WHAT IS AN INCONSISTENT TRUTH TABLE?

Zach Weber, Guillermo Badia, and Patrick Girard

Abstract

Do truth tables—the ordinary sort that we use to teach and explain basic propositional logic—require an assumption of *consistency* for their construction? In this essay we show that truth tables can be built in a consistency-independent paraconsistent setting, without any appeal to classical logic. This is evidence for a more general claim: that when we write down the orthodox semantic clauses for a logic, whatever logic we presuppose in the background will be the logic that appears in the foreground. Rather than any one logic being privileged, then, on this count partisans across the logical spectrum are in relatively similar dialectical positions.

Keywords: Paraconsistent logic, non-classical mathematics, dialetheism

1. Don't inconsistent tables fall down?

There are nowadays many non-classical logics, and some have been studied extensively for both their philosophical and mathematical value [Beall and Restall 2006; Shapiro 2014]. Some, such as intuitionistic or dialetheic paraconsistent logic, have even been argued for as the *correct* logic.[1] Nevertheless, classical logic remains a dominant force. According to the PhilPapers Survey of professional philosophers and others carried out in November 2009, a majority (51.6%) indicated that they accept or lean toward classical logic, while a slim minority (only 15.4%) accept or lean toward non-classical logic [Bourget and Chalmers 2014]. Gone are the days when non-classical logic can be responsibly dismissed out of hand as 'deviant' or 'merely changing the subject' à la Quine, but it is fair to say that non-classicality is still

[1] Intuitionism is a famous *paracomplete* logic, in which the law of excluded middle fails. *Dialetheism* is the thesis that some contradictions are true; a *paraconsistent* logic is one in which a contradiction does not necessarily imply everything. All these and more are detailed in [Priest 2008].

eyed with suspicion.[2] Expressions of concern are especially pressed on strong paraconsistent—dialetheic—proposals [Shapiro 2002; Field 2008: 377]. The idea that a logico-mathematical apparatus might allow some *inconsistency* just seems to engender worries about airplanes falling out of the sky and collapsing bridges. Or, apropos, collapsing tables.

At the risk of doing armchair psychology, we suspect that at least one reason for continued classical privilege is simply pragmatic. Classical logic is (basically) well understood, with a century-old battery of results and methods. Classical propositional truth tables are often taught and learned as uncontroversial; aligning with the classical tradition is safe and sensible. By contrast, much about the philosophical and technical aspects of non-classical logics remains to be developed. So serious scholars who are tempted to work on, say, inconsistent mathematics, may worry the area is not well established enough, the rules of the game too murky, for it to be a conscientious use of research time. Life is short, everyone wants to get some work done, and to date it has been unclear what it would mean to adopt a non-classical logic as a logic that we really do use in our thinking, writing, teaching, etc. As a basic example, no one has really known what a *truth table*, constructed using only paraconsistent methods, might look like. And so the main purpose of this essay is pragmatic, too—just to show the answer.

In doing so, our results provide evidence for a deeper claim about the nature of logical theorizing, that should give some pause to even those well-outside inter-logical debates. The presuppositions we bring to the whiteboard when we describe a logic show up directly in the logic.[3] Write down the standard clauses for the semantics of the logical connectives, assume bivalence and non-contradiction, and voilà, the connectives turn out to be classical. Metatheory determines object theory.[4]

In the 20th century it was found that set theory and logic can be used to co-write each other: the (boolean) logic of sets generates the desired (boolean)

---

[2] See [Williamson 2014] for a nice example of balance between these two attitudes.

[3] As argued in [Priest: 1990; Sylvan 1992]; our work aims to continue and extend these programs.

[4] 'To show the validity of certain rules of inference, truth-conditions alone are not enough. Inference must be made from those conditions: and these inferences *may* be just those whose validity one is trying to demonstrate' [Priest 1990: 210-11].

semantics of logic, and vice versa. Under classical conditions, propositions do not hold in conjunction with their negations *in exactly the same way* that sets do not overlap with their complements.[5] But this is just one approach. A de Morgan algebra can also be used to develop set theory [Routley 1977; Brady 2006; Weber 2012]. We will see that the assumptions made there carry over in just the same way: paraconsistent set theory naturally generates a paraconsistent semantics. On this count, partisans from across the logical spectrum are in roughly equal dialectical positions. Logic implies logic.

2. Logic for talking about logic

Despite vociferous arguments in favor of non-classical logic, it is almost always presupposed that our informal discourse *about* logic must be understood against a background of classical logic.[6] The syntax and semantics of paraconsistent and paracomplete logics—their grammar and truth tables—are almost always taken to be 'classically behaved'. And indeed, most prominent examples in the literature encourage this assumption, from Kripke [1975] to Field [2008] to Beall [2009].

Reliance on classical theory is pretty peculiar, though, if one takes the rhetoric that often accompanies these logics seriously. Someone who is philosophically committed to the need for a paraconsistent logic, say because of contradictions in naive set theory, seems vulnerable to at least a nasty ad hominem if they fall back on inconsistency-intolerant logic at the metalevel. The question is asked by Burgess [2005: 740]:

> How far can a logician who *professes* to hold that [paraconsistency] is the correct criterion of a valid argument, but who freely accepts and offers standard mathematical proofs, in particular for theorems about [paraconsistent] logic itself, be regarded as *sincere* or *serious* in objecting to classical logic?

---

[5] Ignoring that classical set theory cannot have an unrestricted complementation operation, because there is no classical universe of sets in which to take complements [Halmos 1960: 7].

[6] Intuitionistic metatheory has been investigated since the 1960s [McCarty 2008], but, at least in the Bishop style, it occurs in the computational fragment of classical mathematics and in that sense does not challenge the classicist's presupposition. A more recent attempt to raise up a non-classical metatheory is [Bacon 2013], conducted in a classical *meta*-metalanguage.

Indeed, much of the argument for naive set/truth theories is that there is no such thing as a metalevel—'the whole *point* of the dialetheic solution to the semantic paradoxes is to get rid of the distinction between object language and meta-language' [Priest 1990: 208]—making a resort to classicality at the metalevel all the more awkward; cf. [Shapiro 2002: 818].

Perhaps the use of classical logic could be charitably taken as the non-classicist 'preaching to the gentiles in their own tongue', to use a phrase of Meyer.[7] Classicality is no more than a ladder to be kicked away. If so, this can only be a limited rationale for resorting to a classical metatheory. Presumably one would still want a decent theory available once the ladder is gone—a semantics to work from *after* all the gentiles are converted!

And this gets to the most basic point: producing some simple objects like truth tables is a reasonable adequacy criterion to hold a proposed logico-mathematical program to. Whether or not a system can stand independently is a basic test of its viability. Tools that cannot even build ordinary tables are of little value. Regardless of the other philosophical costs and benefits of such systems, and regardless of one's own predispositions about which ones are preferable to use, it is good to know what the serious options really are. The present paper is a contribution to that task—of providing some reassurance that paraconsistent metatheory is at least a live option.[8]

3. A background (paraconsistent) theory

To get going, we need some paraconsistent tools: logic and set theory. The presentation here is minimal so that we can get on to the truth tabling; interested readers are encouraged to consult the cited literature.

---

[7] Meyer was particularly sensitive to these issues; see (if you can) his typescript 'The Completeness of R Proved Relevantly'. The concern is echoed by Brady [Brady 1989: 443].

[8] There are certainly more reasons one could give for sticking with classical metatheory. Our goal here is to provide motivation, not a knockout argument. For more discussion, see [Priest 2006: sec.18.5; Field 2008: sec 5.6; Beall 2009: 39, 112], and [Girard and Weber 2015].

3.1. Logic

A hybrid Hilbert system for our logic is given in Appendix 1. There are two features of the logic we wish to emphasize.

First, because we are in a paraconsistent setting, *disjunctive syllogism fails* [Priest 2008: 154]. If φ is a true contradiction, then the argument from $\varphi \vee \psi$ and $\neg\varphi$ to ψ has true premises but may lead to an absurd (or at least untrue) conclusion, and so cannot be valid in general. However, modus ponens *is* valid: if φ, and φ implies ψ, then ψ. So 'implies' cannot be reduced to negation and disjunction. The material conditional is out, and non-truth-functional conditionals are required ([Beall 2013] notwithstanding.) We need two such 'ponenable' conditionals to get some work done: a connective $\rightarrow$ that respects relevance [Mares 2004; Routley et al 1982], and a 'deductive' $\Longrightarrow$ that respects the rule of conditional-introduction [Caret and Weber 2015]. In this essay, the words `if...then' can be assumed to mean the $\Longrightarrow$ conditional.

Second, *contraction fails*, basically because of Curry's paradox [Meyer et al 1978; Restall 1994]. The principle of contraction is that if φ was used *twice* to get ψ then that is the same as using φ once to get ψ. It turns out to be dangerous—indeed, incoherent in the context of the set theory given in section 3.2 below—to repeat premises without keeping track of how many times they are repeated. Repetitions matter; just ask the boy who cried wolf. Contraction must be avoided at the level of both the 'inner' implication connective(s) and the 'outer' consequence relation [Beall and Murzi 2013]. Any principle that yields contraction must go, too, such as $\varphi \Longrightarrow \varphi \wedge \varphi$. Working without contraction is one of the main technical accomplishments of this essay: building a table without reusing any resources.

3.2. Sets

The non-logical apparatus underwriting all this (to say nothing of *motivating* it) is expressed with set membership. Because for this essay we want to know about how much natural, 'classroom-level' reasoning is possible in a paraconsistent metatheory, without clouding the issue with any accidents of set theoretic reductionism, and for the sake of being self-contained, all of the following is simply assumed without delay; see [Brady 2006; Badia 201x].

Axiom 1 [Extensionality] *Sets with exactly the same members are identical*,

$$\forall z(z \in x \leftrightarrow z \in y) \leftrightarrow x = y$$

Axiom 2 [Comprehension] *Every property has a set extension*,

$$\exists y \, \forall x(x \in y \leftrightarrow \varphi)$$

Basic set algebra is given by comprehension,[9] as is the use of ordered pairs,

$$\langle x, y \rangle \in \{z: \varphi(z)\} \leftrightarrow \varphi(\langle x, y \rangle)$$

which obey the law of ordered pairs $\langle x, y \rangle = \langle u, v \rangle \Leftrightarrow x = u \wedge y = v$, following Brady [2006: 178, 251].[10]

The comprehension axiom—a reconstruction of Frege's Basic Law V—is the most unrestricted version of set formation, from Brady and Routley [1989: 419], where y can appear free in φ. This allows for highly impredicative sets, which can appear in their own defining property [Brady 2006: 176-7]. Having circularly defined sets on hand is not only extremely useful in what follows, suggesting a deeply important role between circularity and proofs by induction; it is also correct, insofar as *every* property determines a set.

The existence of sets is automatic given comprehension, but leads to difficulty with *uniqueness* (dually to constructive mathematics). E.g. the (bizarre) instance of comprehension $\mathcal{D} = \{x: \mathcal{D} \in x\}$, 'the set of all sets containing the very set I am now describing', does not necessarily describe one unique set. Due to the circularity of the definition, to show that two sets satisfying this property are identical using the axiom of extensionality, one would already have to know that they are identical. This is the same as the situation in (consistent) non-well-founded set theory [Barwise and Moss 1996: 15], which must simply assert that equations have unique solutions. Here, we solve the problem with

[9] $x \in \{z: \varphi\} \cup \{z: \psi\} \leftrightarrow \varphi \vee \psi$, and $x \in \{z: \varphi\} \cap \{z: \psi\} \leftrightarrow \varphi \wedge \psi$; and $x \in \overline{\{z: \varphi\}} \leftrightarrow \neg\varphi$.

[10] This is to ensure contrapositive entailments $\langle x, y \rangle \notin \{z: \varphi(z)\} \leftrightarrow \neg\varphi\langle x, y \rangle$ without the failure of disjunctive syllogism getting in the way.

Axiom 3 [Choice] *Let Y be a set of non-empty sets. There is a function*[11] *f on Y such that $f(X) \in X$, for all X in Y.*

Cf. [Brady 2006: 253]. So if we have a circularly defined set but uniqueness is required, the functionality of choice can select one, e.g. fixing *the* set I am currently describing.[12]

The last axiom says that one can give well-behaved proofs about well-behaved structures:

Axiom 4 [Induction] *Any recursively defined structure satisfies induction*.

Cf. [Brady 2006: 318]. For example, the set of all (well-formed) formulae is recursively defined, and e.g. one may prove that *all* formulas are finite strings by showing that all the propositional atoms are finite, and the negation, conjunction, etc of finite strings is finite.

3.3. Validity

For expressive completeness, two relations are added: syntactic validity $\vdash$ and semantic consequence $\vDash$. For $\vdash$, the inductive definition, supported by Axiom 4, is

Definition 1 *With Γ a set of premises,*

$$\Gamma \vdash \varphi$$

iff φ follows from some subset of Γ by valid rules.

The set of *theorems* is made up either of axioms deducible from no premises, or deducible from the axioms via the operational or structural rules. The definition of $\vDash$ is the topic of the next section. If the definition of a syntactic derivation sounds (comfortingly? suspiciously?) familiar, this is prelude for what is to come.

---

[11] `Function' means that $\langle X, x\rangle \in f \wedge \langle X, y\rangle \in f \Rightarrow x = y$.

[12] For (controversial) derivations of choice from comprehension, see [Routley 1977; Weber 2012]. For an alternative way to deal with the uniqueness problem, see [Brady 1989: 419].

4. True vs true only: two values or three?

With suitable groundwork laid, we approach an answer to our title question. But first a crucial methodological issue must be faced.

Everyone knows Tarski's theorem: no consistent language can contain its own truth predicate. Put more motivationally, given that our language *does* contain its own truth predicate, there can be no exclusive and exhaustive partitioning of all the sentences into all-and-only the truths, versus all-and-only the non-truths. It would have been nice, but it is impossible. (See [Field 2008; Beall 2009].) This leaves us with two basic options:

- to have only truths, some must be left out (the *incomplete* strategy);
- to have all the truths, some untruths must be kept in (the *overcomplete* strategy).

We must choose which is more important. Should we focus on untruth-avoidance at all costs, or emphasize truth-seeking, come what may? Both classicists and intuitionists/paracompleteists choose the former. The dialetheist chooses the latter.

Now, standard presentations of dialetheic paraconsistent logic, going back to the foundational works [Priest 1979], are via a three-valued functional semantics, with values {*t, f, b*}, the third value read as 'both'. Put this way, a dialetheic logic looks like a species of multi-valued logic. This is a nice way to present the idea to unfamiliar audiences, but it does not do justice to the little story told in the previous paragraph. For a start, it does not really capture the idea of either the incomplete or overcomplete strategies, since in either case some sui generis item has been introduced, that is not defined in terms of *t* or *f*.

But maybe this problem is just aesthetic. More damningly, a three-valued semantics makes it appear that there is after all an exclusive and exhaustive partitioning of the universe of truths, into three categories: into the all-and-only truths, all-and-only untruths, and the all-and-only 'both's. If the original Tarski problem was insoluble, a little thought about infinite regress will show that this new, three-tiered approach will be no better. It is susceptible to revenge, of the sort: 'the (only) truth-value of this very sentence is *f*'.

The three-valued approach encourages a common criticism of dialetheism—that some important expressive power has been lost, namely, the ability to demarcate the truths (t valued) from the true contradictions (b valued) [Restall 2010; Rossberg 2013; cf. Martin 2015]. The dialetheist cannot pin down all and only the just-trues, on pain of absurdity. *But surely*, says the critic, *this distinction is available—there it is in your semantics!—and yet the object language cannot express it*. Indeed, this would be a devastating criticism ... if not, again, for the *original* problem, which is that no one can in fact make this demarcation, by Tarski. The dialetheist is at no expressive loss, unless we are all at the same loss. With this in mind, the dialetheic paraconsistentist should lead the discussion away from pre-Tarskian ideation, and (except perhaps for pedagogy) use a formalism that does not invite or suggest such criticism.

For this principled reason, then, the presentation in this paper is entirely in a two-valued *relational* semantics, as pioneered by Dunn in the 1960s and developed in the relevant logic literature [Priest 2008: 161]. Classically, functional and relational semantics are equivalent, but we stress that this is not so in a paraconsistent setting [Priest 2008: 151]. The major suggestion of this paper is to move from classical to paraconsistent reasoning about relational semantics. We will see how closely such relations can approximate the otherwise-desirable functional three-valued semantics; the answer will be: very closely, but not exactly. To reiterate, this is not really a *decision* on our part, but rather a *requirement* from the Tarski problem for any logic that can express its own metatheory.

Henceforth, there are two truth values, t and f, which are exclusive, on pain of absurdity: if t and f are identical, then all is lost,

$$(t = f) \Longrightarrow \varphi$$

for any φ. (Those who hold that every sentence is true have no need of a paraconsistent logic, or any logic for that matter; see section 7.3 below.)

The main work of the paper is on the propositional fragment of the language, using the truth-table-apt extensional connectives: conjunction, disjunction, and negation. Giving a paraconsistent semantics of the whole first-order language, including the non-truth-table-apt conditionals, remains for another day; for some starting hints, see [Allwein and Dunn 1993; Restall 2000].

5. Relational Truth Conditions

A truth-value assignment on the propositions is any two-place relation R^0 taking every proposition to at least one truth value t or f under the exclusion condition that sentences are true iff not false and false iff not true:

$$\langle p, t\rangle \in R^0 \Leftrightarrow \langle p, f\rangle \notin R^0$$

$$\langle p, f\rangle \in R^0 \Leftrightarrow \langle p, t\rangle \notin R^0$$

While asking a valuation to be exclusive about truth and falsity seems to go against the title purpose of this essay, notice that R^0 is a set in a background naive paraconsistent set theory. So it may well be that, while R^0 is always exclusive about its assignments, it can sometimes also not be exclusive, in which case $R^0 \neq R^0$. Some sets are inconsistent, after all, as Cantor told Dedekind and Russell told Frege (with apologies to Cantor [1899] and Russell [1902]).

In fact, here is an example, which shows that at least one such relation exists. Take the (inconsistent) assignment on just one proposition p making it both true and false, $\{\langle p, t\rangle, \langle p, f\rangle\}$. Intersect the assignment with Routley's set

$$\mathcal{Z} = \{x: x \notin \mathcal{Z}\}$$

from [Routley 1977], a set for which it is true that $x \in \mathcal{Z}$ and $x \notin \mathcal{Z}$ for any x whatsoever (!). For *any* set X, its intersection with $\mathcal{Z}$ will therefore both include and not include every member of X. So the assignment $\{\langle p, t\rangle, \langle p, f\rangle\} \cap \mathcal{Z}$ both contains and does *not* contain $\{\langle p, t\rangle, \langle p, f\rangle\}$. Therefore the required biconditionals hold, just because their four components do.[13]

Less tendentiously, for any such relation, by the law of excluded middle, it is not empty: either $\langle p, f\rangle \in R^0$, or else $\langle p, f\rangle \notin R^0$, in which case $\langle p, t\rangle \in R^0$.

Now that all the atoms are assigned value(s), extend this relation recursively to all formulas with a relation R giving the stock-standard clauses for the logical connectives (in the more usual infix notation): reading 'φRt' as 'φ is true', then

[13] This 'trivial' valuation shows up again in the soundness theorem below. It may be possible to show the existence of models in less diabolical ways, but we can't think of any. Full circular comprehension is naturally very well suited for getting inductive processes going ex nihilo, since recursion is in some sense circular. Full comprehension continues to be essential for what follows.

$$\neg\varphi Rt \Leftrightarrow \varphi Rf$$

$$\neg\varphi Rf \Leftrightarrow \varphi Rt$$

$$(\varphi \wedge \psi)Rt \Leftrightarrow \varphi Rt \wedge \psi Rt \qquad (\varphi \vee \psi)Rt \Leftrightarrow \varphi Rt \vee \psi Rt$$

$$(\varphi \wedge \psi)Rf \Leftrightarrow \varphi Rf \vee \psi Rf \qquad (\varphi \vee \psi)Rf \Leftrightarrow \varphi Rf \wedge \psi Rf$$

Read aloud, these are completely familiar semantics: a conjunction is true iff both conjuncts are, and so forth.[14] Moreover, the semantic relation will satisfy an exclusivity condition, lifted from the propositional case: φ is true iff it is not false, and false iff it is not true,

$$\varphi Rt \Leftrightarrow \neg(\varphi Rf)$$

$$\varphi Rf \Leftrightarrow \neg(\varphi Rt)$$

In the case of a contradiction, φ both *t* and *f*, then it will also be the case that φ is not *t* and not *f*. Negation facts inside the semantics ($\neg\varphi$ is true) permeate outside the semantics (it is not true that φ)—a main departure from usual presentations of dialetheic semantics [Priest 2006: 70-1], and a main feature of using a paraconsistent metatheory. This is most dramatic in the case of dialetheism itself, in which all contradictions are false: $(\neg(\varphi \wedge \neg\varphi))Rt$ and $\neg((\varphi \wedge \neg\varphi)Rt)$. Dialetheism is, according to dialetheism, false [Priest 1979], and the semantics should bear this out.

A relation *R* of this sort is called a *model* for extensional propositional logic. Following Tarski, $\langle \varphi, t\rangle \in R$ can be read as 'the relation *R satisfies* formula φ'. More generally we can give the Tarskian thesis on semantic consequence:

Definition 2 *A sentence ψ is a* valid consequence *of φ_0, …, φ_n,*

$$\varphi_0, \dots, \varphi_n \vDash \psi$$

iff $\langle \varphi_0, t\rangle \in R \wedge \dots \wedge \langle \varphi_n, t\rangle \in R \Longrightarrow \langle \psi, t\rangle \in R$ *for all models R. A sentence φ is a* tautology

$$\vDash \varphi$$

iff $\langle \varphi, t\rangle \in R$ *for all R.*

[14] 'The truth conditions for Boolean negation ... actually use the notion of negation ['not'] and hence are ambiguous, depending on whether this is itself De Morgan or Boolean negation' [Priest 1990: 207].

Now the thing to do is check that models exist.

Theorem 1 *Any truth-value assignment on a set of propositions can be extended to a model for propositional logic*.

Proofs are in Appendix 2, but the idea is straightforward. As above, we have an R^0 on propositional atoms that is non-empty, in which true sentences are not false and false sentences not true. The trick now is to extend R^0 by a circular definition from unrestricted comprehension, as follows:

$$
\begin{aligned}
R = \quad & \{\langle p, t\rangle : pR^0t\} \\
\cup \quad & \{\langle p, f\rangle : pR^0f\} \\
\\
\cup \quad & \{\langle \neg\varphi, t\rangle : \langle\varphi, f\rangle \in R\} \\
\cup \quad & \{\langle \neg\varphi, f\rangle : \langle\varphi, t\rangle \in R\} \\
\\
\cup \quad & \{\langle \varphi \wedge \psi, t\rangle : \langle\varphi, t\rangle \in R \wedge \langle\psi, t\rangle \in R\} \\
\cup \quad & \{\langle \varphi \wedge \psi, f\rangle : \langle\varphi, f\rangle \in R \vee \langle\psi, f\rangle \in R\} \\
\\
\cup \quad & \{\langle \varphi \vee \psi, t\rangle : \langle\varphi, t\rangle \in R \vee \langle\psi, t\rangle \in R\} \\
\cup \quad & \{\langle \varphi \vee \psi, t\rangle : \langle\varphi, f\rangle \in R \wedge \langle\psi, f\rangle \in R\}
\end{aligned}
$$

Use the axiom of choice to fix a *unique* such *R*. Then an induction on formulas shows that *R* satisfies the conditions for being a model. In essence, there are the truth tables—built standing on themselves.

6. So what does an inconsistent truth table look like?

The answer to our titular question is bluntly simple. An inconsistent truth table looks like a truth table. Here are some:

	$\neg$
t	f
f	t

$\wedge$	t	f
t	t	f
f	f	f

$\vee$	t	f
t	t	t
f	t	f

Such two-dimensional displays are often implicitly assumed to be functional look-up tables. But no such assumption is explicitly displayed or otherwise enforced by what is on the page; cf. [Sylvan 1992]. As we've been saying, you get out what you put in. It is simply presupposing classicality to see these as functions. Diagrams must be used with great care in mathematical argument, as gaps from Euclid's very first proposition—encouraged by an overly-suggestive diagram—through 'proofs' of the intermediate value theorem have taught us. Without assuming consistency, these are displays of semantic relations, and allow overdetermination of truth value.[15]

To be very careful about it, the tables can be read as, 'if t is among the values of φ, then f is among the values of $\neg\varphi$'. But the more concise reading will do, too: 'if φ is true, then $\neg\varphi$ is false'. Such a reading is perfectly acceptable here, provided that additional presuppositions (e.g. that truth *rules out* falsity) are not being made. The copula—the 'is' of predication—is not univocal in general, and it is not here.

Where is the inconsistency, then? Take the liar sentence, λ. (The liar would come from truth theory, more than the (propositional) logic alone; it is just being used as an easy example here.) Consider the conjunction $\lambda \wedge \neg\lambda$. Since λ is true and $\neg\lambda$ is thereby false, we look on the table for $\wedge$ and find that $\lambda \wedge \neg\lambda$ is false. Also, since the liar is true and its negation is true too (it is the liar, after all), the table tells us that $\lambda \wedge \neg\lambda$ is true, too. And from this positive information, it is also correct to say that it is *not* the case that $\lambda \wedge \neg\lambda$ is false, and $\lambda \wedge \neg\lambda$ is not true, again just by looking at the table.

There is nothing, in short, particularly flashy about inconsistent truth tables; they are just truth tables after all. What is at issue is the best *description* of the objects in question. We have offered one, that accords with a thoroughgoing dialetheic approach. Whether or not these are the best

[15] Sylvan argued in [1992] that having only two values when building truths tables (even if the underlying meta-theory is entirely classical) does not suffice to pin down classical logic.

descriptions of truth relations is debatable—a going philosophical concern. Our point, however, is that it cannot be shunted off as a *technical* issue, by which consistency is an inarguable precondition for stating basic semantics.

7. Theorems

Logicians will now want us to check that the semantical consequence relation given in Definition 2 is appropriately connected to the proof relation of Definition 1—that not only do the semantics exist, but they work. Proofs are again inductions, and left to the appendix. As our focus is on truth tables, these theorems apply only to the extensional part of the language, formulae with truth-functional connectives.

7.1. Soundness, and not soundness

Theorem 2 [Soundness] *All derivable theorems are semantic tautologies*:

$$(\vdash \varphi) \Longrightarrow (\vDash \varphi)$$

for all φ.

A strong contrapositive version of this theorem also holds: if $(\vDash \varphi) \Longrightarrow \bot$ then $(\vdash \varphi) \Longrightarrow \bot$. That is, if the semantics are trivial, then so too is the proof relation. Proof: suppose that $(\vDash \varphi) \Longrightarrow \bot$. Assume further that $\vdash \varphi$. From the theorem, $\vDash \varphi$, so by hypothesis, $\bot$.

So the logic is sound. But also not. Pick a propositional variable p. We have that $p \vee \neg p$ is a theorem by the law of excluded middle. However, there is a model where p both holds and does not: consider again the valuation on p making it both true and false, intersected with $\mathcal{Z}$ as in section 5 above, so that $\langle p, t\rangle$ and $\langle p, f\rangle$ both are and are *not* in the assignment. Thus $p \wedge \neg p$ holds, and by de Morgan laws, $\neg(p \vee \neg p)$ holds, which means that $p \vee \neg p$ doesn't hold. Hence, $p \vee \neg p$ is also *invalid*.

Corollary 3 $\neg((\vdash \varphi) \Longrightarrow (\vDash \varphi))$ *for some* φ.

So the logic is sound and not sound. The proof is a short step from the well-known fact that the logic *LP* has a valuation in which every proposition is both true and false [Beall, Forster, and Seligman 2013: Remark 2.1.]; no sentence of *LP* fails in every model. We are simply letting the object level facts affect the meta-level.

Now, since 'sound and unsound' is a bit of a startling conclusion, one might want to pull focus back to non-trivial models, in the hope that the pure *logic* part of the theory might be sound only. Otherwise, the soundness theorem does not provide the reassurance one expects. Can't we prove that inferences never fall from truth to untruth? Well, to the extent that we can legitimately say and prove such things, then yes, we already have; to the extent that the question is asking for a further demarcation of a 'middle' category of truth value, to isolate a stronger notion of soundness for the 'true-only', then no, for the reasons already given in section 4, and one more time in section 7.3 below.

In any case, we already know that in the system more widely—the logic, plus some paradox-generating assumption—there are true contradictions like the liar that can be proved and have true negations. For proven contradictions λ we will have $\vdash \lambda$ but $\nvDash \lambda$. This system will be easily sound and not sound. *Any inconsistent system is unsound*, after all.

7.2. Completeness

For completeness, the proof uses Kalmar's method [1935], a direct (weak) completeness proof based on the idea of transforming truth tables into proofs; cf. [Kleene 1952]. Each row of a truth table contains a model which we systematically encode in a proof. The logic is complete, and as far as we can tell, only complete—though it may turn out that the negation of completeness holds, too. (Open question for paraconsistent mathematics students.)

Theorem 4 [Completeness] $(\vDash \varphi) \Longrightarrow (\vdash \varphi)$

7.3. Non-triviality

Classically, soundness already implies consistency; but not so here. The replacement notion to assure some basic level of coherence is *non-triviality*: a theory is non-trivial iff there is at least one sentence that is not part of the theory. (This is classically equivalent to consistency.) Non-triviality is a deal-breaker; if it were not, as we said above, a dialetheist would not need to bother with paraconsistent logic. Now, this essay has been overweeningly concerned with giving proofs that are independent of classical logic. Non-triviality proofs are usually done as relative consistency proofs, though, showing that paraconsistent objects can be represented as classical. Much as in the case of non-Euclidean geometry, there is nothing at all wrong with having a relative consistency proof, a model of a novel theory represented in a familiar theory. But it is in the spirit of this essay to ask, is there an *internal* demonstration of non-triviality?

There are at least two candidate notions for showing, from within a paraconsistent metatheory, that the $\vDash$ relation is not trivial:

- There is a sentence φ such that $\neg(\vDash \varphi)$
- There is a sentence φ such that $(\vDash \varphi) \Longrightarrow \psi$, for any ψ

It is easy to show that $\vDash$ is not trivial in the first sense, just by producing some theorem $\neg\varphi$. But as the soundness theorems suggest above, this does not preclude $\vDash$ from being trivial as well. So the second sense seems to be more what is wanted, e.g. it should be the case that disjunctive syllogism, $\neg\varphi, \varphi \vee \psi \vDash \psi$, is invalid and invalid *only*. To show it is invalid (only), we want a model in which φ and $\neg\varphi$ are both true, but where ψ is not true on pain of absurdity, and which is a model in the sense of section 5. One might think that, with so much expressive power at our disposal, a suitable countermodel might be at least constructible by hand. But it is not. (The reader is invited to give it a try.) From within the paraconsistent framework, one cannot guarantee that the framework is absolutely non-trivial.

Seeing why is not a matter of pondering the technical details, but appreciating a more subtle point: why *bother* showing internal non-triviality? After all, we can do so easily:

Theorem 5 *Naive set theory is not trivial.*

Proof. Either naive set theory is trivial or not. If not, we are done. If trivial, then, since this very proof is in naive set theory, it follows that the system is not trivial—since, after all, anything follows, QED. □

If this seems like a joke, notice that there is nothing special here about paraconsistency. One must constantly recall the situation in mathematics overall: where consistency (and so non-triviality) is not provable [Voedovsky 2010]. Using a paraconsistent metatheory constitutes no expressive loss; it makes more explicit the epistemic situation that we are in, calling attention to the faith we *all* put in the integrity of logical systems. One does not change the odds of incoherence just by endorsing or disavowing any particular logical system. From this vantage, the debate has not reached stalemate, but the dialectical field is leveled.

8. Conclusion

In logic, you get out what you put in. Logic does not tell us what is true; it tells us what is true given some other truths. If you bring to the uninterpreted propositional connectives a presupposition of classical logic, then the connectives will be classical. But this hardly shows that metatheory 'must' be conducted in classical language. Any such claim is transparently circular—for the classicist, paraconsistentist, or otherwise. In this essay, we have not established that metatheory determines object theory, but the exercise marshals considerable evidence in that direction. One can take all this in two ways. Either paraconsistency is more secure, because able to produce commonplace objects like truth tables; or classicality is less secure, because no better off than paraconsistency. Or both.[16]

[16] Thanks to audiences at the New Zealand Association of Philosophy December 2014 meeting and the Prague Seminar on Non-Classical Mathematics June 2015. Thanks to two anonymous referees and the editor for very helpful comments.

Appendix 1: Logic

A substructural dialetheic logic based on the relevant logic DK from [Routley and Meyer 1976; Routley 1977] is given here as a hybrid Hilbert system (see [Restall 2000: ch 4]). The language has individual constants *a*, *b*, …, variables *x*, *y*, …, function symbols *f*, *g*, …, and predicate symbols P^n, Q^n, …, and =. The letters *p*, *q*, …, for predicate symbols of arity 0 are called *propositions*. Well-formed formulas are produced in the usual way out of atomic formulas and the logical constants ∧, ∨, ¬, ∃, and ∀.

Axioms

$\vdash \varphi \rightarrow \varphi$

$\vdash (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \chi) \rightarrow (\varphi \rightarrow \chi)$

$\vdash \varphi \vee \neg\varphi$

$\vdash \neg\neg\varphi \rightarrow \varphi$

$\vdash (\varphi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \neg\varphi)$

$\vdash \varphi \wedge \psi \rightarrow \varphi$

$\vdash \varphi \wedge \psi \rightarrow \psi \wedge \varphi$

$\vdash \varphi \vee \psi \leftrightarrow \neg(\neg\varphi \wedge \neg\psi)$

$\vdash \varphi \wedge (\psi \vee \chi) \leftrightarrow (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$

$\vdash (\varphi \rightarrow \psi) \Longrightarrow (\varphi \Longrightarrow \psi)$

$\vdash \neg(\varphi \Longrightarrow \psi) \Longrightarrow \neg(\varphi \rightarrow \psi)$

$\vdash (\varphi \Longrightarrow \psi) \wedge (\chi \Longrightarrow \psi) \Longrightarrow$
$(\varphi \vee \chi \Longrightarrow \psi)$

$\vdash x = y \Longrightarrow (\varphi(x) \rightarrow \varphi(y))$

$\vdash \forall x\varphi \rightarrow \varphi$

$\vdash \forall x(\varphi \Longrightarrow \psi) \Longrightarrow (\exists x\varphi \Longrightarrow \psi)$
(x *not free in* ψ)

$\vdash \forall x\varphi \leftrightarrow \neg\exists x\neg\varphi$

Rules

$\varphi, \varphi \Longrightarrow \psi \vdash \psi$

$\varphi, \neg\psi \vdash \neg(\varphi \Rightarrow \psi)$

$\varphi(x) \vdash \forall x\varphi(x)$

$$\begin{array}{c} \Gamma, \varphi \vdash \psi \\ \hline\hline \Gamma \vdash \varphi \Longrightarrow \psi \end{array} \qquad \frac{\Gamma, \varphi, \chi \vdash \psi}{\Gamma, \chi, \varphi \vdash \psi} \qquad \frac{\Gamma, \varphi, \chi \vdash \psi}{\Gamma, \varphi \wedge \chi \vdash \psi}$$

$$\frac{\Gamma \vdash \varphi \qquad \Delta \vdash \psi}{\Gamma, \Delta \vdash \varphi \wedge \psi} \qquad \frac{\Gamma \vdash \psi}{\Gamma, \varphi \vdash \psi} \qquad \frac{\Gamma \vdash \varphi \qquad \Delta, \varphi \vdash \psi}{\Gamma, \Delta \vdash \psi}$$

where Γ, Δ are (multi)sets of formulae.

Appendix 2: Proofs of theorems

Theorem 1 *Any truth-value assignment R^0 on a set of propositions can be extended to a model R for propositional logic.*

Proof. Let R^0 be any assignment on propositional atoms. Given the definition of R in section 5, we show by induction that for any formula

$$\langle\varphi, t\rangle \in R \Leftrightarrow \langle\varphi, f\rangle \notin R$$
$$\langle\varphi, f\rangle \in R \Leftrightarrow \langle\varphi, t\rangle \notin R$$

This will be the inductive hypothesis. To avoid contraction, we don't use the very same hypothesis for each inductive case. They are rather *hypothesis schemata*, each instance used once.

BASE CASE: For any p and $x \in \{t, f\}$, $pRx \Leftrightarrow pR^0x$ and $\neg(pRx) \Leftrightarrow \neg(pR^0x)$.

NEGATION: For left to right, suppose $\neg\varphi Rt$. Then φRf by definition of R. Then using the induction hypothesis, $\neg(\varphi Rt)$. So $\langle\varphi, t\rangle \notin R$, which means again by definition of R that $\neg(\neg\varphi Rf)$. The second case is computed similarly: $\neg\varphi Rf \Rightarrow \varphi Rt \Rightarrow \neg(\varphi Rf) \Rightarrow \neg(\neg\varphi Rt)$. For right to left, suppose $\neg(\neg\varphi Rt)$. Then $\langle\neg\varphi, t\rangle$ is not in R, and in particular, not in the case covering negation, so $\langle\varphi, f\rangle \notin R$. Then by inductive hypothesis, φRt, which is the condition for membership of $\langle\neg\varphi, f\rangle$ in R. The proof that $\neg(\neg\varphi Rf) \Rightarrow \neg\varphi Rt$ is similar.

CONJUNCTION: Left to right, if $(\varphi \wedge \psi)Rt$ then by definition $\varphi Rt \wedge \psi Rt$. Using the induction hypothesis, then $\neg(\varphi Rf) \wedge \neg(\psi Rf)$, which means by duality of conjunction and disjunction that $\neg(\langle\varphi, f\rangle \in R \vee \langle\psi, f\rangle \in R)$, which means by definition of R that $\neg((\varphi \wedge \psi)Rf)$. The other case $(\varphi \wedge \psi)Rf \Rightarrow \neg((\varphi \wedge \psi)Rt)$ is the same. Right to left, suppose $\neg((\varphi \wedge \psi)Rf)$. Then in particular $\langle\varphi \wedge \psi, f\rangle$ is not in the case covering conjunction, so $\neg(\varphi Rf)$and $\neg(\psi Rf)$ by duality of $\wedge$ and $\vee$. By hypothesis, then φRt and ψRt, which gives $(\varphi\wedge\psi)Rt$ by definition of R. And the other case is the same.

DISJUNCTION Similar to conjunction; left to the imagination of the reader. □

For soundness, we isolate the *rule* fragment of the logic with truth-functional connectives. With Γ and Δ finite sets of propositions, take the following rules: excluded middle, $\vdash \varphi \vee \neg\varphi$; distribution, $\varphi \wedge (\psi \vee \xi) \vdash$

$(\varphi \wedge \psi) \vee (\varphi \wedge \xi)$; double negation, $\varphi \dashv\vdash \neg\neg\varphi$; de Morgans, $\neg(\varphi \wedge \psi) \dashv\vdash \neg\varphi \vee \neg\psi$ and $\neg(\varphi \vee \psi) \dashv\vdash \neg\varphi \wedge \neg\psi$; the disjunction rules

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \qquad \frac{\Gamma, \varphi \vdash \xi \qquad \Delta, \psi \vdash \xi}{\Gamma, \Delta, \varphi \vee \psi \vdash \xi}$$

called (∨I) and (∨IL) respectively; and conjunction rules, weakening, and cut, as above.

Theorem 2 $\vdash \varphi \Longrightarrow \vDash \varphi$ *for all* φ.

Proof. We give the proof for distribution. Take a model R in which $\langle \varphi \wedge (\psi \vee \xi), t \rangle \in R$. Using the relational truth conditions from section 5, $\langle \varphi, t \rangle \in R \wedge (\langle \psi, t \rangle \in R \vee \langle \xi, t \rangle \in R)$. By the distribution axiom, $(\langle \varphi, t \rangle \in R) \wedge \langle \psi, t \rangle \in R) \vee (\langle \varphi, t \rangle \in R \wedge \langle \xi, t \rangle \in R)$. By the relational truth definition again, $\langle (\varphi \wedge \psi) \vee (\varphi \wedge \xi), t \rangle \in R$.

To show that rules preserve truth, we illustrate with the rule for conjunction introduction on the right. We assume that the two premises of the rule, and show that the consequent holds. Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ and $\Delta = \{\delta_1, \dots, \delta_m\}$; assume that (1) for every model R, $\langle \gamma_1, t \rangle \in R \wedge \dots \wedge \langle \gamma_n, t \rangle \in R \Rightarrow \langle \varphi, t \rangle \in R$; and (2) for all R, $\langle \delta_1, t \rangle \in R \wedge \dots \wedge \langle \delta_m, t \rangle \in R \Rightarrow \langle \psi, t \rangle \in R$. Combine (1) and (2) to get: for all R, $\langle \gamma_1, t \rangle \in R \wedge \dots \wedge \langle \delta_m, t \rangle \in R \Rightarrow \langle \varphi, t \rangle \in R \wedge \langle \psi, t \rangle \in R$. So: for every model R, $\langle \gamma_1, t \rangle \in R \wedge \dots \wedge \langle \delta_m, t \rangle \in R \Rightarrow \langle \varphi \wedge \psi, t \rangle \in R$. □

Next, we prove completeness. Notation: take a formula φ whose propositional variables are amongst $\{p_1, \dots, p_n\}$. Given a model R, we build a (multi)set $\Theta_R^\varphi = \{p_i : \langle p_i, t \rangle \in R\} \cup \{\neg p_i : \langle p_i, f \rangle \in R\}$. To avoid using contraction in the proof, we ensure that Θ contains as many copies of p_i (or $\neg p_i$, depending on R) as there are occurrences of p_i in φ. For example, suppose $\varphi = \neg p \vee (q \wedge p)$ and take a model R such that $\langle p, t \rangle \in R$ and $\langle q, f \rangle \in R$. Then $\Theta_R^\varphi = \{p, p, \neg q\}$.

Lemma *For any model R and formula* φ,

$$\langle \varphi, t \rangle \in R \Longrightarrow \Theta_R^\varphi \vdash \varphi$$

$$\langle \varphi, f \rangle \in R \Longrightarrow \Theta_R^\varphi \vdash \neg\varphi$$

Proof. Induction on φ. If φ is a propositional variable p, the result is immediate. NEGATION: First, assume $\langle \neg\psi, t\rangle \in R$. Then $\langle \psi, f\rangle \in R$, so $\Theta \vdash \neg\psi$ by inductive hypothesis (omitting indices to ease notation). Now assume $\langle \neg\psi, f\rangle \in R$. Then $\langle \neg\neg\psi, t\rangle \in R$, so $\Theta \vdash \neg\neg\psi$, so $\Theta \vdash \psi$, using double negation and cut.
CONJUNCTION: First assume $\langle \psi \wedge \xi, t\rangle \in R$. Then $\langle \psi, t\rangle \in R$ and $\langle \xi, t\rangle \in R$. By inductive hypothesis, $\Theta^R_\psi \vdash \psi$ and $\Theta^R_\xi \vdash \xi$. By right conjunction intro, $\Theta^R_{\psi\wedge\xi} \vdash \psi \wedge \xi$, since $\Theta^R_\psi \cup \Theta^R_\xi = \Theta^R_{\psi\wedge\xi}$. Now assume $\langle \psi \wedge \xi, f\rangle \in R$. Then $\langle \psi, f\rangle \in R$ or $\langle \xi, f\rangle \in R$. By inductive hypothesis, $\Theta^R_\psi \vdash \neg\psi$ or $\Theta^R_\xi \vdash \neg\xi$. Without loss of generality, assume that $\Theta^R_\psi \vdash \neg\psi$. By rule $\vee$I, $\Theta^R_\psi \vdash \neg\psi \vee \neg\xi$. By weakening, $\Theta^R_{\psi\wedge\xi} \vdash \neg\psi \vee \neg\xi$. By De Morgan and cut, $\Theta^R_{\psi\wedge\xi} \vdash \neg(\psi \wedge \neg\xi)$.
DISJUNCTION Similar to conjunction, omitted. □

Theorem 4 $\vDash \varphi \Longrightarrow \vdash \varphi$

If $\vDash \varphi$, then $\langle \varphi, t\rangle \in R$ for every R. Let $\Phi = \{p_1, ..., p_n\}$ be the propositional variables occurring in φ. Since there are only finitely many propositional variables in φ, there are finitely many different assignments that differ on Φ. Choose representative assignments—think of each representative assignment as a row in a truth table—$R_1, \ldots, R_m$ and consider $\Theta^\varphi_{R_1}, \ldots, \Theta^\varphi_{R_m}$. By the previous Lemma, $\Theta^\varphi_i \vdash \varphi$ for every $1 \leq i \leq m$. By successive applications of left conjunction intro, $\bigwedge \Theta^\varphi_{R_i} \vdash \varphi$ for every $1 \leq i \leq m$. By successive applications of rule ($\vee$IL) and left conjunction intro, $\bigvee_{1\leq i\leq m} \bigwedge \Theta^\varphi_{R_i} \vdash \varphi$. Now, by successive applications of excluded middle, distribution and cut, $\vdash \bigvee_{1\leq i\leq m} \bigwedge \Theta^\varphi_{R_i}$. Finally, by cut, $\vdash \varphi$. □

University of Otago, University of Auckland

References

Allwein, G. and Dunn, J. 1993. Kripke semantics for linear logic, *Journal of Symbolic Logic* 58/2: 514-545.

Bacon, A. 2013. Non-classical metatheory for non-classical logics, *Journal of Philosophical Logic* 42/2: 335-355.

Badia, G. 201x. A remark on induction in paraconsistent set theory, *ms*.

Barwise, J. and Moss, L. 1996. *Vicious Circles*, Stanford: CSLI Publications.

Beall, J. 2009. *Spandrels of Truth*, Oxford: Oxford University Press.

Beall, J. 2013. Free of detachment: Logic, rationality, and gluts, *Noûs* 49/2: 410-423.

Beall, J., Forster, T. and Seligman, J. 2013. A note on freedom from detachment in the logic of paradox, *Notre Dame Journal of Formal Logic* 54/1: 15–20.

Beall, J. and Murzi, J. 2013. Two flavors of curry paradox, *Journal of Philosophy*, 110/3: 143–165.

Beall, J. and Restall, G. 2006. *Logical Pluralism*, Oxford: Oxford University Press.

Bourget D. and Chalmers, D. 2014. What do philosophers believe? *Philosophical Studies*, 170:465–500.

Brady, R. 2006. Universal Logic. Stanford: CSLI Publications.

Brady, R. 1989. The non-triviality of dialectical set theory, in *Paraconsistent Logic: Essays on the Inconsistent*, ed. Graham Priest, Richard Routley, and Jean Norman, Munich: Philosophia Verlag: 437–470.

Brady, R. and Routley, R. 1989. The non-triviality of extensional dialectical set theory, in *Paraconsistent Logic: Essays on the Inconsistent*, ed. Graham Priest, Richard Routley, and Jean Norman, Munich: Philosophia Verlag: 415–436.

Burgess, J. 2005. No requirement of relevance, in the *Oxford Handbook of Philosophy of Mathematics and Logic*, ed. Stewart Shapiro, Oxford: Oxford University Press.

Cantor, G. 1899. Letter to Dedekind, in *From Frege to Gödel: a source book in mathematical logic, 1879–1931*, ed. Jean van Heijenoort, Cambridge: Harvard University Press 1967.

Caret, C. and Weber, Z. 2015. A note on contraction-free logic for validity, *Topoi* 31/1: 63-74.

Field, H. 2008. *Saving Truth from Paradox*, Oxford: Oxford University Press.

Girard, P. and Weber, Z. 2015. Bad Worlds, *Thought* 4: 93-101.

Halmos, P. 1960. *Naive Set Theory*, Princeton: D. Van Nostrand Company Inc.

Kalmár, L. 1935. Über die axiomatisierbarkeit des aussagenkalküls, *Acta scientiarum mathematicarum*, 7: 222–243.

Kleene, S.C. 1952. *Introduction to Metamathematics*, Amsterdam: North Holland.

Kripke, S. 1975. Outline of a theory of truth, *Journal of Philosophy* 72/19:690–716.

Mares, E. 2004. *Relevant Logic*, Cambridge: Cambridge University Press.

Martin, B. 2015. Dialetheism and the Impossibility of the World, *Australasian Journal of Philosophy* 93/1: 61–76.

McCarty, C. 2008. Completeness and incompleteness for intuitionistic logic, *Journal of Symbolic Logic* 73/4: 1315–1327.

Meyer, R., Routley, R., and Dunn, J. 1978. Curry's paradox, *Analysis* 39: 124–128.

Priest, G. 1979. The logic of paradox, *Journal of Philosophical Logic* 8: 219–241.

Priest, G. 1990. Boolean negation and all that, *Journal of Philosophical Logic*, 19/2: 201–215.

Priest, G. 2006. *In Contradiction: A Study of the Transconsistent*, Oxford: Oxford University Press. Second edition.

Priest, G. 2008. *An Introduction to Non-Classical Logic*, Cambridge: Cambridge University Press. Second Edition.

Restall, G. 1994. *On Logics Without Contraction*, PhD thesis, The University of Queensland.

Restall, G. 2000. *An Introduction to Substructural Logics*, London: Routledge.

Restall, G. 2010. On t and u and what they can do, *Analysis* 70/4:673–676.

Rossberg, M. 2013. Too good to be 'just true', *Thought* 2/1:1–8.

Routley, R. 1977. Ultralogic as universal? *Relevance Logic Newsletter*, 2:51–89.

Routley, R., and Meyer, R. 1976. Dialectical logic, classical logic and the consistency of the world, *Studies in Soviet Thought* 16:1–25.

Routley, R., Plumwood, V., Meyer, R., and Brady, R. 1982. *Relevant Logics and their Rivals*, Ridgeview.

Russell, B. 1902. Letter to Frege, in *From Frege to Gödel: a source book in mathe- matical logic, 1879–1931*, ed. Jean van Heijenoort, Cambridge: Harvard University Press 1967.

Shapiro, S. 2002. Incompleteness and inconsistency, *Mind* 111: 817–832.

Shapiro, S. 2014. *Varieties of Logic*, Oxford: Oxford University Press.

Sylvan, R. 1992. On interpreting truth tables and relevant truth table logic, *Notre Dame Journal of Formal Logic* 33:207–215.

Voevodsky, V. 2010. What if current foundations of mathematics are inconsistent? Talk presented at the Institute for Advanced Study, Princeton, September 25.

Weber, Z. 2012. Transfinite Cardinals in Paraconsistent Set Theory, *Review of Symbolic Logic* 5/2: 269-293.

Williamson, T. 2014. Logic, metalogic and neutrality, *Erkenntnis* 79/2: 211– 231.