_x & \beta_x & \beta_y \\ 1 & \beta_x & \beta_y & 1 & \beta_{xy} & \beta_{xy} & 1 \\ \beta_{xy} & \beta_y & \beta_x & \beta_{xy} & 1 & \beta_{xyxy} & \beta_{xy} \\ \beta_{xy} & \beta_y & \beta_x & \beta_{xy} & \beta_{xyxy} & 1 & \beta_{xy} \\ 1 & \beta_x & \beta_y & 1 & \beta_{xy} & \beta_{xy} & 1 \end{pmatrix}.$$

The RG relations, which must hold if $\mathcal{M}(2)$ admits a measure, are

$$X^3 = XY^2 = Y^2X = X, \quad \text{and} \quad X^2Y = YX^2 = Y^3 = Y.$$

Then $B(3)$ is uniquely determined and of the form

$$B(3) = \begin{pmatrix} \beta_x & \beta_y & \beta_y & \beta_x & \beta_y & \beta_x & \beta_x & \beta_y \\ 1 & \beta_{xy} & \beta_{xy} & 1 & \beta_{xy} & \beta_{xyxy} & 1 & \beta_{xy} \\ \beta_{xy} & 1 & \beta_{xyxy} & \beta_{xy} & 1 & \beta_{xy} & \beta_{xy} & 1 \\ \beta_x & \beta_y & \beta_y & \beta_x & \beta_y & \beta_x & \beta_x & \beta_y \\ \beta_y & \beta_x & \beta_x & \beta_y & \beta_x & \beta_y & \beta_y & \beta_x \\ \beta_y & \beta_x & \beta_x & \beta_y & \beta_x & \beta_y & \beta_y & \beta_x \\ \beta_x & \beta_y & \beta_y & \beta_x & \beta_y & \beta_x & \beta_x & \beta_y \end{pmatrix}.$$

We now show that $\text{Ran}(B(3)) \subseteq \text{Ran}(\mathcal{M}(2))$.

Proposition 4.4.2. *In $B(3)$ we have that*

$$\{X^3, X^2Y, XY^2, YX^2, Y^2X, Y^3\} \in \operatorname{Ran}\mathcal{M}(2).$$

Proof. This follows from *Recursive Generation*. □

Since $[\mathcal{M}(2)]_{\{\mathbb{1},X,Y,XY,YX\}}$ is positive definite, it follows that

$$\begin{pmatrix} \beta_y \\ \beta_{xy} \\ \beta_{xyxy} \\ \beta_x \\ \beta_x \end{pmatrix} = A_1\mathbb{1} + B_1X + C_1Y + D_1XY + E_1YX \quad \text{for some } A_1, B_1, C_1, D_1, E_1 \in \mathbb{R}.$$

By symmetry in rows XY, YX it follows that $D_1 = E_1$. Observing the row $\mathbb{1}$ we get

$$\beta_y = A_1\beta_1 + B_1\beta_x + C_1\beta_y + 2D_1\beta_{xy}.$$

By the form of $\mathcal{M}(2)$ we get

$$XYX = A_1\mathbb{1} + B_1X + C_1Y + D_1(XY + YX).$$

By an analogous reasoning we obtain the following.

Proposition 4.4.3. *In* $B(3)$ *we have that*

$$\{XYX, YXY\} \in \operatorname{Ran}\mathcal{M}(2).$$

Hence, $\operatorname{Ran}(B(3)) \subseteq \operatorname{Ran}(\mathcal{M}(2))$.

The candidate for $C \equiv C(3)$ in the flat extension $\mathcal{M}(3)$ of $\mathcal{M}(2)$ is $W^*B(3)$, where $B(3) = \mathcal{M}(2)W$. For $\mathcal{M}(3)$ to be a flat extension, the C block must satisfy the Hankel system

$$\begin{aligned} C_{12} = C_{13} = C_{15}, \\ C_{18} = C_{24} = C_{57}, \\ C_{28} = C_{58} = C_{44} = C_{77}, \\ C_{38} = C_{46} = C_{67}, \\ C_{48} = C_{68} = C_{78}, \\ C_{14} = C_{17} = C_{22} = C_{55}, \\ C_{16} = C_{23} = C_{35}, \\ C_{25} = C_{33}, \\ C_{26} = C_{27} = C_{34} = C_{37} = C_{54} = C_{56}, \\ C_{47} = C_{66}. \end{aligned}$$

Proposition 4.4.4. *We have that*

$$\begin{gathered} C_{12} = C_{13} = C_{15}, \\ C_{18} = C_{24} = C_{57}, \\ C_{28} = C_{58} = C_{44} = C_{77}, \\ C_{38} = C_{46} = C_{67}, \\ C_{48} = C_{68} = C_{78}, \\ C_{14} = C_{17} = C_{22} = C_{55}, \\ C_{16} = C_{23} = C_{35}, \\ C_{26} = C_{27} = C_{34} = C_{37} = C_{54} = C_{56}. \end{gathered}$$

Proof. To prove the first equation, notice that

$$\begin{aligned} C_{12} &= \langle X^2Y, X^3 \rangle \\ &= \langle X^2Y, X \rangle \quad \text{from RG relations} \\ &= \langle XYX, X \rangle \quad \text{from the Hankel structure of } \mathcal{M}(2) \\ &= \langle XYX, X^3 \rangle \\ &= C_{13}, \end{aligned}$$

and

$$\begin{aligned} C_{15} &= \langle YX^2, X^3 \rangle \\ &= \langle X^3, X^2Y \rangle \quad \text{Proposition 4.1.4} \\ &= \langle X^2Y, X^3 \rangle \quad \text{using symmetry of inner product} \\ &= C_{12}. \end{aligned}$$

The remaining equations can be easily checked using similar techniques. □

Theorem 4.4.5. *In Case 1, if $\mathcal{M}(2)$ is positive semi-definite and satisfies the RG relations, then it admits a flat extension and hence a measure, if and only if*

$$C_{25} = C_{33} \quad \textit{and} \quad C_{66} = C_{47}.$$

Theorem 4.4.6. *In Case 1, if $\mathcal{M}(2)$ is positive semi-definite and satisfies the RG relations, then it admits a flat extension and hence a measure, if and only if one of the following is true*

(i) $\beta_y = -\beta_x$ *and* $\beta_{xyxy} = \frac{-\beta_x(4\beta_{xy}+3)-2\beta_{xy}^2+1}{\beta_x-1}$

(ii) $\beta_y = -\beta_x$ *and* $\beta_{xyxy} = -\frac{4\beta_x\beta_{xy}+3\beta_x-2\beta_{xy}^2+1}{\beta_x+1}$

(iii) $\beta_y = \beta_x$ *and* $\beta_{xyxy} = \frac{\beta_x(4\beta_{xy}-3)-2\beta_{xy}^2+1}{\beta_x-1}$

(iv) $\beta_y = \beta_x$ *and* $\beta_{xyxy} = \frac{\beta_x(4\beta_{xy}-3)+2\beta_{xy}^2-1}{\beta_x+1}$

(v) $\beta_x = -1$ *and* $\beta_y = -1$ *and* $\beta_{xy} = 1$

(vi) $\beta_x = -1$ *and* $\beta_y = 1$ *and* $\beta_{xy} = -1$

(vii) $\beta_x = 1$ *and* $\beta_y = -1$ *and* $\beta_{xy} = -1$

(viii) $\beta_x = 1$ *and* $\beta_y = 1$ *and* $\beta_{xy} = 1$

Proof. Using *Mathematica* we compute the C block of the hypothetical flat extension. We use the command

```
Solve[Simplify[Numerator[Simplify[C[[6,6]]-C[[4,7]]]]]]==0
      &&Simplify[Numerator[Simplify[C[[3,3]]-C[[2,5]]]]]]==0]
```

to solve for possible solutions. Using *Simplify* (again), and eliminating the solutions that set $\beta_{xyxy} = 1$ (since $\beta_{x^2y^2} = 1$) proves the claim. □

4.4.2 Case 2: The columns $\{\mathbb{1}, X, Y, X^2, XY\}$ are linearly independent.

The relations are

$$YX = a\mathbb{1} + bX + cY + dX^2 + eXY, \quad a,b,c,d,e \in \mathbb{R}$$
$$Y^2 = \alpha\mathbb{1} + \xi X + \gamma Y + \delta X^2 + \epsilon XY, \quad \alpha,\xi,\gamma,\delta,\epsilon \in \mathbb{R}.$$

By comparing the rows XY, YX we conclude that $e = -1$ and $\epsilon = 0$, i.e.,

$$YX + XY = a\mathbb{1} + bX + cY + dX^2, \quad a,b,c,d \in \mathbb{R}$$
$$Y^2 = \alpha\mathbb{1} + \xi X + \gamma Y + \delta X^2, \quad \alpha,\xi,\gamma,\delta \in \mathbb{R}.$$

We separate three possibilities according to the sign of $\delta \in \mathbb{R}$, i.e., $\delta > 0, \delta = 0, \delta < 0$.

4.4.2.1 Subcase 1: $\delta > 0$.

The relation $Y^2 = \alpha\mathbb{1} + \xi X + \gamma Y + \delta X^2$ can be rewritten as

$$\left(Y - \frac{\gamma}{2}\right)^2 = \delta\left(X - \frac{\xi}{2\delta}\right)^2 + \left(\alpha + \frac{\gamma^2}{4} - \frac{\xi^2}{4\delta}\right)\mathbb{1}.$$

With the transformation

$$X \mapsto \frac{\sqrt{|\delta|}\left(X - \frac{\xi}{2\delta}\right)}{\sqrt{\left|\alpha + \frac{\gamma^2}{4} - \frac{\xi^2}{4\delta}\right|}}, \quad Y \mapsto \frac{Y - \frac{\gamma}{2}}{\sqrt{\left|\alpha + \frac{\gamma^2}{4} - \frac{\xi^2}{4\delta}\right|}},$$

we get

$$Y^2 = X^2 \pm \mathbb{1}.$$

If we have

$$Y^2 = X^2 + \mathbb{1},$$

then the transformation

$$X \mapsto Y, \quad Y \mapsto X$$

gives

$$Y^2 = X^2 - \mathbb{1}.$$

Conversely, if we have

$$Y^2 = X^2 - \mathbb{1},$$

then the same transformation as before gives

$$Y^2 = X^2 + \mathbb{1}.$$

Therefore, it suffices to only study one of these cases. We will suppose that $Y^2 = X^2 + \mathbb{1}$.

Lemma 4.4.7. *If $\mathcal{M}(2)$ satisfies $Y^2 = \mathbb{1} + X^2$ and admits a measure, then in $B(3)$ we have $X^2Y = YX^2$ and $XY^2 = Y^2X$.*

Proof. Multiplying $Y^2 = \mathbb{1} + X^2$ by Y (resp. X) from the left and right gives

$$X^2Y = Y^3 - Y = YX^2, \quad \text{(resp.} \quad Y^2X = X + X^3 = XY^2\text{)},$$

from which the conclusion follows. □

Observe that the other relation is still of the form

$$XY + YX = a\mathbb{1} + bX + cY + dX^2. \tag{4.14}$$

Proposition 4.4.8. *If $\mathcal{M}(2)$ satisfies $Y^2 = \mathbb{1} + X^2$ and admits a measure, then in (4.14) we must have $b = c = 0$.*

Proof. Multiplying $XY + YX = a\mathbb{1} + bX + cY + dX^2$ from left by X (resp. Y) gives

$$X^2Y + XYX = aX + bX^2 + cXY + dX^3, \quad \text{(resp. } YXY + Y^2X = aY + bYX + cY^2 + dYX^2\text{)}$$

By Lemma 4.4.7 and comparing rows XY and YX, it follows that $b = c = 0$. □

So the remaining relations to consider are

$$\begin{aligned} XY + YX &= a\mathbb{1} + dX^2 \\ Y^2 &= \mathbb{1} + X^2. \end{aligned}$$

The relations give us the following system of equations:

$$\begin{aligned}
\beta_{xy} &= a\beta_1 + d\beta_{x^2} - \beta_{xy} \\
\beta_{x^2y} &= a\beta_x + d\beta_{x^3} - \beta_{x^2y} \\
\beta_{xy^2} &= a\beta_y + d\beta_{x^2y} - \beta_{xy^2} \\
\beta_{x^3y} &= a\beta_{x^2} + d\beta_{x^4} - \beta_{x^3y} \\
\beta_{xyxy} &= a\beta_{xy} + d\beta_{x^3y} - \beta_{x^2y^2} \\
\beta_{x^2y^2} &= a\beta_{xy} + d\beta_{x^3y} - \beta_{xyxy} \\
\beta_{xy^3} &= a\beta_{y^2} + d\beta_{x^2y^2} - \beta_{xy^3} \\
\beta_{y^2} &= \beta_1 + \beta_{x^2} \\
\beta_{xy^2} &= \beta_x + \beta_{x^3} \\
\beta_{y^3} &= \beta_y + \beta_{x^2y} \\
\beta_{x^2y^2} &= \beta_{x^2} + \beta_{x^4} \\
\beta_{xy^3} &= \beta_{xy} + \beta_{x^3y} \\
\beta_{xy^3} &= \beta_{xy} + \beta_{x^3y} \\
\beta_{y^4} &= \beta_{y^2} + \beta_{x^2y^2}
\end{aligned} \tag{4.15}$$

Proposition 4.4.9. *There are two possibilities.*

1. *If* $a = 0$, *then*

$$\begin{aligned}
\beta_x &= \frac{1}{4}(-4 + d^2)\beta_{x^3}, & \beta_{xyxy} &= (\frac{d^2}{2} - 1)\beta_{x^4} - \beta_{x^2}, \\
\beta_{xy} &= \frac{d\beta_{x^2}}{2}, & \beta_{xy^3} &= \frac{1}{2}d(\beta_{x^2} + \beta_{x^4}), \\
\beta_{x^2y} &= \frac{d\beta_{x^3}}{2}, & \beta_{y^2} &= \beta_1 + \beta_{x^2}, \\
\beta_{xy^2} &= \frac{d^2\beta_{x^3}}{4}, & \beta_{y^3} &= \beta_y + \frac{d\beta_{x^3}}{2}, \\
& & \beta_{x^2y^2} &= \beta_{x^2} + \beta_{x^4}, \\
\beta_{x^3y} &= \frac{d\beta_{x^4}}{2}, & \beta_{y^4} &= \beta_1 + 2\beta_{x^2} + \beta_{x^4}.
\end{aligned}$$

2. *If* $a \neq 0$, *then*

$$\begin{aligned}
\beta_y &= \frac{(4-ad)\beta_x + (4-d^2)\beta_{x^3}}{2a}\\
\beta_{xy} &= \frac{1}{2}(a\beta_1 + d\beta_{x^2}),\\
\beta_{x^2y} &= \frac{1}{2}(a\beta_x + d\beta_{x^3}),\\
\beta_{xy^2} &= \beta_x + \beta_{x^3},\\
\beta_{x^3y} &= \frac{1}{2}(a\beta_{x^2} + d\beta_{x^4}),\\
\beta_{xyxy} &= \frac{a^2}{2}\beta_1 + (ad-1)\beta_{x^2} + (-1+\frac{d^2}{2})\beta_{x^4},\\
\beta_{xy^3} &= \frac{a}{2}\beta_1 + \frac{a+d}{2}\beta_{x^2} + \frac{d}{2}\beta_{x^4},\\
\beta_{y^2} &= \beta_1 + \beta_{x^2},\\
\beta_{y^3} &= -\frac{(4+a^2-ad)\beta_x + (4+ad-d^2)\beta_{x^3}}{2a},\\
\beta_{x^2y^2} &= \beta_{x^2} + \beta_{x^4},\\
\beta_{y^4} &= \beta_1 + 2\beta_{x^2} + \beta_{x^4}.
\end{aligned}$$

The form of $\mathcal{M}(2)$ *in both cases can be found in Appendix C (see section C.2).*

Proof. Using *Mathematica*. □

RG relations and $B(3)$.

If $\mathcal{M}(2)$ has a measure, then it must satify the following RG relations.

$$XYX + X^2Y = aX + dX^3, \tag{4.16}$$
$$YX^2 + XYX = aX + dX^3, \tag{4.17}$$
$$Y^2X + YXY = aY + dYX^2, \tag{4.18}$$
$$YXY + XY^2 = aY + dX^2Y, \tag{4.19}$$
$$Y^3 = Y + YX^2, \tag{4.20}$$
$$Y^3 = Y + X^2Y, \tag{4.21}$$
$$XY^2 = X + X^3, \tag{4.22}$$
$$Y^2X = X + X^3. \tag{4.23}$$

From (4.16), (4.20) and (4.22) we get the following system.

$$\begin{aligned}
\beta_{x^4y} &= a\beta_{x^3} + d\beta_{x^5} - \beta_{x^4y}, \\
\beta_{x^2yxy} &= a\beta_{x^2y} + d\beta_{x^4y} - \beta_{x^3y^2}, \\
\beta_{x^2yxy} &= a\beta_{x^2y} + d\beta_{x^4y} - \beta_{x^2yxy}, \\
\beta_{y^5} &= \beta_{y^3} - \beta_{x^2y^3}, \\
\beta_{x^3y^2} &= \beta_{x^3} + \beta_{x^5}, \\
\beta_{x^2y^3} &= \beta_{x^2y} + \beta_{x^4y}, \\
\beta_{xy^2xy} &= \beta_{x^2y} + \beta_{x^4y}, \\
\beta_{xy^4} &= \beta_{xy^2} + \beta_{x^3y^2}.
\end{aligned} \tag{4.24}$$

4.4.2.1.1 Subsubcase 1: $a = 0$.

Proposition 4.4.10. *If $a = 0$, then the solution of (4.24) is unique:*

$$\begin{aligned}
\beta_{x^3y^2} = \beta_{x^2yxy} &= \frac{d^2\beta_{x^3}}{d^2 - 4} \\
\beta_{x^2y^3} = \beta_{xyxy^2} &= \frac{d^3\beta_{x^3}}{2d^2 - 8} \\
\beta_{xy^4} &= \frac{d^4\beta_{x^3}}{4d^2 - 16} \\
\beta_{x^4y} &= \frac{2d\beta_{x^3}}{d^2 - 4} \\
\beta_{x^5} &= \frac{4\beta_{x^3}}{d^2 - 4} \\
\beta_{y^5} &= \beta_y + d\frac{\left(d^2 - 2\right)\beta_{x^3}}{d^2 + 4}.
\end{aligned}$$

Now we have to check all the remaining equations obtained from (4.16)-(4.23), i.e.,

$$\begin{aligned}
\beta_{x^2y^3} + \beta_{xy^2xy} &= d\beta_{x^3y^2}, \\
\beta_{x^3y^2} + \beta_{x^2yxy} &= d\beta_{x^4y}, \\
2\beta_{xy^2xy} &= d\beta_{x^2yxy}, \\
2\beta_{xy^4} &= d\beta_{x^2y^3}, \\
\beta_{xy^4} - \beta_{x^2yxy} &= \frac{d^2\beta_{x^3}}{4}.
\end{aligned} \tag{4.25}$$

Proposition 4.4.11. *The system (4.25) is satisfied.*

Flat extensions.

First we have to check that $\operatorname{Ran}(B(3)) \subseteq \operatorname{Ran}(\mathcal{M}(2))$. To show $X^3 \in \operatorname{Ran}(\mathcal{M}(2))$ notice that by positive semi-definiteness of $\mathcal{M}(2)$ and linear independence of $\{\mathbb{1}, X, Y, X^2\}$, it follows that $[\mathcal{M}(2)]_{\{\mathbb{1},X,Y,X^2\}}$ is invertible. Hence, there are scalars $A_1, B_1, C_1, D_1 \in \mathbb{R}$ such that

$$[X^3]_{\{\mathbb{1},X,Y,X^2\}} = A_1[\mathbb{1}]_{\{\mathbb{1},X,Y,X^2\}} + B_{\{\mathbb{1},X,Y,X^2\}}[X]_4 + C_1[Y]_{\{\mathbb{1},X,Y,X^2\}} + D_1[X^2]_{\{\mathbb{1},X,Y,X^2\}}. \tag{4.26}$$

In particular, in the row X^2 we get

$$\frac{4\beta_{x^3}}{d^2-4} = A_1\beta_{x^2} + B_1\beta_{x^3} + C_1\frac{d\beta_{x^3}}{2} + D_1\beta_{x^4}.$$

Multiplying by $\frac{d}{2}$ we get

$$\frac{2d\beta_{x^3}}{d^2-4} = A_1\frac{d\beta_{x^2}}{2} + B_1\frac{d\beta_{x^3}}{2} + C_1\frac{d^2\beta_{x^3}}{4} + D_1\frac{d\beta_{x^4}}{2}.$$

Using (4.15) and (4.26) we calculate

$$\begin{aligned}
&A_1\beta_{y^2} + B_1\beta_{xy^2} + C_1\beta_{y^3} + D_1\beta_{x^2y^2} \\
&= A_1(\beta_1 + \beta_{x^2}) + B_1(\beta_x + \beta_{x^3}) + C_1(\beta_y + \beta_{x^3y}) + D_1(\beta_{x^2} + \beta_{x^4}) \\
&= \beta_{x^3} + \beta_{x^5} \\
&= \frac{d^2}{d^2-4}\beta_{x^3}.
\end{aligned}$$

The forms of $\mathcal{M}(2)$ and $B(3)$ imply

$$[X^3] = A_1[\mathbb{1}] + B_1[X] + C_1[Y] + D_1[X^2].$$

This gives the following.

Proposition 4.4.12. *We have* $\{X^3, XY^2, Y^2X\} \subseteq \operatorname{Ran}(\mathcal{M}(2))$.

Proof. $X^3 \in \operatorname{Ran}(\mathcal{M}(2))$ follows by the above reasoning, while $\{XY^2, Y^2X\} \subseteq \operatorname{Ran}(\mathcal{M}(2))$ follows by (4.22), (4.23). □

Similarly as for X^3 one check that $X^2Y \in \operatorname{Ran}(\mathcal{M}(2))$ and hence $\{XYX, YX^2, YXY, Y^3\} \subseteq \operatorname{Ran}(\mathcal{M}(2))$ follows by (4.16), (4.17), (4.18), (4.19).

Proposition 4.4.13. *We have* $\operatorname{Ran}(B(3)) \subseteq \operatorname{Ran}(\mathcal{M}(2))$.

A necessary condition for the existence of a flat extension of $\mathcal{M}(2)$ is given by the following proposition.

Proposition 4.4.14. *If there exists a flat extension of* $\mathcal{M}(2)$*, then in* $B(3)$ *we must have*

$$X^3 = A\mathbb{1} + BX + DX^2 \quad \textit{for some } A, B, D \in \mathbb{R}. \tag{4.27}$$

Proof. We know that in $B(3)$,

$$X^3 = A\mathbb{1} + BX + CY + DX^2 \quad \text{for some } A, B, C, D \in \mathbb{R}.$$

If there exists a flat extension of $\mathcal{M}(2)$, the same is true in $\begin{pmatrix} B(3) \\ C(3) \end{pmatrix}$, by the uniqueness of A, B, C, D (and $E = 0$ if we allow EXY). But then by *Recursive Generation*,

$$X^4 = AX + BX^2 + CXY + DX^3.$$

Hence $C = 0$. □

Now the candidate for $C \equiv C(3)$ in the flat extension $\mathcal{M}(3)$ is $W^*\mathcal{M}(3)$, where $B(3) = \mathcal{M}(2)W$. In order to have a flat extension C must satisfy the Hankel system

$$\begin{gathered}
C_{18} = C_{24} = C_{57}, \\
C_{28} = C_{58} = C_{44} = C_{77}, \\
C_{38} = C_{46} = C_{67}, \\
C_{48} = C_{68} = C_{78}, \\
C_{12} = C_{13} = C_{15}, \\
C_{14} = C_{17} = C_{22} = C_{55}, \\
C_{16} = C_{23} = C_{35}, \\
C_{25} = C_{33}, \\
C_{26} = C_{27} = C_{34} = C_{37} = C_{54} = C_{56}, \\
C_{47} = C_{66}.
\end{gathered}$$

With a similar analysis as that in *Case 1*, we have the following theorem.

Theorem 4.4.15. $M(2)$ *admits a flat extension if and only if*

$$\begin{gathered}
C_{44} = C_{58}, \\
C_{38} = C_{46}, \\
C_{48} = C_{68}, \\
C_{12} = C_{13}, \\
C_{26} = C_{27}, \\
C_{27} = C_{37}.
\end{gathered}$$

4.4.2.1.2 Subsubcase 2: $a \neq 0$.

Proposition 4.4.16. *If $a \neq 0$, then the solution of the system (4.24) is unique:*

$$\begin{aligned}
\beta_{x^3y^2} = \beta_{x^2yxy} &= \frac{-\beta_x a^2 - 2d\beta_{x^3}a + d^2\beta_{x^3}}{d^2-4}, \\
\beta_{x^2y^3} = \beta_{xyxy^2} &= \frac{\beta_{x^3}d^3 - a^2\beta_x d + a\left(\left(d^2-4\right)\beta_x - \left(d^2+4\right)\beta_{x^3}\right)}{2\left(d^2-4\right)}, \\
\beta_{xy^4} &= \frac{\left(-a^2+d^2-4\right)\beta_x + 2\left(d^2-ad-2\right)\beta_{x^3}}{d^2-4}, \\
\beta_{x^4y} &= \frac{d\beta_x a^2 + \left(d^2+4\right)\beta_{x^3}a - 4d\beta_{x^3}}{2\left(4-d^2\right)}, \\
\beta_{x^5} &= \frac{\beta_x a^2 + 2d\beta_{x^3}a - 4\beta_{x^3}}{4-d^2}, \\
\beta_{y^5} &= \frac{\left(da^3 - 2\left(d^2-4\right)a^2 + d\left(d^2-4\right)a - 4\left(d^2-4\right)\right)\beta_x}{2a\left(4-d^2\right)} \\
&\quad + \frac{\left(\left(d^2+4\right)a^2 - 2d\left(d^2-2\right)a + \left(d^2-4\right)^2\right)\beta_{x^3}}{2a\left(4-d^2\right)}.
\end{aligned}$$

Now we have to check all the remaining equations obtained from (4.16)-(4.23), i.e.,

$$\begin{aligned}
\beta_{x^2y^3} + \beta_{xy^2xy} &= a\beta_{xy^2} + d\beta_{x^3y^2}, \\
\beta_{x^3y^2} + \beta_{x^2yxy} &= a\beta_{x^2y} + d\beta_{x^4y}, \\
2\beta_{xy^2xy} &= a\beta_{xy^2} + d\beta_{x^2yxy}, \\
2\beta_{xy^4} &= a\beta_{y^3} + d\beta_{x^2y^3}, \\
\beta_{xy^4} - \beta_{x^2yxy} &= \beta_{xy^2}.
\end{aligned} \tag{4.28}$$

Proposition 4.4.17. *The system (4.28) is satisfied.*

Proof. Using *Mathematica*. □

Flat extensions.

First we have to check that $\mathrm{Ran}(B(3)) \subseteq \mathrm{Ran}(\mathcal{M}(2))$. To show $X^3 \in \mathrm{Ran}(\mathcal{M}(2))$ first notice that by positive semidefiniteness of $\mathcal{M}(2)$ and linear independence of $\{\mathbb{1}, X, Y, X^2\}$, it follows that $[\mathcal{M}(2)]_{\{\mathbb{1},X,Y,X^2\}}$ is invertible. Hence, there are scalars $A_1, B_1, C_1, D_1 \in \mathbb{R}$ such that

$$[X^3]_{\{\mathbb{1},X,Y,X^2\}} = A_1[\mathbb{1}]_{\{\mathbb{1},X,Y,X^2\}} + B_1[X]_{\{\mathbb{1},X,Y,X^2\}} + C_1[Y]_{\{\mathbb{1},X,Y,X^2\}} + D_1[X^2]_{\{\mathbb{1},X,Y,X^2\}}. \tag{4.29}$$

Using (4.15) and (4.29)

$$\begin{aligned}
&A_1\beta_{xy} + B_1\beta_{x^2y} + C_1\beta_{xy^2} + D_1\beta_{x^3y}\\
&= A_1\frac{1}{2}(a\beta_1 + d\beta_{x^2}) + B_1\frac{1}{2}(a\beta_x + d\beta_{x^3}) + C_1\frac{1}{2}(a\beta_y + d\beta_{x^2y}) + D_1\frac{1}{2}(a\beta_{x^2} + d\beta_{x^4})\\
&= \frac{a}{2}\beta_{x^3} + \frac{d}{2}\beta_{x^5}\\
&= \beta_{x^4y}.
\end{aligned}$$

Using (4.15) and (4.29)

$$\begin{aligned}
&A_1\beta_{y^2} + B_1\beta_{xy^2} + C_1\beta_{y^3} + D_1\beta_{x^2y^2}\\
&= A_1(\beta_1 + \beta_{x^2}) + B_1(\beta_x + \beta_{x^3}) + C_1(\beta_y + \beta_{x^3}y) + D_1(\beta_{x^2} + \beta_{x^4})\\
&= \beta_{x^3} + \beta_{x^5}\\
&= \beta_{x^3y^2}.
\end{aligned}$$

Therefore

$$[X^3] = A_1[\mathbb{1}] + B_1[X] + C_1[Y] + D_1[X^2].$$

Likewise one can check that $X^2Y \in \operatorname{Ran}(\mathcal{M}(2))$

Proposition 4.4.18. *We have* $\operatorname{Ran}(B(3)) \subseteq \operatorname{Ran}(\mathcal{M}(2))$.

Proof. Using (4.16)-(4.23). □

A necessary condition for the existence of a flat extension of $\mathcal{M}(2)$ is given by the following proposition.

Proposition 4.4.19. *If there exists a flat extension of* $\mathcal{M}(2)$*, then in* $B(3)$ *we must have*

$$X^3 = A\mathbb{1} + BX + DX^2 \quad \textit{for some } A, B, D \in \mathbb{R}.$$

Proof. The same proof as for Proposition 4.4.14 applies. □

Now the candidate for $C := C(3)$ in the flat extension $\mathcal{M}(3)$ is $W^*\mathcal{M}(2)$, where $B(3) =$

$\mathcal{M}(2)W$. We follow the analysis as in the subsubcase $a = 0$: The Hankel system is again

$$\begin{gathered} C_{18} = C_{24} = C_{57}, \\ C_{28} = C_{58} = C_{44} = C_{77}, \\ C_{38} = C_{46} = C_{67}, \\ C_{48} = C_{68} = C_{78}, \\ C_{12} = C_{13} = C_{15}, \\ C_{14} = C_{17} = C_{22} = C_{55}, \\ C_{16} = C_{23} = C_{35}, \\ C_{25} = C_{33}, \\ C_{26} = C_{27} = C_{34} = C_{37} = C_{54} = C_{56}, \\ C_{47} = C_{66}. \end{gathered}$$

With a similar analysis as that in *Case 1*, we have the following theorem.

Theorem 4.4.20. *$M(2)$ admits a flat extension if and only if*

$$\begin{gathered} C_{44} = C_{58}, \\ C_{38} = C_{46}, \\ C_{48} = C_{68}, \\ C_{12} = C_{13}, \\ C_{26} = C_{27}, \\ C_{27} = C_{37}. \end{gathered}$$

4.4.2.2 Subcase 2: $\delta = 0$.

The second relation is

$$Y^2 = \alpha\mathbb{1} + \xi X + \gamma Y, \quad \alpha, \xi, \gamma \in \mathbb{R}. \tag{4.30}$$

Lemma 4.4.21. *If $\mathcal{M}(2)$ has a measure, then $\xi = 0$ in (4.30).*

From now on we assume $\xi = 0$ in (4.30), which can be rewritten as

$$(Y - \frac{\gamma}{2})^2 = (\alpha + \frac{\gamma^2}{4})\mathbb{1}, \quad \alpha, \gamma \in \mathbb{R}.$$

After the transformation $Y \mapsto Y - \frac{\gamma}{2}$, $\mathcal{M}(2)$ satisfies the relations

$$YX + XY = \bar{a}\mathbb{1} + \bar{b}X + \bar{c}Y + \bar{d}X^2, \quad \bar{a}, \bar{b}, \bar{c}, \bar{d} \in \mathbb{R} \tag{4.31}$$

$$Y^2 = \bar{\alpha}\mathbb{1}, \quad \bar{\alpha} \in \mathbb{R}. \tag{4.32}$$

Note that since $\mathcal{M}(2)$ has linearly independent columns $\{\mathbb{1}, X, Y, X^2, XY\}$, $\beta_{y^2} > 0$ and hence $\bar{\alpha} > 0$. Therefore (4.32) can be rewrittten as

$$(\frac{1}{\sqrt{\bar{\alpha}}}Y)^2 = \mathbb{1}, \quad \bar{\alpha} \in \mathbb{R}.$$

Hence after the transformation $Y \mapsto \frac{1}{\sqrt{\bar{\alpha}}}Y$ we may assume $\bar{\alpha} = 1$ in (4.32). Now (4.31) can be rewritten as

$$Y(X - \frac{\bar{c}}{2}) + (X - \frac{\bar{c}}{2})Y = \bar{a}\mathbb{1} + \bar{b}X + \bar{d}X^2, \quad \bar{a}, \bar{b}, \bar{c}, \bar{d} \in \mathbb{R}.$$

Thus after the transformation $X \mapsto X - \frac{\bar{c}}{2}$, the relations become

$$YX + XY = \hat{a}\mathbb{1} + \hat{b}X + \hat{d}X^2, \quad \hat{a}, \hat{b}, \hat{d} \in \mathbb{R} \tag{4.33}$$

$$Y^2 = \mathbb{1}. \tag{4.34}$$

We will denote the transformed multivariate Hankel matrix by $\widehat{\mathcal{M}(2)}$.

Lemma 4.4.22. *If $\widehat{\mathcal{M}(2)}$ has a measure, then*

$$Y^2X = XY^2 \quad \textit{and} \quad X^2Y = YX^2$$

and hence the rows XY and YX in $\widehat{B(3)}$ are the same, where $\widehat{\mathcal{M}(3)} = \begin{pmatrix} \widehat{\mathcal{M}(2)} & \widehat{B(3)} \\ \widehat{B(3)}^* & \widehat{C(3)} \end{pmatrix}$.

Proof. From (4.34) we have $Y^2X = XY^2 = X$ and from (4.33) we have

$$YX^2 + XYX = \hat{a}X + \hat{b}X^2 + \hat{d}X^3 = XYX + X^2Y,$$

which implies $X^2Y = YX^2$. The equality of rows XY and YX in $\widehat{B(3)}$ now follows by considering the form of $\widehat{B(3)}$. □

Lemma 4.4.23. *If $\widehat{\mathcal{M}(2)}$ has a measure, then $\hat{b} = 0$ in (4.33).*

Proof. From (4.33) we have

$$YXY + XY^2 - \hat{a}Y - \hat{d}X^2Y = \hat{b}XY.$$

By Lemma 4.4.22, the left-hand side has the same entries in the rows XY, YX. Hence $\hat{b}$ must be 0. □

So we see that we have relations of the form,

$$YX + XY = \hat{a}\mathbb{1} + \hat{d}X^2, \quad \hat{a}, \hat{d} \in \mathbb{R} \tag{4.35}$$

$$Y^2 = \mathbb{1}, \tag{4.36}$$

and we may now assume that the relations we start with are of this form. We now separate cases depending on d in (4.35).

4.4.2.2.1 Subsubcase 1: $d \neq 0$ In this subsubcase, the relations we have are

$$\begin{aligned} YX + XY &= a\mathbb{1} + dX^2, \quad a, d \in \mathbb{R} \\ Y^2 &= \mathbb{1}. \end{aligned}$$

Using the transformation $X \mapsto dX$ and $Y \mapsto Y$ we get the relations

$$\begin{aligned} YX + XY &= \tilde{a}\mathbb{1} + X^2, \quad \tilde{a}, \in \mathbb{R} \\ Y^2 &= \mathbb{1}. \end{aligned}$$

Now we have

$$\begin{aligned} (X - Y)^2 &= Y^2 + X^2 - XY - YX, \\ &= \mathbb{1} + X^2 - X^2 - \tilde{a}\mathbb{1}, \\ &= (1 - \tilde{a})\mathbb{1}. \end{aligned}$$

If $\tilde{a} = 1$, then we have $X = Y$ and we would be in the commutative case. So we may assume that $\tilde{a} \neq 1$. Then the transformation $X \mapsto (X - Y)$ and $Y \mapsto Y$ gives us the relations

$$\begin{aligned} X^2 &= (1 - \tilde{a})\mathbb{1}, \\ Y^2 &= \mathbb{1}. \end{aligned} \tag{4.37}$$

If $\tilde{a} > 1$ then (4.37) implies that $\beta_{x^2} = (1 - \tilde{a})\beta_1 < 0$. However this would contradict positive semi-definiteness of $\mathcal{M}(2)$. So, $\tilde{a} < 1$, and with appropriate scaling we have the relations

$$\begin{aligned} X^2 &= \mathbb{1}, \\ Y^2 &= \mathbb{1}. \end{aligned}$$

This leads us back to *Case 1*.

Special Case: $a = 0$

We present now a special case which highlights the difference between the classical and tracial moment problems. We give exact criteria for flat extensions, and an example where despite the existence of a measure for $\mathcal{M}(2)$, a flat extension of $\mathcal{M}(2)$ does not exist.

Theorem 4.4.24. *If $\mathcal{M}(2)$ is a normalized (i.e., $\beta_1 = 1$) positive semi-definite Hankel matrix of rank 5 satisfying the relations $XY + YX = X^2$ and $Y^2 = \mathbb{1}$, then it admits a flat extension if and only if one of the following is true*

(i)

$$(-8 < \beta_{x^3} < 0) \wedge \left(-\frac{\beta_{x^3}}{2} < \beta_{x^2} < 4\right) \wedge$$

$$((A < \beta_y < B + C \wedge \quad \beta_{x^4} = D) \vee \ (B + C < \beta_y < E \wedge \quad \beta_{x^4} = F))$$

(ii)

$$\left(\beta_{x^3} = 0 \wedge 0 < \beta_{x^2} < 4 \wedge \left(\left(\frac{1}{8}(2\beta_{x^2} - 8) < \beta_y < 0 \wedge \beta_{x^4} = \frac{4\beta_{x^2}^2}{4\beta_y + 4}\right) \vee \right.\right.$$

$$\left.\left.\left(0 < \beta_y < \frac{1}{8}(8 - 2\beta_{x^2}) \wedge \beta_{x^4} = -\frac{4\beta_{x^2}^2}{4\beta_y - 4}\right)\right)\right)$$

(iii)

$$(0 < \beta_{x^3} < 8) \wedge \left(\frac{\beta_{x^3}}{2} < \beta_{x^2} < 4\right) \wedge$$

$$((A < \beta_y < B - C \wedge \quad \beta_{x^4} = D) \vee \ (B - C < \beta_y < E \wedge \quad \beta_{x^4} = F))$$

where

$$A = \frac{1}{8}(2\beta_{x^2} + \beta_{x^3} - 8)$$

$$B = \frac{4\beta_{x^2}^2 + \beta_{x^3}^2}{16\beta_{x^3}}$$

$$C = \frac{1}{16}\sqrt{\frac{16\beta_{x^2}^4 + 8\beta_{x^2}^2\beta_{x^3}^2 - 128\beta_{x^2}\beta_{x^3}^2 + \beta_{x^3}^4 + 256\beta_{x^3}^2}{\beta_{x^3}^2}}$$

$$D = \frac{-8\beta_y\beta_{x^3} + 4\beta_{x^2}^2 + 4\beta_{x^2}\beta_{x^3} + \beta_{x^3}^2 - 8\beta_{x^3}}{4\beta_y + 4}$$

$$E = \frac{1}{8}(-2\beta_{x^2} + \beta_{x^3} + 8)$$

$$F = \frac{8\beta_y\beta_{x^3} - 4\beta_{x^2}^2 + 4\beta_{x^2}\beta_{x^3} - \beta_{x^3}^2 - 8\beta_{x^3}}{4\beta_y - 4}$$

Proof. Using *Reduce* in *Mathematica*. □

Example 3. $\beta_{x^3} = \beta_y = 0$.

$\mathcal{M}(2)$ is of the form

$$\mathcal{M}(2) = \begin{pmatrix} \beta_1 & 0 & 0 & \beta_{x^2} & \frac{\beta_{x^2}}{2} & \frac{\beta_{x^2}}{2} & \beta_1 \\ 0 & \beta_{x^2} & \frac{\beta_{x^2}}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{\beta_{x^2}}{2} & \beta_1 & 0 & 0 & 0 & 0 \\ \beta_{x^2} & 0 & 0 & \beta_{x^4} & \frac{\beta_{x^4}}{2} & \frac{\beta_{x^4}}{2} & \beta_{x^2} \\ \frac{\beta_{x^2}}{2} & 0 & 0 & \frac{\beta_{x^4}}{2} & \beta_{x^2} & -\beta_{x^2} + \frac{\beta_{x^4}}{2} & \frac{\beta_{x^2}}{2} \\ \frac{\beta_{x^2}}{2} & 0 & 0 & \frac{\beta_{x^4}}{2} & -\beta_{x^2} + \frac{\beta_{x^4}}{2} & \beta_{x^2} & \frac{\beta_{x^2}}{2} \\ \beta_1 & 0 & \beta_y & \beta_{x^2} & \frac{\beta_{x^2}}{2} & \frac{\beta_{x^2}}{2} & \beta_1 \end{pmatrix}.$$

Proposition 4.4.25. *If $\mathcal{M}(2)$ is normalized (i.e., $\beta_1 = 1$) and $\beta_{x^3} = \beta_y = 0$, then $[\mathcal{M}(2)]_{\{\mathbb{1},X,Y,X^2,XY\}}$ is positive definite if and only if*

$$0 < \beta_{x^2} < 4,$$
$$\beta_{x^2}^2 < \beta_{x^4} < 4\beta_{x^2}.$$

Proof. Checked using *Reduce* in *Mathematica*. □

Proposition 4.4.26. *If $\mathcal{M}(2)$ is in a normalized form (i.e., $\beta_1 = 1$) and $\beta_{x^3} = \beta_y = 0$, then $\mathcal{M}(2)$ never admits a flat extension.*

Proof. Follows by Theorem 4.4.24. □

Theorem 4.4.27. *If $\mathcal{M}(2)$ is in a normalized form (i.e., $\beta_1 = 1$), and $\beta_{x^3} = \beta_y = 0$, then $\mathcal{M}(2)$ admits a measure. This measure is either:*

1. *2-atomic nc measure with atoms $(X_1, Y_1), (X_2, Y_2) \in (\mathbb{SR}^{2\times 2})^2$,*
2. *3-atomic measure with atoms $(X_1, Y_1) \in (\mathbb{SR}^{2\times 2})^2, (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$,*
3. *4-atomic measure with atoms $(X_1, Y_1) \in (\mathbb{SR}^{2\times 2})^2, (x_2, y_2), (x_3, y_3), (x_4, y_4) \in \mathbb{R}^2$,*
4. *5-atomic measure with atoms $(X_1, Y_1) \in (\mathbb{SR}^{2\times 2})^2, (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5) \in \mathbb{R}^2$,*

Proof. Normalized (i.e., $\beta_1 = 1$) positive semi-definite Hankel matrices of rank 4 satisfying the relations

$$X^2 = \mathbb{1},\ XY + YX = X^2,\ Y^2 = \mathbb{1}$$

have a 1-atomic nc measure with atoms $(X, Y) \in (\mathbb{SR}^{2\times 2})^2$ by Theorem 4.3.1 and are of the form

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 1 & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 1 & \frac{1}{2} \\ 1 & 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 1 \end{pmatrix}.$$

Defining $B(\alpha) = \mathcal{M}(2) - \alpha A$ it is equal to

$$\begin{pmatrix} 1-\alpha & 0 & 0 & \beta_{x^2}-\alpha & \frac{1}{2}(\beta_{x^2}-\alpha) & \frac{1}{2}(\beta_{x^2}-\alpha) & 1-\alpha \\ 0 & \beta_{x^2}-\alpha & \frac{1}{2}(\beta_{x^2}-\alpha) & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(\beta_{x^2}-\alpha) & 1-\alpha & 0 & 0 & 0 & 0 \\ \beta_{x^2}-\alpha & 0 & 0 & \beta_{x^4}-\alpha & \frac{1}{2}(\beta_{x^4}-\alpha) & \frac{1}{2}(\beta_{x^4}-\alpha) & \beta_{x^2}-\alpha \\ \frac{1}{2}(\beta_{x^2}-\alpha) & 0 & 0 & \frac{1}{2}(\beta_{x^4}-\alpha) & \beta_{x^2}-\alpha & \frac{1}{2}\alpha-\beta_{x^2}+\frac{\beta_{x4}}{2} & \frac{1}{2}(\beta_{x^2}-\alpha) \\ \frac{1}{2}(\beta_{x^2}-\alpha) & 0 & 0 & \frac{1}{2}(\beta_{x^4}-\alpha) & \frac{1}{2}\alpha-\beta_{x^2}+\frac{\beta_{x4}}{2} & \beta_{x^2}-\alpha & \frac{1}{2}(\beta_{x^2}-\alpha) \\ 1-\alpha & 0 & 0 & \beta_{x^2}-\alpha & \frac{1}{2}(\beta_{x^2}-\alpha) & \frac{1}{2}(\beta_{x^2}-\alpha) & 1-\alpha \end{pmatrix}.$$

We will prove that for the smallest $\alpha_0 > 0$ such that $\text{rank}(B(\alpha_0)) < 5$ (and hence $B(\alpha_0) \geq 0$), the matrix $B(\alpha_0)$ admits a measure. If α_0 is such that the first three columns are linearly independent, then either $B(\alpha_0)$ is a commutative Hankel matrix satisfying RG relations or $B(\alpha_0)$ is a non-commutative Hankel matrix satisfying the relations $X^2 = t\mathbb{1}$, $t \in \mathbb{R}$ (this follows from the form of $B(\alpha_0)$), $XY + YX = X^2$, $Y^2 = \mathbb{1}$. In both cases it admits a measure. It only remains to study the case when $\det([B(\alpha_0)]_{\{1,2,3\}}) = 0$. Solving $\det([B(\alpha)]_{\{1,2,3\}}) = 0$ one gets

$$\alpha_{0,1} = 1, \quad \alpha_{0,2} = \beta_{x^2} \quad \text{or} \quad \alpha_{0,3} = \frac{4-\beta_{x^2}}{3}.$$

It is easy to see that we never have $\alpha_{0,1} < \min(\alpha_{0,2}, \alpha_{0,3})$.

Case 1: $\beta_{x^2} = \min(\alpha_{0,1}, \alpha_{0,2}, \alpha_{0,3}) \Rightarrow \beta_{x^2} \in (0, 1]$.

From the form of $B(\beta_{x^2})$ we see that the entry $(5, 5)$ of $B(\beta_{x^2})$ is 0. Since $B(\beta_{x^2}) \geq 0$ it follows that the entry $(5, 4)$ of $B(\beta_{x^2})$, i.e., $\frac{1}{2}(\beta_{x^4} - \beta_{x^2})$, must be 0 and hence $\beta_{x^2} = \beta_{x^4}$. Hence $B(\beta_{x^2})$ is of the form

$$\begin{pmatrix} 1-\beta_{x^2} & 0 & 0 & 0 & 0 & 0 & 1-\beta_{x^2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\beta_{x^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1-\beta_{x^2} & 0 & 0 & 0 & 0 & 0 & 1-\beta_{x^2} \end{pmatrix}.$$

So $B(\beta_{x^2})$ is a commutative positive semi-definite Hankel matrix of rank 2 and satisfies RG relations, it has a a 2-atomic commutative measure.

Case 2: $\frac{4-\beta_{x^2}}{3} = \min(\alpha_{0,1}, \alpha_{0,2}, \alpha_{0,3}) \Rightarrow \beta_{x^2} \in (1,4)$ ($\beta_{x^2} = 1$ is covered by Case 1).

The upper left corner 5×5 submatrix of $B(\frac{4-\beta_{x^2}}{3})$ is

$$S = \begin{pmatrix} \frac{\beta_{x^2}-1}{3} & 0 & 0 & \frac{4(\beta_{x^2}-1)}{3} & \frac{2(\beta_{x^2}-1)}{3} \\ 0 & \frac{4(\beta_{x^2}-1)}{3} & \frac{2(\beta_{x^2}-1)}{3} & 0 & 0 \\ 0 & \frac{2(\beta_{x^2}-1)}{3} & \frac{\beta_{x^2}-1}{3} & 0 & 0 \\ \frac{4(\beta_{x^2}-1)}{3} & 0 & 0 & \frac{1}{3}(-4+\beta_{x^2})+\beta_{x^4} & \frac{1}{6}(-4+\beta_{x^2}+3\beta_{x^4}) \\ \frac{2(\beta_{x^2}-1)}{3} & 0 & 0 & \frac{1}{6}(-4+\beta_{x^2}+3\beta_{x^4}) & \frac{4}{3}(-1+\beta_{x^2}) \end{pmatrix}.$$

Using *Reduce* in *Mathematica* for S being a positive semi-definite Hankel matrix under the condition that $\mathcal{M}(2)$ is a positive semi-definite Hankel matrix gives that we must have $\beta_{x^4} = -4 + 5\beta_{x^2}$. But then $B(-4 + 5\beta_{x^2})$ is a commutative Hankel matrix satifying the relations

$$Y^2 = \mathbb{1}, \quad XY = YX = 2\mathbb{1}, \quad X^2 = 4\mathbb{1}, \quad Y = \frac{1}{2}X.$$

Hence $B(-4+5\beta_{x^2})$ is a commutative positive semi-definite Hankel matrix of rank 2 satisfying RG relations and therefore admits a 2-atomic commutative measure. □

Returning to our analysis, we now consider the subsubcase when $d = 0$.

4.4.2.2.2 Subsubcase 2: $d = 0$ In this subsubcase, the relations we have are

$$XY + YX = a\mathbb{1},$$
$$Y^2 = \mathbb{1}.$$

If $a \neq 0$ then the transformation $X \mapsto \frac{1}{a}X$ and $Y \mapsto Y$ gives

$$XY + YX = \mathbb{1},$$
$$Y^2 = \mathbb{1}.$$

On the other hand, if $a = 0$, then the transformation $X \mapsto (X + \frac{1}{2}Y)$ and $Y \mapsto Y$ gives

$$XY + YX = \mathbb{1},$$
$$Y^2 = \mathbb{1}.$$

So we may assume $a = 1$ and that the relations we work with are

$$XY + YX = \mathbb{1}, \tag{4.38}$$
$$Y^2 = \mathbb{1}. \tag{4.39}$$

We will attempt to construct a flat extension $\mathcal{M}(3)$ where

$$\mathcal{M}(3) = \begin{pmatrix} \mathcal{M}(2) & B(3) \\ B(3)^* & C(3) \end{pmatrix}.$$

(4.38) and (4.39) give us the following system in $\mathcal{M}(2)$

$$\begin{aligned} 2\beta_{xy} &= \beta_1, \\ 2\beta_{x^2y} &= \beta_x, \\ 2\beta_{xy^2} &= \beta_y, \\ 2\beta_{x^3y} &= \beta_{x^2}, \\ \beta_{x^2y^2} + \beta_{xyxy} &= \beta_{xy}, \\ 2\beta_{xy^3} &= \beta_{y^2}, \\ \beta_{y^2} &= \beta_1, \\ \beta_{xy^2} &= \beta_x, \\ \beta_{y^3} &= \beta_y, \\ \beta_{x^2y^2} &= \beta_{x^2}, \\ \beta_{xy^3} &= \beta_{xy}, \\ \beta_{y^4} &= \beta_{y^2}. \end{aligned} \tag{4.40}$$

Proposition 4.4.28. *The solution to (4.40) is given by*

$$\begin{aligned} \beta_y = \beta_{y^3} &= 2\beta_x, \\ \beta_{xy^3} = \beta_{xy} &= \frac{1}{2}\beta_1, \\ \beta_{y^2} = \beta_{y^4} &= \beta_1, \\ \beta_{x^2y} &= \frac{1}{2}\beta_x, \\ \beta_{xy^2} &= \beta_x, \\ \beta_{x^2y^2} &= \beta_{x^2}, \\ \beta_{x^3y} &= \frac{1}{2}\beta_{x^2}, \\ \beta_{xyxy} &= \frac{1}{2}\beta_1 - \beta_{x^2}. \end{aligned} \tag{4.41}$$

Proof. Using *Mathematica*. □

Recursive generation implies that in the B block we have

$$\begin{aligned} X^2Y + XYX &= X, \\ XYX + YX^2 &= X, \\ YXY + Y^2X &= Y, \\ XY^2 + YXY &= Y, \\ XY^2 &= X, \\ Y^2X &= X, \\ Y^3 &= Y. \end{aligned} \tag{4.42}$$

Lemma 4.4.29. *If $\mathcal{M}(2)$ has a flat extension, then*

$$X^2Y = YX^2 \quad \text{and} \quad Y^2X = XY^2.$$

Hence the rows XY and YX are the same in $B(3)$.

Proof. From the (4.42) we have $Y^2X = XY^2$, and

$$\begin{aligned} X^2Y - YX^2 &= X^2Y + XYX - XYX - YX^2, \\ &= X - X, \\ &= 0. \end{aligned}$$

□

The RG relations (4.42) give the following system to be solved in $B(3)$.

$$\begin{aligned} 2\beta_{x^4y} &= \beta_{x^3}, \\ \beta_{x^3y^2} + \beta_{x^2yxy} &= \beta_{x^2y}, \\ 2\beta_{x^2yxy} &= \beta_{x^2y}, \\ \beta_{x^2y^3} + \beta_{xyxy^2} &= \beta_{xy^2}, \\ 2\beta_{xyxy^2} &= \beta_{xy^2}, \\ 2\beta_{xy^4} &= \beta_{y^3}, \\ \beta_{x^3y^2} &= \beta_{x^3}, \\ \beta_{xyxy^2} &= \beta_{x^2y}, \\ \beta_{x^2y^3} &= \beta_{x^2y}, \\ \beta_{xy^4} &= \beta_{xy^2}, \\ \beta_{y^5} &= \beta_{y^3}. \end{aligned} \tag{4.43}$$

Proposition 4.4.30. *The RG system (4.43) has the solution,*

$$\begin{aligned} \beta_{xy^2} = \beta_{xy^4} &= 2\beta_{x^2y}, \\ \beta_{y^3} = \beta_{y^5} &= 4\beta_{x^2y}, \\ \beta_{x^3} = \beta_{x^3y^2} = \beta_{x^2yxy} &= \frac{1}{2}\beta_{x^2y}, \\ \beta_{x^2y^3} = \beta_{xyxy^2} &= \beta_{x^2y}, \\ \beta_{x^4y} &= \frac{1}{4}\beta_{x^2y}, \end{aligned} \tag{4.44}$$

and is consistent with (4.41).

Proof. Using *Mathematica*. □

Next, we want to show that $\mathrm{Ran}(B(3)) \subseteq \mathrm{Ran}(\mathcal{M}(2))$. To do this we need only show that $X^3, X^2Y \in \mathrm{Ran}(\mathcal{M}(2))$. Since $\mathcal{M}(2)$ is positive semi-definite and $\{\mathbb{1}, X, Y, X^2, XY\}$ is linearly independent, it follows that there exists scalars $A_1, A_2, \ldots, E_1, E_2$ such that

$$\begin{aligned} [X^3]_{\{\mathbb{1},X,Y,X^2,XY\}} &= A_1[\mathbb{1}]_{\{\mathbb{1},X,Y,X^2,XY\}} + \cdots + E_1[XY]_{\{\mathbb{1},X,Y,X^2,XY\}}, \\ [X^2Y]_{\{\mathbb{1},X,Y,X^2,XY\}} &= A_2[\mathbb{1}]_{\{\mathbb{1},X,Y,X^2,XY\}} + \cdots + E_2[XY]_{\{\mathbb{1},X,Y,X^2,XY\}}. \end{aligned}$$

Since the rows XY and YX are the same, and the rows Y^2 and 1 are the same in $B(3)$, it easily follows that $X^3, X^2Y \in \mathrm{Ran}(\mathcal{M}(2))$. We have now proven the following.

Lemma 4.4.31. $\mathrm{Ran}(B(3)) \subseteq \mathrm{Ran}(\mathcal{M}(2))$.

From this we know there exists a matrix W, such that $B(3) = \mathcal{M}(2)W$. We define $C \equiv C(3) := W^*\mathcal{M}(2)W$. For the extension $\mathcal{M}(3)$ to be a flat extension, we must have C satisfy the Hankel system

$$\begin{gathered} C_{18} = C_{24} = C_{57}, \\ C_{28} = C_{58} = C_{44} = C_{77}, \\ C_{38} = C_{46} = C_{67}, \\ C_{48} = C_{68} = C_{78}, \\ C_{12} = C_{13} = C_{15}, \\ C_{14} = C_{17} = C_{22} = C_{55}, \\ C_{16} = C_{23} = C_{35}, \\ C_{25} = C_{33}, \\ C_{26} = C_{27} = C_{34} = C_{37} = C_{54} = C_{56}, \\ C_{47} = C_{66}. \end{gathered}$$

Proposition 4.4.32. *We have*

$$\begin{aligned}
C_{18} &= C_{24} = C_{57},\\
C_{28} &= C_{58} = C_{44} = C_{77},\\
C_{38} &= C_{46} = C_{67},\\
C_{48} &= C_{68} = C_{78},\\
C_{12} &= C_{13} = C_{15},\\
C_{14} &= C_{17},\\
C_{22} &= C_{55},\\
C_{23} &= C_{35},\\
C_{25} &= C_{33},\\
C_{26} &= C_{27} = C_{34} = C_{37} = C_{54} = C_{56},\\
C_{47} &= C_{66},\\
C_{16} - C_{23} &= C_{22} - C_{17}.
\end{aligned}$$

Proof. To prove the first equation notice,

$$\begin{aligned}
C_{18} = \langle Y^3, X^3\rangle = \langle Y, X^3\rangle &\quad \text{from (4.42)},\\
= \langle X, X^2Y\rangle &\quad \text{from } \mathcal{M}(2),\\
= \langle XY^2, X^2Y\rangle &\quad \text{from (4.42)},\\
= C_{24},
\end{aligned}$$

and

$$\begin{aligned}
C_{24} &= \langle XY^2, X^2Y\rangle,\\
&= \langle YX^2, Y^2X\rangle \quad \text{using Proposition (4.1.4)},\\
&= C_{57}.
\end{aligned}$$

The rest can be checked similarly. □

This gives us a criteria for when the extension $\mathcal{M}(3)$ will be a flat extension.

Theorem 4.4.33. *If a positive semi-definite, multivariate Hankel matrix $\mathcal{M}(2)$ satisfies the relations*

$$\begin{aligned}
XY + YX &= \mathbb{1},\\
Y^2 &= \mathbb{1},
\end{aligned}$$

then, $\mathcal{M}(2)$ has a flat extension if and only if

$$C_{16} = C_{23}.$$

4.4.2.3 Special Case: $a = 0, \quad \beta_{x^3} = 0$

We give precise conditions for when a multivariate Hankel matrix satisfying the relations $XY + YX = 0$ and $Y^2 = \mathbb{1}$ (equivalent to our case), and having $\beta_{x^3} = 0$ has a flat extension.

Theorem 4.4.34. *If $\mathcal{M}(2)$ is a normalized (i.e., $\beta_1 = 1$) positive semi-definite Hankel matrix of rank 5 with $\beta_{x^3} = 0$ satisfying the relations $XY + YX = 0$ and $Y^2 = \mathbb{1}$, then it admits a flat extension if and only if*

$$\beta_{x^2} > 0 \quad \wedge \left(\left(-1 < \beta_y < 0 \quad \wedge \quad \beta_{x^4} = \frac{\beta_{x^2}^2}{1 + \beta_y} \right) \quad \vee \right.$$
$$\left. \left(0 < \beta_y < 1 \quad \wedge \quad \beta_{x^4} = -\frac{\beta_{x^2}^2}{-1 + \beta_y} \right) \right)$$

Example 4.

Theorem 4.4.35. *If $\beta_y = \beta_{x^3} = 0$, then $\mathcal{M}(2)$ does not admit a flat extension but admits a 2-atomic non-commutative measure with atoms $(X_1, Y_1), (X_2, Y_2) \in (\mathbb{SR}^{2\times 2})^2$.*

Proof. $\mathcal{M}(2)$ does not admit a flat extension by Theorem 4.4.34. $\mathcal{M}(2)$ is of the form

$$\mathcal{M}(2) = \begin{pmatrix} \beta_1 & 0 & 0 & \beta_{x^2} & 0 & 0 & \beta_1 \\ 0 & \beta_{x^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1 & 0 & 0 & 0 & 0 \\ \beta_{x^2} & 0 & 0 & \beta_{x^4} & 0 & 0 & \beta_{x^2} \\ 0 & 0 & 0 & 0 & \beta_{x^2} & -\beta_{x^2} & 0 \\ 0 & 0 & 0 & 0 & -\beta_{x^2} & \beta_{x^2} & 0 \\ \beta_1 & 0 & 0 & \beta_{x^2} & 0 & 0 & \beta_1 \end{pmatrix}.$$

Normalized (i.e., $\beta_1 = 1$) positive semi-definite Hankel matrices of rank 4 satisfying the relations

$$X^2 = t\mathbb{1},\ XY + YX = 0,\ Y^2 = \mathbb{1} \quad \text{for } t > 0$$

are of the form

$$A(t) = \begin{pmatrix} 1 & 0 & 0 & t & 0 & 0 & 1 \\ 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ t & 0 & 0 & t^2 & 0 & 0 & t \\ 0 & 0 & 0 & 0 & t & -t & 0 \\ 0 & 0 & 0 & 0 & -t & t & 0 \\ 1 & 0 & 0 & t & 0 & 0 & 1 \end{pmatrix}.$$

and have a 1-atomic non-commutative measure with an atom $(X, Y) \in (\mathbb{SR}^{2\times 2})^2$ by Theorem 4.3.1 (and a translation $X \mapsto \frac{1}{\sqrt{t}}X$).

Defining $B(\alpha,t)=\mathcal{M}(2)-\alpha A(t)$ we have

$$B(\alpha,t)=\begin{pmatrix} \beta_1-\alpha & 0 & 0 & \beta_{x^2}-t\alpha & 0 & 0 & \beta_1-\alpha \\ 0 & \beta_{x^2}-t\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \beta_1-\alpha & 0 & 0 & 0 & 0 \\ \beta_{x^2}-t\alpha & 0 & 0 & \beta_{x^4}-t^2\alpha & 0 & 0 & \beta_{x^2}-t\alpha \\ 0 & 0 & 0 & 0 & \beta_{x^2}-t\alpha & -\beta_{x^2}+t\alpha & 0 \\ 0 & 0 & 0 & 0 & -\beta_{x^2}+t\alpha & \beta_{x^2}-t\alpha & 0 \\ \beta_1-\alpha & 0 & 0 & \beta_{x^2}-t\alpha & 0 & 0 & \beta_1-\alpha \end{pmatrix}.$$

The determinant of the submatrix

$$B(\alpha,t)_{\{1,4\}}=\begin{pmatrix} \beta_1-\alpha & \beta_{x^2}-t\alpha \\ \beta_{x^2}-t\alpha & \beta_{x^4}-t^2\alpha \end{pmatrix}$$

is 0 if and only if

$$(\beta_1-\alpha)(\beta_{x^4}-t^2\alpha)-(\beta_{x^2}-t\alpha)^2=0 \Leftrightarrow \alpha=\frac{\beta_1\beta_{x^4}-\beta_{x^2}^2}{\beta_1 t^2-2t\beta_{x^2}+\beta_{x^4}}.$$

Let us define the function

$$f:(0,\infty)\to\mathbb{R}, \quad f(t):=\frac{\beta_1\beta_{x^4}-\beta_{x^2}^2}{\beta_1 t^2-2t\beta_{x^2}+\beta_{x^4}}.$$

Since

$$\det(\mathcal{M}(2)_{\{1,4\}})=\beta_1\beta_{x^4}-\beta_{x^2}^2>0,$$

and

$$\beta_1 t^2-2t\beta_{x^2}+\beta_{x^4}\to\infty \quad \text{as } t\to\infty,$$

it follows that

$$f(t)\searrow 0 \quad \text{and} \quad tf(t)\searrow 0 \quad t\to\infty.$$

Hence also

$$(B(f(t),t))_{\{1,2,3,5\}}\to\mathcal{M}(2)_{\{1,2,3,5\}}>0 \quad t\to\infty.$$

Thus for $t>0$ sufficiently large

$$B(f(t),t)$$

becomes a positive semi-definite Hankel matrix of rank 4 satisfying the relations

$$X^2=A\mathbb{1},\ XY+YX=0,\ Y^2=\mathbb{1} \quad \text{where } A>0.$$

By Theorem 4.3.1, it has a 1-atomic measure with atoms $(X,Y)\in(\mathbb{SR}^{2\times 2})^2$. Therefore

$$\mathcal{M}(2)=f(t)A(t)+B(f(t),t)$$

has a 2-atomic measure with atoms $(X,Y)\in(\mathbb{SR}^{2\times 2})^2$. $\square$

4.4.2.4 Subcase 3: $\delta < 0$

This is similar to *Subcase 2*.

4.4.3 Case 3: The columns $\{\mathbb{1}, X, Y, Y^2, YX\}$ are linearly independent.

This case is analogous to *Case 1* by symmetry.

4.5 Rank 6

Lemma 4.5.1. *Let $\mathcal{M}(2)$ be the Hankel matrix of rank 6 with linearly independent columns $\{\mathbb{1}, X, Y, XY\}$. We have to consider the following two possibilities:*

1. *The columns $\{\mathbb{1}, X, Y, X^2, XY, YX\}$ are linearly independent.*
2. *The columns $\{\mathbb{1}, X, Y, X^2, XY, Y^2\}$ are linearly independent.*

Proof. By symmetry this covers all different options. □

Lemma 4.5.2. *Let $\mathcal{M}(2)$ be the Hankel matrix of rank 6 with linearly independent columns $\{\mathbb{1}, X, Y, XY\}$. We have to consider the following three possibilities for the only relation:*

1. $Y^2 = \mathbb{1}$,
2. $Y^2 = \mathbb{1} - X^2$,
3. $Y^2 = \mathbb{1} + X^2$.

Proof. Let us say we are in case 1 of Lemma 4.5.1. Then

$$Y^2 = a\mathbb{1} + bX + cY + dX^2 + eXY + fYX.$$

By symmetry it follows $e = f$. We rewrite this as

$$(Y - eX)^2 = a\mathbb{1} + bX + cY + (d + e^2)X^2.$$

We do a substitution $Y \mapsto Y - eX$ and get the relation of the form

$$Y^2 = \tilde{a}\mathbb{1} + \tilde{b}X + \tilde{c}Y + \tilde{d}X^2.$$

Now by the same analysis as in Rank 5 this can be further reduced to

$$Y^2 \in \{\mathbb{1}, \mathbb{1} - X^2, \mathbb{1} + X^2\}.$$

Let us say we are in case 2 of Lemma 4.5.1. Then

$$YX = a\mathbb{1} + bX + cY + dX^2 + eXY + fY^2.$$

By symmetry we have $e = -1$. Let us say that $d^2 + f^2 \neq 0$. By symmetry we may assume that $f \neq 0$. But then we rewrite the equality as

$$Y^2 = \tilde{a}\mathbb{1} + \tilde{b}X + \tilde{c}Y + \tilde{d}X^2 + \tilde{e}(XY + YX).$$

By the same reasoning as in case 1 this can be further reduced to

$$Y^2 \in \{\mathbb{1}, \mathbb{1} - X^2, \mathbb{1} + X^2\}.$$

Otherwise $d = f = 0$. But then we make a substitution $X \mapsto X + Y$ and get a relation of the form

$$Y^2 = \tilde{a}\mathbb{1} + \tilde{b}X + \tilde{c}Y + \tilde{d}X^2 + \tilde{e}(XY + YX)$$

which can be reduced to one of the cases in the statement of the lemma. □

4.5.1 Case 1: $Y^2 = \mathbb{1}$.

$\mathcal{M}(2)$ is of the form

$$\mathcal{M}(2) = \begin{pmatrix} \beta_1 & \beta_x & \beta_y & \beta_{x^2} & \beta_{xy} & \beta_{xy} & \beta_1 \\ \beta_x & \beta_{x^2} & \beta_{xy} & \beta_{x^3} & \beta_{x^2y} & \beta_{x^2y} & \beta_x \\ \beta_y & \beta_{xy} & \beta_1 & \beta_{x^2y} & \beta_x & \beta_x & \beta_y \\ \beta_{x^2} & \beta_{x^3} & \beta_{x^2y} & \beta_{x^4} & \beta_{x^3y} & \beta_{x^3y} & \beta_{x^2} \\ \beta_{xy} & \beta_{x^2y} & \beta_x & \beta_{x^3y} & \beta_{x^2} & \beta_{xyxy} & \beta_{xy} \\ \beta_{xy} & \beta_{x^2y} & \beta_x & \beta_{x^3y} & \beta_{xyxy} & \beta_{x^2} & \beta_{xy} \\ \beta_1 & \beta_x & \beta_y & \beta_{x^2} & \beta_{xy} & \beta_{xy} & \beta_1 \end{pmatrix}.$$

RG relations and B(3).

If $\mathcal{M}(2)$ has a measure then it must satisfy the following RG relations.

$$\begin{aligned} Y^3 &= Y \\ XY^2 &= X \\ Y^2X &= X. \end{aligned}$$

We get the following system.

$$\begin{aligned}
\beta_{x^2y^3} &= \beta_{x^2y}, \\
\beta_{xy^4} &= \beta_{xy}, \\
\beta_{y^5} &= \beta_1, \\
\beta_{x^3y^2} &= \beta_{x^3}, \\
\beta_{xy^2xy} &= \beta_{x^2y}, \\
\beta_{xy^4} &= \beta_x.
\end{aligned}$$

The remaining moments of degree 5 are then β_{x^5}, β_{x^4y}, β_{x^2yxy}. Setting $(p,q,r) := (\beta_{x^5}, \beta_{x^4y}, \beta_{x^2yxy})$ we have a candidate for $B(3)(p,q,r)$ given by the following proposition.

Proposition 4.5.3. *If $\mathcal{M}(2)$ has a measure, then there is a three-parametric solution for $B(3)(p,q,r)$ given by*

$$B(3)(p,q,r) = \begin{pmatrix}
\beta_{x^3} & \beta_{x^2y} & \beta_{x^2y} & \beta_x & \beta_{x^2y} & \beta_x & \beta_x & \beta_y \\
\beta_{x^4} & \beta_{x^3y} & \beta_{x^3y} & \beta_{x^2} & \beta_{x^4y} & \beta_{xyxy} & \beta_{x^2} & \beta_{xy} \\
\beta_{x^3y} & \beta_{x^2} & \beta_{xyxy} & \beta_{xy} & \beta_{x^2} & \beta_{xy} & \beta_{xy} & \beta_1 \\
p & q & q & \beta_{x^3} & q & r & \beta_{x^3} & \beta_{x^2y} \\
q & \beta_{x^3} & r & \beta_{x^2y} & r & \beta_{x^2y} & \beta_{x^2y} & \beta_x \\
q & r & r & \beta_{x^2y} & \beta_{x^3} & \beta_{x^2y} & \beta_{x^2y} & \beta_x \\
\beta_{x^3} & \beta_{x^2y} & \beta_{x^2y} & \beta_x & \beta_{x^2y} & \beta_x & \beta_x & \beta_y
\end{pmatrix}.$$

Flat extensions.

First we have to check for which triples $(p,q,r) \in \mathbb{R}^3$ we have $\operatorname{Ran} B(3)(p,q,r) \subseteq \operatorname{Ran} \mathcal{M}(2)$. Let $\{\mathbb{1}, X, Y, X^2, XY, YX\} = Q$. Since $[\mathcal{M}(2)]_Q$ is invertible, there are scalars $A, B, C, D, E, F \in \mathbb{R}$ such that

$$[X^3(p,q,r)]_Q = A[\mathbb{1}]_Q + B[X]_Q + C[Y]_Q + D[X^2]_Q + E[XY]_Q + F[YX]_Q.$$

Since the first and the last row of $\mathcal{M}(3)$ and $X^3(p,q,r)$ are the same, it follows that

$$[X^3(p,q,r)] = A[\mathbb{1}] + B[X] + C[Y] + D[X^2] + E[XY] + F[YX].$$

The same reasoning can by applied to all the other columns of $B(3)(p,q,r)$. We get the following.

Proposition 4.5.4. $\operatorname{Ran} B(3)(p,q,r) \subseteq \operatorname{Ran} \mathcal{M}(2)$ *for every* $(p,q,r) \in \mathbb{R}^3$.

The candidate for $C(p,q,r) \equiv C(3)(p,q,r)$ in the flat extension $\mathcal{M}(3)(p,q,r)$ of $\mathcal{M}(2)$ is $W(p,q,r)^* B(3)(p,q,r)$, where $B(3)(p,q,r) = \mathcal{M}(2)W(p,q,r)$. We analyze the Hankel system in the same way as for Rank 5.

Theorem 4.5.5. *$\mathcal{M}(2)$ admits a flat extension if and only if*

$$\begin{aligned} C_{12} &= C_{13}, \\ C_{22} &= \beta_{x^4}, \\ C_{16} &= C_{23}, \\ C_{25} &= C_{33}, \\ C_{26} &= \beta_{x^3y}, \\ C_{66} &= \beta_{x^2}. \end{aligned}$$

The following matrices are positive semi-definite Hankel matrix of rank 6 satisfying $Y^2 = \mathbb{1}$, but just one of them has a flat extension.

Example 5. For $\beta_{x^4} > 1$,

$$\mathcal{M}(2, \beta_{x^4}) = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & \beta_{x^4} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

is a positive semi-definite Hankel matrix of rank 6 satisfying $Y^2 = \mathbb{1}$. Using *Mathematica* we calculate possible candidates for $C(3)(\beta_{x^4}, p, q, r)$

$$\begin{pmatrix} \frac{p^2}{\beta_{x^4}-1} + 2q^2 + \beta_{x^4}^2 & rq + \frac{pq}{\beta_{x^4}-1} & 2rq + \frac{pq}{\beta_{x^4}-1} & \beta_{x^4} & rq + \frac{pq}{\beta_{x^4}-1} & \frac{pr}{\beta_{x^4}-1} & \beta_{x^4} & 0 \\ rq + \frac{pq}{\beta_{x^4}-1} & \frac{q^2}{\beta_{x^4}-1} + r^2 + 1 & \frac{q^2}{\beta_{x^4}-1} + r^2 & 0 & \frac{q^2}{\beta_{x^4}-1} + 1 & \frac{qr}{\beta_{x^4}-1} & 0 & 1 \\ 2rq + \frac{pq}{\beta_{x^4}-1} & \frac{q^2}{\beta_{x^4}-1} + r^2 & \frac{q^2}{\beta_{x^4}-1} + 2r^2 & 0 & \frac{q^2}{\beta_{x^4}-1} + r^2 & \frac{qr}{\beta_{x^4}-1} & 0 & 0 \\ \beta_{x^4} & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ rq + \frac{pq}{\beta_{x^4}-1} & \frac{q^2}{\beta_{x^4}-1} + 1 & \frac{q^2}{\beta_{x^4}-1} + r^2 & 0 & \frac{q^2}{\beta_{x^4}-1} + r^2 + 1 & \frac{qr}{\beta_{x^4}-1} & 0 & 1 \\ \frac{pr}{\beta_{x^4}-1} & \frac{qr}{\beta_{x^4}-1} & \frac{qr}{\beta_{x^4}-1} & 0 & \frac{qr}{\beta_{x^4}-1} & \frac{r^2}{\beta_{x^4}-1} & 0 & 0 \\ \beta_{x^4} & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

For the flat extension we must have, by Theorem 4.5.5,

$$\begin{aligned} rq + \frac{pq}{\beta_{x^4}-1} &= 2rq + \frac{pq}{\beta_{x^4}-1}, \\ \frac{q^2}{\beta_{x^4}-1} + r^2 + 1 &= \beta_{x^4}, \\ \frac{pr}{\beta_{x^4}-1} &= \frac{q^2}{\beta_{x^4}-1} + r^2, \\ \frac{q^2}{\beta_{x^4}-1} + 1 &= \frac{q^2}{\beta_{x^4}-1} + 2r^2, \\ \frac{qr}{\beta_{x^4}-1} &= 0, \\ \frac{r^2}{\beta_{x^4}-1} &= 1. \end{aligned}$$

Using *Mathematica* we see that these equations are satisfied if and only if $\beta_{x^4} = \frac{3}{2}$ in which case $p = \pm\frac{1}{2\sqrt{2}}, q = 0, r = \pm\frac{1}{\sqrt{2}}$.

However, for every $\beta_{x^4} > 1$, $\mathcal{M}(2, \beta_{x^4})$ admits a measure consisting of 1 atom of the form $(X_1, Y_1) \in (\mathbb{SR}^{2\times 2})^2$ and 5 atoms of the form $(x_j, y_j) \in \mathbb{R}^2$, $j = 2, \ldots, 6$. This can be seen as follows. We define

$$D(\beta_{x^4}, \alpha) := \mathcal{M}(2, \beta_{x^4}) - \alpha A,$$

where

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

A is a positive semi-definite Hankel matrix of rank 4 satisfying the relations

$$X^2 = \mathbb{1}, \quad XY + YX = 0, \quad Y^2 = \mathbb{1}$$

and thus admits a measure. Let $Q = \{\mathbb{1}, X, Y, X^2, XY, YX\}$. Calculating the smallest $\alpha > 0$ such that

$$\det([D(\beta_{x^4}, \alpha)]_Q) = 0$$

under the condition that $\beta_{x^4} > 1$ gives $\alpha = \frac{1}{2}$. We have

$$D(\beta_{x^4}, \frac{1}{2}) = \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \beta_{x^4} - \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Hence $D(\beta_{x^4}, \frac{1}{2})$ is positive semi-definite commutative Hankel matrix of rank at most 5 admitting a 5-atomic measure of the from $(x_j, y_j) \in \mathbb{R}^2$, $j = 2, \ldots, 6$.

4.5.2 Case 2: $Y^2 = \mathbb{1} - X^2$.

$\mathcal{M}(2)$ is of the form

$$\mathcal{M}(2) = \begin{pmatrix} \beta_1 & \beta_x & \beta_y & \beta_{x^2} & \beta_{xy} & \beta_{xy} & \beta_1 - \beta_{x^2} \\ \beta_x & \beta_{x^2} & \beta_{xy} & \beta_{x^3} & \beta_{x^2y} & \beta_{x^2y} & \beta_x - \beta_{x^3} \\ \beta_y & \beta_{xy} & \beta_1 - \beta_{x^2} & \beta_{x^2y} & \beta_x - \beta_{x^3} & \beta_x - \beta_{x^3} & \beta_y - \beta_{x^2y} \\ \beta_{x^2} & \beta_{x^3} & \beta_{x^2y} & \beta_{x^4} & \beta_{x^3y} & \beta_{x^3y} & \beta_{x^2} - \beta_{x^4} \\ \beta_{xy} & \beta_{x^2y} & \beta_x - \beta_{x^3} & \beta_{x^3y} & \beta_{x^2} - \beta_{x^4} & \beta_{xyxy} & \beta_{xy} - \beta_{x^3y} \\ \beta_{xy} & \beta_{x^2y} & \beta_x - \beta_{x^3} & \beta_{x^3y} & \beta_{xyxy} & \beta_{x^2} - \beta_{x^4} & \beta_{xy} - \beta_{x^3y} \\ \beta_1 - \beta_{x^2} & \beta_x - \beta_{x^3} & \beta_y - \beta_{x^2y} & \beta_{x^2} - \beta_{x^4} & \beta_{xy} - \beta_{x^3y} & \beta_{xy} - \beta_{x^3y} & \beta_1 - 2\beta_{x^2} + \beta_{x^4} \end{pmatrix}.$$

RG relations and B(3).

If $\mathcal{M}(2)$ has a measure then it must satisfy the following RG relations.

$$\begin{aligned} Y^3 &= Y - X^2Y, \\ Y^3 &= Y - YX^2, \\ XY^2 &= X - X^3, \\ Y^2X &= X - X^3. \end{aligned}$$

We get the following system.

$$\begin{aligned} \beta_{x^2y^3} &= \beta_{x^2y} - \beta_{x^4y}, \\ \beta_{xy^4} &= (\beta_x - \beta_{x^3}) - \beta_{x^3y^2}, \\ \beta_{xy^4} &= (\beta_x - \beta_{x^3}) - \beta_{x^2yxy}, \\ \beta_{y^5} &= (\beta_y - \beta_{x^2y}) - \beta_{x^2y^3}, \\ \beta_{x^3y^2} &= \beta_{x^3} - \beta_{x^5}, \\ \beta_{xy^2xy} &= \beta_{x^2y} - \beta_{x^4y}. \end{aligned} \tag{4.45}$$

We have 8 unknown parameters and 6 equations. Setting $(p, q) := (\beta_{x^5}, \beta_{x^4y})$ we have a candidate for $B(3)(p, q)$ given by the following proposition.

Proposition 4.5.6. *If $\mathcal{M}(2)$ has a measure, then the solution to the system (4.45) is*

$$\begin{aligned} \beta_{x^2y^3} = \beta_{xy^2xy} &= \beta_{x^2y} - q, \\ \beta_{xy^4} &= \beta_x - 2\beta_{x^3} + p, \\ \beta_{y^5} &= \beta_y - 2\beta_{x^2y} + q, \\ \beta_{x^3y^2} = \beta_{x^2yxy} &= \beta_{x^3} - p. \end{aligned}$$

and a two-parametric solution for $B(3)(p, q)$ is given in Appendix C (C2).

Flat extensions.

First we have to check for which pairs $(p, q) \in \mathbb{R}^2$ we have $\operatorname{Ran} B(3)(p, q) \subseteq \operatorname{Ran} \mathcal{M}(2)$. Let $\{\mathbb{1}, X, Y, X^2, XY, YX\} = Q$. Since $[\mathcal{M}(2)]_Q$ is invertible, there are scalars $A, B, C, D, E, F \in \mathbb{R}$ such that

$$[X^3(p, q)]_Q = A[\mathbb{1}]_Q + B[X]_Q + C[Y]_Q + D[X^2]_Q + E[XY]_Q + F[YX]_Q.$$

First notice that $E = F$. We calculate

$$\begin{aligned} &A\beta_{y^2} + B\beta_{xy^2} + C\beta_{y^3} + D\beta_{x^2y2} + 2E\beta_{xy^3} \\ &= A(\beta_1 - \beta_{x^2}) + B(\beta_x - \beta_{x^3}) + C(\beta_y - \beta_{x^2y}) + D(\beta_{x^2} - \beta_{x^4}) + 2E(\beta_{xy} - \beta_{x^3y}) \\ &= \beta_{x^3} - p. \end{aligned}$$

By the form of $B(3)(p, q)$ it follows that

$$[X^3(p, q)] = A[\mathbb{1}] + B[X] + C[Y] + D[X^2] + E([XY] + [YX]).$$

The same reasoning can by applied to all the other columns of $B(3)(p, q)$. We get the following.

Proposition 4.5.7. $\operatorname{Ran} B(3)(p, q) \subseteq \operatorname{Ran} \mathcal{M}(2)$ *for every* $(p, q) \in \mathbb{R}^2$.

The candidate for $C(p, q) \equiv C(3)(p, q)$ in the flat extension $\mathcal{M}(3)(p, q)$ of $\mathcal{M}(2)$ is $W(p, q)^*B(3)(p, q)$, where $B(3)(p, q) = \mathcal{M}(2)W(p, q)$. We analyze the Hankel system in the same way as for Rank 5.

Theorem 4.5.8. $\mathcal{M}(2)$ *admits a flat extension if and only if*

$$\begin{aligned} C_{28} &= C_{58}, \\ C_{38} &= C_{46}, \\ C_{48} &= C_{68}, \\ C_{12} &= C_{13}, \\ C_{14} &= C_{22}, \\ C_{25} &= C_{33}, \\ C_{26} &= C_{27}, \\ C_{47} &= C_{66}. \end{aligned}$$

Example 6. For $\beta_{x^4} \in (\frac{1}{4}, \frac{1}{2})$,

$$\mathcal{M}(2, \beta_{x^4}) = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \beta_{x^4} & 0 & 0 & \frac{1}{2} - \beta_{x^4} \\ 0 & 0 & 0 & 0 & \frac{1}{2} - \beta_{x^4} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} - \beta_{x^4} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} - \beta_{x^4} & 0 & 0 & \beta_{x^4} \end{pmatrix}$$

is a positive semi-definite Hankel matrix of rank 6 satisfying $Y^2 = \mathbb{1} - X^2$. Using *Mathematica* we check, that for any β_{x^4} the Hankel system given by Theorem 4.5.8 does not admit a solution. Hence $\mathcal{M}(2, \beta_{x^4})$ does not admit a flat extension.

However, for every $\beta_{x^4} > 1$, $\mathcal{M}(2, \beta_{x^4})$ admits a measure consisting of 1 atom of the form $(X_1, Y_1) \in (\mathbb{SR}^{2\times 2})^2$ and 5 atoms of the form $(x_j, y_j) \in \mathbb{R}^2$, $j = 2, \ldots, 6$. This can be seen as follows. We define

$$D(\beta_{x^4}, \alpha) := \mathcal{M}(2, \beta_{x^4}) - \alpha B,$$

where

$$B = \begin{pmatrix} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

B is a positive semi-definite Hankel matrix of rank 4 satisfying the relations

$$X^2 = \frac{1}{2}\mathbb{1}, \quad XY + YX = 0, \quad Y^2 = \frac{1}{2}\mathbb{1} \quad \Rightarrow \quad Y^2 = \mathbb{1} - X^2.$$

and thus admits a measure. Calculating the smallest $\alpha > 0$ such that

$$\det([D(\beta_{x^4}, \alpha)]_Q) = 0$$

under the condition that $\beta_{x^4} \in (\frac{1}{4}, \frac{1}{2})$ gives $\alpha = 1 - 2\beta_{x^4}$. We have that $D(\beta_{x^4}, 1 - 2\beta_{x^4})$ equals

$$(A, B),$$

where

$$A = \begin{pmatrix} 2\beta_{x^4} & 0 & 0 & \frac{1}{2}(2\beta_{x^4}-1)+\frac{1}{2} \\ 0 & \frac{1}{2}(2\beta_{x^4}-1)+\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2}(2\beta_{x^4}-1)+\frac{1}{2} & 0 \\ \frac{1}{2}(2\beta_{x^4}-1)+\frac{1}{2} & 0 & 0 & \beta_{x^4}+\frac{1}{4}(2\beta_{x^4}-1) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2}(2\beta_{x^4}-1)+\frac{1}{2} & 0 & 0 & -\beta_{x^4}+\frac{1}{4}(2\beta_{x^4}-1)+\frac{1}{2} \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 & \beta_{x^4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4}(1-2\beta_{x^4}) \\ \frac{1}{4}(1-2\beta_{x^4}) & \frac{1}{4}(1-2\beta_{x^4}) & 0 \\ \frac{1}{4}(1-2\beta_{x^4}) & \frac{1}{4}(1-2\beta_{x^4}) & 0 \\ 0 & 0 & \frac{1}{4}(6\beta_{x^4}-1) \end{pmatrix}.$$

Hence $D(\beta_{x^4}, 1 - 2\beta_{x^4})$ is a positive semi-definite commutative Hankel matrix of rank at most 5 admitting a 5-atomic measure of the from $(x_j, y_j) \in \mathbb{R}^2$, $j = 2, \ldots, 6$.

4.5.3 Case 3: $Y^2 = \mathbb{1} + X^2$.

$\mathcal{M}(2)$ is of the form

$$\mathcal{M}(2) = \begin{pmatrix} \beta_1 & \beta_x & \beta_y & \beta_{x^2} & \beta_{xy} & \beta_{xy} & \beta_1+\beta_{x^2} \\ \beta_x & \beta_{x^2} & \beta_{xy} & \beta_{x^3} & \beta_{x^2y} & \beta_{x^2y} & \beta_x+\beta_{x^3} \\ \beta_y & \beta_{xy} & \beta_1+\beta_{x^2} & \beta_{x^2y} & \beta_x+\beta_{x^3} & \beta_x+\beta_{x^3} & \beta_y+\beta_{x^2y} \\ \beta_{x^2} & \beta_{x^3} & \beta_{x^2y} & \beta_{x^4} & \beta_{x^3y} & \beta_{x^3y} & \beta_{x^2}+\beta_{x^4} \\ \beta_{xy} & \beta_{x^2y} & \beta_x+\beta_{x^3} & \beta_{x^3y} & \beta_{x^2}+\beta_{x^4} & \beta_{xyxy} & \beta_{xy}+\beta_{x^3y} \\ \beta_{xy} & \beta_{x^2y} & \beta_x+\beta_{x^3} & \beta_{x^3y} & \beta_{xyxy} & \beta_{x^2}+\beta_{x^4} & \beta_{xy}+\beta_{x^3y} \\ \beta_1+\beta_{x^2} & \beta_x+\beta_{x^3} & \beta_y+\beta_{x^2y} & \beta_{x^2}+\beta_{x^4} & \beta_{xy}+\beta_{x^3y} & \beta_{xy}+\beta_{x^3y} & \beta_1+2\beta_{x^2}+\beta_{x^4} \end{pmatrix}.$$

RG relations and B(3).

If $\mathcal{M}(2)$ has a measure then it must satisfy the following RG relations.

$$\begin{aligned} Y^3 &= Y + X^2Y, \\ Y^3 &= Y + YX^2, \\ XY^2 &= X + X^3, \\ Y^2X &= X + X^3. \end{aligned}$$

We get the following system.

$$\begin{aligned} \beta_{x^2y^3} &= \beta_{x^2y} + \beta_{x^4y}, \\ \beta_{xy^4} &= (\beta_x + \beta_{x^3}) + \beta_{x^3y^2}, \\ \beta_{xy^4} &= (\beta_x + \beta_{x^3}) + \beta_{x^2yxy}, \\ \beta_{y^5} &= (\beta_y + \beta_{x^2y}) + \beta_{x^2y^3}, \\ \beta_{x^3y^2} &= \beta_{x^3} + \beta_{x^5}, \\ \beta_{xy^2xy} &= \beta_{x^2y} + \beta_{x^4y}. \end{aligned} \tag{4.46}$$

We have 8 unknown parameters and 6 equations. Setting $(p,q) := (\beta_{x^5}, \beta_{x^4y})$ we have a candidate for $B(3)(p,q)$ given by the following proposition.

Proposition 4.5.9. *If $\mathcal{M}(2)$ has a measure, then the solution to the system (4.46) is*

$$\begin{aligned} \beta_{x^2y^3} = \beta_{xy^2xy} &= \beta_{x^2y} + q, \\ \beta_{xy^4} &= \beta_x + 2\beta_{x^3} + p, \\ \beta_{y^5} &= \beta_y + 2\beta_{x^2y} + q, \\ \beta_{x^3y^2} = \beta_{x^2yxy} &= \beta_{x^3} + p. \end{aligned}$$

and a two-parametric solution for $B(3)(p,q)$ is given in Appendix C (see section C.2).

Flat extensions.

First we have to check for which pairs $(p,q) \in \mathbb{R}^2$ we have $\operatorname{Ran} B(3)(p,q) \subseteq \operatorname{Ran} \mathcal{M}(2)$. Let $\{\mathbb{1}, X, Y, X^2, XY, YX\} = Q$ Since $[\mathcal{M}(2)]_Q$ is invertible, there are scalars $A, B, C, D, E, F \in \mathbb{R}$ such that

$$[X^3(p,q)]_Q = A[\mathbb{1}]_Q + B[X]_Q + C[Y]_Q + D[X^2]_Q + E[XY]_Q + F[YX]_Q.$$

First notice that $E = F$. We calculate

$$\begin{aligned} &A\beta_{y^2} + B\beta_{xy^2} + C\beta_{y^3} + D\beta_{x^2y2} + 2E\beta_{xy^3} \\ &= A(\beta_1 + \beta_{x^2}) + B(\beta_x + \beta_{x^3}) + C(\beta_y + \beta_{x^2y}) + D(\beta_{x^2} + \beta_{x^4}) + 2E(\beta_{xy} + \beta_{x^3y}) \\ &= \beta_{x^3} + p. \end{aligned}$$

By the form of $B(3)(p,q)$ it follows that

$$[X^3(p,q)] = A[\mathbb{1}] + B[X] + C[Y] + D[X^2] + E([XY] + [YX]).$$

The same reasoning can by applied to all the other columns of $B(3)(p,q)$. We get the following.

Proposition 4.5.10. $\operatorname{Ran} B(3)(p,q) \subseteq \operatorname{Ran} \mathcal{M}(2)$ *for every* $(p,q) \in \mathbb{R}^2$.

The candidate for $C(p,q) \equiv C(3)(p,q)$ in the flat extension $\mathcal{M}(3)(p,q)$ of $\mathcal{M}(2)$ is $W(p,q)^* B(3)(p,q)$, where $B(3)(p,q) = \mathcal{M}(2)W(p,q)$. We follow the same analysis as for $Y^2 = 1 + X^2$ in Rank 5.

Theorem 4.5.11. $\mathcal{M}(2)$ *admits a flat extension if and only if*

$$\begin{aligned} C_{28} &= C_{58}, \\ C_{38} &= C_{46}, \\ C_{48} &= C_{68}, \\ C_{12} &= C_{13}, \\ C_{14} &= C_{22}, \\ C_{25} &= C_{33}, \\ C_{26} &= C_{27}, \\ C_{47} &= C_{66}. \end{aligned}$$

Chapter 5

References

[1] NI Akhiezer. The classical moment problem and some related questions in analysis. 1965. *Hafner, New York*, 1965.

[2] Daniel Bessis, Pierre Moussa, and Matteo Villani. Monotonic converging variational approximations to the functional integrals in quantum statistical mechanics. *Journal of Mathematical Physics*, 16(11):2318–2325, 1975.

[3] Grigoriy Blekherman, Pablo A Parrilo, and Rekha R Thomas. *Semidefinite optimization and convex algebraic geometry*, volume 13. SIAM, 2013.

[4] Sabine Burgdorf, Kristijan Cafuta, Igor Klep, and Janez Povh. The tracial moment problem and trace-optimization of polynomials. *Mathematical Programming*, 137(1-2):557–578, 2013.

[5] Sabine Burgdorf and Igor Klep. Trace-positive polynomials and the quartic tracial moment problem. *Comptes Rendus Mathematique*, 348(13):721–726, 2010.

[6] Sabine Burgdorf and Igor Klep. The truncated tracial moment problem. *J. Oper. Theory*, 68(1):141–163, 2012.

[7] Alain Connes. Classification of injective factors cases ii1, ii, iiiλ, $\lambda \neq 1$. *Annals of Mathematics*, pages 73–115, 1976.

[8] R Curto, L Fialkow, et al. Solution of the singular quartic moment problem. *J. Operator Theory*, 48(2):315–354, 2002.

[9] Raúl Curto and Seonguk Yoo. Concrete solution to the nonsingular quartic binary moment problem. *Proceedings of the American Mathematical Society*, 144(1):249–258, 2016.

[10] Raúl E Curto and Lawrence A Fialkow. *Solution of the truncated complex moment problem for flat data*, volume 568. American Mathematical Soc., 1996.

[11] Raúl E Curto and Lawrence A Fialkow. Solution of the truncated parabolic moment problem. *Integral Equations and Operator Theory*, 50(2):169–196, 2004.

[12] Raúl E Curto and Lawrence A Fialkow. Solution of the truncated hyperbolic moment problem. *Integral Equations and Operator Theory*, 52(2):181–218, 2005.

[13] Lawrence Fialkow. Positivity, extensions and the truncated complex moment problem. *Contemporary Mathematics*, 185:133–133, 1995.

[14] Laurent Gosse, Olof Runborg, et al. Finite moment problems and applications to multiphase computations in geometric optics. *Communications in Mathematical Sciences*, 3(3):373–392, 2005.

[15] Hans Hamburger. Über eine Erweiterung des Stieltjesschen Momentenproblems. *Mathematische Annalen*, 81(2-4):235–319, 1920.

[16] Felix Hausdorff. Momentprobleme für ein endliches Intervall. *Mathematische Zeitschrift*, 16(1):220–248, 1923.

[17] Igor Klep and Markus Schweighofer. Connes' embedding conjecture and sums of hermitian squares. *Advances in Mathematics*, 217(4):1816–1837, 2008.

[18] Igor Klep and Markus Schweighofer. Sums of hermitian squares and the bmv conjecture. *Journal of Statistical Physics*, 133(4):739–760, 2008.

[19] David A Mazziotti. Realization of quantum chemistry without wave functions through first-order semidefinite programming. *Physical review letters*, 93(21):213001, 2004.

[20] Marcel Riesz. Sur le probleme des moments et le théoreme de parseval correspondant. *Scandinavian Actuarial Journal*, 1924(1):54–74, 1924.

[21] Yurii L'vovich Shmul'yan. An operator hellinger integral. *Matematicheskii Sbornik*, 91(4):381–430, 1959.

[22] James Alexander Shohat and Jacob David Tamarkin. *The problem of moments*. Number 1. American Mathematical Soc., 1943.

[23] T-J Stieltjes. Recherches sur les fractions continues. In *Annales de la Faculté des sciences de Toulouse: Mathématiques*, volume 8, pages 1–122, 1894.

[24] FH Vasilescu. Spectral measures and moment problems. *Spectral Theory and Its Applications, Theta*, pages 173–215, 2003.

Appendix A

Proof of Theorem 2.0.2 for Hyperbolas and Parabolas

Theorem 2.0.2, which is restated below for the readers convenience, is a monumental achievement that completely describes when the classic truncated quartic moment problem has a solution. The proof of this theorem is quite difficult for hyperbolas and parabolas. So, for the readers convenience, we replicate the proof from [12] for hyperbolas and from [11] for parabolas.

Theorem 2.0.2 ([12], Theorem 1.2)**.** *Let $p \in \mathbb{R}[x, y]$, with* $\deg p(x, y) \leq 2$. *Then $\beta^{(2n)}$ has a representing measure supported in the curve $p(x, y) = 0$ if and only if $\mathcal{M}(n)$ has a column dependence relation $p(X, Y) = 0$, $\mathcal{M}(n) \geq 0$, $\mathcal{M}(n)$ is recursively generated and* rank $\mathcal{M}(n) \leq$ card $\mathcal{V}(\beta)$.

The proof of Theorem 2.0.2 requires several preliminary Lemma's and Theorems. In the following results, we define the variety of $M(n)$ (or of β) as $\mathcal{V}(\beta) := \bigcap_{p \in \mathbb{R}_n[x,y], p(X,Y)=0} \mathcal{Z}(p)$, and given a vector, $\mathbf{v}$, we define $[\mathbf{v}]_K$ to be the compression of $\mathbf{v}$ to the rows (or perhaps columns) indexed by the set K. The size of $\mathcal{M}(n)$ is $m(n) = (n + 1)(n + 2)/2$.

A.1 Hyperbolas

The necessity of Theorem 2.0.2 is shown in the earlier work of Curto and Fialkow on the truncated moment problem. We replicate below the proof for the sufficiency of Theorem 2.0.2.

Theorem A.1.1. *Let $\beta \equiv \beta^{(2n)}$ be a family of real numbers, $\beta_{00} > 0$, and let $\mathcal{M}(n)$ be the associated multivariate Hankel matrix. Assume that $\mathcal{M}(n)$ is positive, recursively generated, and satisfies $YX = \mathbb{1}$ and* rank $\mathcal{M}(n) \leq$ card $\mathcal{V}(\beta)$. *Then* rank $\mathcal{M}(n) \leq 2n + 1$. *If* rank $\mathcal{M}(n) \leq 2n$, *then $\mathcal{M}(n)$ admits a flat extension $\mathcal{M}(n + 1)$ (so β admits a* rank $\mathcal{M}(n)$-*atomic representing measure supported in the curve $yx = 1$). If* rank $\mathcal{M}(n) = 2n + 1$, *then $\mathcal{M}(n)$ admits an extension to a positive, recursively generated $\mathcal{M}(n+1)$, satisfying $2n+1 \leq \text{rank}\mathcal{M}(n+1) \leq 2n+2$, and $\mathcal{M}(n+1)$ admits a flat extension $\mathcal{M}(n+2)$ (so β admits a representing measure μ supported in $yx=1$ with $2n+1 \leq \text{card supp}\mu \leq 2n+2$).*

We require several preliminary results to prove A.1.1. It is well known that $\mathcal{M}(n)$ (positive semi-definite and recursively generated) admits a flat extension when $\{\mathbb{1}, X, Y\}$ is linearly dependent in $\mathcal{C}_{\mathcal{M}(n)}$ (the column space of $\mathcal{M}(n)$). Thus, hereafter we will assume that the set $\{\mathbb{1}, X, Y\}$ is linearly independent. We begin with an elementary result which exploits the fact that $\mathcal{M}(n)$ is recursively generated. For $1 \leq k \leq n$ let

$$\mathcal{S}_n(k) := \{\mathbb{1}, X, Y, X^2, Y^2, \ldots, X^k, Y^k\} \subseteq \mathcal{C}_{\mathcal{M}(n)}.$$

Lemma A.1.2. *For $n \geq 2$, let $\mathcal{M}(n)$ be positive and recursively generated, and assume that $YX = \mathbb{1}$ in $\mathcal{C}_{\mathcal{M}(n)}$. Then each column of $\mathcal{M}(n)$ is equal to a column in $\mathcal{S}_n(n)$; in particular,* rank $\mathcal{M}(n) \leq 2n+1$.

Proof. The proof is by induction on $n \geq 2$. For $n=2$ the statement is clearly true, so assume that it holds for $n = k$ (≥ 2). Suppose $\mathcal{M}(k+1)$ is positive and recursively generated, with $YX = \mathbb{1}$ in $\mathcal{C}_{\mathcal{M}(k+1)}$. Let $i,j \geq 0$, $i+j \leq k$. By the induction hypothesis, each column of the form $[Y^iX^j]_{m(k)}$ (the compression to entries contained in $\mathcal{M}(k)$) is in $\mathcal{S}_k(k)$, and since $\mathcal{M}(k+1) \geq 0$, the extension principle ([13], Proposition 2.4) shows that $Y^iX^j \in \mathcal{S}_k(k+1)$ $(\subseteq \mathcal{S}_{k+1}(k+1))$. Since $X^{k+1}, Y^{k+1} \in \mathcal{S}_{k+1}(k+1)$, it now suffices to consider a column in $\mathcal{M}(k+1)$ of the form $Y^{k+1-j}X^j$, with $1 \leq j \leq k$. Let $q(x,y) := yx - 1$ and let $p_{ij}(x,y) := y^ix^j$, so that $Y^{k+1-j}X^j = p_{k+1-j,j}(X,Y)$. Also, let $r_{ij}(x,y) := y^{k+1-j}x^j - y^{k-j}x^{j-1}$. Now $r_{ij}(x,y) = y^{k-j}x^{j-1}(yx-1) = p_{k-j,j-1}(x,y)q(x,y)$; since $\mathcal{M}(k+1)$ is recursively generated and $q(X,Y) = 0$, it follows that $r_{ij}(X,Y) = 0$, that is, $Y^{k+1-j}X^j = Y^{k-j}X^{j-1}$ in $\mathcal{C}_{\mathcal{M}(k+1)}$. By induction, $[Y^{k-j}X^{j-1}]_{m(k)} \in \mathcal{S}_k(k)$, and since $\mathcal{M}(k+1) \geq 0$, it follows again by the extension principle that $Y^{k-j}X^{j-1} \in \mathcal{S}_{k+1}(k)$. Thus $Y^{k+1-j}X^j \in \mathcal{S}_{k+1}(k) \subseteq \mathcal{S}_{k+1}(k+1)$, as desired. □

We next have two supplementary lemmas that will be used frequently in the following results. For $i+j, k+\ell \leq n$, $\langle Y^iX^j, Y^kX^\ell \rangle$ denotes the entry of $\mathcal{M}(n)$ in row Y^kX^ℓ column Y^iX^j, namely $\beta_{i+k,j+\ell}$. This inner product notation is extended from monomials to polynomials as follows. For $p = \sum_{0 \leq i+j \leq n} a_{ij}y^ix^j$ and $q = \sum_{0 \leq k+\ell \leq n} b_{k\ell}y^kx^\ell$, we define $\langle p(X,Y), q(X,Y) \rangle := \sum_{0 \leq i+j, k+\ell \leq n} a_{ij}b_{k\ell}\beta_{i+k,j+\ell}$. Further, if deg $p+$deg p', deg $q+$ deg $q' \leq n$, by $\langle p(X,Y)p'(X,Y), q(X,Y)q'(X,Y) \rangle$ we mean $\langle (pp')(X,Y), (qq')(X,Y) \rangle$. The following result is immediate from the definitions.

Lemma A.1.3. *(i) For $p, q \in \mathbb{R}_n[x,y]$,*

$$\langle p(X,Y), q(X,Y) \rangle = \langle q(X,Y), p(X,Y) \rangle.$$

(ii) For $p, q \in \mathbb{R}_n[x, y], i, j \geq 0, i + j \leq n$ *and* $\deg p, \deg q \leq n - (i + j)$,

$$\left\langle p(X,Y)Y^jX^i, q(X,Y)\right\rangle = \left\langle p(X,Y), q(X,Y)Y^jX^i\right\rangle.$$

(iii) If $p, q, r \in \mathbb{R}_n[x, y]$ *with* $p(X, Y) = q(X, Y)$ *in* $\mathcal{C}_{\mathcal{M}(n)}$ *then*

$$\langle r(X,Y), p(X,Y)\rangle = \langle r(X,Y), q(X,Y)\rangle.$$

Lemma A.1.4. *Let* $\mathcal{M}(n)$ *be positive, recursively generated with* $YX = \mathbb{1}$*, and assume* $p, q \in \mathbb{R}_{n-1}[x, y]$*. Then*

$$\langle p(X,Y), q(X,Y)\rangle = \langle Yp(X,Y), Xq(X,Y)\rangle \tag{A.1}$$

$$= \langle Xp(X,Y), Yq(X,Y)\rangle. \tag{A.2}$$

Proof. The definition of $\langle p(X,Y), q(X,Y)\rangle$ implies that, without loss of generality we may assume that $p(X, Y) = Y^iX^j$ and $q(X, Y) = Y^kX^\ell$. Assume first that $k \geq 1$. We have

$$\begin{aligned}\left\langle Y^iX^j, Y^kX^\ell\right\rangle &= \left\langle Y^{i+1}X^j, Y^{k-1}X^\ell\right\rangle \text{ (by A.1.3(ii))}\\ &= \left\langle Y^{i+1}X^j, Y^{k-1}X^\ell YX\right\rangle \text{ (by Lemma A.1.3(iii), using } YX = 1)\\ &= \left\langle Y^{i+1}X^j, Y^kX^{\ell+1}\right\rangle.\end{aligned}$$

If $k = 0$ and $j \geq 1$, we have

$$\begin{aligned}\left\langle Y^iX^j, X^\ell\right\rangle &= \left\langle Y^iX^{j-1}, X^{\ell+1}\right\rangle \text{ (using Lemma A.1.3(ii))}\\ &= \left\langle Y^iX^{j-1}YX, X^{\ell+1}\right\rangle \text{ (using Lemma A.1.3(iii))}\\ &= \left\langle Y^{i+1}X^j, X^{\ell+1}\right\rangle.\end{aligned}$$

If $k = j = 0$, we have $p(X, Y) = Y^i$ and $q(X, Y) = X^\ell$, so we need to prove that $\left\langle Y^i, X^\ell\right\rangle = \left\langle Y^{i+1}, X^{\ell+1}\right\rangle$. If $i \geq 0$, we have

$$\begin{aligned}\left\langle Y^i, X^\ell\right\rangle &= \left\langle Y^{i-1}, YX^\ell\right\rangle \text{ (by Lemma A.1.3(ii))}\\ &= \left\langle Y^iX, YX^\ell\right\rangle \text{ (by Lemma A.1.3(iii))}\\ &= \beta_{i+1,\ell+1} = \left\langle Y^{i+1}, X^{\ell+1}\right\rangle.\end{aligned}$$

If $i = 0$, then

$$\begin{aligned}\left\langle Y^i, X^\ell\right\rangle = \left\langle \mathbb{1}, X^\ell\right\rangle &= \left\langle YX, X^\ell\right\rangle \text{ (by Lemma A.1.3(iii))}\\ &= \left\langle Y, X^{\ell+1}\right\rangle \text{ (by Lemma A.1.3(ii)).}\end{aligned}$$

We have now completed the proof of the first equation. The proof of the second is a straightforward consequence of the first equation and Lemam A.1.3(i). □

We divide the proof of Theorem A.1.1 into four cases, based on possible dependence relations among the elements of $\mathcal{S}_n(n)$. In each case we ultimately obtain some flat extension $\mathcal{M}$; the existence of a corresponding rank $\mathcal{M}$ atomic representing measure is immediate. Until the proof of Theorem A.1.1, we are assuming that $\mathcal{M}(n)$ is positive, recursively generated, $\{\mathbb{1}, X, Y\}$ is linearly independent, $YX = \mathbb{1}$ and rank $\mathcal{M}(n) \leq$ card $\mathcal{V}(\beta)$.

Proposition A.1.5. *In $\mathcal{S}_n(n)$, assume that the first dependence relation occurs at X^k, with $2 \leq k \leq n$. Then $\mathcal{M}(n)$ is flat and, a fortiori, it admits a unique flat extension $\mathcal{M}(n+1)$.*

Proof. Write $X^k = p_{k-1}(X) + q_{k-1}(Y)$, where deg p_{k-1}, deg $q_{k-1} \leq k-1$. It follows that $\mathcal{V}(\beta) \subseteq (yx = 1) \cap (p_{k-1}(x) + q_{k-1}(y) = x^k) \subseteq (yx = 1) \cap (p_{k-1}(x) + q_{k-1}(\frac{1}{x}) = x^k)$. Since $p_{k-1}(x) + q_{k-1}(\frac{1}{x}) = x^k$ leads to a polynomial equation in x of degree at most $2k-1$, it follows that card $\mathcal{V}(\beta) \leq 2k-1$, so rank $\mathcal{M}(n) \leq 2k-1$. Then $\mathcal{S}_n(k-1) \equiv \{\mathbb{1}, X, Y, X^2, Y^2, \ldots, X^{k-1}, Y^{k-1}\}$ is a basis for $\mathcal{C}_{\mathcal{M}(n)}$, whence $\mathcal{M}(n)$ is flat. □

Proposition A.1.6. *In $\mathcal{S}_n(n)$ assume that the first dependence relation occurs at Y^k with $1 \leq k < n$. Then $\mathcal{M}(n)$ is flat and admits a unique flat extension $\mathcal{M}(n+1)$.*

Proof. Write $Y^k = p_k(X) + q_{k-1}(Y)$, where deg $p_k \leq k$ and deg $q_{k-1} \leq k-1$. Since Y^k corresponds to a monomial of degree at most $n-1$, and since $YX = \mathbb{1}$ and $\mathcal{M}(n)$ is recursively generated, we must have

$$Y^{k-1} = XY^k = Xp_k(X) + Xq_{k-1}(Y). \tag{A.3}$$

Since $\mathcal{M}(n)$ is recursively generated and $YX = \mathbb{1}$, $Xq_{k-1}(Y)$ is clearly a linear combination of columns corresponding to monomials of degree at most $k-2$. Let a_k be the coefficient of X^k in $p_k(X)$. If $a_k = 0$, then it follows from (A.3) that $\mathcal{S}_n(k-1) \cup \{X^k\}$ is linearly dependent, a contradiction. Hence we must have that $a_k \neq 0$, thus (A.3) implies that X^{k+1} is a linear combination of previous columns. Moreover, $Y^{k+1} = Yp_k(X) + Yq_{k-1}(Y)$, and $Yp_k(X)$ has degree $k-1$ in X, so $\mathcal{M}(k+1)$ is flat. It now follows from the extension principle [13] and recursiveness that $\mathcal{M}(n)$ is flat, i.e., $\mathcal{M}(n)$ is a flat extension of $\mathcal{M}(k)$. □

Proposition A.1.7. *In $\mathcal{S}_n(n)$ assume the first dependence relation occurs at Y^n. Then $\mathcal{M}(n)$ admits a flat extension $\mathcal{M}(n+1)$.*

The proof of Proposition A.1.7 requires several preliminary results. Under the hypothesis of Proposition A.1.7, write

$$Y^n = a_n X^n + p_{n-1}(X) + q_{n-1}(Y), \tag{A.4}$$

with $\deg p_{n-1}, \deg q_{n-1} \leq n-1$. We claim that $a_n \neq 0$. Assume instead that $a_n = 0$, i.e., $Y^n = p_{n-1}(X) + q_{n-1}(Y)$. Then $\mathcal{V}(\beta) \subseteq (yx = 1) \cap (p_{n-1}(x) + q_{n-1}(y) = y^n) \subseteq (yx = 1) \cap (p_{n-1}(\frac{1}{y}) + q_{n-1}(y) = y^n)$. Since $p_{n-1}(\frac{1}{y}) + q_{n-1}(y) = y^n$ leads to a polynomial equation in y of degree at most $2n-1$, it follows that card $\mathcal{V}(\beta) \leq 2n-1$, so rank $\mathcal{M}(n) \leq 2n-1$. Then $\mathcal{S}_n(n-1) \equiv \{\mathbb{1}, X, Y, X^2, Y^2, \ldots, X^{n-1}, Y^{n-1}\}$ is a basis for $\mathcal{C}_{\mathcal{M}(n)}$, whence X^n is a linear combination of the columns in $\mathcal{S}_n(n-1)$, a contradiction as the first linear dependency is at Y^n. Thus $a_n \neq 0$, so in particular

$$X^n = \frac{1}{a_n}[Y^n - p_{n-1}(X) - q_{n-1}(Y)]. \tag{A.5}$$

To build a flat extension

$$\mathcal{M}(n+1) = \begin{pmatrix} \mathcal{M}(n) & B(n+1) \\ B(n+1) & C(n+1) \end{pmatrix},$$

we define the middle n columns of a prospective block $B \equiv B(n+1)$, by exploiting recursiveness and the relation $YX = \mathbb{1}$, as follows;

$$YX^n := X^{n-1}; Y^2X^{n-1} := YX^{n-2}; \ldots, Y^nX := Y^{n-1}. \tag{A.6}$$

Also, motiated by (A.5) and, respectively, (A.4), we let

$$X^{n+1} := \frac{1}{a_n}[Y^{n-1} - Xp_{n-1}(X) - Xq_{n-1}(Y)], \tag{A.7}$$

and

$$Y^{n+1} := a_nX^{n-1} + Yp_{n-1}(X) + Yq_{n-1}(Y). \tag{A.8}$$

(The expressions $Y^{n-1} - Xp_{n-1}(X) - Xq_{n-1}(Y)$ and $a_nX^{n-1} + Yp_{n-1}(X) + Yq_{n-1}(Y)$ are shorthand notation for $(y^{n-1} - xp_{n-1}(x) - xq_{n-1}(y))(X, Y)$ and $(a_nx^{n-1} + yp_{n-1}(x) + yq_{n-1}(y))(X, Y)$ in $\mathcal{C}_{\mathcal{M}(n)}$, respectively. Observe that these defining relations are all neseccary if one is to obtain a positive recursively generated extension $\mathcal{M}(n+1)$.) Since the columns defined by (A.6)-(A.8) belong to $\mathcal{C}_{\mathcal{M}(n)}$, we have that $B = \mathcal{M}(n)W$ for some matrix W. Thus a flat extension $M := [\mathcal{M}(n); B]$ is uniquely determined by defining the C-block as $C := W^*\mathcal{M}(n)W$ (see Smul'jan's Lemma). To complete the proof that M is a multivariate Hankel matrix $\mathcal{M}(n+1)$ it suffices to show that block B is of the form $(B_{i,n+1})_{i=0}^n$ and that block C is of the form $B_{n+1,n+1}$. To this end we require some additional notation and several preliminary results.

We next extend the notation $\langle p(X,Y), q(X,Y) \rangle$ to the case when $\deg p = n+1$ and $\deg q \leq n$. Indeed using the definitions of the columns of B, for $i, j \geq 0, i+j = n+1$, there exists $p_{ij} \in \mathbb{R}_n[x,y]$ with $Y^iX^j = p_{ij}(X,Y)$, and we define

$$\left\langle Y^iX^j, q(X,Y) \right\rangle := \langle p_{ij}(X,Y), q(X,Y) \rangle.$$

Now if $p(x,y) = \sum_{0\le k+\ell\le n+1} a_{k\ell} x^\ell y^k$, we define

$$\langle p(X,Y), q(X,Y)\rangle := \sum_{0\le k+\ell\le n+1} a_{k\ell} \left\langle Y^k X^\ell, q(X,Y)\right\rangle .$$

It is easy to check that Lemma A.1.3(iii) holds with deg $r = n+1$.

Lemma A.1.8. *Under the hypothesis of Proposition A.1.7, assume $i,j \ge 0$ with $i+j=n+1$ and $r,s\ge 1$ with $r+s\le n$. Then*

$$\left\langle Y^i X^j, Y^r X^s\right\rangle = \left\langle Y^i X^j, Y^{r-1} X^{s-1}\right\rangle . \tag{A.9}$$

Proof. Fix i and j with $i+j=n+1$. We know from (A.6)-(A.8) that there exists a polynomial $p\in\mathbb{R}_n[x,y]$ such that $Y^iX^j = p(X,Y) = \sum_{0\le k+\ell\le n} a_{k,\ell} Y^k X^\ell$. Then

$$\begin{aligned}
\left\langle Y^i X^j, Y^r X^s\right\rangle &= \sum_{0\le k+\ell\le n} a_{k,\ell}\left\langle Y^k X^\ell, Y^r X^s\right\rangle \\
&= \sum_{0\le k+\ell\le n} a_{k,\ell}\left\langle Y^r X^s, Y^k X^\ell\right\rangle \text{ (becuase } \mathcal{M}(n) \text{ is self-adjoint)} \\
&= \sum_{0\le k+\ell\le n} a_{k,\ell}\left\langle Y^{r-1} X^{s-1}, Y^k X^\ell\right\rangle \text{ (using } YX = \mathbb{1} \text{ and recursiveness)} \\
&= \sum_{0\le k+\ell\le n} a_{k,\ell}\left\langle Y^k X^\ell, Y^{r-1} X^{s-1}\right\rangle \text{ (using the self-adjointness again)} \\
&= \left\langle Y^i X^j, Y^{r-1} X^{s-1}\right\rangle
\end{aligned}$$

as desired. □

The next result provides a reduction for the proof that $B(n+1)$ has the Hankel property.

Lemma A.1.9. *Under the hypothesis of Proposition A.1.7, assume $i+j=n+1$, with $j\ge 1$, $i\ge 0$, and assume the Hankel property*

$$\left\langle Y^i X^j, Y^r X^s\right\rangle = \left\langle Y^{i+1} X^{j-1}, Y^{r-1} X^{s+1}\right\rangle \tag{A.10}$$

holds with $1\le r\le n$ and $s=0$. Then (A.10) holds for all r and s such that $1\le r+s\le n$, $r\ge 1$, $s\ge 0$.

Proof. Fix i and j with $i+j=n+1$. We use induction on $t := r+s$, where $1\le r+s\le n, r\ge 1, s\ge 0$. For $t=1$ the result follows from the hypothesis, since $r=1, s=0$. Assume now that $t=2$. By hypothesis we may assume that $r=s=1$, so we consider the equation

$$\left\langle Y^i X^j, YX\right\rangle = \left\langle Y^{i+1} X^{j-1}, X^2\right\rangle , \tag{A.11}$$

with $j \geq 1, i \geq 0, i+j = n+1$. Since $YX = \mathbb{1}$, the left hand side of (A.11) equals $\langle Y^i X^j, \mathbb{1}\rangle$ by Lemma A.1.8. For $j \geq 2$ and $i \geq 1$, the right hand side of (A.11) equals $\langle Y^i X^{j-2}, X^2\rangle$ (by (A.6)), which in turn equals $\langle Y^i X X^{j-2}, X\rangle = \langle Y^{i-1} X^{j-2}, X\rangle = \langle Y^{i-1} X^{j-1}, \mathbb{1}\rangle = \langle Y^i X^j, \mathbb{1}\rangle$ (by (A.6) for the last step). When $j \geq 2$ and $i = 0$ (which then implies that $j = n+1$), we have

$$\begin{aligned}
\langle X^{n+1}, YX\rangle &= \left\langle \frac{1}{a_n}[Y^{n-1} - Xp_{n-1}(X) - Xq_{n-1}(Y)], YX \right\rangle \text{ (by (A.7))} \\
&= \frac{1}{a_n}[\langle Y^n, X\rangle - \langle p_{n-1}(X), XYX\rangle - \langle q_{n-1}(Y), XYX\rangle] \\
&\text{(by Lemma A.1.3(ii) for the first term and Lemma A.1.4 for the last two terms)} \\
&= \left\langle \frac{1}{a_n}[Y^n - p_{n-1}(X) - q_{n-1}(Y)], X \right\rangle \text{ (by Lemma A.1.3(iii))} \\
&= \langle X^n, X\rangle \text{ (by (A.5))} \\
&= \langle X^{n-1}, X^2\rangle \\
&= \langle YX^n, X^2\rangle \text{ (by (A.6))}
\end{aligned}$$

When $j = 1$ (so that $i = n$), the right hand side of (A.11) is

$$\begin{aligned}
\langle Y^{n+1}, X^2\rangle &= \langle a_n X^{n-1} + Yp_{n-1}(X) + Yq_{n-1}(Y), X^2\rangle \text{ (by (A.8))} \\
&= \langle a_n X^n + p_{n-1}(X) + q_{n-1}(Y), X\rangle \text{ (using Lemma A.1.3(ii) and Lemma A.1.4, as above)} \\
&= \langle Y^n, X\rangle \text{ (by (A.4))} \\
&= \langle Y^{n-1}, \mathbb{1}\rangle \text{ (again using Lemma A.1.4)}
\end{aligned}$$

On the other hand, the left hand side of (A.11) is

$$\begin{aligned}
\langle Y^n X, YX\rangle &= \langle Y^{n-1}, YX\rangle \text{ (by (A.6))} \\
&= \langle Y^{n-1}, \mathbb{1}\rangle \text{ (by Lemma A.1.3(iii))}
\end{aligned}$$

This completes the case when $t = 2$.

Assume now that (A.10) is true for $t \leq u$ with $u \geq 2$, and consider the case $t = u + 1$. Thus $r + s = u + 1$ ($\leq n$) and we may assume $r, s \geq 1$. When $r \geq 2$,

$$\begin{aligned}
\langle Y^i X^j, Y^r X^s\rangle &= \langle Y^i X^j, Y^{r-1} X^{s-1}\rangle \text{ (by Lemma A.1.8)} \\
&= \langle Y^{i+1} X^{j-1}, Y^{r-2} X^s\rangle \text{ (by the inductive step)} \\
&= \langle Y^{i+1} X^{j-1}, Y^{r-1} X^{s+1}\rangle \text{ (by Lemma A.1.8),}
\end{aligned}$$

as desired. When $r = 1$ and $j \geq 2$, we have $s \leq n - r = n - 1$, and we consider three subcases.

Subcase 1. For $j = 2, i = n - 1$,

$$\begin{aligned}
\left\langle Y^{n-1}X^2, YX^s\right\rangle &= \left\langle Y^{n-2}X, YX^s\right\rangle \text{ (by (A.6))}\\
&= \left\langle Y^{n-2}X, X^{s-1}\right\rangle \text{ (by Lemma A.1.3(iii))}\\
&= \left\langle Y^{n-2}, X^s\right\rangle\\
&= \left\langle Y^{n-1}, X^{s+1}\right\rangle \text{ (by Lemma A.1.4, since } s \le n-1)\\
&= \left\langle Y^{n}X, X^{s+1}\right\rangle \text{ (by (A.6)).}
\end{aligned}$$

Subcase 2. For $j \geq 3, i \geq 1$,

$$\begin{aligned}
\left\langle Y^iX^j, YX^s\right\rangle &= \left\langle Y^{i-1}X^{j-1}, YX^s\right\rangle \text{ (by (A.6))}\\
&= \left\langle Y^{i-1}X^{j-1}, X^{s-1}\right\rangle \text{ (by Lemma A.1.3(iii))}\\
&= \left\langle Y^{i-1}X^{j-3}, X^{s+1}\right\rangle = \left\langle Y^{i}X^{j-2}, X^{s+1}\right\rangle \text{ (since } YX = \mathbb{1} \text{ in } \mathcal{M}(n))\\
&= \left\langle Y^{i+1}X^{j-1}, X^{s+1}\right\rangle \text{ (by (A.6)).}
\end{aligned}$$

Subcase 3. For $j = n + 1$ and $i = 0$,

$$\begin{aligned}
\left\langle X^{n+1}, YX^s\right\rangle &= \left\langle \frac{1}{a_n}[Y^{n-1} - Xp_{n-1}(X) - Xq_{n-1}(Y)], YX^s\right\rangle\\
&= \left\langle \frac{1}{a_n}[Y^{n} - p_{n-1}(X) - q_{n-1}(Y)], X^s\right\rangle \text{ (by Lemma A.1.4)}\\
&= \left\langle X^n, X^s\right\rangle = \left\langle X^{n-1}, X^{s+1}\right\rangle\\
&= \left\langle YX^n, X^{s+1}\right\rangle \text{ (by (A.6)).}
\end{aligned}$$

Finally, when $r = 1$ and $j = 1$, we have $i = n$ and $s \leq n - 1$, so

$$\begin{aligned}
\left\langle Y^nX, YX^s\right\rangle &= \left\langle Y^{n-1}, YX^s\right\rangle \text{ (by (A.6))}\\
&= \left\langle Y^n, X^s\right\rangle = \left\langle a_nX^n + p_{n-1}(X) + q_{n-1}(Y), X^s\right\rangle \text{ (by (A.4))}\\
&= \left\langle a_nX^{n-1} + Yp_{n-1}(X) + Yq_{n-1}(Y), X^{s+1}\right\rangle\\
&\text{(by Lemma A.1.3(ii) and Lemma A.1.4, as above)}\\
&= \left\langle Y^{n+1}, X^{s+1}\right\rangle.
\end{aligned}$$

□

Recall from (A.6) that columns $YX^n, \ldots, Y^nX$ are taken as a block from consecutive columns of degree $n - 1$ in $\mathcal{M}(n)$, so these columns satisfy the Hankel property. Thus, in view of Lemma A.1.9, the next two results complete the proof that $B(n + 1)$ has the Hankel property.

Lemma A.1.10. *For $k = 1, \ldots, n$,*

$$\left\langle X^{n+1}, Y^k \right\rangle = \left\langle Y X^n, Y^{k-1} X \right\rangle . \tag{A.12}$$

Proof. We have

$$\begin{aligned}
\left\langle X^{n+1}, Y^k \right\rangle &= \left\langle \frac{1}{a_n}[Y^{n-1} - Xp_{n-1}(X) - Xq_{n-1}(Y)], Y^k \right\rangle \text{ (by (A.7))} \\
&= \left\langle \frac{1}{a_n}[Y^n - p_{n-1}(X) - q_{n-1}(Y)], Y^{k-1} \right\rangle \\
&\text{(by Lemma A.1.3(ii) and Lemma A.1.4, as above)} \\
&= \left\langle X^n, Y^{k-1} \right\rangle = \left\langle X^{n-1}, Y^{k-1} X \right\rangle \\
&= \left\langle Y X^n, Y^{k-1} X \right\rangle \text{ (by (A.6))}
\end{aligned}$$

□

Lemma A.1.11. *For $k = 1, \ldots, n$,*

$$\left\langle Y^n X, Y^k \right\rangle = \left\langle Y^{n+1}, Y^{k-1} X \right\rangle .$$

Proof. We have

$$\begin{aligned}
\left\langle Y^n X, Y^k \right\rangle &= \left\langle Y^{n-1}, Y^k \right\rangle \text{ (by (A.6))} \\
&= \left\langle Y^n, Y^{k-1} \right\rangle = \left\langle a_n X^n + p_{n-1}(X) + q_{n-1}(Y), Y^{k-1} \right\rangle \text{ (by (A.4))} \\
&= \left\langle a_n X^{n-1} + Y p_{n-1}(X) + Y q_{n-1}(Y), Y^{k-1} X \right\rangle \\
&\text{(by Lemma A.1.4 for the last two terms)} \\
&= \left\langle Y^{n+1}, Y^{k-1} X \right\rangle \text{ (by (A.8))}
\end{aligned}$$

□

The proof that block B is of the form $\{B_{i,n+1}\}_{i=0}^n$ is now complete. To finish the proof of Proposition A.1.7 it now suffices to show that $C = W^* \mathcal{M}(n) W$ is Hankel. To do this, observe that in the C block of $M := [\mathcal{M}(n); B] = \begin{pmatrix} \mathcal{M}(n) & B \\ B^* & C \end{pmatrix}$, we need to compute inner products of the form $\langle Y^i X^j, Y^k X^\ell \rangle$ $(i+j = k+\ell = n+1)$. For this, we require an auxiliary lemma. For $i+j = k+\ell = n+1$, by $\langle Y^i X^j, Y^k X^\ell \rangle$ we mean, as usual, the entry in row $Y^k X^\ell$ of column $Y^i X^j$; by self-adjointness of $\mathcal{M}(n)$ we have $\langle Y^i X^j, Y^k X^\ell \rangle = \langle Y^k X^\ell, Y^i X^j \rangle$. Now suppose $Y^i X^j = p(X, Y)$ and $Y^k X^\ell = q(X, Y)$, where $p(x, y) = \sum_{0 \le r+s \le n} a_{rs} y^r x^s$ and $q(x, y) = \sum_{0 \le t+u \le n} b_{tu} y^t x^u$; we define $\langle p(X, Y), q(X, Y) \rangle := \sum_{0 \le r+s, t+u \le n} a_{rs} b_{tu} \left\langle Y^r X^s, Y^t X^u \right\rangle$.

Lemma A.1.12. *For $i+j, k+\ell = n+1$, $\left\langle Y^i X^j, Y^k X^\ell \right\rangle = \langle p(X,Y), q(X,Y)\rangle$.*

Proof.

$$\begin{aligned}
\langle p(X,Y), q(X,Y)\rangle &= \sum_{0\le r+s, t+u\le n} a_{rs} b_{tu} \left\langle Y^r X^s, Y^t X^u \right\rangle \\
&= \sum_{0\le t+u\le n} b_{tu} \left\langle \sum_{0\le r+s\le n} a_{rs} Y^r X^s, Y^t X^u \right\rangle \\
&= \sum_{0\le t+u\le n} b_{tu} \left\langle p(X,Y), Y^t X^u \right\rangle = \sum_{0\le t+u\le n} b_{tu} \left\langle Y^i X^j, Y^t X^u \right\rangle \\
&= \sum_{0\le t+u\le n} b_{tu} \left\langle Y^t X^u, Y^i X^j \right\rangle \text{ (by self-adjointness)} \\
&= \left\langle \sum_{0\le t+u\le n} b_{tu} Y^t X^u, Y^i X^j \right\rangle = \left\langle q(X,Y), Y^i X^j \right\rangle \\
&= \left\langle Y^k X^\ell, Y^i X^j \right\rangle = \left\langle Y^i X^j, Y^k X^\ell \right\rangle \text{ (by self-adjointness).}
\end{aligned}$$

□

*Proof of Proposition **A.1.7**.* Note that since M is a flat extension, dependence relations in columns of $(\mathcal{M}(n)\ B)$ extend to column relation in $(B^*\ C)$. In particular, the middle n columns of C coincide with columns of degree $n-1$ of B^*; since B has the Hankel property, so does B^*, and thus the middle n columns of C have the Hankel property. To verify that C is Hankel, it now suffices to focus on the first two and last two columns of C, namely X^{n+1} and YX^n, and Y^nX and Y^{n+1}. Since C is self-adjoint, and the middle n columns have the Hankel property, to check that C is Hankel it only remains to show that $C_{n+2,1} = C_{n+1,2}$, i.e., $\left\langle X^{n+1}, Y^{n+1} \right\rangle = \langle YX^n, Y^nX\rangle$. Now, by (A.7), (A.8) and Lemma A.1.12 we have

$$\begin{aligned}
\left\langle X^{n+1}, Y^{n+1} \right\rangle &= \Big\langle \frac{1}{a_n}\{Y^{n-1} - X[p_{n-1}(X) + q_{n-1}(Y)]\}\ , \\
&\qquad a_n X^{n-1} + Y[p_{n-1}(X) + q_{n-1}(Y)] \Big\rangle \\
&= \left\langle Y^{n-1}, X^{n-1} \right\rangle + \frac{1}{a_n} \left\langle Y^{n-1}, Y[p_{n-1}(X) + q_{n-1}(Y)] \right\rangle \\
&\quad - \left\langle X[p_{n-1}(X) + q_{n-1}(Y)], X^{n-1} \right\rangle \\
&\quad - \frac{1}{a_n} \left\langle X[p_{n-1}(X) + q_{n-1}(Y)], Y[p_{n-1}(X) + q_{n-1}(Y)] \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle Y^n, X^n \rangle + \frac{1}{a_n} \langle Y^n, p_{n-1}(X) + q_{n-1}(Y) \rangle \\
&\quad - \langle p_{n-1}(X) + q_{n-1}(Y), X^n \rangle \\
&\quad - \frac{1}{a_n} \langle p_{n-1}(X) + q_{n-1}(Y), p_{n-1}(X) + q_{n-1}(Y) \rangle \\
&\quad \text{(by Lemma A.1.4 for the first and fourth terms, and Lemma A.1.3(ii)} \\
&\quad \text{for the second and third terms)} \\
&= \Big\langle \frac{1}{a_n} \{Y^n - [p_{n-1}(X) + q_{n-1}(Y)]\}, \\
&\quad a_n X^n + p_{n-1}(X) + q_{n-1}(Y) \Big\rangle \\
&= \langle X^n, Y^n \rangle \ \text{(by (A.5), (A.4) and Lemma A.3(i),(iii))} \\
&= \langle X^{n-1}, Y^{n-1} \rangle \ \text{(by Lemma A.1.4)} \\
&= \langle Y X^n, Y^n X \rangle \ \text{(by (A.6) and Lemma A.1.12)}
\end{aligned}
$$

This concludes the proof of Proposition A.1.7. □

Remark. It is important to note that for the proof of Proposition A.1.7, the variety condition rank $\mathcal{M}(n) \le$ card $\mathcal{V}(\beta)$ was used only to show that $a_n \neq 0$ in (A.4). Thus if $\mathcal{M}(n)$ is positive, recursively generated, satisfies $YX = \mathbb{1}$ in $\mathcal{C}_{\mathcal{M}(n)}$, and the first dependence relation in $\mathcal{S}_n(n)$ is of the form (A.4) with $a_n \neq 0$, then we may conclude that $\mathcal{M}(n)$ has a flat extension $\mathcal{M}(n+1)$.

Proposition A.1.13. *Assume that $\mathcal{S}_n(n)$ is a basis for $\mathcal{C}_{\mathcal{M}(n)}$. Then $\mathcal{M}(n)$ admits a flat extension $\mathcal{M}(n+1)$ or $\mathcal{M}(n)$ admits a positive, recursively generated extension $\mathcal{M}(n+1)$ with* rank $\mathcal{M}(n+1) = 2n+2$*, and $\mathcal{M}(n+1)$ admits a flat extension $\mathcal{M}(n+2)$.*

Proof. Since $YX = \mathbb{1}$ and to guarantee that $\mathcal{M}(n+1)$ is recursively generated, we define the middle n columns of a proposed B block for $\mathcal{M}(n+1)$ as $[YX^n]_{m(n)} := X^{n-1} \in \mathcal{C}_{\mathcal{M}(n)}$, $[Y^2X^{n-1}]_{m(n)} := YX^{n-2} \in \mathcal{C}_{\mathcal{M}(n)}, \dots, [Y^nX]_{m(n)} := Y^{n-1} \in \mathcal{C}_{\mathcal{M}(n)}$. Moreover, if we wish to make $B_{n,n+1}$ Hankel, it is clear that all but the entry in $\langle X^{n+1}, X^n \rangle$ in the column $[X^{n+1}]_{m(n)}$ must be given in terms of entries in $\mathcal{M}(n)$, and that all but the entry in $\langle Y^{n+1}, Y^n \rangle$ in $[Y^{n+1}]_{m(n)}$ must be given in terms of entries in $\mathcal{M}(n)$. To handle the remaining entries we introduce two parameters p and q; concretely, for $i + j = 0, \dots, n$,

$$
\langle X^{n+1}, Y^i X^j \rangle := \begin{cases} \langle Y X^n, Y^{i-1} X^{j+1} \rangle & (1 \le i \le n) \\ \beta_{0,n+j+1} & (i = 0,\ 0 \le j \le n-1) \\ p & (i = 0,\ j = n) \end{cases}, \tag{A.13}
$$

$$\left\langle Y^{n+1}, Y^i X^j \right\rangle := \begin{cases} \left\langle Y^n X, Y^{i+1} X^{j-1} \right\rangle & (1 \leq j \leq n) \\ \beta_{n+i+1,0} & (j = 0,\ 0 \leq i \leq n-1) \\ q & (j = 0,\ i = n) \end{cases} . \tag{A.14}$$

A positive extension $\mathcal{M}(n+1)$ entails Ran $B \subseteq$ Ran $\mathcal{M}(n)$, so in particular, we must show that $[X^{n+1}]_{m(n)}, [Y^{n+1}]_{m(n)} \in$ Ran $\mathcal{M}(n)$. To this end, note that since $N := [\mathcal{M}(n)]_{\mathcal{S}_n(n)} > 0$, there exists vectors $\mathbf{f}, \mathbf{g} \in \mathbb{R}^{2n+1}$ such that $N\mathbf{f} = [X^{n+1}]_{\mathcal{S}_n(n)}$ and $N\mathbf{g} = [Y^{n+1}]_{\mathcal{S}_n(n)}$. Let $\mathbf{F}, \mathbf{G} \in \mathbb{R}^{m(n)}$ be given by

$$\left\langle \mathbf{F}, Y^i X^j \right\rangle := \begin{cases} \left\langle \mathbf{f}, Y^i X^j \right\rangle & \text{if } Y^i X^j \in \mathcal{S}_n(n) \\ 0 & \text{otherwise} \end{cases} \tag{A.15}$$

and

$$\left\langle \mathbf{G}, Y^i X^j \right\rangle := \begin{cases} \left\langle \mathbf{g}, Y^i X^j \right\rangle & \text{if } Y^i X^j \in \mathcal{S}_n(n) \\ 0 & \text{otherwise} \end{cases} . \tag{A.16}$$

We observe for future reference, that since $\mathbf{f} = N^{-1}[X^{n+1}]_{\mathcal{S}_n(n)}$, $\mathbf{f}$ is a linear map in p (and independent of q), and so also is $\mathbf{F}$; similarly $\mathbf{g}$ and $\mathbf{G}$ are linear in q and independent of p.
Claim. $\mathcal{M}(n)\mathbf{F} = [X^{n+1}]_{m(n)}$; equivalently,

$$\left\langle \mathcal{M}(n)\mathbf{F}, Y^i X^j \right\rangle = \left\langle [X^{n+1}]_{m(n)}, Y^i X^j \right\rangle$$

for each $(i, j) \in I_u := \{(i, j) : i + j \leq n \text{ and } ((i = u \leq j \leq n) \text{ or } (j = u \leq i \leq n))\}$, $u = 0, 1, \ldots, [\frac{n}{2}]$. The proof of the claim is by induction on u. For $u = 0$ we consider $Z = Y^i$ or $Z = X^j$ in $\mathcal{S}_n(n)$, so

$$\begin{aligned} \langle \mathcal{M}(n)\mathbf{F}, Z \rangle &= \sum_{Y^k X^\ell \in \mathcal{C}_{\mathcal{M}(n)}} \left\langle Y^k X^\ell, Z \right\rangle \left\langle \mathbf{F}, Y^k X^\ell \right\rangle \\ &= \sum_{Y^k X^\ell \in \mathcal{S}_n(n)} \left\langle Y^k X^\ell, Z \right\rangle \left\langle \mathbf{f}, Y^k X^\ell \right\rangle + \sum_{Y^k X^\ell \notin \mathcal{S}_n(n)} \left\langle Y^k X^\ell, Z \right\rangle \cdot 0 \\ &= \langle N\mathbf{f}, Z \rangle = \left\langle [X^{n+1}]_{m(n)}, Z \right\rangle , \end{aligned}$$

as desired. We must now deal with rows of the form $Y^i X^j (i, j \geq 1)$; that is, we must prove that $\left\langle \mathcal{M}(n)\mathbf{F}, Y^i X^j \right\rangle = \left\langle [X^{n+1}]_{m(n)}, Y^i X^j \right\rangle$ for $i, j \geq 1$ and $i + j \leq n$. Assume that the claim is true for $u = k$ $(0 \leq k \leq [\frac{n}{2}] - 1)$, and consider $(i, j) \in I_{k+1}$. We have

$$\begin{aligned} \left\langle [X^{n+1}]_{m(n)}, Y^i X^j \right\rangle &\equiv \left\langle X^{n+1}, Y^i X^j \right\rangle = \left\langle Y X^n, Y^{i-1} X^{j+1} \right\rangle \text{ (by (A.13))} \\ &= \left\langle X^{n-1}, Y^{i-1} X^{j+1} \right\rangle . \end{aligned}$$

On the other hand,

$$\begin{aligned}\left\langle \mathcal{M}(n)\mathbf{F}, Y^iX^j\right\rangle &= \left\langle \mathcal{M}(n)\mathbf{F}, Y^{i-1}X^{j-1}\right\rangle \text{ (by Lemma A.1.3(iii))}\\ &= \left\langle [X^{n+1}]_{m(n)}, Y^{i-1}X^{j-1}\right\rangle \text{ (by the inductive step)}\\ &= \left\langle X^{n+1}, Y^{i-1}X^{j-1}\right\rangle.\end{aligned}$$

It thus suffices to prove that $\left\langle X^{n+1}, Y^{i-1}X^{j-1}\right\rangle = \left\langle X^{n-1}, Y^{i-1}X^{j+1}\right\rangle$. For $i=1$,

$$\begin{aligned}\left\langle X^{n+1}, X^{j-1}\right\rangle &= \beta_{0,n+j} \text{ (by (A.13))}\\ &= \left\langle X^{n-1}, X^{j+1}\right\rangle,\end{aligned}$$

and for $i>1$,

$$\begin{aligned}\left\langle X^{n+1}, Y^{i-1}X^{j-1}\right\rangle &= \left\langle YX^n, Y^{i-2}X^j\right\rangle \text{ (by (A.13))}\\ &= \left\langle X^{n-1}, Y^{i-2}X^j\right\rangle = \left\langle X^{n-1}, Y^{i-1}X^{j+1}\right\rangle \text{ (by Lemma A.1.3(iii))}\end{aligned}$$

This completes the proof of the claim. Using an entirely similar argument using $\mathbf{g}$ instead of $\mathbf{f}$ and (A.14) instead of (A.13) shows that $\mathcal{M}(n)\mathbf{G} = [Y^{n+1}]_{m(n)}$. Moreover, by definition, $[Y^iX^j]_{m(n)} = Y^{i-1}X^{j-1} \in \mathcal{C}_{\mathcal{M}(n)}$ $(i+j=n+1; i,j\geq 1)$, so we now have Ran $B \subseteq$ Ran $\mathcal{M}(n)$; in particular, there exists W such that $\mathcal{M}(n)W = B$.

We note the following for future reference. From Lemma A.1.2 and the fact that $\mathcal{M}(n) = \mathcal{M}(n)^T$, each row of $\mathcal{M}(n)$ coincides with a row indexed by an element of $\mathcal{S}_n(n)$. Since $B = \mathcal{M}(n)W$, it now follows that each row of $(\mathcal{M}(n)\ B)$ coincides with a row of $(\mathcal{M}(n)\ B)$ indexed by an element of $\mathcal{S}_n(n)$.

We now form the flat extension $M := [\mathcal{M}(n); B] = \begin{pmatrix}\mathcal{M}(n) & B\\ B^* & C\end{pmatrix}$, where $C := W^*\mathcal{M}(n)W$. Exactly as in the proof of Proposition A.1.7, C is of the form

$$C = \begin{pmatrix}\tau & \beta_{0,2n} & \cdots & \beta_{02} & \eta\\ \beta_{0,2n} & \beta_{0,2n-2} & \cdots & \beta_{00} & \beta_{20}\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \beta_{02} & \beta_{00} & \cdots & \beta_{2n-2,0} & \beta_{2n,0}\\ \eta & \beta_{20} & \cdots & \beta_{2n,0} & \rho\end{pmatrix},$$

where $C_{11} \equiv \tau := [X^{n+1}]^T_{\mathcal{S}_n(n)} N^{-1} [X^{n+1}]_{\mathcal{S}_n(n)}$ and $C_{1,n+2} = C_{n+2,1} \equiv \eta := [X^{n+1}]^T_{\mathcal{S}_n(n)} N^{-1}[Y^{n+1}]_{\mathcal{S}_n(n)}$. Thus, if $\eta = \beta_{00}$, then M is a flat moment matric extension of the form $\mathcal{M}(n+1)$, and we are done.

Assume now that $\eta \neq \beta_{00}$. Let $u > \tau$ be arbitrary, and consider the moment matrix $M' \equiv \mathcal{M}(n+1)'$ obtained from M by replacing τ by u and η by β_{00}. We partition M' as $M' \equiv \begin{pmatrix}\tilde{\mathcal{M}} & \tilde{B}\\ \tilde{B}^* & \tilde{C}\end{pmatrix}$ where $\tilde{\mathcal{M}}$ is the compression of M' to rows and columns indexed by $\tilde{\mathcal{B}} :=$

$\{\mathbb{1}, X, Y, X^2, Y^2, \ldots, X^n, YX^{n-1}, \ldots, Y^{n-1}X, Y^n, X^{n+1}\} \subseteq \mathcal{C}_{M'}$ (i.e., $\tilde{\mathcal{M}}$ is extension of $\mathcal{M}(n)$ by row X^{n+1} and column X^{n+1} of M').

We claim that Ran $\tilde{B} \subseteq$ Ran $\tilde{\mathcal{M}}$. By the flat construction of M the middle n columns of $\begin{pmatrix} B \\ C \end{pmatrix}$ are borrowed from columns of $\begin{pmatrix} \mathcal{M}(n) \\ B^* \end{pmatrix}$ of degree $n-1$, so, in particular, the columns of $\tilde{B}$ (except the rightmost column) are borrowed from columns in $\tilde{\mathcal{M}}$. To prove the claim, it thus suffices to show that $[Y^{n+1}]_{\tilde{\mathcal{B}}} \in$ Ran $\mathcal{M}$. Since $u > \tau$, $\tilde{\mathcal{S}} := \{\mathbb{1}, X, Y, X^2, Y^2, \ldots, X^n, Y^n, X^{n+1}\}$ is a basis for $\mathcal{C}_{\tilde{\mathcal{M}}}$, and $[\tilde{\mathcal{M}}]_{\tilde{\mathcal{S}}}$ is positive and invertible. Thus there exists unique scalars $a_1, a_2, \ldots, a_{2n+2}$ such that in $\mathcal{C}_{(\tilde{\mathcal{M}}\ \tilde{B})}$, we have

$$[Y^{n+1}]_{\tilde{\mathcal{S}}} = a_1[\mathbb{1}]_{\tilde{\mathcal{S}}} + a_2[X]_{\tilde{\mathcal{S}}} + \cdots + a_{2n+2}[X^{n+1}]_{\tilde{\mathcal{S}}}.$$

From the first part of the proof (concerning block B), we know that each row of $(\tilde{\mathcal{M}}\ \tilde{B})$ coincides with a row indexed by an element of $\tilde{\mathcal{S}}$, so it now follows that, in $\mathcal{C}_{(\tilde{\mathcal{M}}\ \tilde{B})}$,

$$[Y^{n+1}]_{\tilde{\mathcal{B}}} = a_1[\mathbb{1}]_{\tilde{\mathcal{B}}} + a_2[X]_{\tilde{\mathcal{B}}} + \cdots + a_{2n+2}[X^{n+1}]_{\tilde{\mathcal{B}}}, \tag{A.17}$$

whence the claim is proved.

Since $\tilde{\mathcal{M}} \geq 0$ and Ran $\tilde{B} \subseteq$ Ran $\tilde{\mathcal{M}}$, we may construct the (positive)flat extension $M^\flat := [\tilde{\mathcal{M}}; \tilde{B}] \equiv \begin{pmatrix} \tilde{\mathcal{M}} & \tilde{B} \\ \tilde{B}^* & D \end{pmatrix}$ which we may repartition as the moment matrix $\mathcal{M}(n+1) = \begin{pmatrix} \mathcal{M}(n) & B \\ B^* & C^\flat \end{pmatrix}$, where $C^\flat$ is obtained from C by replacing τ by u, η by β_{00} and ρ by some $\rho^\flat$ (determined by extending (A.17) to the full columns of $M^\flat$).

Now $\mathcal{M}(n+1)$ is positive, recursively generated, satisfies $YX = \mathbb{1}$ and (by the flatness of $M^\flat$), rank $\mathcal{M}(n+1) =$ rank $\tilde{\mathcal{M}} = 1 +$ rank $\mathcal{M}(n)$. In $\mathcal{S}_{n+1}(n+1)$, the first dependence relation is of the form $Y^{n+1} = a_1\mathbb{1} + a_2X + \cdots + a_{2n+2}X^{n+1}$, and we assert that $a_{2n+2} \neq 0$. Indeed, if $a_{2n+2} = 0$, then $[Y^{n+1}]_{\mathcal{S}_n(n)} = a_1[\mathbb{1}]_{\mathcal{S}_n(n)} + a_2[X]_{\mathcal{S}_n(n)} + \cdots + a_{2n+1}[Y^n]_{\mathcal{S}_n(n)}$, whence $(a_1, \ldots, a_{2n+1})^T = N^{-1}[Y^{n+1}]_{\mathcal{S}_n(n)}$. Now we have

$$\begin{aligned}
\beta_{00} &= \left\langle [Y^{n+1}]_{\tilde{\mathcal{B}}}, X^{n+1} \right\rangle \\
&= a_1 \left\langle [\mathbb{1}]_{\tilde{\mathcal{B}}}, X^{n+1} \right\rangle + a_2 \left\langle [X]_{\tilde{\mathcal{B}}}, X^{n+1} \right\rangle + \cdots + \ a_{2n+1} \left\langle [Y^n]_{\tilde{\mathcal{B}}}, X^{n+1} \right\rangle \\
&= [X^{n+1}]^T_{\mathcal{S}_n(n)} \cdot (a_1, \ldots, a_{2n+1})^T \\
&= [X^{n+1}]^T_{\mathcal{S}_n(n)} N^{-1} [Y^{n+1}]_{\mathcal{S}_n(n)} = \eta,
\end{aligned}$$

a contradiction. Since $a_{2n+2} \neq 0$, we may now proceed exactly as in the proof of Proposition A.1.7 (beginning at (A.5) and replacing n by $n+1$) to conclude that $\mathcal{M}(n+1)$ admits a flat extension $\mathcal{M}(n+2)$. □

Proof of Theorem ***A.1.1***. Straightforward from Propositions A.1.5, A.1.6, A.1.7 and A.1.13. □

Theorem A.1.1 shows that Theorem 2.0.2 is true for any hyperbolic polynomial. Next we present the proof of Theorem 2.0.2 for parabolas.

A.2 Parabolas

We replicate in this section the proof of Theorem 2.0.2 for parabolas. We've restated below the sufficiency required.

Theorem A.2.1. *Let $\beta \equiv \beta^{(2n)} : \beta_{00}, \beta_{01}, \beta_{10}, \ldots, \beta_{0,2n}, \ldots, \beta_{2n,0}$ be a family of real numbers with $\beta_{00} > 0$, and let $\mathcal{M}(n)$ be the associated multivariate Hankel matrix. β admits a representing measure supported in $y = x^2$ if and only if $\mathcal{M}(n)$ is positive, recursively generated, satisfies $Y = X^2$ and* rank $\mathcal{M}(n) \leq$ card $\mathcal{V}(\beta)$. *In this case, $\mathcal{M}(n)$ admits a flat extension $\mathcal{M}(n+1)$ and β admits a* rank $\mathcal{M}(n)$*-atomic (minimal) representing measure.*

The necessity of these conditions is a clear consequence of results previously attained by Curto and Fialkow. We next prove the sufficiency, which is the following result.

Theorem A.2.2. *Let $\beta \equiv \beta^{(2n)} : \beta_{00}, \beta_{01}, \beta_{10}, \ldots, \beta_{0,2n}, \ldots, \beta_{2n,0}$ be a family of real numbers with $\beta_{00} > 0$, and let $\mathcal{M}(n)$ be the associated multivariate Hankel matrix. Assume that $\mathcal{M}(n)$ is positive, recursively generated and satisfies $Y = X^2$ and* rank $\mathcal{M}(n) \leq$ card $\mathcal{V}(\beta)$. *Then $\mathcal{M}(n)$ admits a flat extension $\mathcal{M}(n+1)$.*

Again, we know that $\mathcal{M}(n)$ admits a flat extension when the set column vectors $\{\mathbb{1}, X, Y\}$ are linearly dependent. So we continue to assume that this set is linearly independent. We begin by exploiting the fact that $\mathcal{M}(n)$ is recursively generated. For $1 \leq k \leq n$ let

$$\mathcal{S}_n(k) := \{\mathbb{1}, X, Y, YX, Y^2, Y^2X, Y^3, \ldots, Y^{k-1}X, Y^k\} \subseteq \mathcal{C}_{\mathcal{M}(n)}.$$

Lemma A.2.3. *For $n \geq 2$ let $\mathcal{M}(n)$ be positive and recursively generated, and assume that $Y = X^2$. Then $\mathcal{S}_n(n)$ spans $\mathcal{C}_{\mathcal{M}(n)}$, and therefore* rank $\mathcal{M}(n) \leq 2n+1$*; moreover, each column of $\mathcal{M}(n)$ is equal to a column in $\mathcal{S}_n(n)$.*

Proof. The proof is by induction on $n \geq 2$. For $n = 2$ the statement is clearly true, so assume it holds for $n = k$. Suppose $\mathcal{M}(k+1)$ is positive and recursively generated with $Y = X^2$ in $\mathcal{C}_{\mathcal{M}(k+1)}$. Consider a column in $\mathcal{M}(k+1)$ of the form $Y^{k+1-j}X^j$, with $2 \leq j \leq k+1$. Let $q(x,y) := y - x^2$ and let $p_{ij}(x,y) := y^i x^j$ so that $Y^{k+1-j}X^j = p_{k+1-j,j}(X,Y)$. Also, let $r_{ij}(x,y) := y^{k+2-j}x^{j-2} - y^{k+1-j}x^j$. Since $j \geq 2$ it is straightforward to verify that $r_{ij}(x,y) = y^{k+1-j}x^{j-2}(y - x^2) = p_{k+1-j,j-2}(x,y)q(x,y)$. Since $\mathcal{M}(k+1)$ is recursively generated and $q(X,Y) = 0$, it follows that $r_{ij}(X,Y) = 0$, that is, $Y^{k+2-j}X^{j-2} = Y^{k+1-j}X^j$ in $\mathcal{C}_{\mathcal{M}(k+1)}$. By induction, $[Y^{k+2-j}X^{j-2}]_{m(k)} \in$ lin.span $\mathcal{S}_k(k)$, and since $\mathcal{M}(k+1) \geq 0$, it follows by the extension principle [13] that $Y^{k+2-j}X^{j-2} \in$ lin.span $\mathcal{S}_{k+1}(k)$. Thus $Y^{k+1-j}X^j$ $(= Y^{k+2-j}X^{j-2}) \in$ lin.span $\mathcal{S}_{k+1}(k) \subseteq$ lin.span $\mathcal{S}_{k+1}(k+1)$, as desired. □

We next divide the proof of Theorem A.2.2 into five cases, based on the possible linear dependencies among the elements of $\mathcal{S}_n(n)$. For the remainder of this section (unless otherwise stated) we will be assuming that $\mathcal{M}(n)$ is positive, recursively generated, satisfies $Y = X^2$ and rank $\mathcal{M}(n) \leq$ card $\mathcal{V}(\beta)$.

Proposition A.2.4. *In $\mathcal{S}_n(n)$ assume the first linear dependence relation occurs at $Y^k X$, with $2 \leq k < n-1$. Then $\mathcal{M}(n)$ is flat and, a fortiori, it admits a unique flat extension $\mathcal{M}(n+1)$.*

Proof. Write

$$Y^k X = p_k(Y) + q_{k-1}(Y)X, \tag{A.18}$$

where $\deg p_k \leq k, \deg q_{k-1} \leq k-1$. Since $Y^k X$ corresponds to a monomial of degree at most $n-1$, and since $Y = X^2$ and $\mathcal{M}(n)$ is recursively generated, we must have that $Y^k X^2 = Y^{k+1} = p_k(Y)X + q_{k-1}(Y)Y$. Substituting from (A.18) for the $Y^k X$ term in $p_k(Y)X$, we see that both $Y^k X$ and Y^{k+1} are linear combinations of columns of corresponding to monomials of degree at most k. It now follows from recursiveness and from Lemma A.2.3 that $\mathcal{M}(n)$ is flat, so there exists a (unique) flat extension $\mathcal{M}(n+1)$. □

Proposition A.2.5. *In $\mathcal{S}_n(n)$, assume that the first dependence relation occurs at Y^k, with $k < n$. Then $\mathcal{M}(n)$ is flat and thus admits a unique flat extension $\mathcal{M}(n+1)$.*

Proof. Write

$$Y^k = p_{k-1}(Y) + q_{k-1}(Y)X, \tag{A.19}$$

where $\deg p_{k-1}, \deg q_{k-1} \leq k-1$. Since $k < n$, $Y = X^2$, and $\mathcal{M}(n)$ is recursively generated, we must have $Y^k X = p_{k-1}(Y)X + q_{k-1}(Y)Y$. We thus see that $Y^k X$ is a linear combination of columns in $\mathcal{S}_n(k)$. On the other hand from (A.19) it follows that $Y^{k+1} = p_{k-1}(Y)Y + q_{k-1}(Y)YX$. If $\deg q_{k-1} = k-1$, we may apply (A.19) to the Y^k term in $q_{k-1}(Y)Y$, and also use $Y = X^2$, to see that Y^{k+1} is a linear combination of columns $\mathcal{S}_n(k)$. It now follows (via recursiveness and Lemma A.2.3) that $\mathcal{M}(n)$ is flat and thus admits a unique flat extension $\mathcal{M}(n+1)$. □

Proposition A.2.6. *In $\mathcal{S}_n(n)$ assume that the first dependency relation occurs at $Y^{n-1}X$. Then $\mathcal{M}(n)$ is flat and thus admits a unique flat extension $\mathcal{M}(n+1)$.*

Proof. Write $Y^{n-1}X = p_{n-1}(Y) + q_{n-2}(Y)X$, with $\deg p_{n-1} \leq n-1$ and $\deg q_{n-2} \leq n-2$, and let $r(x,y) := y^{n-1}x - (p_{n-1}(y) + q_{n-2}(y)x)$ and $s(x,y) := y - x^2$. It follows that $\mathcal{V}(\beta) \subseteq \mathcal{Z}(r) \cap \mathcal{Z}(s)$. Now observe that if we substitute $y = x^2$ in $r(x,y) = 0$, we obtain a polynomial equation in x of degree at most $2n-1$. It then follows that card $\mathcal{V}(\beta) \leq 2n-1$, so that rank $\mathcal{M}(n) \leq$ card $\mathcal{V}(\beta) \leq 2n-1$. Then $\mathcal{S}_n(n-1) = \{\mathbb{1}, X, Y, YX, Y^2, Y^2X, \ldots, Y^{n-2}X, Y^{n-1}\}$ is a basis for $\mathcal{C}_{\mathcal{M}(n)}$, whence Y^n is a linear combination of columns in $\mathcal{S}_n(n-1)$. Since, by recursiveness, the columns $Y^i X^j$, with $i+j=n$ and $j \geq 2$, coincide with columns of lower degree, it now follows that $\mathcal{M}(n)$ is flat, and thus admits a unique flat extension $\mathcal{M}(n+1)$. □

Proposition A.2.7. *In $\mathcal{S}_n(n)$ assume the first dependency relation occurs at Y^n. Then $\mathcal{M}(n)$ admits a unique flat extension $\mathcal{M}(n+1)$.*

Under the hypothesis of Proposition A.2.7, write

$$Y^n = p_{n-1}(Y) + q_{n-1}(Y)X, \tag{A.20}$$

where $\deg p_{n-1}, \deg q_{n-1} \leq n-1$. (The expression $p_{n-1}(Y) + q_{n-1}(Y)X$ is shorthand notation for $(p_{n-1} + xq_{n-1})(X,Y)$.) To build a flat extension $\mathcal{M}(n+1)$, we define the first n columns of a prospective B block by exploiting the relation $Y = X^2$, as follows: $X^{n+1} := YX^{n-1}, YX^n := Y^2X^{n-2}, \ldots, Y^{n-1}X^2 := Y^n$. Also, using (A.20), we let

$$Y^nX := p_{n-1}(Y)X + q_{n-1}(Y)Y \in \mathcal{C}_{\mathcal{M}(n)} \tag{A.21}$$

(where $p_{n-1}(Y)X + q_{n-1}(Y)Y = (xp_{n-1} + yq_{n-1})(X,Y)$), and using (A.20) and (A.21), we let

$$Y^{n+1} := p_{n-1}(Y)Y + q_{n-1}(Y)YX \in \mathcal{C}_{\mathcal{M}(n)} \tag{A.22}$$

(where $p_{n-1}(Y)Y + q_{n-1}(Y)YX = (yp_{n-1} + xyq_{n-1})(X,Y)$). (Observe that these defining relations are all required if one is to obtain a positive recursively generated multivariate Hankel matrix extension for $\mathcal{M}(n)$.) Since the columns (A.20)-(A.22) belong to $\mathcal{C}_{\mathcal{M}(n)}$, we have that $B = \mathcal{M}(n)W$ for some matrix W. Thus a flat extension $\mathcal{M} := [\mathcal{M}(n); B]$ is uniquely determined by defining the C block as $C := W^*\mathcal{M}(n)W$. To complete the proof that $\mathcal{M}$ is a multivariate Hankel matrix $\mathcal{M}(n+1)$, is suffices to show that the B block is of the form $(B_{i,n+1})_{i=0}^n$ and that block C is of the form $B_{n+1,n+1}$.

To this end we require some additional notation and several preliminary results. Recall that for $i+j, k+\ell \leq n$, we have

$$\left\langle Y^iX^j, Y^kX^\ell \right\rangle \equiv \left\langle Y^iX^j, Y^kX^\ell \right\rangle_{\mathcal{M}(n)} = \beta_{i+k,j+\ell}.$$

For $p, q \in \mathbb{R}[x,y]_n$, $p(x,y) \equiv \sum_{0\leq i+j\leq n} a_{ij}y^ix^j$, $q(x,y) \equiv \sum_{0\leq k+\ell\leq n} b_{k\ell}y^kx^\ell$, we define

$$\langle p(X,Y), q(X,Y)\rangle = \sum_{0\leq i+j,k+\ell\leq n} a_{ij}b_{k\ell}\left\langle Y^iX^j, Y^kX^\ell \right\rangle = \sum_{0\leq i+j,k+\ell\leq n} a_{ij}b_{k\ell}\beta_{i+k,j+\ell}.$$

The following result follows directly from the preceding definitions.

Lemma A.2.8. *(i) For $p, q \in \mathbb{R}[x,y]_n$,*

$$\langle p(X,Y), q(X,Y)\rangle = \langle q(X,Y), p(X,Y)\rangle.$$

(ii) For $p, q \in \mathbb{R}[x,y]_n$, $i, j \geq 0$, $i+j \leq n$, and $\deg p, \deg q \leq n-(i+j)$,

$$\left\langle p(X,Y)Y^jX^i, q(X,Y)\right\rangle = \left\langle p(X,Y), q(X,Y)Y^jX^i\right\rangle.$$

(iii) If $p, q, r \in \mathbb{R}[x, y]_n$ with $p(X, Y) = q(X, Y)$ in $\mathcal{C}_{\mathcal{M}(n)}$, then

$$\langle r(X, Y), p(X, Y)\rangle = \langle r(X, Y), q(X, Y)\rangle .$$

We next extend the notation $\langle p(X, Y), q(X, Y)\rangle$ to the case when $\deg p = n+1$, $\deg q \le n$. Indeed, using the definitions of the columns of B, for $i, j \ge 0$, $i + j = n + 1$, there exists $p_{ij} \in \mathbb{R}[x, y]_n$ with $Y^i X^j = p_{ij}(X, Y)$, and we define

$$\left\langle Y^i X^j, q(X, Y)\right\rangle := \langle p_{ij}(X, Y), q(X, Y)\rangle .$$

Now if $p(x, y) \equiv \sum_{0\le k+\ell\le n+1} a_{k\ell} y^k x^\ell$, we define

$$\langle p(X, Y), q(X, Y)\rangle := \sum_{0\le k+\ell\le n+1} a_{k\ell} \left\langle Y^k X^\ell, q(X, Y)\right\rangle .$$

It is easy to check that Lemma A.2.8(iii) hold with $\deg r = n + 1$.

Lemma A.2.9. *Assume $i + j = n + 1$, $s \ge 2$ and $r + s \le n$. Then*

$$\left\langle Y^i X^j, Y^r X^s\right\rangle = \left\langle Y^i X^j, Y^{r+1} X^{s-2}\right\rangle . \tag{A.23}$$

Proof. Fix i and j with $i + j = n + 1$. We know that there exists a polynomial $p \in \mathbb{R}[x, y]_n$ such that $Y^i X^j = p(X, Y) \equiv \sum_{0\le k+\ell\le n} a_{k\ell} Y^k X^\ell$. Then

$$\begin{aligned}\left\langle Y^i X^j, Y^r X^s\right\rangle &= \sum_{k+\ell\le n} a_{k\ell} \left\langle Y^k X^\ell, Y^r X^s\right\rangle = \sum_{k+\ell\le n} a_{k\ell} \left\langle Y^r X^s, Y^k X^\ell\right\rangle \\ &\text{(because } \mathcal{M}(n) \text{ is self-adjoint)} \\ &= \sum_{k+\ell\le n} a_{k\ell} \left\langle Y^{r+1} X^{s-1}, Y^k X^\ell\right\rangle \text{ (using } Y = X^2\text{)} \\ &= \sum_{k+\ell\le n} a_{k\ell} \left\langle Y^k X^\ell, Y^{r+1} X^{s-1}\right\rangle \\ &\text{(using self-adjointness again)} \\ &= \left\langle Y^i X^j, Y^{r+1} X^{s-1}\right\rangle ,\end{aligned}$$

as desired. □

Corollary A.2.10. *Assume $i + j = n + 1$, with $j \ge 1$, and assume that the Hankel property*

$$\left\langle Y^i X^j, Y^r X^s\right\rangle = \left\langle Y^{i+1} X^{j-1}, Y^{r-1} X^{s+1}\right\rangle \tag{A.24}$$

holds for all $Y^r X^s \in \mathcal{S}_n(n)$ with $r \ge 1$. Then (A.24) holds for all r and s such that $1 \le r + s \le n$, $r \ge 1$.

Proof. Fix i and j with $i+j=n+1$. We do induction on $t:=r+s$, where $1 \leq r+s \leq n$, $r \geq 1$. For $t=1$ the result is clear, since $Y \in \mathcal{S}_n(n)$, and for $t=2$ the result follows from the fact that YX and Y^2 are in $\mathcal{S}_n(n)$. Assume the statement is true for $t=u \geq 2$, and consider the case $t=u+1$. For Y^rX^s with $r+s=u+1$, we may assume that $Y^rX^s \notin \mathcal{S}_n(n)$, whence $s \geq 2$. Now

$$\begin{aligned}\left\langle Y^iX^j, Y^rX^s\right\rangle &= \left\langle Y^iX^j, Y^{r+1}X^{s-2}\right\rangle \text{ (by (A.23))}\\ &= \left\langle Y^{i+1}X^{j-1}, Y^rX^{s-1}\right\rangle \text{ (by the inductive step and (A.24))}\\ &= \left\langle Y^{i+1}X^{j-1}, Y^{r-1}X^{s+1}\right\rangle \text{ (by (A.23) again),}\end{aligned}$$

as desired. □

Lemma A.2.11. *For* $k=0,\ldots,n-2$,

$$\left\langle Y^nX, Y^k\right\rangle = \left\langle Y^n, Y^kX\right\rangle. \tag{A.25}$$

Proof.

$$\begin{aligned}\left\langle Y^nX, Y^k\right\rangle &= \left\langle p_{n-1}(Y)X + q_{n-1}(Y)Y, Y^k\right\rangle \text{ (by (A.21))}\\ &= \left\langle p_{n-1}(Y), Y^kX\right\rangle + \left\langle q_{n-1}(Y), Y^{k+1}\right\rangle \text{ (by Lemma A.2.8(ii))}\\ &= \left\langle p_{n-1}(Y), Y^kX\right\rangle + \left\langle q_{n-1}(Y), Y^kX^2\right\rangle\\ &\text{(using } Y=X^2\text{, since } k \leq n-2\text{, and Lemma A.2.8(iii))}\\ &= \left\langle p_{n-1}(Y), Y^kX\right\rangle + \left\langle q_{n-1}(Y)X, Y^kX\right\rangle \text{ (by Lemma A.2.8(ii))}\\ &= \left\langle p_{n-1}(Y) + q_{n-1}(Y)X, Y^kX\right\rangle\\ &= \left\langle Y^n, Y^kX\right\rangle \text{ (by (A.20)),}\end{aligned}$$

as desired. □

Proof of Proposition A.2.7. The first part of the proof is devoted to showing that the B block, as defined above, is of the form $\{B_{i,n+1}\}_{i=0}^n$. To this end, and since the first n columns of B are taken, as a package, from columns in $\mathcal{M}(n)$, it suffices to prove that the last three columns of B, namely $Y^{n-1}X^2, Y^nX$, and Y^{n+1} satisfy the proper Hankel conditions. From Corollary A.2.10, we can restrict out attention to rows corresponding to monomials of the form Y^kX $(k=1,\ldots,n-1)$ and Y^{k+1}, for $k=0,\ldots,n-1$. We shall establish that

$$\begin{cases}\text{(i) } \left\langle Y^{n-1}X^2, Y^kX\right\rangle = \left\langle Y^nX, Y^{k-1}X^2\right\rangle & \text{if } 1 \leq k \leq n-1\\ \text{(ii) } \left\langle Y^{n-1}X^2, Y^{k+1}\right\rangle = \left\langle Y^nX, Y^kX\right\rangle & \text{if } 0 \leq k \leq n-1\\ \text{(iii) } \left\langle Y^nX, Y^kX\right\rangle = \left\langle Y^{n+1}, Y^{k-1}X^2\right\rangle & \text{if } 1 \leq k \leq n-1\\ \text{(iv) } \left\langle Y^nX, Y^{k+1}\right\rangle = \left\langle Y^{n+1}, Y^kX\right\rangle & \text{if } 0 \leq k \leq n-1.\end{cases} \tag{A.26}$$

We first consider rows of B corresponding to monomials of total degree at most $n-1$. To establish (A.26)(i) for $k \le n-2$, we calculate

$$\left\langle Y^{n-1}X^2, Y^kX \right\rangle = \left\langle Y^n, Y^kX \right\rangle = \left\langle Y^nX, Y^k \right\rangle \text{ (since } Y^{n-1}X^2 = Y^n\text{, and by (A.25))}$$
$$= \left\langle Y^nX, Y^{k-1}X^2 \right\rangle \text{ (by (A.23)).}$$

To verify (A.26)(ii) for $k \le n-2$, we have

$$\left\langle Y^{n-1}X^2, Y^{k+1} \right\rangle = \left\langle Y^n, Y^{k+1} \right\rangle = \left\langle p_{n-1}(Y) + q_{n-1}(Y)X, Y^{k+1} \right\rangle \text{ (by (A.20))}$$
$$= \left\langle p_{n-1}(Y), Y^{k+1} \right\rangle + \left\langle q_{n-1}(Y)X, Y^{k+1} \right\rangle$$
$$= \left\langle Y^{k+1}, p_{n-1}(Y) \right\rangle + \left\langle q_{n-1}(Y)X, Y^{k+1} \right\rangle \text{ (by Lemma A.2.8(i))}$$
$$= \left\langle Y^kX^2, p_{n-1}(Y) \right\rangle + \left\langle q_{n-1}(Y)X, Y^{k+1} \right\rangle \text{ (using } Y = X^2\text{)}$$
$$= \left\langle p_{n-1}(Y), Y^kX^2 \right\rangle + \left\langle q_{n-1}(Y)X, Y^{k+1} \right\rangle \text{ (by Lemma A.2.8(i))}$$
$$= \left\langle p_{n-1}(Y)X, Y^kX \right\rangle + \left\langle q_{n-1}(Y), Y^{k+1}X \right\rangle$$
$$= \left\langle p_{n-1}(Y)X, Y^kX \right\rangle + \left\langle q_{n-1}(Y)Y, Y^kX \right\rangle$$
$$= \left\langle p_{n-1}(Y)X + q_{n-1}(Y)Y, Y^kX \right\rangle$$
$$= \left\langle Y^nX, Y^kX \right\rangle \text{ (by (A.21)).}$$

Next, consider (A.26)(iii) with $k \le n-2$. Write $q_{n-1}(Y) \equiv r_{n-2}(Y) + c_{n-1}Y^{n-1}$, with $\deg r_{n-2} \le n-2$; then

$$\left\langle Y^{n+1}, Y^{k-1}X^2 \right\rangle = \left\langle p_{n-1}(Y)Y + q_{n-1}(Y)YX, Y^{k-1}X^2 \right\rangle \text{ (by (A.22))}$$
$$= \left\langle p_{n-1}(Y), Y^kX^2 \right\rangle + \left\langle [r_{n-2}(Y) + c_{n-1}Y^{n-1}]YX, Y^k \right\rangle \text{ (by (A.23))}$$
$$= \left\langle p_{n-1}(Y)X, Y^kX \right\rangle + \left\langle r_{n-2}(Y)Y, Y^kX \right\rangle + c_{n-1}\left\langle Y^nX, Y^k \right\rangle$$
$$= \left\langle p_{n-1}(Y)X, Y^kX \right\rangle + \left\langle r_{n-2}(Y)Y, Y^kX \right\rangle + c_{n-1}\left\langle Y^n, Y^kX \right\rangle$$
(by (A.25))
$$= \left\langle p_{n-1}(Y)X + [r_{n-2}(Y) + c_{n-2}Y^{n-1}]Y, Y^kX \right\rangle$$
$$= \left\langle p_{n-1}(Y)X + q_{n-1}(Y)Y, Y^kX \right\rangle$$
$$= \left\langle Y^nX, Y^kX \right\rangle \text{ (by (A.21)).}$$

Now we prove (A.26)(iv) for $k \le n-2$. We have

$$\begin{aligned}\left\langle Y^nX, Y^{k+1}\right\rangle &= \left\langle p_{n-1}(Y)X + q_{n-1}(Y)Y, Y^{k+1}\right\rangle \\ &= \left\langle p_{n-1}(Y)X, Y^{k+1}\right\rangle + \left\langle q_{n-1}(Y)Y, Y^{k+1}\right\rangle \\ &= \left\langle p_{n-1}(Y), Y^{k+1}X\right\rangle + \left\langle q_{n-1}(Y)Y, Y^{k+1}\right\rangle \\ &\left\langle p_{n-1}(Y)Y, Y^kX\right\rangle + \left\langle q_{n-1}(Y), Y^{k+2}\right\rangle\end{aligned}$$

and

$$\begin{aligned}\left\langle Y^{n+1}, Y^kX\right\rangle &= \left\langle p_{n-1}(Y)Y + q_{n-1}(Y)YX, Y^kX\right\rangle \\ &= \left\langle p_{n-1}(Y)Y, Y^kX\right\rangle + \left\langle q_{n-1}(Y)YX, Y^kX\right\rangle.\end{aligned}$$

Thus, the Hankel condition $\left\langle Y^nX, Y^{k+1}\right\rangle = \left\langle Y^{n+1}, Y^kX\right\rangle$ is satisfied if and only if

$$\left\langle Y^j, Y^{k+2}\right\rangle = \left\langle Y^jYX, Y^kX\right\rangle \ (0 \le j \le n-1).$$

For $j \le n-2$ we have

$$\left\langle Y^j, Y^{k+2}\right\rangle = \left\langle Y^jX^2, Y^{k+1}\right\rangle = \left\langle Y^jX, Y^{k+1}X\right\rangle = \left\langle Y^jYX, Y^kX\right\rangle,$$

and for $j = n-1$,

$$\begin{aligned}\left\langle Y^{n-1}, Y^{k+2}\right\rangle &= \left\langle Y^n, Y^{k+1}\right\rangle = \left\langle Y^{n-1}X^2, Y^{k+1}\right\rangle \text{ (by the definition of } Y^{n-1}X^2\text{)} \\ &= \left\langle Y^nX, Y^kX\right\rangle \text{ (by (A.26)(ii) for } k \le n-2\text{)} \\ &= \left\langle Y^{n-1}YX, Y^kX\right\rangle,\end{aligned}$$

as desired.

We now consider the case of (A.26) when the rows have total degree n, i.e., $k = n-1$. To establish (A.26)(i) for $k = n-1$, we calculate

$$\begin{aligned}\left\langle Y^{n-1}X^2, Y^{n-1}X\right\rangle &= \left\langle Y^n, Y^{n-1}X\right\rangle = \left\langle p_{n-1}(Y) + q_{n-1}(Y)X, Y^{n-1}X\right\rangle \text{ (by (A.20))} \\ &= \left\langle p_{n-1}(Y)X, Y^{n-1}\right\rangle + \left\langle q_{n-1}(Y)X, Y^{n-1}X\right\rangle. \qquad \text{(A.27)}\end{aligned}$$

We next verify that $\left\langle Y^{n-1}X, Y^{n-1}X\right\rangle = \left\langle Y^{n-1}X^2, Y^{n-1}\right\rangle$; indeed,

$$\begin{aligned}\left\langle Y^{n-1}X, Y^{n-1}X\right\rangle &= \left\langle Y^n, Y^{n-2}X^2\right\rangle \text{ (since } B_{n,n} \text{ is Hankel)} \\ &= \left\langle Y^n, Y^{n-1}\right\rangle \text{ (by Lemma A.2.8(iii))} \\ &= \left\langle Y^{n-1}X^2, Y^{n-1}\right\rangle \text{ (by the definition of } Y^{n-1}X^2\text{)}.\end{aligned}$$

Now, the expression in (A.27) coincides with

$$\begin{aligned}\left\langle p_{n-1}(Y)X, Y^{n-1}\right\rangle + \left\langle q_{n-1}(Y)X^2, Y^{n-1}\right\rangle& \\ &= \left\langle p_{n-1}(Y)X + q_{n-1}(Y)Y, Y^{n-1}\right\rangle \\ &= \left\langle Y^n X, Y^{n-1}\right\rangle = \left\langle Y^n X, Y^{n-2}X^2\right\rangle \text{ (by (A.23)).}\end{aligned}$$

For (A.26)(ii) with $k = n-1$, note first that

$$\begin{aligned}\left\langle Y^{n-1}X^2, Y^n\right\rangle &= \langle Y^n, Y^n\rangle = \langle p_{n-1}(Y) + q_{n-1}(Y)X, Y^n\rangle \text{ (by (A.20))} \\ &= \langle p_{n-1}(Y), Y^n\rangle + \langle q_{n-1}(Y)X, Y^n\rangle\,. \qquad \text{(A.28)}\end{aligned}$$

Next, we claim

$$\left\langle p_{n-1}(Y)X, Y^{n-1}X\right\rangle = \langle p_{n-1}(Y), Y^n\rangle\,. \qquad \text{(A.29)}$$

Indeed, for $j \le n-2$, $\left\langle Y^j X, Y^{n-1}X\right\rangle = \left\langle Y^j X^2, Y^{n-1}\right\rangle = \left\langle Y^j Y, Y^{n-1}\right\rangle = \left\langle Y^j, Y^n\right\rangle$, while $\left\langle Y^{n-1}X, Y^{n-1}X\right\rangle = \left\langle Y^n, Y^{n-1}\right\rangle$ (as shown above, in the proof of (A.26) with $k = n-1$). It follows that

$$\begin{aligned}\left\langle Y^n X, Y^{n-1}X\right\rangle &= \left\langle p_{n-1}(Y)X + q_{n-1}(Y)Y, Y^{n-1}X\right\rangle \\ &= \left\langle p_{n-1}(Y)X, Y^{n-1}X\right\rangle + \left\langle q_{n-1}(Y)Y, Y^{n-1}X\right\rangle \\ &= \langle p_{n-1}(Y), Y^n\rangle + \left\langle q_{n-1}(Y)Y, Y^{n-1}X\right\rangle \text{ (by (A.29)).} \qquad \text{(A.30)}\end{aligned}$$

We thus see from (A.28) and (A.30) that

$$\left\langle Y^{n-1}X^2, Y^n\right\rangle = \left\langle Y^n X, Y^{n-1}X\right\rangle \qquad \text{(A.31)}$$

if and only if $\langle q_{n-1}(Y)X, Y^n\rangle = \left\langle q_{n-1}(Y)Y, Y^{n-1}X\right\rangle$; thus this reduces to verifying that $\left\langle Y^{n-1}X, Y^n\right\rangle = \left\langle Y^n, Y^{n-1}X\right\rangle$, which follows from the self-adjointness of $\mathcal{M}(n)$.

To verify (A.26)(iii) for $k = n-1$ we need to show that $\left\langle Y^n X, Y^{n-1}X\right\rangle = \left\langle Y^{n+1}, Y^{n-2}X^2\right\rangle$. Observe that

$$\begin{aligned}\left\langle Y^{n+1}, Y^{n-2}X^2\right\rangle &= \left\langle p_{n-1}(Y)Y + q_{n-1}(Y)YX, Y^{n-2}X^2\right\rangle \text{ (by (A.22))} \\ &= \left\langle p_{n-1}(Y)Y + q_{n-1}(Y)YX, Y^{n-1}\right\rangle \text{ (by (A.23)).} \qquad \text{(A.32)}\end{aligned}$$

We claim that $\left\langle Y^{n-1}X, Y^n\right\rangle = \left\langle Y^n X, Y^{n-1}\right\rangle$; indeed,

$$\begin{aligned}\left\langle Y^{n-1}X, Y^n\right\rangle &= \left\langle Y^n, Y^{n-1}X\right\rangle = \left\langle Y^{n-1}X^2, Y^{n-1}X\right\rangle \\ &\quad \text{(by the definition of } Y^{n-1}X^2\text{)} \\ &= \left\langle Y^n X, Y^{n-2}X^2\right\rangle \text{ (by (A.26)(i) with } k = n-1\text{)} \\ &= \left\langle Y^n X, Y^{n-1}\right\rangle \text{ (by (A.23)).}\end{aligned}$$

Now, the expression in (A.32) coincides with

$$\begin{aligned}\left\langle p_{n-1}(Y)Y, Y^{n-1}\right\rangle + \left\langle q_{n-1}(Y)X, Y^{n}\right\rangle &= \left\langle p_{n-1}(Y) + q_{n-1}(Y)X, Y^{n}\right\rangle \\ &= \left\langle Y^{n}, Y^{n}\right\rangle = \left\langle Y^{n-1}X^{2}, Y^{n}\right\rangle \\ &= \left\langle Y^{n}X, Y^{n-1}X\right\rangle \text{ (by (A.31))},\end{aligned}$$

as desired. To complete the case $k = n-1$ we need to show condition (iv) in (A.26) holds, that it, $\left\langle Y^{n}X, Y^{n}\right\rangle = \left\langle Y^{n+1}, Y^{n-1}X\right\rangle$. We do this as follows:

$$\begin{aligned}\left\langle Y^{n}X, Y^{n}\right\rangle &= \left\langle p_{n-1}(Y)X + q_{n-1}(Y)Y, Y^{n}\right\rangle \text{ (by (A.21))} \\ &= \left\langle p_{n-1}(Y)X, Y^{n}\right\rangle + \left\langle q_{n-1}(Y)Y, Y^{n}\right\rangle\end{aligned}$$

and

$$\begin{aligned}\left\langle Y^{n+1}, Y^{n-1}X\right\rangle &= \left\langle p_{n-1}(Y)Y + q_{n-1}(Y)YX, Y^{n-1}X\right\rangle \text{ (by (A.22))} \\ &= \left\langle p_{n-1}(Y)Y, Y^{n-1}X\right\rangle + \left\langle q_{n-1}(Y)YX, Y^{n-1}X\right\rangle.\end{aligned}$$

It thus suffices to verify that

$$\left\langle p_{n-1}(Y)X, Y^{n}\right\rangle = \left\langle p_{n-1}(Y)Y, Y^{n-1}X\right\rangle$$

and

$$\left\langle q_{n-1}(Y)Y, Y^{n}\right\rangle = \left\langle q_{n-1}(Y)YX = Y^{n-1}X\right\rangle.$$

The first equality follows from $\left\langle Y^{j}X, Y^{n}\right\rangle = \left\langle Y^{j+1}, Y^{n-1}X\right\rangle$ $(0 \le j \le n-2)$ and from $\left\langle Y^{n-1}X, Y^{n}\right\rangle = \left\langle Y^{n}, Y^{n-1}X\right\rangle$ (by the self-adjointness of $\mathcal{M}(n)$). To prove the second equality, note first that in $\mathcal{C}_{\mathcal{M}(n)}$,

$$\left\langle Y^{j+1}X, Y^{n-1}X\right\rangle = \left\langle Y^{j+1}X^{2}, Y^{n-1}\right\rangle = \left\langle Y^{j+1}Y, Y^{n-1}\right\rangle = \left\langle Y^{j+1}, Y^{n}\right\rangle$$

for $0 \le j \le n-3$. Further,

$$\left\langle Y^{n-1}, Y^{n}\right\rangle = \left\langle Y^{n}, Y^{n-1}\right\rangle = \left\langle Y^{n}, Y^{n-2}X^{2}\right\rangle = \left\langle Y^{n-1}X, Y^{n-1}X\right\rangle \tag{A.33}$$

(by the Hankel property in $\mathcal{M}(n)$). Finally,

$$\begin{aligned}\left\langle Y^{n}X, Y^{n-1}X\right\rangle &= \left\langle Y^{n-1}X^{2}, Y^{n}\right\rangle \text{ (by (A.31))} \\ &= \left\langle Y^{n}, Y^{n}\right\rangle \text{ (by the definition of } Y^{n-1}X^{2}\text{)}.\end{aligned} \tag{A.34}$$

The proof that block B is of the form $\{B_{i,n+1}\}_{i=0}^{n}$ is now complete.

To finish the proof of Proposition A.2.7 it now suffices to show that $C := W^{*}\mathcal{M}(n)W$ is Hankel. Observe that in $\mathcal{M} := [\mathcal{M}(n); B] = \begin{pmatrix}\mathcal{M}(n) & B \\ B^{*} & C\end{pmatrix}$ we may compute inner products of the form $\langle p(X,Y), q(X,Y)\rangle$ where $p, q \in \mathbb{R}[x,y]_{n+1}$. Note also that since $\mathcal{M}$ is a flat extension, dependence relations in the columns of $\begin{pmatrix}\mathcal{M}(n) & B\end{pmatrix}$ extend to column relations in $\begin{pmatrix}B^* & C\end{pmatrix}$. In particular, the first n columns of C coincide with the last n columns of B^*; since B has the Hankel property, so does B^*, and thus the first n columns of C have the Hankel property. Further, columns Y^nX and Y^{n+1} of C are defined as in (A.21) and (A.22) respectively. To verify that C is Hankel it now suffices to focus on the last three columns of C, namely $Y^{n-1}X^2, Y^nX$ and Y^{n+1}. We will first compare the entries of $Y^{n-1}X^2$ and Y^nX, and later those of Y^nX and Y^{n+1}. To this end, we need three preliminary facts.

Claim 1. For $0 \le i \le n-1$,

$$\left\langle Y^{n-1}X^2, Y^iX^{n+1-i}\right\rangle = \left\langle Y^{n-1}X^2, Y^{i+1}X^{n-1-i}\right\rangle. \tag{A.35}$$

Proof.

$$\begin{aligned}\left\langle Y^{n-1}X^2, Y^iX^{n+1-i}\right\rangle &= \left\langle Y^n, Y^iX^{n+1-i}\right\rangle \\ &\text{(by the definition of the columns of } C) \\ &= \left\langle Y^iX^{n+1-i}, Y^n\right\rangle \text{ (since } \mathcal{M}=\mathcal{M}^*) \\ &= \left\langle Y^{i+1}X^{n-1-i}, Y^n\right\rangle \\ &\text{(by the definition of the columns of } C) \\ &= \left\langle Y^n, Y^{i+1}X^{n-1-i}\right\rangle \text{ (since } \mathcal{M}=\mathcal{M}^*) \\ &= \left\langle Y^{i-1}X^2, Y^{i+1}X^{n-1-i}\right\rangle \\ &\text{(by the definition of columns of } C).\end{aligned}$$

Claim 2. For $0 \le i \le n-1$,

$$\left\langle Y^nX, Y^iX^{n+1-i}\right\rangle = \left\langle Y^nX, Y^{i+1}X^{n-1-i}\right\rangle. \tag{A.36}$$

Proof.

$$\begin{aligned}\left\langle Y^nX, Y^iX^{n+1-i}\right\rangle &= \left\langle p_{n-1}(Y)X + q_{n-1}(Y)Y, Y^iX^{n+1-i}\right\rangle \text{ (by (A.21))} \\ &= \left\langle Y^iX^{n+1-i}, p_{n-1}(Y)X + q_{n-1}(Y)Y\right\rangle \text{ (since } \mathcal{M}=\mathcal{M}^*) \\ &= \left\langle Y^{i+1}X^{n-1-i}, p_{n-1}(Y)X + q_{n-1}(Y)Y\right\rangle \\ &\text{(by the definition of the columns of } C) \\ &= \left\langle p_{n-1}(Y)X + q_{n-1}(Y)Y, Y^{i+1}X^{n-1-i}\right\rangle \text{ (since } \mathcal{M}=\mathcal{M}^*) \\ &= \left\langle Y^nX, Y^{i+1}X^{n-1-i}\right\rangle \text{ (by (A.21)).}\end{aligned}$$

Claim 3. For $0 \le i \le n-1$,

$$\left\langle Y^{n+1}, Y^iX^{n+1-i}\right\rangle = \left\langle Y^{n+1}, Y^{i+1}X^{n-1-i}\right\rangle. \tag{A.37}$$

Proof.

$$\begin{aligned}\left\langle Y^{n+1}, Y^iX^{n+1-i}\right\rangle &= \left\langle p_{n-1}(Y)Y + q_{n-1}(Y)YX, Y^iX^{n+1-i}\right\rangle \text{ (by (A.22))}\\ &= \left\langle Y^iX^{n+1-i}, p_{n-1}(Y)Y + q_{n-1}(Y)YX\right\rangle \text{ (since } \mathcal{M}=\mathcal{M}^*)\\ &= \left\langle Y^{i+1}X^{n-1-i}, p_{n-1}(Y)Y + q_{n-1}(Y)YX\right\rangle\\ &\text{(by the definition of the columns of } C)\\ &= \left\langle p_{n-1}(Y)Y + q_{n-1}(Y)YX, Y^{i+1}X^{n-1-i}\right\rangle \text{ (since } \mathcal{M}=\mathcal{M}^*)\\ &= \left\langle Y^{n+1}, Y^{i+1}X^{n-1-i}\right\rangle \text{ (by (A.22)).}\end{aligned}$$

Comparison of $Y^{n-1}X^2$ and Y^nX. We will establish that

$$\left\langle Y^{n-1}X^2, Y^iX^{n+1-i}\right\rangle = \left\langle Y^nX, Y^{i-1}X^{n+2-i}\right\rangle \ (1 \le i \le n+1).$$

Case 1. $(1 \le i \le n-1)$:

$$\begin{aligned}\left\langle Y^{n-1}X^2, Y^iX^{n+1-i}\right\rangle &= \left\langle Y^{n-1}X^2, Y^{i+1}X^{n-1-i}\right\rangle \quad \text{(by (A.35))}\\ &= \left\langle Y^nX, Y^iX^{n-i}\right\rangle \quad \text{(because } B \text{ is Hankel)}\\ &= \left\langle Y^nX, Y^{i-1}X^{n+2-i}\right\rangle \quad \text{(by (A.36)).}\end{aligned}$$

Case 2. $(i = n)$ This is straightforward from the self-adjointness of C.
Case 3. $(i = n+1)$ We need to prove that

$$\left\langle Y^{n-1}X^2, Y^{n+1}\right\rangle = \left\langle Y^nX, Y^nX\right\rangle \tag{A.38}$$

Observe that

$$\begin{aligned}\left\langle Y^{n-1}X^2, Y^{n+1}\right\rangle &= \left\langle Y^n, Y^{n+1}\right\rangle \quad \text{(by the definition of the columns of } C)\\ &= \left\langle Y^{n+1}, Y^n\right\rangle \quad \text{(since } \mathcal{M}=\mathcal{M}^*)\\ &= \left\langle p_{n-1}(Y)Y + q_{n-1}(Y)YX, Y^n\right\rangle \quad \text{(by (A.22))}\end{aligned}$$

Similarly,

$$\left\langle Y^nX, Y^nX\right\rangle = \left\langle p_{n-1}(Y)X + q_{n-1}(Y)Y, Y^nX\right\rangle.$$

It follows that to verify (A.38) it suffices to establish

$$\left\langle p_{n-1}(Y)Y, Y^n\right\rangle = \left\langle p_{n-1}(Y)X, Y^nX\right\rangle \tag{A.39}$$

and

$$\left\langle q_{n-1}(Y)YX, Y^n\right\rangle = \left\langle q_{n-1}(Y)Y, Y^nX\right\rangle. \tag{A.40}$$

For (A.39), we will verify that for $0 \leq j \leq n-1$,

$$\left\langle Y^{j+1}, Y^{n}\right\rangle = \left\langle Y^{j}X, Y^{n}X\right\rangle . \tag{A.41}$$

We have

$$\left\langle Y^{j+1}, Y^{n}\right\rangle = \left\langle Y^{n}, Y^{j+1}\right\rangle = \left\langle p_{n-1}(Y) + q_{n-1}(Y)X, Y^{j+1}\right\rangle$$

and

$$\left\langle Y^{j}X, Y^{n}X\right\rangle = \left\langle Y^{n}X, Y^{j}X\right\rangle = \left\langle p_{n-1}(Y)X + q_{n-1}(Y)Y, Y^{j}X\right\rangle .$$

To establish (A.41), it now suffices to prove that

$$\left\langle Y^{k}, Y^{j+1}\right\rangle = \left\langle Y^{k}X, Y^{j}X\right\rangle \quad (0 \leq k \leq n-1, 0 \leq j \leq n-1) \tag{A.42}$$

and

$$\left\langle Y^{k}X, Y^{j+1}\right\rangle = \left\langle Y^{k+1}, Y^{j}X\right\rangle \quad (0 \leq k \leq n-1, 0 \leq j \leq n-1). \tag{A.43}$$

For (A.42), we have

$$\begin{aligned}\left\langle Y^{k}X, Y^{j}X\right\rangle &= \left\langle Y^{j}X, Y^{k}X\right\rangle = \left\langle Y^{j+1}, Y^{k-1}X^{2}\right\rangle \\ &\text{(by the Hankel property in } \mathcal{M}(n)) \\ &= \left\langle Y^{j+1}, Y^{k}\right\rangle \quad \text{(via recursiveness and self-adjointness in } \mathcal{M}(n)) \\ &= \left\langle Y^{k}, Y^{j+1}\right\rangle .\end{aligned}$$

For (A.43), first consider the case when $j, k \leq n-2$; then

$$\begin{aligned}\left\langle Y^{k}X, Y^{j+1}\right\rangle &= \left\langle Y^{k}X, Y^{j}X^{2}\right\rangle \\ &= \left\langle Y^{k}X^{2}, Y^{j}X\right\rangle = \left\langle Y^{k+1}, Y^{j}X\right\rangle .\end{aligned}$$

Now consider (A.43) with $j = n-1$, $k \leq n-1$; we have

$$\begin{aligned}\left\langle Y^{k+1}, Y^{n-1}X\right\rangle &= \left\langle Y^{n-1}X, Y^{k+1}\right\rangle = \left\langle Y^{n}, Y^{k}X\right\rangle \\ &\text{(by the Hankel property in } \mathcal{M}(n)) \\ &= \left\langle Y^{k}X, Y^{n}\right\rangle .\end{aligned}$$

Finally, for (A.43) with $k = n-1$, $j \leq n-1$, note that $\left\langle Y^{n-1}X, Y^{j+1}\right\rangle = \left\langle Y^{n}, Y^{j}X\right\rangle$, by the Hankel property in $\mathcal{M}(n)$. Thus, we have established (A.43), whence (A.39) follows. We next prove (A.40); to do so, it suffices to establish that

$$\left\langle Y^{j+1}X, Y^{n}\right\rangle = \left\langle Y^{j+1}, Y^{n}X\right\rangle \quad (0 \leq j \leq n-1). \tag{A.44}$$

Consider first the case when $j \le n-3$; then

$$\begin{aligned}\left\langle Y^{j+1}, Y^n X\right\rangle &= \left\langle Y^n X, Y^{j+1}\right\rangle \\ &= \left\langle Y^n, Y^{j+1} X\right\rangle \quad \text{(by (A.25))} \\ &= \left\langle Y^{j+1} X, Y^n\right\rangle .\end{aligned}$$

For (A.44) with $j = n-2$, we have

$$\begin{aligned}\left\langle Y^{n-1}, Y^n X\right\rangle = \left\langle Y^n X, Y^{n-1}\right\rangle &= \left\langle p_{n-1}(Y)X + q_{n-1}(Y)Y, Y^{n-1}\right\rangle \\ &= \left\langle p_{n-1}(Y)X, Y^{n-1}\right\rangle + \left\langle q_{n-1}(Y)Y, Y^{n-1}\right\rangle \\ &= \left\langle p_{n-1}(Y), Y^{n-1}X\right\rangle + \left\langle q_{n-1}(Y)X, Y^{n-1}X\right\rangle \quad \text{(using (A.33))} \\ &= \left\langle p_{n-1}(Y) + q_{n-1}(Y)X, Y^{n-1}X\right\rangle \\ &= \left\langle Y^n, Y^{n-1}X\right\rangle = \left\langle Y^{n-1}X, Y^n\right\rangle .\end{aligned}$$

Finally, (A.44) with $j = n-1$ follows from self-adjointness in $\mathcal{M}$: $\langle Y^n X, Y^n\rangle = \langle Y^n, Y^n X\rangle$. This concludes the proof of (A.44); thus (A.40) is established and the proof of Case 3 is complete.

Comparison of $Y^n X$ and Y^{n+1}. We will establish that

$$\left\langle Y^n X, Y^i X^{n+1-i}\right\rangle = \left\langle Y^{n+1}, Y^{i-1} X^{n+2-i}\right\rangle \quad (1 \le i \le n+1).$$

Case 1. $(1 \le i \le n-1)$:

$$\begin{aligned}\left\langle Y^n X, Y^i X^{n+1-i}\right\rangle &= \left\langle Y^n X, Y^{i+1} X^{n-1-i}\right\rangle \quad \text{(by (A.36))} \\ &= \left\langle Y^{n+1}, Y^i X^{n-i}\right\rangle \quad \text{(by the Hankel property in } B\text{)} \\ &= \left\langle Y^{n+1}, Y^{i-1} X^{n+2-i}\right\rangle \quad \text{(by (A.37))}.\end{aligned}$$

Case 2. $(i = n)$ This is (A.38).

Case 3. $(i = n+1)$ This is straightforward from the self-adjointness of C.

This concludes the proof of Proposition A.25. □

Proposition A.2.12. *Assume that $\mathcal{M}(n)$ is positive, recursively generated, and satisfies $Y = X^2$. Assume also that $\mathcal{S}_n(n)$ is a basis for $\mathcal{C}_{\mathcal{M}(n)}$. Then $\mathcal{M}(n)$ admits a one-parameter family of flat extensions $\mathcal{M}(n+1)$.*

Proof. Since $Y = X^2$, and to guarantee that $\mathcal{M}(n+1)$ is recursively generated, we define the first n columns of a proposed B block for $\mathcal{M}(n+1)$ as $[X^{n+1}]_{m(n)} := YX^{n-1} \in \mathcal{C}_{\mathcal{M}(n)}, [YX^n]_{m(n)} := Y^2 X^{n-2} \in \mathcal{C}_{\mathcal{M}(n)}, \ldots, [Y^{n-1}X^2]_{m(n)} := Y^n \in \mathcal{C}_{\mathcal{M}(n)}$. Moreover, if we wish to make $B_{n,n+1}$ Hankel, it is clear that all but the last entry in column $[Y^n X]_{m(n)}$

must be given in terms of the entries in $\mathcal{M}(n)$, and that all but the last entry in $[Y^{n+1}]_{m(n)}$ must be given in terms of the entries in $[Y^nX]_{m(n)}$; concretely,

$$\left\langle Y^nX, Y^k\right\rangle := \beta_{n+k,1} \quad (0 \leq k \leq n-1)$$

and

$$\begin{aligned}\langle Y^nX, Y^{i+1}X^j\rangle &:= \langle Y^{n-1}X^2, Y^{i+1}X^{j-1}\rangle \\ &= \langle Y^n, Y^{i+1}X^{j-1}\rangle \\ &\equiv \beta_{n+i+1,j-1} \quad (i \geq 0, j \geq 1, i+j \leq n).\end{aligned}$$

To handle the last entry of $[Y^nX]_{m(n)}$ we introduce the parameter $p \equiv \langle Y^nX, Y^n\rangle_{\mathcal{M}(n+1)}$. Similarly, we let

$$\left\langle Y^{n+1}, Y^k\right\rangle := \beta_{n+1+k,0} \quad (0 \leq k \leq n-1),$$

$$\langle Y^{n+1}, Y^iX^j\rangle := \langle Y^nX, Y^{i+1}X^{j-1}\rangle \quad (i \geq 0, j \geq 1, i+j \leq n),$$

and $q := \langle Y^{n+1}, Y^n\rangle$.

Claim 1.

$$\langle Y^nX, Y^iX^{j+2}\rangle = \langle Y^nX, Y^{i+1}X^j\rangle \quad (i+j+2 \leq n).$$

Proof. Assume first $j \geq 1$. Then

$$\begin{aligned}\langle Y^nX, Y^{i+1}X^j\rangle &= \langle Y^n, Y^{i+1}X^{j+1}\rangle \\ &= \langle Y^n, Y^{i+1}X^{j-1}\rangle \quad (\text{since } Y = X^2) \\ &= \langle Y^nX, Y^{i+1}X^j\rangle.\end{aligned}$$

If $j = 0$

$$\langle Y^nX, Y^iX^2\rangle = \langle Y^n, Y^{i+1}X\rangle = \beta_{n+i+1,1} = \langle Y^nX, Y^{i+1}\rangle.$$

Claim 2.

$$\langle Y^{n+1}, Y^iX^{j+2}\rangle = \langle Y^{n+1}, Y^{i+1}X^j\rangle \quad (i+j+2 \leq n).$$

Proof. Assume first $j \geq 1$. Then

$$\begin{aligned}\langle Y^{n+1}, Y^iX^{j+2}\rangle &= \langle Y^nX, Y^{i+1}X^{j+1}\rangle \\ &= \langle Y^nX, Y^{i+2}X^{j-1}\rangle \quad (\text{by Claim 1}) \\ &= \langle Y^{n+1}, Y^{i+1}X^j\rangle.\end{aligned}$$

If $j = 0$,

$$\langle Y^{n+1}, Y^iX^2\rangle = \langle Y^nX, Y^{i+1}X\rangle = \langle Y^n, Y^{i+2}\rangle = \beta_{n+i+2,0} = \langle Y^{n+1}, Y^{i+1}\rangle.$$

Repeated application of Claims 1 and 2 show that each row of B is identical to a row whose associated monomial corresponds to a column in the basis $\mathcal{S}_n(n)$, a property clearly present in $\mathcal{M}(n)$. This will be crucial in establishing that both $[Y^nX]_{m(n)}$ and $[Y^{n+1}]_{m(n)}$ are in the range of $\mathcal{M}(n)$.

Since $N := \mathcal{M}(n)_{\mathcal{S}_n(n)} > 0$, there exist vectors $\mathbf{f}, \mathbf{g} \in \mathbb{R}^{2n+1}$ such that $N\mathbf{f} = [Y^nX]_{\mathcal{S}_n(n)}$ and $N\mathbf{g} = [Y^{n+1}]_{\mathcal{S}_n(n)}$. Let $\mathbf{F}, \mathbf{G} \in \mathbb{R}^{m(n)}$ be given by

$$\left\langle \mathbf{F}, Y^iX^j \right\rangle := \begin{cases} \left\langle \mathbf{f}, Y^iX^j \right\rangle & \text{if } Y^iX^j \in \mathcal{S}_n(n) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\left\langle \mathbf{G}, Y^iX^j \right\rangle := \begin{cases} \left\langle \mathbf{g}, Y^iX^j \right\rangle & \text{if } Y^iX^j \in \mathcal{S}_n(n) \\ 0 & \text{otherwise} \end{cases}.$$

We claim that $\mathcal{M}(n)\mathbf{F} = [Y^nX]_{m(n)}$. Indeed, for $Y^iX^j \in \mathcal{S}_n(n)$, we have

$$\begin{aligned} \left\langle \mathcal{M}(n)\mathbf{F}, Y^iX^j \right\rangle &= \sum_{Y^kX^\ell \in \mathcal{C}_{\mathcal{M}(n)}} \left\langle Y^kX^\ell, Y^iX^j \right\rangle \left\langle \mathbf{F}, Y^kX^\ell \right\rangle \\ &= \sum_{Y^kX^\ell \in \mathcal{S}_n(n)} \left\langle Y^kX^\ell, Y^iX^j \right\rangle \left\langle \mathbf{f}, Y^kX^\ell \right\rangle \\ &\quad + \sum_{Y^kX^\ell \notin \mathcal{S}_n(n)} \left\langle Y^kX^\ell, Y^iX^j \right\rangle \cdot 0 \\ &= \left\langle [Y^nX]_{\mathcal{S}_n(n)}, Y^iX^j \right\rangle = \left\langle [Y^nX]_{m(n)}, Y^iX^j \right\rangle. \end{aligned}$$

Further, for $Y^iX^j \in \mathcal{C}_{\mathcal{M}(n)} \backslash \mathcal{S}_n(n)$, there exist i', j' such that $Y^iX^j = Y^{i'}X^{j'} \in \mathcal{C}_{\mathcal{M}(n)}$ and $Y^{i'}X^{j'} \in \mathcal{S}_n(n)$. Since $\mathcal{M}(n)$ is self-adjoint, row Y^iX^j of $\mathcal{M}(n)$ coincides with row $Y^{i'}X^{j'}$. Now,

$$\begin{aligned} \left\langle \mathcal{M}(n)\mathbf{F}, Y^iX^j \right\rangle &= \sum_{Y^kX^\ell \in \mathcal{C}_{\mathcal{M}(n)}} \left\langle Y^kX^\ell, Y^iX^j \right\rangle \left\langle \mathbf{F}, Y^kX^\ell \right\rangle \\ &= \sum_{Y^kX^\ell \in \mathcal{C}_{\mathcal{M}(n)}} \left\langle Y^kX^\ell, Y^{i'}X^{j'} \right\rangle \left\langle \mathbf{F}, Y^kX^\ell \right\rangle \\ &= \left\langle \mathcal{M}(n)\mathbf{F}, Y^{i'}X^{j'} \right\rangle \\ &= \left\langle [Y^nX]_{m(n)}, Y^{i'}X^{j'} \right\rangle \quad \text{(from the preceding case)} \\ &= \left\langle [Y^nX]_{m(n)}, Y^iX^j \right\rangle \quad \text{(by Claim 1).} \end{aligned}$$

Thus $\mathcal{M}(n)\mathbf{F} = [Y^nX]_{m(n)}$, and a similar argument (using Claim 2) shows that $\mathcal{M}(n)\mathbf{G} = [Y^{n+1}]_{m(n)}$. Since $[Y^iX^j]_{m(n)} = Y^{i+1}X^{j-2} \in \mathcal{C}_{\mathcal{M}(n)}$ for $i + j = n + 1$, $j \geq 2$, it follows that $\operatorname{Ran} B \subseteq \operatorname{Ran} \mathcal{M}(n)$; thus there exists W such that $\mathcal{M}(n)W = B$.

To show that the flat extension $\mathcal{M} \equiv [\mathcal{M}(n); B]$ is of the form $\mathcal{M}(n + 1)$, it now suffices to show that $C := W^*\mathcal{M}(n)W$ is Hankel. We have

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}(n) & B \\ B^* & C \end{pmatrix};$$

recall that dependence relations in $\begin{pmatrix} \mathcal{M}(n) & B \end{pmatrix}$ extend to corresponding relations in $\begin{pmatrix} B^* & C \end{pmatrix}$. Now $B^* = \begin{pmatrix} * & B^*_{n,n+1} \end{pmatrix}$, where

$$B^*_{n,n+1} = \begin{pmatrix} * & \dots & * & * & * \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ * & \dots & * & * & * \\ * & \dots & * & * & p \\ * & \dots & * & p & q \end{pmatrix}_{(n+2)\times(n+1)} .$$

Since, in the column space of $\begin{pmatrix} \mathcal{M}(n) & B \end{pmatrix}$, we have $X^{n+1} = YX^{n-1}, \dots, Y^{n-1}X^2 = Y^n$, it follows that C is of the form $\begin{pmatrix} \tilde{B}^*_{n,n+1} & * \end{pmatrix}_{(n+2)\times(n+2)}$, where $\tilde{B}^*_{n,n+1}$ is obtained from $B^*_{n,n+1}$ by deleting it's leftmost column. Thus, since $\tilde{B}^*_{n,n+1}$ is Hankel and $C = C^*$, we have

$$C = \begin{pmatrix} * & \dots & * & * & * & * \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ * & \dots & * & * & * & p \\ * & \dots & * & * & p & q \\ * & \dots & * & p & C_{n+1,n+1} & u \\ * & \dots & p & q & u & v \end{pmatrix},$$

for some $u, v \in \mathbb{R}$, and where each cross-diagonal that is not shown is constant. Observe that by the flat extension construction, $C_{n+1,n+1} = [Y^nX]^t_{m(n)}\mathbf{F}$. Since $[Y^nX]_{m(n)}$ is independent of q, so is $\mathbf{f} = N^{-1}[Y^nX]_{\mathcal{S}_n(n)}$, whence $\mathbf{F}$ is also independent of q. Thus $C_{n+1,n+1}$ does not depend on q (though it does depend on p). For each p, if we let $q := C_{n+1,n+1}(p)$, it follows that $\mathcal{M} = [\mathcal{M}(n); B]$ is a flat moment matrix extension of the form $\mathcal{M}(n + 1)$. The proof of Poposition A.2.12 is now complete. □

Proof of Theorem A.2.2. This is now just a matter of applying one of Proposition A.2.4, Proposition A.2.5, Proposition A.2.6, Proposition A.2.7 or Proposition A.2.12. □

Appendix B

6-atomic measures for Lines, Parabolas and Hyperbolas

Let $\mathcal{M}(2)$ be the multivariate Hankel matrix for the truncated sequence $\beta \equiv \beta^{(4)}$. We replicate the proof from [8], of β having a measure with at most 6 atoms for parabolas, hyperbolas and lines.

Proposition B.0.1. *If* $\mathcal{M}(2) \geq 0$, rank $\mathcal{M}(2) = 5$ *and* $Y = X^2$ *in* $\mathcal{C}_{\mathcal{M}(2)}$, *then* $\mathcal{M}(2)$ *has a flat extension* $\mathcal{M}(3)$ *and* β *admits a* 5*-atomic representing measure.*

Proof. Since $Y = X^2$, $\mathcal{M}(2)$ is of the form

$$\begin{bmatrix} 1 & a & b & b & d & e \\ a & b & d & d & e & f \\ b & d & e & e & f & g \\ b & d & e & e & f & g \\ d & e & f & f & g & h \\ e & f & g & g & h & k \end{bmatrix},$$

where $N := \mathcal{M}(2)_{\{\mathbb{1},X,Y,YX,Y^2\}} > 0$. The B-block of a positive, recursively generated extension $\mathcal{M}(3)$ satisfies $X^3 = YX$ and $YX^2 = Y^2$, and thus assumes the form

$$B(3;p,q) = \begin{bmatrix} d & e & f & g \\ e & f & g & h \\ f & g & h & k \\ f & g & h & k \\ g & h & k & p \\ h & k & p & q \end{bmatrix},$$

where p and q are new moments, corresponding to the monomials y^4x and y^5, respectively. Since N is invertible, it follows that there exists a matrix W such that $\mathcal{M}(2)W = B(3;p,q)$, and a calculation of the C-block of the flat extension $[\mathcal{M}(2);B(3;p,q)]$ reveals it is of the form

$$C(3;p,q) := B(3;p,q)^T W = \begin{bmatrix} g & h & k & p \\ h & k & p & q \\ k & p & C_{33} & u \\ p & q & u & v \end{bmatrix},$$

for some $u, v \in \mathbb{R}$. A further calculation shows that $C_{33} \equiv C_{33}(p)$ is independent of q. Thus given a choice of p, we can let $q = C_{33}(p)$, and $C(3;p,q)$ then becomes a Hankel matrix which implies that $[\mathcal{M}(2);B(3;p,q)]$ is of the form $\mathcal{M}(3)$. □

Proposition B.0.2. *If* $\mathcal{M}(2) \geq 0$*,* rank $\mathcal{M}(2) = 5$*, and* $YX = \mathbb{1}$ *in* $\mathcal{C}_{\mathcal{M}(2)}$*, then* $\mathcal{M}(2)$ *has a flat extenion* $\mathcal{M}(3)$ *and* β *admits a* 5*-atomic representing measure.*

Proof. Since $YX = \mathbb{1}$, $\mathcal{M}(2)$ can be expressed as

$$\begin{bmatrix} 1 & a & b & c & 1 & d \\ a & c & 1 & e & a & b \\ b & 1 & d & a & b & f \\ c & e & a & g & c & 1 \\ 1 & a & b & c & 1 & d \\ d & b & f & 1 & d & h \end{bmatrix},$$

where $N := \mathcal{M}(2)_{\{\mathbb{1},X,Y,X^2,Y^2\}} > 0$. The B-block of a positive, recursively generated extension $\mathcal{M}(3)$ satisfies $YX^2 = X$ and $Y^2X = Y$, and thus may be represented as

$$B(3;p,q) = \begin{bmatrix} e & a & b & f \\ g & c & 1 & d \\ c & 1 & d & h \\ p & e & a & b \\ e & a & b & f \\ a & b & f & q \end{bmatrix},$$

where p and q are new moments corresponding to the monomials x^5 and y^5, respectively. Since N is invertible, there exists a matrix W such that $\mathcal{M}(2)W = B(3;p,q)$, and a calculation of the C-block of the flat extension $[\mathcal{M}(2);B(3;p,q)]$ reveals it has the form

$$C(3;p,q) = \begin{bmatrix} u & g & c & C_{1,4} \\ g & c & 1 & d \\ c & 1 & d & h \\ C_{41} & d & h & v \end{bmatrix},$$

for some $u, v \in \mathbb{R}$, where $C_{14} = C_{41}$. Thus $\mathcal{M}(2)$ admits a flat extension if and only if $C_{14} = 1$ for some real numbers p and q. A *Mathematica* calculation shows that $C_{14} = Num/Den$, where Num and Den are polynomials in the moments (including p and q). Further, $\Delta := Num - Den$ can be expressed as

$$\Delta \equiv \Delta(p, q) \equiv \delta_0 + \delta_1 p + \delta_2 q + \delta_{12} pq,$$

where $\delta_0, \delta_1, \delta_2, \delta_{12}$ are independent of p and q. Observe that $\Delta = \delta_0 + \delta_1 p + (\delta_2 + \delta_{12} p)q$, so if for some value of p, $\delta_2 + \delta_{12} p \neq 0$, then $q := -(\delta_0 + \delta_1 p)/(\delta_2 + \delta_{12} p)$ satisfies $\Delta(p, q) = 0$. Similarly, $\Delta = \delta_0 + \delta_2 q + (\delta_1 + \delta_{12} q)p$, so if for some value of q, $\delta_1 + \delta_{12} q \neq 0$, then $p := -(\delta_0 + \delta_2 q)/(\delta_1 + \delta_{12} q)$ satisfies $\Delta(p, q) = 0$. Thus, if δ_1, δ_2 or δ_{12} is non-zero, then $\mathcal{M}(2)$ admits a flat extension.

Let us assume therefore that $\mathcal{M}(2)$ admits no flat extension and derive a contradiction; that is, we shall assume that $\delta_1 = \delta_2 = \delta_{12} = 0$. A calculation using *Mathematica* shows that $\delta_{12} = \eta f + F$, where

$$\eta := -a^3 + 2ac - bc^2 - e + abe$$

and

$$\begin{aligned} F := {} & 1 - 3ab + a^2 b^2 + 2b^2 c + 2a^2 d - 2cd \\ & - 2abcd + c^2 d^2 - b^3 e + 2bde - ad^2 e, \end{aligned}$$

so each of η and F is independent of f. We claim that $\eta = 0$. Indeed, if $\eta \neq 0$, then $\delta_{12} = 0$ implies $f = f_0 := -F/\eta$. A *Mathematica* calculation using $f = f_0$ now reveals that in this case $\det N$ admits a factorization

$$\det N = \frac{1}{\eta^2} \det \mathcal{M}(2)_{\{\mathbb{1}, X, Y, X^2\}} G,$$

where G is a polynomial in $a, b, c, d, e,$ and h of degree 7, and that δ_1 admits a factorization of the form

$$\delta_1 \equiv -\frac{1}{\eta^3} G^2.$$

Since $\delta_1 = 0$ it follows that $G = 0$, whence $\det N = 0$, a contradiction. Thus $\eta = 0$. Now let

$$\phi := -b^3 + 2bd - ad^2 - f + abf$$

(for general f). If $\mathcal{M}(2)$ admits no flat extension, then $\delta_{12} = \eta = 0$ contradicts the condition $\det \mathcal{M}(2)_{\{\mathbb{1}, X, Y\}} > 0$, via the formula

$$(\det \mathcal{M}(2)_{\{\mathbb{1}, X, Y\}})^2 = (1 - ab)\delta_{12} + \eta\phi.$$

$\square$

Proposition B.0.3 ([8], Proposition 5.5)**.** *If* $\mathcal{M}(2) \geq 0$*, if* rank $\mathcal{M}(2) = 5$*, and if* $XY = 0$ *in the column space of* $\mathcal{M}(2)$*, then* $\mathcal{M}(2)$ *admits a representing measure* μ *with* card supp $\mu \leq 6$.

Proof. With the hypothesis $XY = 0$, $\mathcal{M}(2)$ can be expressed as

$$\begin{pmatrix} 1 & a & b & c & 0 & d \\ a & c & 0 & e & 0 & 0 \\ b & 0 & d & 0 & 0 & f \\ c & e & 0 & g & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ d & 0 & f & 0 & 0 & h \end{pmatrix},$$

where $N := \mathcal{M}(2)_{\{\mathbb{1},X,Y,X^2,Y^2\}} > 0$ (here $\mathcal{M}(2)_{\{\mathbb{1},X,Y,X^2,Y^2\}}$ is the compression of $\mathcal{M}(2)$ to the rows and columns in the set $\{\mathbb{1}, X, Y, X^2, Y^2\}$). For the matrix $\mathcal{M}(2)$, we wish to find a block extension

$$\mathcal{M}(3) = \begin{pmatrix} \mathcal{M}(2) & B \\ B & C \end{pmatrix},$$

such that the rank of $\mathcal{M}(3)$ is equal to the rank of $\mathcal{M}(2)$ (a *flat* extension).

The B-block of a positive, recursively generated extension $\mathcal{M}(3)$ satisfies $XY^2 = 0 = X^2Y$, and thus has the form

$$B(\alpha, \beta) := \begin{pmatrix} e & 0 & 0 & f \\ g & 0 & 0 & 0 \\ 0 & 0 & 0 & h \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix},$$

where α and β are new moments corresponding to the monomials x^5 and y^5 respectively. Since N is invertible, there exists a matrix W such that $\mathcal{M}(2)W = B$. A calculation of the C-block of the flat extension $[\mathcal{M}(2); B]$ reveals it had the form

$$C(\alpha, \beta) \equiv \begin{pmatrix} C_{11} & 0 & 0 & C_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ C_{41} & 0 & 0 & C_{44} \end{pmatrix},$$

where $C_{14} = C_{41} \equiv F_1F_2$, with $F_1 \equiv F_1(\alpha) := H + (c^2 - ae)\alpha$ ($H := e^3 - 2ecg + ag^2$) and $F_2 \equiv F_2(\beta) := L + (d^2 - bf)\beta$ ($L := f^3 - 2fdh + bh^2$). Thus $\mathcal{M}(2)$ admits a flat extension $\mathcal{M}(3)$ if and only if $C_{14} = 0$, i.e., if and only if for some value of α we have $F_1(\alpha) = 0$, or for some value of β we have $F_2(\beta) = 0$. Equivalently, there is a flat extension if and only if $H = 0$, or $c^2 - ae \neq 0$, or $L = 0$, or $d^2 - bf \neq 0$.

So assume that $H \neq 0$, $c^2 - ae = 0$, $L \neq 0$ and $d^2 - bf = 0$. Choose any real numbers α and β. Clearly, there exists $u, v > 0$ such that

$$C'(u,v) := \begin{pmatrix} u & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v \end{pmatrix}$$

satisfies $u > C_{11}$, $C'(u,v) - C(\alpha,\beta) \geq 0$ and $\det[C'(u,v) - C(\alpha,\beta)] = 0$, i.e.,

$$(u - C_{11})(v - C_{44}) = C_{14}^2.$$

This uniquely determines v in terms of u and previous moments (including possibly α and β, although the choice of u is independent of β), so that

$$\mathcal{M}(3) := \begin{pmatrix} \mathcal{M}(2) & B(\alpha,\beta) \\ B(\alpha,\beta)^* & C'(u,v) \end{pmatrix}$$

is a recursively generated positive moment matrix extension of $\mathcal{M}(2)$ having rank 6, with column basis $\{\mathbb{1}, X, Y, X^2, Y^2, X^3\}$. It can be shown that there are unique values of r and s so that

$$B(r,s) \equiv \begin{pmatrix} g & 0 & 0 & 0 & h \\ \alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta \\ u & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & v \\ r & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s \end{pmatrix}$$

satisfies $\text{Im}B(r,s) \subseteq \text{Im}\mathcal{M}(3)$, i.e., $\mathcal{M}(3)W' = B(r,s)$ for some W'. With this value of r a calculation shows that the C-block of the flat extension $[\mathcal{M}(3); B(r,s)]$ is of the form

$$\tilde{C} \equiv \begin{pmatrix} D_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} \end{pmatrix},$$

that is, $\tilde{C}$ is Hankel. Thus, for each value of α and β, and for u sufficiently large, we get a uniquely determined flat extension and a corresponding 6-atomic representing measure. □

Appendix C

Large Matrices

C.1 Positive Definite Case

To attempt replicating Curto and Yoo's method, we must construct a flat extension of the matrix

$$U = \begin{pmatrix} \varepsilon_1 & 0 & 0 & \varepsilon_1 & 0 & 0 & u \\ 0 & \varepsilon_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & u & 0 & 0 & 0 & 0 \\ \varepsilon_1 & 0 & 0 & \varepsilon_1 & 0 & 0 & u \\ 0 & 0 & 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u & 0 \\ u & 0 & 0 & u & 0 & 0 & \varepsilon_2 \end{pmatrix}.$$

Which has a hypothetical B block of the form

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon_1 & 0 & 0 & 0 & u & 0 & u & 0 \\ 0 & u & 0 & u & 0 & 0 & 0 & \varepsilon_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b_1 & b_1 & b_2 \\ 0 & 0 & 0 & 0 & b_1 & b_1 & 0 & b_2 \\ 0 & 0 & b_1 & 0 & b_2 & b_2 & b_2 & b_3 \end{pmatrix},$$

for some $b_1, b_2, b_3 \in \mathbb{R}$. Computing the C block for an extension shows it to be

$$C = \begin{pmatrix} \varepsilon_1 & 0 & 0 & 0 & u & 0 & u & 0 \\ 0 & u & 0 & u & 0 & 0 & 0 & \varepsilon_2 \\ 0 & 0 & \frac{b_1^2\varepsilon_1}{\varepsilon_1\varepsilon_2-u^2} & 0 & \frac{b_1b_2\varepsilon_1}{\varepsilon_1\varepsilon_2-u^2} & \frac{b_1b_2\varepsilon_1}{\varepsilon_1\varepsilon_2-u^2} & \frac{b_1b_2\varepsilon_1}{\varepsilon_1\varepsilon_2-u^2} & \frac{b_1b_3\varepsilon_1}{\varepsilon_1\varepsilon_2-u^2} \\ 0 & u & 0 & u & 0 & 0 & 0 & \varepsilon_2 \\ u & 0 & \frac{b_1b_2\varepsilon_1}{\varepsilon_1\varepsilon_2-u^2} & 0 & \frac{b_1^2}{u}+\frac{u^2}{\varepsilon_1}+\frac{b_2^2\varepsilon_1}{\varepsilon_1\varepsilon_2-u^2} & \frac{b_1^2}{u}+\frac{b_2^2\varepsilon_1}{\varepsilon_1\varepsilon_2-u^2} & \frac{\varepsilon_1 b_2^2}{\varepsilon_1\varepsilon_2-u^2}+\frac{u^2}{\varepsilon_1} & \frac{b_1b_2}{u}+\frac{b_3\varepsilon_1b_2}{\varepsilon_1\varepsilon_2-u^2} \\ 0 & 0 & \frac{b_1b_2\varepsilon_1}{\varepsilon_1\varepsilon_2-u^2} & 0 & \frac{b_1^2}{u}+\frac{b_2^2\varepsilon_1}{\varepsilon_1\varepsilon_2-u^2} & \frac{2b_1^2}{u}+\frac{b_2^2\varepsilon_1}{\varepsilon_1\varepsilon_2-u^2} & \frac{b_1^2}{u}+\frac{b_2^2\varepsilon_1}{\varepsilon_1\varepsilon_2-u^2} & \frac{2b_1b_2}{u}+\frac{b_3\varepsilon_1b_2}{\varepsilon_1\varepsilon_2-u^2} \\ u & 0 & \frac{b_1b_2\varepsilon_1}{\varepsilon_1\varepsilon_2-u^2} & 0 & \frac{\varepsilon_1 b_2^2}{\varepsilon_1\varepsilon_2-u^2}+\frac{u^2}{\varepsilon_1} & \frac{b_1^2}{u}+\frac{b_2^2\varepsilon_1}{\varepsilon_1\varepsilon_2-u^2} & \frac{b_1^2}{u}+\frac{u^2}{\varepsilon_1}+\frac{b_2^2\varepsilon_1}{\varepsilon_1\varepsilon_2-u^2} & \frac{b_1b_2}{u}+\frac{b_3\varepsilon_1b_2}{\varepsilon_1\varepsilon_2-u^2} \\ 0 & \varepsilon_2 & \frac{b_1b_3\varepsilon_1}{\varepsilon_1\varepsilon_2-u^2} & \varepsilon_2 & \frac{b_1b_2}{u}+\frac{b_3\varepsilon_1b_2}{\varepsilon_1\varepsilon_2-u^2} & \frac{2b_1b_2}{u}+\frac{b_3\varepsilon_1b_2}{\varepsilon_1\varepsilon_2-u^2} & \frac{b_1b_2}{u}+\frac{b_3\varepsilon_1b_2}{\varepsilon_1\varepsilon_2-u^2} & \frac{2b_2^2}{u}+\frac{\varepsilon_2^2}{u}+\frac{b_3^2\varepsilon_1}{\varepsilon_1\varepsilon_2-u^2} \end{pmatrix}$$

C.2 Positive Semi-definite Cases

C.2.1 Rank 5, Case 2, $\delta > 0$

For Proposition 4.4.9, if $a = 0$ then $\mathcal{M}(2)$ is of the form

$$\begin{pmatrix}
\beta_1 & \frac{1}{4}\left(d^2-4\right)\beta_{x^3} & \beta_y & \beta_{x^2} & \frac{d\beta_{x^2}}{2} & \frac{d\beta_{x^2}}{2} & \beta_1+\beta_{x^2} \\
\frac{1}{4}\left(d^2-4\right)\beta_{x^3} & \beta_{x^2} & \frac{d\beta_{x^2}}{2} & \beta_{x^3} & \frac{d\beta_{x^3}}{2} & \frac{d\beta_{x^3}}{2} & \frac{d^2\beta_{x^3}}{4} \\
\beta_y & \frac{d\beta_{x^2}}{2} & \beta_1+\beta_{x^2} & \frac{d\beta_{x^3}}{2} & \frac{d^2\beta_{x^3}}{4} & \frac{d^2\beta_{x^3}}{4} & \beta_y+\frac{d\beta_{x^3}}{2} \\
\beta_{x^2} & \beta_{x^3} & \frac{d\beta_{x^3}}{2} & \beta_{x^4} & \frac{d\beta_{x^4}}{2} & \frac{d\beta_{x^4}}{2} & \beta_{x^2}+\beta_{x^4} \\
\frac{d\beta_{x^2}}{2} & \frac{d\beta_{x^3}}{2} & \frac{d^2\beta_{x^3}}{4} & \frac{d\beta_{x^4}}{2} & \beta_{x^2}+\beta_{x^4} & -\beta_{x^2}+\frac{1}{2}(-2+d^2)\beta_{x^4} & \frac{1}{2}d(\beta_{x^2}+\beta_{x^4}) \\
\frac{d\beta_{x^2}}{2} & \frac{d\beta_{x^3}}{2} & \frac{d^2\beta_{x^3}}{4} & \frac{d\beta_{x^4}}{2} & -\beta_{x^2}+\frac{1}{2}(-2+d^2)\beta_{x^4} & \beta_{x^2}+\beta_{x^4} & \frac{1}{2}d(\beta_{x^2}+\beta_{x^4}) \\
\beta_1+\beta_{x^2} & \frac{d^2\beta_{x^3}}{4} & \beta_y+\frac{d\beta_{x^3}}{2} & \beta_{x^2}+\beta_{x^4} & \frac{1}{2}d(\beta_{x^2}+\beta_{x^4}) & \frac{1}{2}d(\beta_{x^2}+\beta_{x^4}) & \beta_1+2\beta_{x^2}+\beta_{x^4}
\end{pmatrix}$$

If $a \neq 0$ in Proposition 4.4.9 then $\mathcal{M}(2)$ is of the form

$$\begin{pmatrix}
\beta_1 & \beta_x & \frac{(4-ad)\beta_x+\left(4-d^2\right)\beta_{x^3}}{2a} & \beta_{x^2} & 0 & 0 & 0 \\
\beta_x & \beta_{x^2} & \frac{1}{2}(a\beta_1+d\beta_{x^2}) & \beta_{x^3} & 0 & 0 & 0 \\
\frac{(4-ad)\beta_x+\left(4-d^2\right)\beta_{x^3}}{2a} & \frac{1}{2}(a\beta_1+d\beta_{x^2}) & \beta_1+\beta_{x^2} & \frac{1}{2}(a\beta_x+d\beta_{x^3}) & 0 & 0 & 0 \\
\beta_{x^2} & \beta_{x^3} & \frac{1}{2}(a\beta_x+d\beta_{x^3}) & \beta_{x^4} & 0 & 0 & 0 \\
\frac{1}{2}(a\beta_1+d\beta_{x^2}) & \frac{1}{2}(a\beta_x+d\beta_{x^3}) & \beta_x+\beta_{x^3} & \frac{1}{2}(a\beta_{x^2}+d\beta_{x^4}) & 0 & 0 & 0 \\
\frac{1}{2}(a\beta_1+d\beta_{x^2}) & \frac{1}{2}(a\beta_x+d\beta_{x^3}) & \beta_x+\beta_{x^3} & \frac{1}{2}(a\beta_{x^2}+d\beta_{x^4}) & 0 & 0 & 0 \\
\beta_1+\beta_{x^2} & \beta_x+\beta_{x^3} & \frac{\left(4+a^2-da\right)\beta_x+\left(-d^2+ad+4\right)\beta_{x^3}}{2a} & \beta_{x^2}+\beta_{x^4} & 0 & 0 & 0
\end{pmatrix}$$

$$+\begin{pmatrix}
0 & 0 & 0 & 0 & \frac{1}{2}(a\beta_1+d\beta_{x^2}) & \frac{1}{2}(a\beta_1+d\beta_{x^2}) & \beta_1+\beta_{x^2} \\
0 & 0 & 0 & 0 & \frac{1}{2}(a\beta_x+d\beta_{x^3}) & \beta_x+\beta_{x^3} & \\
0 & 0 & 0 & 0 & \beta_x+\beta_{x^3} & \beta_x+\beta_{x^3} & \frac{\left(4+a^2-ad\right)\beta_x+\left(4+ad-d^2\right)\beta_{x^3}}{2a} \\
0 & 0 & 0 & 0 & \frac{1}{2}(a\beta_{x^2}+d\beta_{x^4}) & \frac{1}{2}(a\beta_{x^2}+d\beta_{x^4}) & \beta_{x^2}+\beta_{x^4} \\
0 & 0 & 0 & 0 & \beta_{x^2}+\beta_{x^4} & \frac{\beta_1a^2}{2}+ad\beta_{x^2}-\beta_{x^2}+\frac{d^2\beta_{x^4}}{2}-\beta_{x^4} & \frac{1}{2}(a(\beta_1+\beta_{x^2})+d(\beta_{x^2}+\beta_{x^4})) \\
0 & 0 & 0 & 0 & \frac{\beta_1a^2}{2}+ad\beta_{x^2}-\beta_{x^2}+\frac{d^2\beta_{x^4}}{2}-\beta_{x^4} & \beta_{x^2}+\beta_{x^4} & \frac{1}{2}(a(\beta_1+\beta_{x^2})+d(\beta_{x^2}+\beta_{x^4})) \\
0 & 0 & 0 & 0 & \frac{1}{2}(a(\beta_1+\beta_{x^2})+d(\beta_{x^2}+\beta_{x^4})) & \frac{1}{2}(a(\beta_1+\beta_{x^2})+d(\beta_{x^2}+\beta_{x^4})) & \beta_1+2\beta_{x^2}+\beta_{x^4}
\end{pmatrix}.$$

C.2.2 Rank 6, Case 2

For Proposition 4.5.6, a two-parametric solution for $B(3)(p,q)$ is given by

$$B(3)(p,q) = \begin{pmatrix} \beta_{x^3} & \beta_{x^2y} & \beta_{x^2y} & \beta_x - \beta_{x^3} & \beta_{x^2y} & \beta_x - \beta_{x^3} & \beta_x - \beta_{x^3} & \beta_y - \beta_{x^2y} \\ \beta_{x^4} & \beta_{x^3y} & \beta_{x^3y} & \beta_{x^2} - \beta_{x^4} & \beta_{x^3y} & \beta_{xyxy} & \beta_{x^2} - \beta_{x^4} & \beta_{xy} - \beta_{x^3y} \\ \beta_{x^3y} & \beta_{x^2} - \beta_{x^4} & \beta_{xyxy} & \beta_{xy} - \beta_{x^3y} & \beta_{x^2} - \beta_{x^4} & \beta_{xy} - \beta_{x^3y} & \beta_{xy} - \beta_{x^3y} & \beta_1 - 2\beta_{x^2} + \beta_{x^4} \\ p & q & q & \beta_{x^3} - p & q & \beta_{x^3} - p & \beta_{x^3} - p & \beta_{x^2y} - q \\ q & \beta_{x^3} - p & \beta_{x^3} - p & \beta_{x^2y} - q & \beta_{x^3} - p & \beta_{x^2y} - q & \beta_{x^2y} - q & p + \beta_x - 2\beta_{x^3} \\ q & \beta_{x^3} - p & \beta_{x^3} - p & \beta_{x^2y} - q & \beta_{x^3} - p & \beta_{x^2y} - q & \beta_{x^2y} - q & p + \beta_x - 2\beta_{x^3} \\ \beta_{x^3} - p & \beta_{x^2y} - q & \beta_{x^2y} - q & p + \beta_x - 2\beta_{x^3} & \beta_{x^3} - p & p + \beta_x - 2\beta_{x^3} & p + \beta_x - 2\beta_{x^3} & q + \beta_y - 2\beta_{x^2y} \end{pmatrix}.$$

C.2.3 Rank 6, Case 3

For Proposition 4.5.9, a two-parametric solution for $B(3)(p,q)$ is

$$B(3)(p,q) = \begin{pmatrix} \beta_{x^3} & \beta_{x^2y} & \beta_{x^2y} & \beta_x + \beta_{x^3} & \beta_{x^2y} & \beta_x + \beta_{x^3} & \beta_x + \beta_{x^3} & \beta_y + \beta_{x^2y} \\ \beta_{x^4} & \beta_{x^3y} & \beta_{x^3y} & \beta_{x^2} + \beta_{x^4} & \beta_{x^3y} & \beta_{xyxy} & \beta_{x^2} + \beta_{x^4} & \beta_{xy} + \beta_{x^3y} \\ \beta_{x^3y} & \beta_{x^2} + \beta_{x^4} & \beta_{xyxy} & \beta_{xy} + \beta_{x^3y} & \beta_{x^2} + \beta_{x^4} & \beta_{xy} + \beta_{x^3y} & \beta_{xy} + \beta_{x^3y} & \beta_1 + 2\beta_{x^2} + \beta_{x^4} \\ p & q & q & \beta_{x^3} + p & q & \beta_{x^3} + p & \beta_{x^3} + p & \beta_{x^2y} + q \\ q & \beta_{x^3} + p & \beta_{x^3} + p & \beta_{x^2y} + q & \beta_{x^3} + p & \beta_{x^2y} + q & \beta_{x^2y} + q & p + \beta_x + 2\beta_{x^3} \\ q & \beta_{x^3} + p & \beta_{x^3} + p & \beta_{x^2y} + q & \beta_{x^3} + p & \beta_{x^2y} + q & \beta_{x^2y} + q & p + \beta_x + 2\beta_{x^3} \\ \beta_{x^3} + p & \beta_{x^2y} + q & \beta_{x^2y} + q & p + \beta_x + 2\beta_{x^3} & \beta_{x^3} + p & p + \beta_x + 2\beta_{x^3} & p + \beta_x + 2\beta_{x^3} & q + \beta_y + 2\beta_{x^2y} \end{pmatrix}.$$