CDMTCS Research Report Series

Exact Constructive and Computable Dimensions

Ludwig Staiger
Martin-Luther-Universität
Halle-Wittenberg

CDMTCS-502
October 2016

Centre for Discrete Mathematics and
Theoretical Computer Science

Exact Constructive and Computable Dimensions*

Ludwig Staiger[†]
Martin-Luther-Universität Halle-Wittenberg
Institut für Informatik
von-Seckendorff-Platz 1, D–06099 Halle (Saale), Germany

Abstract

In this paper we derive several results which generalise the constructive dimension of (sets of) infinite strings to the case of exact dimension. We start with proving a martingale characterisation of exact Hausdorff dimension. Then using semi-computable super-martingales we introduce the notion of exact constructive dimension of (sets of) infinite strings. This allows us to derive several bounds on the complexity functions of infinite strings, that is, functions assigning to every finite prefix its Kolmogorov complexity. In particular, it is shown that the exact Hausdorff dimension of a set of infinite strings lower bounds the maximum complexity function of strings in this set. Furthermore, we show a result bounding the exact Hausdorff dimension of a set of strings having a certain computable complexity function as upper bound.

Obviously, the Hausdorff dimension of a set of strings alone without any computability constraints cannot yield upper bounds on the complexity of strings in the set. If we require, however, the set of strings to be Σ_2-definable several results upper bounding the complexity by the exact Hausdorff dimension hold true. First we prove that for a Σ_2-definable set with computable dimension function one can construct a computable – not only semi-computable – martingale succeeding on this set. Then, using this result, a tight upper bound on the prefix complexity function for all strings in the set is obtained.

*The results of this paper were presented at the conference “Varieties of Algorithmic Information 2015”, June 15 - 18, 2015, Heidelberg, Germany

[†]email: `ludwig.staiger@informatik.uni-halle.de`

Contents

The paper addresses a problem from Algorithmic Information Theory. In his papers [Lut00, Lut03b] Lutz came up with an effectivisation of Hausdorff dimension, called constructive dimension. Constructive dimension characterises the algorithmic complexity of (sets of) infinite strings as real numbers. It turned out to be equivalent to asymptotic Kolmogorov complexity (cf. [Sta05]) and is related to the concept of partial randomness of infinite strings [Tad02, CST06]. However, the results of Reimann and Stephan [RS06] show, unlike the case of random infinite strings, different notions of Kolmogorov complexity (cf. [DH10, Usp92, US96]) yield different notions of partial randomness.

To distinguish these types of partial randomness requires a refinement of the complexity scale of (sets of) infinite strings. The present paper shows that an effectivisation of Hausdorff's original concept of dimension [Hau18], referred to as exact Hausdorff dimension in [MGW87, GMW88, MM09], is possible and leads, similarly to the case of "usual" dimensions (cf. [Rya84, Rya86, Sta93, Sta98, Lut00, Lut03b]), to close connections between exact Hausdorff dimension and exact constructive dimension. In contrast to the "usual" constructive or Hausdorff dimension an exact dimension of a string or a set of strings is a real function, referred to as dimension function [Fal90, Section 2.5] or gauge function [GMW88]. This makes it more difficult to specify uniquely 'the' exact Hausdorff dimension of set of strings.

After introducing some notation and some necessary concepts related to Kolmogorov complexity we proceed in Section 1.3 with Hausdorff's original approach [Hau18] and give a definition of what is an exact Hausdorff dimension of a set of infinite words. The subsequent Section 1.4 presents a brief account on some known results relating "classical" Hausdorff dimension and (asymptotic) Kolmogorov complexity.

Then the subsequent part consisting of four sections generalises the "classical" results mentioned in Section 1.4 to the case of exact Hausdorff dimension and Kolmogorov complexity functions. First, in Section 2 we deal with the martingale characterisation of exact Hausdorff dimension. The following section defines the exact constructive dimension and derives a lower complexity bound by the principle "large sets contain complex elements", and in Section 4 we use a dilution principle to construct sets of infinite strings having a prescribed computable dimension function.

Upper complexity bounds for Σ_2-definable sets of infinite strings are the subject of Section 5. It should be mentioned here that, as already proved in [Sta98] these bounds cannot be extended to the class of Π_2-definable sets.

Finally, in Section 6, we apply our results to the family of functions of the logarithmic scale, which was also considered by Hausdorff [Hau18]. This family on the one hand refines the "usual" scale of asymptotic dimension and is, on the other hand, – in contrast to the general case – linearly ordered.

Some of the results are contained in the conference papers [Sta11] and [Sta12]

1 Notation and Preliminaries

1.1 Notation

In this section we introduce the notation used throughout the paper. By $\mathbb{N} = \{0,1,2,\ldots\}$ we denote the set of natural numbers and by $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ are the real numbers and $\mathbb{R}_+$ the non-negative real numbers.

Let X be an alphabet of cardinality $|X| = r \geq 2$. By X^* we denote the set of finite words on X, including the *empty word* e, and X^ω is the set of infinite strings (ω-words) over X. Subsets of X^* will be referred to as *languages* and subsets of X^ω as *ω-languages*[1].

For $w \in X^*$ and $\eta \in X^* \cup X^\omega$ let $w \cdot \eta$ be their *concatenation*. This concatenation product extends in an obvious way to subsets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$.

We denote by $|w|$ the *length* of the word $w \in X^*$ and $\mathbf{pref}(B)$ is the set of all finite prefixes of strings in $B \subseteq X^* \cup X^\omega$. We shall abbreviate $w \in \mathbf{pref}(\eta)$ ($\eta \in X^* \cup X^\omega$) by $w \sqsubseteq \eta$, and $\eta \restriction n$ is the n-length prefix of η provided $|\eta| \geq n$. A language $W \subseteq X^*$ is referred to as *prefix-free* if $w, v \in W$ and $w \sqsubseteq v$ imply $w = v$. If $W \subseteq X^*$ then $\mathrm{Min}_\sqsubseteq W := \{w : w \in W \wedge \forall v(v \in W \to v \not\sqsubset w)\}$ is the (prefix-free) set of minimal w.r.t. $\sqsubseteq$ elements of W.

It is sometimes convenient to regard X^ω as a topological space (Cantor space). Here *open* sets in X^ω are those of the form $W \cdot X^\omega$ with $W \subseteq X^*$. *Closed* are sets $F \subseteq X^\omega$ which satsify the condition $F = \{\xi : \mathbf{pref}(\xi) \subseteq \mathbf{pref}(F)\}$. In this space the following compactness theorem (or König's lemma) holds.

Compactness Theorem. *If $F \subseteq X^\omega$ is closed and $F \subseteq W \cdot X^\omega$ for some $W \subseteq X^*$ then there is a finite subset $W' \subseteq W$ such that $F \subseteq W' \cdot X^\omega$.*

For a computable domain $\mathcal{D}$, such as $\mathbb{N}$, $\mathbb{Q}$ or X^*, we refer to a function $f : \mathcal{D} \to \mathbb{R}$ as *left computable* (or *approximable from below*) provided the set $\{(d,q) : d \in \mathcal{D} \wedge q \in \mathbb{Q} \wedge q < f(d)\}$ is computably enumerable. Accordingly, a function $f : \mathcal{D} \to \mathbb{R}$ is called *right computable* (or *approximable from above*) if the set $\{(d,q) : d \in \mathcal{D} \wedge q \in \mathbb{Q} \wedge q > f(d)\}$ is computably enumerable, and f is *computable* if f is right and left computable. Accordingly, a real number $\alpha \in \mathbb{R}$ is left computable, right computable or computable provided the constant function $c_\alpha(t) = \alpha$ is left computable, right computable or computable, respectively.

If we refer to a function $f : \mathcal{D} \to \mathbb{Q}$ as computable we usually mean that it maps the domain $\mathcal{D}$ to the domain $\mathbb{Q}$, that is, it returns the exact value $f(d) \in \mathbb{Q}$.

[1] In the recent monograph [DH10] ω-words are referrred to as *sets* and ω-languages as *classes*. In this paper we reserve the term "set" to the original meaning as introduced by Georg Cantor.

1.2 Semi-measures and Kolmogorov complexity

In this part we introduce those variants of Kolmogorov complexity which we will consider in the subsequent parts. The first two of them can be related to left computable semi-measures on X^* For the third one—monotone complexity—we use the relation based approach (see [USS90, US96]).

1.2.1 Prefix complexity

A function $\nu : X^* \to \mathbb{R}_+$ is referred to as a (*discrete*) *semi-measure* provided $\sum_{w \in X^*} \nu(w) < \infty$. It is well-known ([LV93, Cal02, DH10]) that there is a *universal* left discrete computable semi-measure $\mathbf{m} : X^* \to \mathbb{R}_+$ such that for every left computable discrete semi-measure ν there is a constant $c_\nu > 0$ such that $\mathbf{m}(w) \geq c_\nu \cdot \nu(w)$ for all $w \in X^*$.

Fix a universal left computable semi-measure $\mathbf{m} : X^* \to \mathbb{R}_+$ with $\mathbf{m}(w) \leq 1$ for all $w \in X^*$. Then $\mathrm{H}(w) := \lceil \log_{|X|} \nu(w) \rceil$ is the *prefix complexity* of the word $w \in X^*$.

1.2.2 *a priori* complexity

A *continuous* (or *cylindrical*) *semi-measure* on X^* is a function $\mu : X^* \to \mathbb{R}_+$ which satisfies $\mu(e) \leq 1$ and $\mu(w) \geq \sum_{x \in X} \mu(wx)$, for $w \in X^*$. If there is no danger of confusion, in the sequel we will refer to continuous (semi)-measures simply as (semi)-measures.

If $\mu(w) = \sum_{x \in X} \mu(wx)$ the function μ is called a *measure*. A continuous semi-measure μ has the following property.

Proposition 1 *If $C \subseteq w \cdot X^*$ is prefix-free then $\mu(w) \geq \sum_{v \in C} \mu(v)$.*

In [ZL70] the existence of a universal left-computable semi-measure $\mathbf{M}$ is proved: There is a left-computable semi-measure $\mathbf{M}$ which satisfies

$$\exists c_m > 0 \, \forall w \in X^* : \mu(w) \leq c_m \cdot \mathbf{M}(w), \tag{1}$$

for all left-computable semi-measures m.

The *a priori complexity* of a word $w \in X^*$ is defined as

$$\mathrm{KA}(w) := \lceil -\log_{|X|} \mathbf{M}(w) \rceil \, . \tag{2}$$

The properties of the semi-measure $\mathbf{M}$ imply $\mathrm{KA}(w) \leq \mathrm{KA}(w \cdot v)$ and, moreover, $\sum_{v \in C} |X|^{-\mathrm{KA}(v)} \leq \mathbf{M}(e)$ when $C \subseteq X^*$ is prefix-free.

A simple computable continuous measure is $\mu_=(w) := |X|^{-|w|}$. Since $\mathbf{M}$ is universal, we have $\mathbf{M}(w) \geq c \cdot |X|^{-|w|}$ for every $w \in X^*$. Thus $\mathrm{KA}(w) \leq |w| + c'$.

1.2.3 Monotone complexity

In this section we introduce the monotone complexity along the lines of [USS90, US96]. To this end let $E \subseteq X^* \times X^*$ be a description mode (a computably enumerable set) which satisfies the condition.

$$(\pi, w), (\pi', v) \in E \wedge \pi \sqsubseteq \pi' \rightarrow w \sqsubseteq v \vee v \sqsubseteq w \tag{3}$$

Then $K_E(w) := \inf\{|\pi| : \exists u(w \sqsubseteq u \wedge (\pi, u) \in E)\}$ is the *monotone complexity* of the word w w.r.t. E. In [USS90, § 3.2] it is shown that there is a description mode $\mathcal{E}$ universal among all description modes, that is, $K_E(w) \leq K_{\mathcal{E}}(w) + c_E$ for all $w \in X^*$ and some constant not depending on w. In the sequel we use the term Km for $K_{\mathcal{E}}$. Since $E_= := \{(w, w) : w \in X^*\}$ is a description mode satisfying Eq. (3) we have $\mathrm{Km}(w) \leq |w| + O(1)$.

Finally we mention some relations between the complexities H, KA and Km (see [DH10, US96]).

$$\mathrm{KA}(w) \leq \mathrm{Km}(w) + O(1) \quad , \quad \mathrm{Km}(w) \leq \mathrm{H}(w) + O(1) \tag{4}$$

$$|K_1(w) - K_2(w)| \leq O(\log |w|) \text{ for } K_i \in \{\mathrm{H}, \mathrm{KA}, \mathrm{Km}\} \tag{5}$$

It should be mentioned that the inequalities in Eq. (4) cannot be reversed, that is, the differences $|\mathrm{Km}(w) - \mathrm{KA}(w)|$ and $|\mathrm{H}(w) - \mathrm{Km}(w)|$ are unbounded.

What concerns transformations with (partially defined) computable mappings we have the following.

Proposition 2 *Let $\varphi : X^* \rightarrow X^*$ be a prefix-monotone partial computable function and let $K \in \{\mathrm{H}, \mathrm{KA}, \mathrm{Km}\}$. Then there is a constant c_φ such that $K(\varphi(w)) \leq K(w) + c_\varphi$ for all $w \in X^*$.*

In the case of prefix complexity H we can drop the requirement that φ be prefix-monotone.

Proof. In the case of H define a discrete semi-measure $\nu(w) := \mathbf{m}(\varphi(w))$. Then ν is left-computable and the assertion follows from $\nu(w) \leq c_\varphi \cdot \mathbf{m}(w)$.

For Km define $E_\varphi := \{(\pi', \varphi(w)) : \exists \pi(\pi \sqsubseteq \pi' \wedge (\pi, w) \in \mathcal{E})\}$. Then E_φ is computably enumerable and satisfies Eq. (3).

Since $\mathcal{E}$ is a universal description mode satisfying Eq. (3), we have $K_{E_\varphi}(\varphi(w)) \leq K_{\mathcal{E}}(w) + c_\varphi$.

The proof for KA is more complicated and can be found in [LV93, Section 4.2.2] or in a more detailed form in [Sta16]. ❑

1.2.4 Complexity of infinite words

A simple and natural way to extend the complexity of finite words to infinite ones is to consider, for $\xi \in X^\omega$, the *Kolmogorov complexity function for infinite words* $K(\xi \upharpoonright n)$ where K is a complexity of finite words as mentioned in the preceding part.

In connection with constructive dimension (see e.g. [Lut03b, DH10]) the following variant of *asymptotic complexity* plays a major rôle.

$$\underline{\kappa}(\xi) := \liminf_{n\to\infty} \frac{K(\xi \upharpoonright n)}{n} \tag{6}$$

In view of Eq. (5) it is apparent that the actual variant of complexity is not essential here (see also [CH94]).

1.3 Gauge functions and Hausdorff's original approach

A function $h : (0,\infty) \to (0,\infty)$ is referred to as a *gauge function* (or *dimension function* [Fal90]) provided h is right continuous and non-decreasing.[2] If not stated otherwise, we will assume that $\lim_{t\to 0} h(t) = 0$.

The h-dimensional outer measure of F on the space X^ω is given by

$$\mathscr{H}^h(F) := \lim_{n\to\infty} \inf\Big\{ \sum_{v\in V} h(r^{-|v|}) : V \subseteq X^* \wedge F \subseteq V\cdot X^\omega \wedge \min_{v\in V} |v| \geq n \Big\}. \tag{7}$$

If $\lim_{t\to 0} h(t) > 0$ then $\mathscr{H}^h(F) < \infty$ if and only if F is finite.

For $\alpha \in \mathbb{R}_+$ the α-dimensional Hausdorff measure $\mathscr{H}^\alpha$ is defined by the gauge functions $h_\alpha(t) = t^\alpha, \alpha \in [0,1]$, that is, $\mathscr{H}^\alpha = \mathscr{H}^{h_\alpha}$.

In this case the (also referred to as "classical" in [DH10, Chapter 13.1]) Hausdorff dimension of a set $F \subseteq X^\omega$ is defined as the change-over point in the plot of Fig. 1.

$$\dim_H F := \sup\{\alpha : \alpha = 0 \vee \mathscr{H}^\alpha(F) = \infty\} = \inf\{\alpha : \wedge \mathscr{H}^\alpha(F) = 0\}. \tag{8}$$

The following properties of gauge functions h and the related measure $\mathscr{H}^h$ are proved in the standard way (see e.g. [Edg08, Fal90] or [Rog98, Theorem 40]).

Property 1 *Let h, h' be gauge functions.*

1. *If $c \cdot h(r^{-n}) \leq h'(r^{-n})$ for some $c > 0$, then $c \cdot \mathscr{H}^h(F) \leq \mathscr{H}^{h'}(F)$.*

[2] In fact, since we are only interested in the values $h(r^{-n})$, $n \in \mathbb{N}$, the requirement of right continuity is just to conform with the usual meaning (cf. [GMW88, Rog98]).

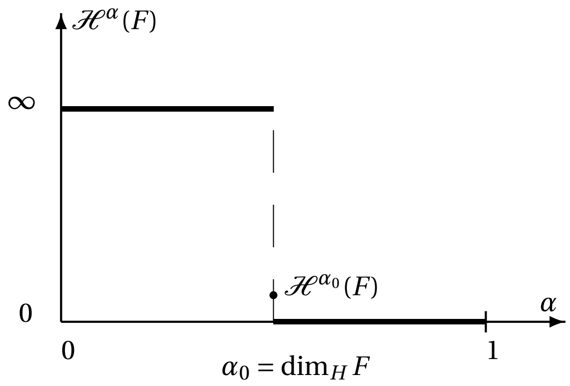

Figure 1: Plot of $\mathscr{H}^\alpha(F)$ as a function of α

2. *If* $\lim_{n\to\infty} \frac{h(r^{-n})}{h'(r^{-n})} = 0$ *then* $\mathscr{H}^{h'}(F) < \infty$ *implies* $\mathscr{H}^h(F) = 0$*, and* $\mathscr{H}^h(F) > 0$ *implies* $\mathscr{H}^{h'}(F) = \infty$.

Here the first property implies a certain equivalence of gauge functions. In fact, if $c \cdot h \le h'$ and $c \cdot h' \le h$ in the sense of Property 1.1 then for all $F \subseteq X^\omega$ the measures $\mathscr{H}^h(F)$ and $\mathscr{H}^{h'}(F)$ are both zero, finite or infinite.

As we see from Eq. (7) for our purposes the behaviour of gauge functions is of interest only for large values of n, that is, in a small vicinity of 0. Moreover, in many cases we are not interested in the exact value of $\mathscr{H}^h(F)$ when $0 < \mathscr{H}^h(F) < \infty$. Thus we can often make use of scaling a gauge function and altering it in a range (ε, ∞) apart from 0.

In the same way the second property gives a partial pre-order of gauge functions (see [Rog98, Chapter 2, § 4]). By analogy to the change-over-point $\alpha_0 = \dim_H F$ for $\mathscr{H}^\alpha(F)$ this partial pre-order yields a suitable notion of Hausdorff dimension in the range of arbitrary gauge functions.

Definition 1 We refer to a gauge function h as an *exact Hausdorff dimension function* for $F \subseteq X^\omega$ provided

$$\mathscr{H}^{h'}(F) = \begin{cases} \infty, & \text{if } \lim_{n\to\infty} \frac{h(r^{-n})}{h'(r^{-n})} = 0\,, \text{ and} \\ 0, & \text{if } \lim_{n\to\infty} \frac{h'(r^{-n})}{h(r^{-n})} = 0\,. \end{cases}$$

Hausdorff [Hau18] called a function h *dimension* of F provided $0 < \mathscr{H}^h(F) < \infty$. This case is covered by our definition and Property 1.

Partitioning the gauge (or dimension) functions into those for which $\mathscr{H}^h(F)$ is finite and those for which $\mathscr{H}^h(F)$ is infinite gives a more precise indication to the ‘dimension’ of F than just the number $\alpha = \dim_H F$ from above.

As in [Rog98, Chapter 2, §4] Definition 1 leads to a partial ordering of gauge functions

$$h \prec h' \iff \lim_{n\to\infty} \frac{h'(r^{-n})}{h(r^{-n})} = 0$$

by saying that h corresponds to a smaller dimension than h'[3]. This partial ordering is not as simple as the one of the classical Hausdorff dimension in Eq. (8), and it seems to be much more difficult to find the exact borderline, if it exists, between gauge functions with $\mathscr{H}^h(F) = 0$ and such with $\mathscr{H}^h(F) = \infty$. In fact, Eggleston [Egg50, Egg51] (see also [Rog98, Theorem 42]) proved that there are sets F which have functions $h, \bar{h}$ both satisfying Definition 1 such that $\limsup_{n\to\infty} \frac{h(r^{-n})}{\bar{h}(r^{-n})} = \limsup_{n\to\infty} \frac{\bar{h}(r^{-n})}{h(r^{-n})} = \infty$. This, in particular, implies that one cannot always compare two sets $E, F \subseteq X^\omega$ and say that one or the other of the two must be the smaller from the point of Hausdorff dimension due to the lack of total ordering among the gauge functions.

One easily observes that $h_1(t) := t$ yields $\mathscr{H}^{h_1}(F) \le 1$, thus $\mathscr{H}^h(F) = 0$ for all h with $h_1 \prec h$. Therefore, we can always assume that a gauge function satisfies $h(t) \ge t,\ t \in (0,1)$.

1.4 Kolmogorov complexity and Hausdorff dimension

In this part we briefly recall known results relating classical Hausdorff dimension and (asymptotic) Kolmogorov complexity. Then in the subsequent sections we generalise them to the case of exact Hausdorff dimension and Kolmogorov complexity functions. For a more detailed account on previous results see the above mentioned books [LV93, Cal02, DH10] or the survey [Sta07].

1.4.1 Martingales and Hausdorff dimension

Closely related to the invention of constructive dimension is the notion of martingale. Martingales had already been used successfully to characterise randomness and in conjunction with order functions the order of randomness [Sch71]. A similar approach was pursued when Lutz [Lut03a, Lut03b] constructivised Hausdorff dimension using what he called s-(super-)gales a combination of (super-)martingales and exponential order functions (see [Ter04, Section 4.2]). We follow SchnorrâĂŹs approach, because it seems that the combination of (super-)martingales with order functions is more flexible at least in two respects: on the one hand, as in the investigation of Hausdorff dimension, it allows for the use of order functions other than exponential ones, and on the

[3] Observe that $h' \le h$ implies $h \preceq h'$.

other hand, as the proof of Theorem 11 in [Sta98] shows, computable martingales may achieve non-computable (exponential) order functions, something which is not possible for s-gales, as computable s-gales exist only for computable reals $s \in \mathbb{R}$.

A *super-martingale* is a function $\mathcal{V} : X^* \to \mathbb{R}_+$ which satisfies $\mathcal{V}(e) \leq 1$ and the super-martingale inequality

$$r \cdot \mathcal{V}(w) \geq \sum_{x \in X} \mathcal{V}(wx) \text{ for all } w \in X^* . \tag{9}$$

If Eq. (9) is satisfied with equality $\mathcal{V}$ is called a martingale. Closely related with (super-)martingales are continuous (or cylindrical) (semi-)measures $\mu : X^* \to [0,1]$ where $\mu(e) \leq 1$ and $\mu(w) \geq \sum_{x \in X} \mu(wx)$ for all $w \in X^*$.

Indeed, if $\mathcal{V}$ is a super-martingale then μ defined by the key equation

$$\mu(w) := r^{-|w|} \cdot \mathcal{V}(w) \tag{10}$$

is a continuous semi-measure, and vice versa. Moreover, from Proposition 1 we obtain

$$r^{-|w|} \cdot \mathcal{V}(w) \geq \sum_{v \in C} r^{-|v|} \cdot \mathcal{V}(v) \tag{11}$$

if $C \subseteq w \cdot X^*$ is prefix-free.

Let

$$S_{c,h}[\mathcal{V}] := \left\{\xi : \xi \in X^\omega \wedge \limsup_{n \to \infty} \frac{\mathcal{V}(\xi \upharpoonright n)}{r^n \cdot h(r^{-n})} \geq c\right\}, \tag{12}$$

for a super-martingale $\mathcal{V} : X^* \to [0,\infty)$, a gauge function h and a threshold $c \in (0,\infty]$. The following relations for gauge functions h, h' and thresholds c, c' are obvious.

Lemma 1 *If $c \geq c'$ then $S_{c,h}[\mathcal{V}] \subseteq S_{c',h}[\mathcal{V}]$, and if $h \prec h'$ then $S_{c,h}[\mathcal{V}] \subseteq S_{c,h'}[\mathcal{V}]$.*

In particular, $S_{\infty,h}[\mathcal{V}]$ is the set of all ω-words on which the super-martingale $\mathcal{V}$ is *successful* w.r.t. the order function $f(n) = r^n \cdot h(r^{-n})$ in the sense of Schnorr [Sch71]. $S_{\infty,h}[\mathcal{V}]$ is also referred to as the *success set* of the super-martingale $\mathcal{V}$ w.r.t. the order function $f(n) = r^n \cdot h(r^{-n})$.

Schnorr [Sch71] required an order function $f : \mathbb{N} \to \mathbb{N}$ to be non-decreasing, unbounded and computable. For our purposes it is more convenient to consider $f : \mathbb{N} \to \mathbb{R}_+$ only as a real-valued non-decreasing function. Nevertheless, this does not guarantee that $f(n) = r^n \cdot h(r^{-n})$ is always an order function whenever h is a gauge function and $h(0) = 0$. The following gives a sufficient condition. We call a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ *upwardly convex* if $(t_1 - t_0) \cdot h(t') \geq h(t_0) + (t' - t_0) \cdot (h(t_1) - h(t_0))$ for $0 \leq t_0 < t' < t_1$.

Lemma 2 *If a gauge function $h : [0,1] \to \mathbb{R}_+$ is upwardly convex then $f(n) := r^n \cdot h(r^{-n})$ is an order function.*

Proof. It suffices to show that $h(t)/t$ is non-increasing.

If h is upwardly convex then $(t_1 - t_0)\cdot h(t') \geq h(t_0) + (t' - t_0)\cdot(h(t_1) - h(t_0))$ when $0 \leq t_0 < t' < t_1 \leq 1$. Thus $t_1 = t$ and $t_0 = 0$ imply $t\cdot h(t') \geq t'\cdot h(t) + (1-t')\cdot h(0) \geq t'\cdot h(t)$. ❏

The converse of Lemma 2 is not true.

Example 1 Set
$$h(t) := \begin{cases} \sqrt{t} & \text{if } 0 \leq t \leq 1/4 \\ 1/2 & \text{if } 1/4 \leq t \leq 1/2, \text{ and} \\ t & \text{otherwise}. \end{cases}$$

Then
$$h(t)/t := \begin{cases} 1/\sqrt{t} & \text{if } 0 \leq t \leq 1/4 \\ 1/2t & \text{if } 1/4 \leq t \leq 1/2, \text{ and} \\ 1 & \text{otherwise}. \end{cases}$$

Thus $h(t)/t$ is non-increasing but $\frac{3}{4}\cdot h(1/2) = \frac{3}{8} < h(1/4) + \frac{1}{4}\cdot(h(1) - h(1/4)) = \frac{5}{8}$. ❏

Lutz [Lut03a, Lut03b] coined the term s-(super-)gale for the combination $d(w) := \mathcal{V}(w)/(r^{|w|}\cdot r^{-s|w|})$ of a (super-)martingale $\mathcal{V}$ with the gauge function $h(t) = t^s$ or, alternatively with the order function $f(n) = r^{(1-s)\cdot n}$. He then proved the following relation between success sets and classical Hausdorff dimension.

Theorem 1 ([Lut03a]) *Let $F \subseteq X^\omega$. Then*
$$\dim_H F < \alpha \;\to\; \exists \mathcal{V}(F \subseteq S_{\infty,t^\alpha}[\mathcal{V}]) \;\to\; \dim_H F \leq \alpha\,.$$

1.4.2 Constructive dimension and asymptotic Kolmogorov complexity

In [Lut03b] Lutz invented constructive dimension by restricting to success sets of left-computable super-martingales. In this case the condition $\exists\mathcal{V}(F \subseteq S_{\infty,t^\alpha}[\mathcal{V}])$ turns out to be simpler because the results of Levin [ZL70] and Schnorr [Sch71] show that there is an optimal left-computable super-martingale $\mathcal{U}$, that is, every other left-computable super-martingale $\mathcal{V}$ satisfies $\mathcal{V}(w) \leq c_{\mathcal{V}}\cdot\mathcal{U}(w)$ for all $w \in X^*$ and some constant $c_{\mathcal{V}} > 0$ not depending on w.

Thus we may define the *constructive dimension* of $F \subseteq X^\omega$ as $\inf\{\alpha : F \subseteq S_{\infty,t^\alpha}[\mathcal{U}]\}$. Using our key equation (10) and the definitions of $\underline{\kappa}$ and KA in Eqs. (6) and (2), respectively, we obtain immediately the coincidence of constructive dimension and asymptotic Kolmogorov complexity (see [May02, Theorem 1] and for a more detailed discussion [Sta05]).

Corollary 1 *Let $F \subseteq X^\omega$. Then*
$$\sup\{\underline{\kappa}(\xi) : \xi \in F\} = \inf\{\alpha : F \subseteq S_{\infty,\alpha}[\mathcal{U}]\}\,.$$

In [Rya84, Rya86] Ryabko proved results which related Hausdorff dimension to asymptotic Kolmogorov complexity.

Theorem 2 ([Rya84]) *Let $F \subseteq X^\omega$. Then*
$$\dim_H\{\xi : \xi \in X^\omega \wedge \underline{\kappa}(\xi) < \alpha\} = \alpha\,.$$

The proof of the inequality $\leq$ is based on the principle "large sets contain complex elements" (cf. also Theorem 6 below).

Lemma 3 ([Rya86]) $\dim_H F \leq \sup\{\underline{\kappa}(\xi) : \xi \in F\}$

The other direction is constructive: for rational $\alpha' < \alpha$, ω-languages $E \subset \{\xi : \underline{\kappa}(\xi) < \alpha\}$ having dimension $\dim_H E \geq \alpha'$ are constructed (cf. Theorem 10 below).

Ryabko's results, however, give no bounds on the actual Kolmogorov complexity functions $K(\xi \upharpoonright n)$ when $\xi \in F$ and $\dim_H F = \alpha$. Those results were derived in [Mie08, Sta93, CHS11] or for the the particular case of ω-languages definable by finite automata in [Sta93, Sta08].

1.4.3 Bounds for Σ_2-definable ω-languages

The lower bound given in Lemma 3 might be quite loose depending on the structure of the ω-language F. In [Sta98] it was shown that this bound is also an upper bound for Σ_2-definable ω-languages. Here, as usual, we refer to an ω-language $F \subseteq X^\omega$ as Σ_2-*definable* provided there is a computable relation $\mathbf{R} \subseteq X^* \times \mathbb{N}$ such that $F = \{\xi : \exists i \forall n((\xi \upharpoonright n, i) \in \mathbf{R})\}$.

Lemma 4 ([Sta98]) *If $F \subseteq X^\omega$ is Σ_2-definable then* $\dim_H F = \sup\{\underline{\kappa}(\xi) : \xi \in F\}$.

Moreover, the proof of Theorem 11 of [Sta98] shows the following.

Theorem 3 *If $F \subseteq X^\omega$ is Σ_2-definable and $\alpha \geq \dim_H F$ is a right-computable real then there is a computable martingale $\mathcal{V}$ such that for all $\xi \in F$ there is a constant $c_\xi > 0$ such that $\mathcal{V}(\xi \upharpoonright n) \geq_{i.o.} c_\xi \cdot r^{(1-\alpha)\cdot n}$.*

So far the reviewed results concern "classical" Hausdorff dimension. The subsequent sections are devoted to generalisations of these results to the case of exact Hausdorff dimension and gauge functions.

2 Exact Hausdorff dimension and martingales

In this section we show the generalisation of Lutz's theorem (Theorem 1) for arbitrary gauge functions. In view of Property 1 we split the assertion into two parts.

Lemma 5 *Let $F \subseteq X^\omega$ and h, h' be gauge functions such that $h \prec h'$ and $\mathcal{H}^h(F) < \infty$. Then $F \subseteq S_{\infty,h'}[\mathcal{V}]$ for some martingale $\mathcal{V}$.*

Proof. First we follow the lines of the proof of Theorem 13.2.3 in [DH10] and show the assertion for $\mathcal{H}^h(F)=0$. Thus there are prefix-free subsets $U_i \subseteq X^*$ such that $F \subseteq \bigcap_{i\in\mathbb{N}} U_i \cdot X^\omega$ and $\sum_{u\in U_i} h(r^{-|u|}) \leq 2^{-i}$.

Define $\mathcal{V}_i(w) := \begin{cases} r^{|w|}\cdot\sum_{wu\in U_i} h(r^{-|wu|}), & \text{if } w\in \mathbf{pref}(U_i)\setminus U_i \text{ , and} \\ \sup\{r^{|v|}\cdot h(r^{-|v|}) : v \sqsubseteq w \wedge v\in U_i\}, & \text{otherwise[4].}\end{cases}$

In order to prove that $\mathcal{V}_i$ is a martingale we consider three cases:

$w\in\mathbf{pref}(U_i)\setminus U_i$**:** Since then $U_i\cap w\cdot X^* = \bigcup_{x\in X} U_i\cap wx\cdot X^*$, we have $\mathcal{V}_i(w) = r^{|w|}\cdot\sum_{wu\in U_i} h(r^{-|wu|}) = r^{-1}\cdot\sum_{x\in X} r^{|wx|}\sum_{wxu\in U_i} h(r^{-|wxu|}) = r^{-1}\cdot\sum_{x\in X}\mathcal{V}_i(wx)$.

$w\in U_i\cdot X^*$**:** Let $w\in v\cdot X^*$ where $v\in U_i$. Then $\mathcal{V}_i(w)=\mathcal{V}_i(wx)=r^{|v|}\cdot h(r^{-|v|})$ whence $\mathcal{V}_i(w) = r^{-1}\cdot\sum_{x\in X}\mathcal{V}_i(wx)$.

$w\notin\mathbf{pref}(U_i)\cup U_i\cdot X^*$**:** Here $\mathcal{V}_i(w)=\mathcal{V}_i(wx)=0$.

Now, set $\mathcal{V}(w) := \sum_{i\in\mathbb{N}}\mathcal{V}_i(w)$.

Then, for $\xi\in\bigcap_{i\in\mathbb{N}} U_i\cdot X^\omega$ there are $n_i\in\mathbb{N}$ such that $\xi\restriction n_i\in U_i$ and we obtain $\frac{\mathcal{V}(\xi\restriction n_i)}{r^{n_i}\cdot h'(r^{-n_i})} \geq \frac{\mathcal{V}_i(\xi\restriction n_i)}{r^{n_i}\cdot h'(r^{-n_i})} = \frac{h(r^{-n_i})}{h'(r^{-n_i})}$ which tends to infinity as i tends to infinity.

Now let $\mathcal{H}^h(F)<\infty$. Then $h\prec\sqrt{h\cdot h'}\prec h'$. Thus $\mathcal{H}^{\sqrt{h\cdot h'}}(F)=0$ and we can apply the first part of the proof to the functions $\sqrt{h\cdot h'}$ and h'. ❑

The next lemma is in some sense a converse to Lemma 5.

Lemma 6 *Let h be a gauge function, $c\in(0,\infty]$ and $\mathcal{V}$ be a super-martingale. Then $\mathcal{H}^h(S_{c,h}[\mathcal{V}]) \leq \frac{\mathcal{V}(e)}{c}$.*

Proof. It suffices to prove the assertion for $c<\infty$.

Define $V_k := \{w : w\in X^* \wedge |w|\geq k \wedge \frac{\mathcal{V}(w)}{r^{|w|}\cdot h(r^{-|w|})} \geq c-2^{-k}\}$ and set $U_k := \mathrm{Min}_{\sqsubseteq} V_k$. Then $S_{c,h}[\mathcal{V}] \subseteq \bigcap_{k\in\mathbb{N}} U_k\cdot X^\omega$.

$$\begin{aligned} \text{Now } \sum_{w\in U_k} h(r^{-|w|}) &\leq \sum_{w\in U_k} h(r^{-|w|})\cdot\frac{\mathcal{V}(w)}{r^{|w|}\cdot h(r^{-|w|})}\cdot\frac{1}{c-2^{-k}} \\ &= \frac{1}{c-2^{-k}}\cdot\sum_{w\in U_k}\frac{\mathcal{V}(w)}{r^{|w|}} \leq \frac{\mathcal{V}(e)}{c-2^{-k}} \text{ (cf. Eq. (11)).}\end{aligned}$$

Thus $\mathcal{H}^h(\bigcap_{k\in\mathbb{N}} U_k\cdot X^\omega) \leq \frac{\mathcal{V}(e)}{c}$. ❑

Lemmata 5 and 6 yield the following martingale characterisation of exact Hausdorff dimension functions.

Theorem 4 *Let $F\subseteq X^\omega$. Then a gauge function h is an exact Hausdorff dimension function for F if and only if*

1. *for all gauge functions h' with $h\prec h'$ there is a super-martingale $\mathcal{V}$ such that $F\subseteq S_{\infty,h'}[\mathcal{V}]$, and*

[4] This yields $\mathcal{V}_i(w)=0$ for $w\notin\mathbf{pref}(U_i)\cup U_i\cdot X^*$.

2. *for all gauge functions h'' with $h'' \prec h$ and all super-martingales $\mathcal{V}$ it holds $F \not\subseteq S_{\infty,h''}[\mathcal{V}]$.*

Proof. Assume h to be exact for F and $h \prec h'$. Then $h \prec \sqrt{h \cdot h'} \prec h'$. Thus $\mathcal{H}^{\sqrt{h \cdot h'}}(F) = 0$ and applying Lemma 5 to $\sqrt{h \cdot h'}$ and h' yields a super-martingale $\mathcal{V}$ such that $F \subseteq S_{\infty,h'}[\mathcal{V}]$.

If $h'' \prec h$ then $\mathcal{H}^{h''}(F) = \infty$ and according to Lemma 6 $F \not\subseteq S_{\infty,h''}[\mathcal{V}]$ for all super-martingales $\mathcal{V}$.

Conversely, let Conditions 1 and 2 be satisfied. Let $h \prec h'$, and let $\mathcal{V}$ be a super-martingale such that $F \subseteq S_{\infty,h'}[\mathcal{V}]$. Now Lemma 6 shows $\mathcal{H}^{h'}(F) \leq \mathcal{H}^{h'}(S_{\infty,h'}[\mathcal{V}]) = 0$.

Finally, suppose $h'' \prec h$ and $\mathcal{H}^{h''}(F) < \infty$. Then $\mathcal{H}^{\sqrt{h \cdot h''}}(F) = 0$ and Lemma 5 shows that there is a super-martingale $\mathcal{V}$ such that $F \subseteq S_{\infty,\sqrt{h \cdot h''}}[\mathcal{V}]$. This contradicts Condition 2. ❑

Lemmata 5 and 6 also show that we can likewise formulate Theorem 4 for martingales instead of super-martingales.

3 Constructive dimension: the exact case

Constructive dimension is a variant of dimension defined analogously to Theorem 4 using only left computable super-martingales. As mentioned above, in this case we can simplify our considerations using an optimal left-computable super-martingale. For definiteness we use $\mathcal{U}(w) := r^{|w|} \cdot \mathbf{M}(w)$ where $\mathbf{M}$ is the universal left-computable semi-measure from Section 1.2.2. Thus we may define

Definition 2 *Let $F \subseteq X^\omega$. We refer to $h : \mathbb{R} \to \mathbb{R}$ as an* exact constructive dimension function *for F provided $F \subseteq S_{\infty,h'}[\mathcal{U}]$ for all h', $h \prec h'$, and $F \not\subseteq S_{\infty,h''}[\mathcal{U}]$ for all h'', $h'' \prec h$.*

The next theorem follows immediately from the identity $\mathcal{U}(w) = r^{|w|} \cdot \mathbf{M}(w)$ and the inequality $\mathbf{M}(w) \geq \mathbf{M}(w \cdot v)$.

Theorem 5 *The function h_ξ defined by $h_\xi(r^{-n}) := \mathbf{M}(\xi \upharpoonright n)$ is an exact constructive dimension function for the set $\{\xi\}$.*

In view of Eq. (2) also $h'_\xi(r^{-n}) := r^{-\mathrm{KA}(\xi \upharpoonright n)}$ is an exact constructive dimension function for the set $\{\xi\}$. In contrast to the asymptotic case, however, this does not hold for other complexities, like the monotone complexity Km, prefix complexity H or plain complexity. For the latter two this is made apparent by considering computable ω-words or Martin-Löf random ω-words. Corollary 4.5.2

of [DH10] shows that also Km and KA differ more than by a constant for certain ω-words.

Next we are going to show that the principle “large sets contain complex elements” holds also for exact Hausdorff dimension. We obtain the following bound from [Mie08].

Theorem 6 *Let $F \subseteq X^\omega$, h be a gauge function and $\mathscr{H}^h(F) > 0$.*

Then for every $c > 0$ with $\mathscr{H}^h(F) > c \cdot \mathbf{M}(e)$ there is a $\xi \in F$ such that $\mathrm{KA}(\xi \restriction n) \geq_{\mathrm{ae}} -\log_r h(r^{-n}) + \log_r c$.

For the sake of completeness we give a short proof. To this end we introduce the δ*-limit* $W^\delta := \{\xi : \xi \in X^\omega \wedge |\mathbf{pref}(\xi) \cap W| = \infty\}$ of a language $W \subseteq X^*$.

Proof. It is readily seen that the set of infinite words not fulfilling the asserted inequality is the δ-limit of $W_c = \{w : \mathrm{KA}(w) \leq -\log_r h(r^{-|w|}) + \log_r c\}$.

Let $V_m = \mathrm{Min}_\sqsubseteq(W_c \cap X^m \cdot X^*)$. Then V_m is prefix-free and $W_c^\delta \subseteq V_m \cdot X^\omega$ for all $m \in \mathbb{N}$. Consequently, $\mathscr{H}^h(W_c^\delta) \leq \sum_{v \in V_m} h(r^{-|v|})$ and in view of Eq. (1) we have $\sum_{v \in V_m} h(r^{-|v|}) \leq c \cdot \sum_{v \in V_m} r^{-\mathrm{KA}(v)} \leq c \cdot \sum_{v \in V_m} \mathbf{M}(v) \leq c \cdot \mathbf{M}(e)$. Now the inequality $\mathscr{H}^h(F) > \mathscr{H}^h(W_c^\delta)$ shows the assertion $F \not\subseteq W_c^\delta$. ❑

This lower bound on the maximum complexity of an infinite string in F yields a set-theoretic lower bound on the success sets $S_{c,h}[\mathscr{U}]$ of $\mathscr{U}$.

Theorem 7 *Let $c \in \mathbb{R}$ and let h be a gauge function. Then there is a $c' > 0$ such that*

$$\{\xi : \exists^\infty n(\mathrm{KA}(\xi \restriction n) \leq -\log_r h(r^{-n}) + c)\} \subseteq S_{c',h}[\mathscr{U}].$$

Proof. If ξ has infinitely many prefixes such that $\mathrm{KA}(\xi \restriction n) \leq -\log_r h(r^{-n}) + c$ then $\mathbf{M}(\xi \restriction n) \geq r^{-\mathrm{KA}(\xi \restriction n)} \geq h(r^{-n}) \cdot r^{-c}$. Since $\mathscr{U}(w) = r^{|w|} \cdot \mathbf{M}(w)$, we obtain $\limsup_{n\to\infty} \frac{\mathscr{U}(\xi \restriction n)}{r^n \cdot h(r^{-n})} = \limsup_{n\to\infty} \frac{\mathbf{M}(\xi \restriction n)}{h(r^{-n})} \geq r^{-c}$. ❑

Corollary 2 *Let h, h' be gauge functions such that $h \prec h'$ and $c \in \mathbb{R}$. Then*

1. $\{\xi : \mathrm{KA}(\xi \restriction n) \leq_{\mathrm{io}} \log_r h(r^{-n}) + c\} \subseteq S_{\infty,h'}[\mathscr{U}]$, *and*
2. $\mathscr{H}^{h'}(\{\xi : \mathrm{KA}(\xi \restriction n) \leq_{\mathrm{io}} -\log_r h(r^{-n}) + c\}) = 0$.

4 Complexity and dilution

In this section we are going to show that, analogously to Ryabko’s proof for the “usual” dimension, the bound given in Corollary 2 is tight for a large class of (computable) gauge functions. To this end we prove that certain sets of infinite strings diluted according to a gauge function h have positive Hausdorff measure $\mathscr{H}^h$.

4.1 A generalised dilution principle

We show that for a large family of gauge functions, a set of finite positive measures can be constructed. Our construction is a generalisation of Hausdorff's 1918 construction. Instead of his method of cutting out middle thirds in the unit interval we use the idea of dilution functions as presented in [Sta08]. In fact dilution appears much earlier (see e.g. [Dal74, Sta93, Lut03b, Rei04])

We consider *prefix-monotone* mappings, that is, mappings $\varphi : X^* \to X^*$ satisfying $\varphi(w) \sqsubseteq \varphi(v)$ whenever $w \sqsubseteq v$. We call a function $g : \mathbb{N} \to \mathbb{N}$ a *modulus function* for φ provided $|\varphi(w)| = g(|w|)$ for all $w \in X^*$. This, in particular, implies that $|\varphi(w)| = |\varphi(v)|$ for $|w| = |v|$ whenever φ has a modulus function.

Every prefix-monotone mapping $\varphi : X^* \to X^*$ defines as a limit a partial mapping $\overline{\varphi} :\subseteq X^\omega \to X^\omega$ in the following way: $\mathbf{pref}(\overline{\varphi}(\xi)) = \mathbf{pref}(\varphi(\mathbf{pref}(\xi)))$ whenever $\varphi(\mathbf{pref}(\xi))$ is an infinite set, and $\overline{\varphi}(\xi)$ is undefined when $\varphi(\mathbf{pref}(\xi))$ is finite.

If, for some strictly increasing function $g : \mathbb{N} \to \mathbb{N}$, the mapping φ satisfies the conditions $|\varphi(w)| = g(|w|)$ and for every $v \in \mathbf{pref}(\varphi(X^*))$ there are $w_v \in X^*$ and $x_v \in X$ such that

$$\varphi(w_v) \sqsubset v \sqsubseteq \varphi(w_v \cdot x_v) \wedge \forall y (y \in X \wedge y \neq x_v \to v \not\sqsubseteq \varphi(w_v \cdot y)) \tag{13}$$

then we call φ a *dilution function* with modulus g. If φ is a dilution function then $\overline{\varphi}$ is a one-to-one mapping.

For the image $\overline{\varphi}(X^\omega)$ we obtain the following bounds on its Hausdorff measure (cf. the mapping theorem [Rog98, Theorem 29]).

Theorem 8 *Let $g : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function, φ a corresponding dilution function and $h : (0, \infty) \to (0, \infty)$ be a gauge function. Then*

1. $\mathscr{H}^h(\overline{\varphi}(X^\omega)) \leq \liminf_{n\to\infty} \frac{h(r^{-g(n)})}{r^{-n}}$

2. *If* $c \cdot r^{-n} \leq_{\mathrm{ae}} h(r^{-g(n)})$ *then* $c \leq \mathscr{H}^h(\overline{\varphi}(X^\omega))$.

Proof. The first assertion follows from $\overline{\varphi}(X^\omega) \subseteq \bigcup_{|w|=n} \varphi(w) \cdot X^\omega$ and $|\varphi(w)| = g(|w|)$.

The second assertion is obvious for $\mathscr{H}^h(\overline{\varphi}(X^\omega)) = \infty$. Let the measure $\mathscr{H}^h(\overline{\varphi}(X^\omega))$ be finite, $\varepsilon > 0$, and $V \cdot X^\omega \supseteq \overline{\varphi}(X^\omega)$ such that $\sum_{v\in V} h(r^{-|v|}) \leq \mathscr{H}^h(\overline{\varphi}(X^\omega)) + \varepsilon$. Without loss of generality we may assume that $V \subseteq X^*$ is prefix-free. Then, since φ is a dilution function, for every $v \in V$ there is at most one $w_v \cdot x_v$ such that $v \sqsubseteq \varphi(w_v \cdot x_v)$.

Now, the set $W_V := \{w_v \cdot x_v : \exists v (v \in V \wedge \varphi(w_v) \sqsubset v \sqsubseteq \varphi(w_v \cdot x_v))\}$ (see Eq. (13)) is prefix-free, and it holds $W_V \cdot X^\omega \supseteq X^\omega$. Thus W_V is finite and $\sum_{w\in W_V} r^{-|w|} = 1$.

Assume now $\min\{|v| : v \in V\}$ large enough such that $c \cdot r^{-|v|} \le h(r^{-g(|v|)})$ for all $v \in V$.

Then $\sum_{v\in V} h(r^{-|v|}) \;\ge\; \sum_{wx\in W_V} h(r^{-|\varphi(wx)|}) = \sum_{wx\in W_V} h(r^{-g(|wx|)})$
$\ge\; \sum_{wx\in W_V} c \cdot r^{-|wx|} = c\,.$

As $\varepsilon > 0$ is arbitrary, the assertion follows. ❑

Corollary 3 *If* $c \cdot r^{-n} \le_{\mathrm{ae}} h(r^{-g(n)}) \le c' \cdot r^{-n}$ *then* $c \le \mathscr{H}^h(\overline{\varphi}(X^\omega)) \le c'$.

In connection with Theorem 8 and Corollary 3 it is of interest which gauge functions allow for a construction of a set of positive finite measure via dilution. Hausdorff's cutting out was demonstrated for upwardly convex gauge functions. We consider the slightly more general case of functions fulfilling the following (cf. also Lemma 2).

Lemma 7 *Let* $h : [0,1] \to \mathbb{R}_+$ *be a gauge function* h *with* $\lim_{t\to\infty} h(t) = 0$. *If* $h(t)/t$ *is non-increasing on* $(0,\varepsilon)$, $\varepsilon \le 1$, *then there is an* $n_0 \in \mathbb{N}$ *such that for all* $n \ge n_0$ *there is an* $m \in \mathbb{N}$ *satisfying*

$$r^{-n} < h(r^{-m}) \le r^{-n+1}\,. \tag{14}$$

In particular, Eq. (14) implies that the gauge function h does not tend faster to 0 than the identity function $\mathrm{id} : \mathbb{R} \to \mathbb{R}$.

Proof. We prove the assertion by induction on m.

Since $\lim_{t\to\infty} h(t) = 0$, we may choose n_0, m_0 such that $r^{-(n_0+1)} < h(r^{-m_0}) \le r^{-n_0}$. Now assume $r^{-(n+1)} < h(r^{-m}) \le r^{-n}$ for $n \ge n_0, m \ge m_0$.

Then $r^{-(n+2)} < \frac{h(r^{-m})}{r} \le r^{-(n+1)}$. Since $\frac{h(t)}{t}$ is non-increasing, we have $r^m \cdot h(r^{-m}) \le r^{(m+1)} \cdot h(r^{-(m+1)})$, that is, $\frac{h(r^{-m})}{r} \le h(r^{-(m+1)}) \le h(r^{-m})$. Consequently, $r^{-(n+2)} < h(r^{-(m+1)}) \le h(r^{-m}) \le r^{-n}$, and we have $r^{-(n+2)} < h(r^{-(m+1)}) \le r^{-(n+1)}$ or $r^{-(n+1)} < h(r^{-(m+1)}) \le r^{-n}$. ❑

Remark 1 Using the scaling factor $c = r^{n_0}$, that is, considering $c \cdot h$ instead of h and taking $h'(t) = \min\{c \cdot h(t), r\}$ one can always assume that $n_0 = 0$ and $h'(1) > 1$. Defining then $g(n) := \max\{m : m \in \mathbb{N} \wedge r^{-n} < h(r^{-m})\}$ we obtain via Property 1 and Corollary 3 that for every gauge function h fulfilling Eq. (14) there is a subset $F_h = \overline{\varphi}(X^\omega)$ of X^ω having finite and positive $\mathscr{H}^h$-measure.

As in Lemma 2 the condition of Lemma 7 is not necessary. We provide an example.

Example 2 Set $h(r^{-n}) := \begin{cases} r^{-n/2} & \text{if } n \text{ is even, and} \\ r^{-(n+3)/2} + r^{-n} & \text{if } n \text{ is odd.} \end{cases}$

and extend h to a continuous non-decreasing function. Then $h(r^{-2n}) = r^{-n}$, $h(r^{-(2n+1)}) = r^{-(n+2)} + r^{-(2n+1)}$ and, consequently, $r^{-n} \ge h(r^{-2n}) > r^{-(n+1)} \ge h(r^{-(2n+1)}) > r^{-(n+2)}$, if $n \ge 1$.

On the other hand, $\frac{h(r^{-2n})}{r^{-2n}} = r^n > \frac{h(r^{-(2n+1)})}{r^{-(2n+1)}} = r^{(n-1)} + 1$.

4.2 Computable gauge functions

The aim of this section is to show that the modulus function g and thus the dilution function φ can be chosen computable if the gauge function h fulfilling Eq. (14) is computable.

Lemma 8 *Let $h : \mathbb{Q} \to \mathbb{R}$ be a computable gauge function satisfying the conditions $1 < h(1) < r$ and for every $n \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that $r^{-n} < h(r^{-m}) \leq r^{-n+1}$. Then there is a computable strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ such that $r^{-n} < h(r^{-g(n)}) < r^{-n+2}$.*

Proof. We define g inductively. To this end we compute for every $n \geq 1$ a closed interval I_n such that $h(r^{-g(n)}) \in I_n \subset (r^{-(n-1)}, \min I_{n-1})$

We start with $g(0) := 0$ and $I_{-1} = [r, r+1]$ and estimate I_0 as an sufficiently small approximating interval of $h(r^{-g(0)}) > 1$ satisfying $I_0 \subseteq (1, r)$.

Assume now that for n the value $g(n)$ and the interval I_n satisfying $h(r^{-g(n)}) \in I_n \subset (r^{-n}, \min I_{n-1})$ are computed.

We search for an m and an approximating interval $I(m)$, $h(r^{-m}) \in I(m)$, such that $I(m) \subset (r^{-n-1}, \min I_n)$. Since $\liminf_{m\to\infty} h(r^{-m}) = 0$ and $\exists m (r^{-n-1} < h(r^{-m}) \leq r^{-n})$ and $r^{-n} < \min I_n$ this search will eventually be successful. Define $g(n+1)$ as the first such m found by our procedure and set $I_{n+1} := I(m)$.

Finally, the monotonicity of h implies $g(n+1) > g(n)$. ❑

With Corollary 3 we obtain the following.

Corollary 4 *Under the hypotheses of Lemma 8 there is a computable dilution function $\varphi : X^* \to X^*$ such that $r^{-2} \leq \mathcal{H}^h(\overline{\varphi}(X^\omega)) \leq 1$.*

4.3 Complexity of diluted infinite strings

In the final part of this section we show that, for a large class of computable gauge functions, sets like $\{\xi : \mathrm{KA}(\xi \restriction n) \leq_{\mathrm{io}} -\log_r h(r^{-n}) + c\}$ (see Corollary 2) have the function h as an exact dimension function, that is, a converse to Corollary 2.2.

We use the following estimate on the monotone complexity of a diluted string analogous to Theorem 3.1 of [Sta08].

Theorem 9 *Let $\varphi : X^* \to X^*$ be a one-to-one prefix-monotone comptable function satisfying Eq (13) with strictly increasing modulus function g. Then*

$$\left|\mathrm{Km}(\overline{\varphi}(\xi)[0..g(n)]) - \mathrm{Km}(\xi \restriction n)\right| \leq O(1) \text{ for all } \xi \in X^\omega .$$

Proof. The function φ has a prefix monotone computable partial inverse. Then the proof follows from Proposition 2. ❑

This auxiliary result yields that certain sets of non-complex strings have non-null h-dimensional Hausdorff measure.

Theorem 10 *If $h : \mathbb{Q} \to \mathbb{R}$ is a computable gauge function satisfying Eq. (14) then there is a $c \in \mathbb{N}$ such that*

$$\mathscr{H}^h(\{\zeta : \mathrm{Km}(\zeta \upharpoonright n) \leq_{\mathrm{ae}} -\log_r h(r^{-n}) + c\}) > 0.$$

Proof. From the gauge function h we construct a computable dilution function φ with modulus function g such that $r^{-(l+k)} < h(r^{-g(l)}) < r^{-(l+k-2)}$ for a suitable constant k (cf. Lemma 8 and Remark 1). Then, according to Corollary 4, $\mathscr{H}^h(\overline{\varphi}(X^\omega)) > 0$.

Using Theorem 9 we obtain $\mathrm{Km}(\overline{\varphi}(\xi) \upharpoonright g(l)) \leq \mathrm{Km}(\xi \upharpoonright l) + c_1 \leq l + c_2$ for suitable constants $c_1, c_2 \in \mathbb{N}$. Let $n \in \mathbb{N}$ satisfy $g(l) < n \leq g(n+1)$. Then $\mathrm{Km}(\overline{\varphi}(\xi) \upharpoonright n) \leq \mathrm{Km}(\overline{\varphi}(\xi) \upharpoonright g(l+1)) \leq l + 1 + c_2$.

Now from $l + k - 1 < -\log_r h(r^{-g(l)}) \leq -\log_r h(r^{-n})$ we obtain the assertion $\mathrm{Km}(\overline{\varphi}(\xi) \upharpoonright n) \leq -\log_r h(r^{-n}) - k + c_2 + 2$. ❑

Now Corollary 2.2 and Theorem 6 prove the following analogue to Ryabko's [Rya84] result.

Lemma 9 *If $h : \mathbb{Q} \to \mathbb{R}$ is a computable gauge function satisfying Eq. (14) then there is a $c \in \mathbb{N}$ such that h is an exact Hausdorff dimension for the sets $\{\xi : \mathrm{KA}(\xi \upharpoonright n) \leq_{\mathrm{io}} -\log_r h(r^{-n}) + c\}$ and $\{\zeta : \mathrm{Km}(\zeta \upharpoonright \ell) \leq_{\mathrm{ae}} -\log_r h(r^{-\ell}) + c\}$.*

Despite the fact that there are ω-words on which KA and Km differ by more that an additive constant Lemma 9 seems to indicate that the set of those ω-words is not too big even in the sense of exact Hausdorff dimension.

5 Exact dimensions for Σ_2-definable ω-languages

In this section we generalise the results of Section 1.4.3 to the case of exact dimension. The proofs follow closely the line of the corresponding proofs in [Sta98]. It is remarkable that the upper complexity bound holds for prefix complexity.

5.1 Constructive Dimension

We start with an auxiliary lemma characterising subsets $F \subseteq X^\omega$ having null measure via the δ-limit of languages V^δ.

Lemma 10 ([Rei04]) *Let $F \subseteq X^\omega$ and h be a gauge function. Then $\mathscr{H}^h(F) = 0$ if and only if there is a language $V \subseteq X^*$ such that $F \subseteq V^\delta$ and $\sum_{v \in V} h(r^{-|v|}) < \infty$.*

The following theorem gives a constructive version of Lemma 10.

Theorem 11 *If $F \subseteq X^\omega$ is a Σ_2-definable ω-language and $h, h(1) \leq 1$, is a right computable gauge function such that $\mathcal{H}^h(F) = 0$ then there are a computable non-decreasing function $\bar{h} : \{r^{-i} : i \in \mathbb{N}\} \to \mathbb{Q}$ and a computable language $V \subseteq X^*$ satisfying*

1. $\bar{h}(r^{-n}) \geq h(r^{-n})$ *for all* $n \in \mathbb{N}$,
2. $F \subseteq V^\delta$ *and* $\sum_{v\in V} \bar{h}(r^{-|v|}) < \infty$.

Proof. Let $h_n : \{r^{-\ell} : \ell \in \mathbb{N}\} \to \mathbb{Q}$, $n \in \mathbb{N}$, be computable approximations of h such that $h_n(t) \geq h_{n+1}(t) \geq h(t)$ and $\lim_{n\to\infty} h_n(t) = h(t)$ for $t \in \{r^{-\ell} : \ell \in \mathbb{N}\}$. The functions h_n are assumed to be non-decreasing on the set $\{r^{-\ell} : \ell \in \mathbb{N}\}$. As $h(t) \geq t$ we have also $h(r^{-n}) \geq r^{-n}$.

Furthermore, let $(U_j)_{j\in\mathbb{N}}$ be an effective enumeration of all finite prefix codes over X such that $\sup\{|v| : v \in U_j\} \leq \sup\{|v| : v \in U_{j+1}\}$, and let $F \in \Sigma_2$, as described in Section 1 of [Sta98] be given by $F = \bigcup_{k\in\mathbb{N}} X^\omega \setminus L_k \cdot X^\omega$ where $M_F := \{(w,k) : w \in L_k\}$ is a computable set and the family of languages $(L_k)_{k\in\mathbb{N}}$ satisfies $L_k := \bigcap_{i=0}^{k} L_i \cdot X^*$.

Define the predicate

$$\mathbf{test}(k,j,n) :\Leftrightarrow \Big((U_j \cup (L_k \cap X^n)) \cdot X^\omega = X^\omega \ \wedge \sum_{v\in U_j} h_n(r^{-|v|}) < r^{-k} \Big) .$$

Observe that $\mathbf{test}(k,j,n)$ is computable and if $\mathbf{test}(k,j,n)$ is true then the conditions $F \subseteq U_j \cdot X^\omega$ and $\forall v (v \in U_j \to k < |v|)$ are satisfied.

The first condition follows from $L_k \cdot X^\omega \cap F = \emptyset$ and the second one from $h_n(r^{-|v|}) > r^{-|v|}$.

Now the following algorithm, when given M_F, computes a finite prefix code C_k and a value m_k satisfying the conditions $F \subseteq C_k \cdot X^\omega$ and $\sum_{v\in C_k} h_{m_k}(r^{-|v|}) < r^{-k}$:

Algorithm C_k

```
input   k
        n = 0
        repeat  j = −1
                repeat  j = j + 1
                until   test(k, j, n) ∨ (sup{|v| : v ∈ U_j} > n)
                n = n + 1
        until   test(k, j, n)
output  C_k := U_j, m_k := n
```

By construction we have $k < |v| \leq m_k$ for $v \in C_k$.

Informally, for every $n \geq 0$ our algorithm successively searches for a U_j satisfying the condition $\mathbf{test}(k, j, n)$, more precisely, it searches until such a U_j is found or else all U_j having $\sup\{|v| : v \in U_j\} \leq n$ fail to satisfy $\mathbf{test}(k, j, n)$.

In the latter case the value of n is increased (thus allowing for a larger maximum codeword length, a larger complementary ω-language $(L_k \cap X^n) \cdot X^\omega$ and a closer approximation h_{n+1} of the gauge function h) and the search starts anew. Consequently, the algorithm terminates if and only if there is a finite prefix code U such that $\sum_{v \in U} h_n(r^{-|v|}) < r^{-k}$ and $U \cdot X^\omega \cup (L_k \cap X^n) \cdot X^\omega = X^\omega$ for some $n \in \mathbb{N}$.

First we show that our algorithm always terminates. Observe that for every $\varepsilon > 0$ there is a $W \subseteq X^*$ such that $F \subseteq W \cdot X^\omega$ and $\sum_{w \in W} h(r^{-|w|}) < \frac{\varepsilon}{2}$.

Since $X^\omega \setminus L_k \cdot X^\omega$ is a closed subset of F, for $\varepsilon \leq r^{-k}$ we find a finite subset $W' \subseteq W$ such that $X^\omega \setminus L_k \cdot X^\omega \subseteq W' \cdot X^\omega$. Then $\sum_{w \in W} h(r^{-|w|}) < \frac{\varepsilon}{2}$ implies that $\sum_{w \in W'} h_n(r^{-|w|}) < \varepsilon$ for $n \geq n_{\varepsilon,k}$, say.

Consequently, there is a finite prefix code $U_j \subseteq W'$ satisfying $(U_j \cup L_k) \cdot X^\omega = X^\omega$ and thus $(U_j \cup (L_k \cap X^n)) \cdot X^\omega = X^\omega$ for $n \geq n'_{j,k}$. This shows that the predicate $\mathbf{test}(k, j, n)$ is satisfied whenever $n \geq \max\{n_{r^{-k},k}, n'_{j,k}\}$.

Now we define $V := \bigcup_{i \in \mathbb{N}} C_i$ and show that V meets the requirements of the theorem. We have $w \in V$ if and only if $\exists i (i < |w| \wedge w \in C_i)$. This predicate is computable, since $i < |w|$ bounds the quantifier $\exists i$ from above. Thus the language V is computable.

Next we show that $F \subseteq V^\delta$. If $\xi \in F$ there is an i_ξ such that $\xi \in X^\omega \setminus L_i \cdot X^\omega$ for all $i \geq i_\xi$. Hence, for every $i \geq i_\xi$ the ω-word ξ has a prefix $w_i \in C_i$. As it was observed above, $|w_i| > i$. Consequently, ξ has infinitely many prefixes in $V = \bigcup_{i \in \mathbb{N}} C_i$.

Finally, in order to define the function $\bar{h}$ we let $\ell_i := \max\{m_k : k < i\}$. Clearly, the value ℓ_i can be computed from i. Define $\bar{h}(r^{-i}) := h_{\ell_i}(r^{-i})$. Then $h_{m_k}(t) \geq h(t)$ implies $\bar{h}(r^{-i}) \geq h(r^{-i})$ and $\ell_i \leq \ell_{i+1}$ shows that $\bar{h}(r^{-i}) \geq \bar{h}(r^{-(i+1)})$.

It remains to show that $\sum_{v \in V} \bar{h}(r^{-|v|}) < \infty$. Taking into account that $k < |v| \leq m_k$, for $v \in C_k$, we have $\bar{h}(r^{-|v|}) = h_{\ell_{|v|}}(r^{-|v|}) \leq h_{m_k}(r^{-|v|})$ for $v \in C_k$ and thus

$$\begin{aligned} \sum_{v \in V} \bar{h}(r^{-|v|}) &\leq \sum_{k \in \mathbb{N}} \sum_{v \in C_k} h_{m_k}(r^{-|v|}) \\ &\leq \sum_{k \in \mathbb{N}} r^{-k} < \infty . \end{aligned}$$ ❑

Interpolating the computable function $\bar{h}$ we obtain the following consequence.

Corollary 5 *If $F \subseteq X^\omega$ is a Σ_2-definable ω-language and h is a right computable gauge function such that $\mathscr{H}^h(F) = 0$ then there is a computable non-decreasing function $\bar{h} : \mathbb{Q} \to \mathbb{Q}$ satisfying $\mathscr{H}^{\bar{h}}(F) = 0$ and $\bar{h}(t) \geq h(t)$ for $t \in \mathbb{Q} \cap (0, 1)$.*

This result corresponds in some sense to a result by Besicovitch [Rog98, Theorem 41] which states that for every $E \subseteq X^\omega$ with $\mathscr{H}^h(E) = 0$ there is a h'' such that $\mathscr{H}^{h''}(E) = 0$ and $\liminf_{n\to\infty} \frac{h(r^{-n})}{h''(r^{-n})} = 0$.

Our Theorem 11 yields the required upper bound for the prefix complexity H, and hence also of the monotone and a priori complexities Km and KA, respectively, of an ω-word in F.

If $V \subseteq X^*$ is computably enumerable and $\bar{h} : \{r^{-n} > n \in \mathbb{N}\} \to \mathbb{R}$ is a left computable function such that $\sum_{v\in V} \bar{h}(r^{-|v|}) < \infty$ then

$$\nu(w) := \begin{cases} \bar{h}(r^{-|w|}), & \text{if } w \in V\text{, and} \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

defines a left computable discrete semi-measure. Thus Theorem 11 implies the following upper bound on the complexity functions of ω-words.

Lemma 11 *If $F \subseteq X^\omega$ is a Σ_2-definable ω-language and h is a right computable gauge function such that $\mathscr{H}^h(F) = 0$ then*

$$\mathrm{H}(\xi \upharpoonright n) \leq_{\text{i.o.}} -\log_r h(r^{-n}) + O(1) \textit{ for all } \xi \in F.$$

Proof. We use the computable subset $V \subseteq X^*$ and the computable function $\bar{h}$ defined in the proof of Theorem 11 and define the discrete semi-measure ν via Eq. (15). Then $\nu(w) \leq c \cdot \mathbf{m}(w)$, for all $w \in X^*$ and, consequently $\mathrm{H}(w) \leq -\log_r \bar{h}(r^{-|w|}) \leq -\log_r h(r^{-|w|})$, for $w \in V$. The assertion follows from $F \subseteq V^\delta$. ❑

Finally, Lemma 11 and Corollary 2 prove the following.

Theorem 12 *If $F \subseteq X^\omega$ is a union of Σ_2-definable sets and h is a right computable gauge function such that $\mathscr{H}^h(F) = 0$ then $F \subseteq S_{\infty,h'}[\mathscr{U}]$ for every gauge function h' such that* $\lim_{t\to 0} \frac{h'(t)}{h(t)} = 0$.

5.2 Computable Dimension

Computable dimension is based on computable super-martingales as constructive dimension was based on left computable super-martingales. In contrast to the latter, for the former there is no universal computable super-martingale (cf. [DH10, Sch71]). Thus we define analogously to Theorem 4

Definition 3 We refer to a gauge function h as an *exact computable dimension function* for $F \subseteq X^\omega$ provided

1. for all gauge functions h' with $\lim\limits_{n\to\infty} \frac{h'(r^{-n})}{h(r^{-n})} = 0$ there is a computable super-martingale $\mathscr{V}$ such that $F \subseteq S_{\infty,h'}[\mathscr{V}]$, and

2. for all gauge functions h'' with $\lim_{n\to\infty} \frac{h(r^{-n})}{h''(r^{-n})} = 0$ and all computable supermartingales $\mathcal{V}$ it holds $F \not\subseteq S_{\infty,h''}[\mathcal{V}]$.

As for the constructive case the second item is fulfilled provided $\mathcal{H}^h(F) > 0$. For Item 1 we prove that for computable gauge functions h and Σ_2^0-definable sets $F \subseteq X^\omega$ with $\mathcal{H}^h(F) = 0$ there is a computable martingale $\mathcal{V}$ such that $F \subseteq \bigcup_{c>0} S_{c,h}[\mathcal{V}]$.

In order to achieve our goal we introduce families of covering codes as in [Sta98]. For a prefix code $C \subseteq X^*$ we define its *minimal complementary code* as

$$\widehat{C} := (X \cup \mathbf{pref}(C) \cdot X) \setminus \mathbf{pref}(C) \,.$$

If $C = \emptyset$ we have $\widehat{C} = X$, and if $C \neq \emptyset$ the set $\widehat{C}$ consists of all words $w \cdot x \notin \mathbf{pref}(C)$ where $w \in \mathbf{pref}(C)$ and $x \in X$. It is readily seen that $C \cup \widehat{C}$ is a maximal prefix code, $C \cap \widehat{C} = \emptyset$, and $\mathbf{pref}(C \cup \widehat{C}) = \{e\} \cup \mathbf{pref}(C) \cup \widehat{C}$.

We call $\mathfrak{C} := (C_w)_{w \in X^*}$ a *family of covering codes* provided each C_w is a finite prefix code. Since then the set $C_w \cup \widehat{C}_w$ is a finite maximal prefix code, every word $u \in X^*$ has a uniquely specified $\mathfrak{C}$-factorisation $u = u_1 \cdots u_n \cdot u'$ where $u_{i+1} \in C_{u_1\cdots u_i} \cup \widehat{C}_{u_1\cdots u_i}$ for $i = 0, \ldots, n-1$ ($u_1 \cdots u_i = e$, if $i = 0$) and $u' \in \mathbf{pref}(C_{u_1\cdots u_n} \cup \widehat{C}_{u_1\cdots u_n})$. Analogously, every $\xi \in X^\omega$ has a uniquely specified $\mathfrak{C}$-factorisation $\xi = u_1 \cdots u_i \cdots$ where $u_{i+1} \in C_{u_1\cdots u_i} \cup \widehat{C}_{u_1\cdots u_i}$ for $i = 1, \ldots$.

In what follows we use martingales derived from prefix codes in the following manner.

Lemma 12 *Let $h : \mathbb{R} \to \mathbb{R}$ a gauge function and $\emptyset \neq C \subseteq X^*$ be a prefix code satisfying $\sum_{v\in C} h(r^{-|v|}) < \infty$. Then there is a martingale $\mathcal{V}_C^{(h)} : X^* \to [0,\infty)$ such that*

$$\mathcal{V}_C^{(h)}(w) = \begin{cases} \dfrac{r^{|w|} \cdot h(r^{(-|w|)})}{\sum_{v\in C} h(r^{(-|v|)}) + \sum_{u\in\widehat{C}} r^{-|u|}} & \text{, for } w \in C \text{, and} \\[2ex] \dfrac{1}{\sum_{v\in C} h(r^{(-|v|)}) + \sum_{u\in\widehat{C}} r^{-|u|}} & \text{, for } w \in \widehat{C} . \end{cases} \tag{16}$$

Proof. Set $\Gamma := \sum_{v\in C} h(r^{-|v|}) + \sum_{u\in\widehat{C}} r^{-|u|}$, and define for $u \in \mathbf{pref}(C \cup \widehat{C})$ and $w \in C \cup \widehat{C}$, $v \in X^*$

$$\begin{aligned} \mathcal{V}_C^{(h)}(u) &:= \frac{r^{|u|}}{\Gamma} \cdot \left(\sum_{u\cdot w\in C} h(r^{-|u\cdot w|}) + \sum_{u\cdot w\in\widehat{C}} r^{-|u\cdot w|} \right) \\ \mathcal{V}_C^{(h)}(w \cdot v) &:= \mathcal{V}_C^{(h)}(w) \,. \end{aligned}$$

Then $\mathcal{V}_C^{(h)}$ fulfils Eq. (16). We still have to show the property $\mathcal{V}_C^{(h)}(u) = \frac{1}{r} \sum_{x\in X} \mathcal{V}_C^{(h)}(ux)$. This identity is obvious if $u \in (C \cup \widehat{C}) \cdot X^*$.

Now, let $u \notin (C \cup \widehat{C}) \cdot X^*$, that is, $u \in \mathbf{pref}(C \cup \widehat{C}) \setminus (C \cup \widehat{C})$. Then

$$\begin{aligned}\sum_{x \in X} \frac{\mathcal{V}_C^{(h)}(ux)}{r} &= \sum_{x \in X} \frac{r^{|ux|}}{r \cdot \Gamma} \cdot \left(\sum_{uxw \in C} h(r^{-|uxw|}) + \sum_{uxw \in \widehat{C}} r^{-|uxw|} \right) \\ &= \frac{r^{|u|}}{\Gamma} \cdot \sum_{x \in X} \left(\sum_{uxw \in C} h(r^{-|uxw|}) + \sum_{uxw \in \widehat{C}} r^{-|uxw|} \right) ,\end{aligned}$$

because for $u \in \mathbf{pref}(C \cup \widehat{C}) \setminus (C \cup \widehat{C})$ the set $\{w : w \in C \cup \widehat{C} \wedge u \sqsubseteq w\}$ partitions into the sets $\{w : w \in C \cup \widehat{C} \wedge ux \sqsubseteq w\}$ $(x \in X)$, and the required equation follows. ❑

Remark 2 If C is a finite prefix code and $h : \mathbb{Q} \to \mathbb{Q}$ is computable then $\mathcal{V}_C^{(h)}$ is a computable martingale.

For a gauge function $h : \mathbb{R} \to \mathbb{R}$ let $h_w(t) := \frac{h(r^{-|w|} \cdot t)}{h(r^{-|w|})}$ and let $\mathfrak{C} := (C_w)_{w \in X^*}$ be a family of covering codes.

Using the martingales $\mathcal{V}_{C_w}^{(h_w)}$ we define a new martingale $\mathcal{V}_{\mathfrak{C}}$ as follows:

For $u \in X^*$ consider the $\mathfrak{C}$-factorisation $u_1 \cdots u_n \cdot u'$, and put

$$\mathcal{V}_{\mathfrak{C}}^{(h)}(u) := \left(\prod_{i=0}^{n-1} \mathcal{V}_{C_{u_1 \cdots u_i}}^{(h_{u_1 \cdots u_i})}(u_{i+1}) \right) \cdot \mathcal{V}_{C_{u_1 \cdots u_n}}^{(h_{u_1 \cdots u_n})}(u') ,$$

that is, $\mathcal{V}_{\mathfrak{C}}^{(h)}$ is in some sense the concatenation of the martingales $\mathcal{V}_{C_w}^{(h_w)}$. Observe that $\mathcal{V}_{\mathfrak{C}}^{(h)}$ is computable if only $h : \mathbb{R} \to \mathbb{R}$ is a computable function, the codes C_w are finite and the function which assigns to every w the corresponding code C_w is computable.

We have the following.

Lemma 13 *Let $h : \mathbb{N} \to \mathbb{Q}$ be a gauge function and let $\mathfrak{C} = (C_w)_{w \in X^*}$ be a family of covering codes such that $\sum_{v \in C_w} \frac{h(r^{-|wv|})}{h(r^{-|v|})} \le r^{-|w|}$ for all $w \in X^*$.*

If the ω-word $\xi \in X^\omega$ has a $\mathfrak{C}$-factorisation $\xi = u_1 \cdots u_i \cdots$ such that for some $n_\xi \in \mathbb{N}$ and all $i \ge n_\xi$ the factors u_{i+1} belong to $C_{u_1 \cdots u_i}$. Then there is a constant $c_\xi > 0$ not depending on i for which

$$\mathcal{V}_{\mathfrak{C}}(u_1 \cdots u_i) \ge c_\xi \cdot r^{|u_1 \cdots u_i|} \cdot h(r^{-|u_1 \cdots u_i|}) .$$

Proof. Since $\widehat{C}_w$ is a code, we have $\sum_{v \in \widehat{C}_w} r^{-|v|} \le 1$, and from the assumption and the definition of the function h_w we obtain

$$\textstyle\sum_{v \in C_w} h_w(r^{-|v|}) + \sum_{v \in \widehat{C}_w} r^{-|v|} \le r^{-|w|} + 1 .$$

Now $|u_j| \geq 1$ implies $|u_1 \cdots u_i| \geq i$, and the above Eq. (16) yields for $w = u_1 \cdots u_i$

$$\mathcal{V}^{(h_w)}_{C_{u_1\cdots u_i}}(u_{i+1}) \geq \begin{cases} \dfrac{1}{r^{-i}+1} = \dfrac{r^i}{1+r^i} & \text{, if } i \leq n_\xi \text{ , and} \\ \dfrac{r^{|u_{i+1}|} \cdot h_w(r^{-|u_{i+1}|})}{r^{-i}+1} & \text{, if } i > n_\xi \text{ .} \end{cases} \tag{17}$$

Put

$$c_\xi := \prod_{i=0}^{\infty} \frac{r^i}{1+r^i} \cdot \prod_{i=0}^{n_\xi} r^{|u_{i+1}|} \cdot h_w(r^{-|u_{i+1}|}) = r^{|u_1\cdots u_{n_\xi}|} \cdot h(r^{-|u_1\cdots u_{n_\xi}|}) \cdot \prod_{i=0}^{\infty} \frac{r^i}{1+r^i} \,.$$

Clearly, $c_\xi > 0$, and using Eq. (17) by induction on i the assertion is easily verified. ❑

Now we derive the announced result.

Theorem 13 *For every Σ_2-definable ω-language $F \subseteq X^\omega$ and every right-computable gauge function $h : \mathbb{Q} \to \mathbb{R}$ such that $\mathscr{H}^h(F) = 0$ there is a computable martingale $\mathcal{V}$ such that $F \subseteq \bigcup_{c>0} S_{c,h}[\mathcal{V}]$.*

Proof. In view of Corollary 5 it suffices to prove the theorem for computable functions $h : \mathbb{Q} \to \mathbb{R}$.

We use computable approximations $h_n : \mathbb{Q} \to \mathbb{Q}$ of h such that $h_n(t) \leq h_{n+1}(t)$ and $h_n(t) \leq h(t) \leq (1 + r^{-n}) \cdot h_n(t)$ for $t \in (0,1) \cap \mathbb{Q}$.

By virtue of Lemma 13 it suffices to construct a computable family of covering codes $\mathfrak{C} = (C_w)_{w \in X^*}$ such that the function which assigns to every w the corresponding finite prefix code C_w is computable.

To this end we modify the predicate **test** introduced in the proof of Theorem 11 as follows:

$$\begin{aligned} \mathbf{test}'(w,j,n) \quad :\Leftrightarrow \quad & \Big(n \geq |w| \wedge (w \cdot U_j \cup (L_{|w|} \cap X^{|w|+n})) \cdot X^\omega \supseteq w \cdot X^\omega \\ & \wedge \sum_{v \in U_j} \frac{(1+r^{-n}) \cdot h_n(r^{-|wv|})}{h_n(r^{-|w|})} < r^{-|w|} \Big) \,. \end{aligned}$$

In the same way we modify the algorithm presented there.

Algorithm C_w

input w
  $n = 0$
  repeat $j = -1$
    repeat $j = j + 1$
    until $\mathbf{test}'(w, j, n) \vee (\sup\{|v| : v \in U_j\} > n)$
    $n = n + 1$
  until $\mathbf{test}'(w, j, n)$
output $C_w := U_j$

Similar to the proof of Theorem 11 this algorithm computes a prefix code C_w with $\sum_{v \in C_w} \frac{h(r^{-|wv|})}{h(r^{-|w|})} < r^{-|w|}$ and $w \cdot C_w \cdot X^\omega \supseteq w \cdot X^\omega \setminus L_{|w|} \cdot X^\omega$.

Next we show that under the hypotheses of the theorem the algorithm always terminates. We have $\mathscr{H}^h(F \cap w \cdot X^\omega) = 0$ for all $w \in X^*$. Thus for $w \in X^*$ and every $\varepsilon > 0$ there is a prefix-free language $W \subseteq X^*$ such that $F \cap w \cdot X^\omega \subseteq W \cdot X^\omega$ and $\sum_{v \in W} h(r^{-|v|}) < r^{-|w|} \cdot \frac{h_0(r^{-|w|})}{1+r}$. As in the proof of Theorem 11, in view of $F \supseteq X^\omega \setminus L_{|w|} \cdot X^\omega$, there is a finite subset $W' \subseteq W \cap w \cdot X^*$ such that $w \cdot X^\omega \setminus L_{|w|} \cdot X^\omega \subseteq W' \cdot X^\omega$. Consequently, if $w \cdot U_j \cdot X^\omega \supseteq W' \cdot X^\omega$ and n is large enough the condition $\mathbf{test}'(w, j, n)$ will be satisfied.

It remains to show that every $\xi \in F$ has a $\mathfrak{C}$-factorisation $\xi = u_1 \cdots u_i \cdots$ such that almost all factors u_{i+1} belong to the corresponding codes $C_{u_1 \cdots u_i}$.

Let $\xi \in F$. Then there is a $k \in \mathbb{N}$ such that $\xi \in X^\omega \setminus L_i \cdot X^\omega$ for all $i \geq k$. Consequently, $w \in \mathbf{pref}(\xi)$ implies $w \notin L_k = L_k \cdot X^*$, and according to the definition of $\mathfrak{C}$ there is a $u \in C_w$ such that $w \cdot u \in \mathbf{pref}(\xi)$ whenever $|w| \geq k$. ❏

6 Functions of the logarithmic scale

As we have seen in Section 1.3 the set of gauge functions is not ordered which makes it difficult to assign a Hausdorff dimension to that set. On the other hand, the scale of the "classical" Hausdorff dimension is sometimes to coarse to distinguish sets.

In this part we consider a generalisation of the "usual" dimension which is finer than the "classical" Hausdorff dimension but is linearly ordered.

To this end we use the family of functions of the logarithmic scale. This family is, similarly to the family $h_\alpha(t) = t^\alpha$, also linearly ordered and, thus, allows for more specific versions of Corollary 2.2 and Theorem 6.

A function of the form where the first non-zero exponent satisfies $p_i > 0$

$$h_{(p_0,\ldots,p_k)}(t) = t^{p_0} \cdot \prod_{i=1}^{k} \left(\frac{1}{\log_r^i \frac{1}{t}}\right)^{p_i} \tag{18}$$

is referred to as a *function of the logarithmic scale* (see [Hau18]). Here we have the convention that $\log_r^i \frac{1}{t} = \max\{\underbrace{\log_r \ldots \log_r}_{i \text{ times}} \frac{1}{t}, 1\}$, that is, $\log_r^i \frac{1}{t} = \underbrace{\log_r \ldots \log_r}_{i \text{ times}} \frac{1}{t}$ if t is sufficiently small.

One observes that the lexicographic order on the tuples $(p_0, \ldots, p_k)$ yields an order of the functions $h_{(p_0,\ldots,p_k)}$ in the sense that $(p_0, \ldots, p_k) >_{\text{lex}} (q_0, \ldots, q_k)$ if and only if $h_{(q_0,\ldots,q_k)}(t) \prec h_{(p_0,\ldots,p_k)}(t)$.

This gives rise to a generalisation of the "usual" Hausdorff dimension as follows.

$$\begin{aligned} \dim_{\mathrm{H}}^{(k)} F &:= \sup\{(p_0, \ldots, p_k) : \mathscr{H}^{h_{(p_0,\ldots,p_k)}}(F) = \infty\} \\ &= \inf\{(p_0, \ldots, p_k) : \mathscr{H}^{h_{(p_0,\ldots,p_k)}}(F) = 0\} \end{aligned} \tag{19}$$

When taking supremum or infimum we admit also values $-\infty$ and ∞ for the last parameter p_k although we did not define the corresponding functions of the logarithmic scale. E.g. $\dim_{\mathrm{H}}^{(1)} F = (0, \infty)$ means that $\mathscr{H}^{h_{(0,\gamma)}}(F) = \infty$ but $\mathscr{H}^{h_{(\alpha,-\gamma)}}(F) = 0$ for all $\gamma \in (0, \infty)$ and all $\alpha > 0$.

The following theorems generalise Theorem 2 of the set of strings having asymptotic Kolmogorov complexity $\underline{\kappa}(\xi) \le p_0$.

Let $h_{(p_0,\ldots,p_k)}$ be a function of the logarithmic scale. We define its first logarithmic truncation as $\beta_h(t) := -\log_r h_{(p_0,\ldots,p_{k-1})}$. Observe that $\beta_h(r^{-n}) = p_0 \cdot n + \sum_{i=1}^{k-1} p_i \cdot \log_r^i n$ and $-\log_r h_{(p_0,\ldots,p_k)}(r^{-n}) = \beta_h(r^{-n}) + p_k \cdot \log_r^k n$, for sufficiently large $n \in \mathbb{N}$.

Then from Corollary 2.2 we obtain the following result.

Theorem 14 ([Mie10]) *Let $k > 0$, $(p_0, \ldots, p_k)$ be a $(k+1)$-tuple and $h_{(p_0,\ldots,p_k)}$ be a function of the logarithmic scale. Then*

$$\dim_{\mathrm{H}}^{(k)} \left\{\xi : \xi \in X^\omega \wedge \liminf_{n\to\infty} \tfrac{\mathrm{KA}(\xi\upharpoonright n) - \beta_h(2^{-n})}{\log_r^k n} < p_k\right\} \le (p_0, \ldots, p_k).$$

Proof. From $\liminf_{n\to\infty} \frac{\mathrm{KA}(\xi\upharpoonright n) - \beta_h(2^{-n})}{\log_r^k n} < p_k$ follows $\mathrm{KA}(\xi\upharpoonright n) \le \beta_h(2^{-n}) + p'_k \cdot \log_r^k n + O(1)$ for some $p'_k < p_k$. Thus $h_{(p_0,\ldots,p'_k)} \prec h_{(p_0,\ldots,p_k)}$ and the assertion follows from Corollary 2.2. ❏

Using Theorem 6 we obtain a partial converse to Theorem 14 slightly refining Satz 4.11 of [Mie10].

Theorem 15 *Let $k > 0$, $(p_0, \ldots, p_k)$ be a $(k+1)$-tuple where $p_0 > 0$ and $p_0, \ldots, p_{k-1}$ are computable numbers. Then for the function $h_{(p_0,\ldots,p_k)}$ it holds*

$$\dim_{\mathrm{H}}^{(k)} \left\{\xi : \xi \in X^\omega \wedge \limsup_{n\to\infty} \frac{\mathrm{KA}(\xi\upharpoonright n) - \beta_h(2^{-n})}{\log_r^k n} \le p_k\right\} = (p_0, \ldots, p_k).$$

Proof. Let $p'_k < p_k$ be a computable number. Then $h_{(p_0,\dots,p'_k)}$ is a computable gauge function, $h_{(p_0,\dots,p'_k)} \prec h_{(p_0,\dots,p_k)}$ and $\mathscr{H}^h(\{\xi : \mathrm{KA}(\xi \restriction n) \le -\log_r h(r^{-n}) + c_h\}) > 0$ for $h = h_{(p_0,\dots,p'_k)}$ and some constant c_h. Moreover the relation $\mathrm{KA}(\xi \restriction n) \le -\log_r h(r^{-n}) + c_h$ implies $\limsup_{n\to\infty} \frac{\mathrm{KA}(\xi\restriction n)-\beta_h(2^{-n})}{\log_r^k n} \le p_k$.

Thus $\dim_{\mathrm{H}}^{(k)} \left\{\xi : \xi \in X^\omega \wedge \limsup_{n\to\infty} \frac{\mathrm{KA}(\xi\restriction n)-\beta_h(2^{-n})}{\log_r^k n} \le p_k\right\} \ge (p_0,\dots,p'_k)$.

As p'_k can be made arbitrarily close to p_k the assertion follows. ❑

It would be desirable to prove Theorem 6 for arbitrary gauge functions or Theorem 15 for arbitrary $(k+1)$-tuples. One obstacle is that, in contrast to the case of real number dimension where the computable numbers are dense in the reals, already the computable pairs (p_0, p_1) are not dense in the above mentioned lexicographical order of pairs. This can be verified by the following fact.

Remark 3 Let $p_0 \in (0,1)$. If $r^{-p_0\cdot n} \le h(r^{-n}) \le n \cdot r^{-p_0\cdot n}$ for a computable function $h : \mathbb{Q} \to \mathbb{R}$ and sufficiently large $n \in \mathbb{N}$. Then p_0 is a computable real. Thus, if p_0 is not a computable number, the interval between $h_{(p_0,0)}$ and $h_{(p_0,1)}$ does not contain a computable gauge function.

References

[Cal02] Cristian S. Calude. *Information and randomness.* Texts in Theoretical Computer Science. An EATCS Series. Springer-Verlag, Berlin, second edition, 2002.

[CH94] Jin-Yi Cai and Juris Hartmanis. On Hausdorff and topological dimensions of the Kolmogorov complexity of the real line. *J. Comput. System Sci.*, 49(3):605–619, 1994.

[CHS11] Cristian S. Calude, Nicholas J. Hay, and Frank Stephan. Representation of left-computable ε-random reals. *J. Comput. System Sci.*, 77(4):812–819, 2011.

[CST06] Cristian S. Calude, Ludwig Staiger, and Sebastiaan A. Terwijn. On partial randomness. *Ann. Pure Appl. Logic*, 138(1-3):20–30, 2006.

[Dal74] Robert P. Daley. The extent and density of sequences within the minimal-program complexity hierarchies. *J. Comput. System Sci.*, 9:151–163, 1974.

[DH10] Rodney G. Downey and Denis R. Hirschfeldt. *Algorithmic Randomness and Complexity*. Theory and Applications of Computability. Springer-Verlag, New York, 2010.

[Edg08] Gerald Edgar. *Measure, topology, and fractal geometry*. Undergraduate Texts in Mathematics. Springer, New York, second edition, 2008.

[Egg50] Harold G. Eggleston. A property of Hausdorff measure. *Duke Math. J.*, 17:491–498, 1950.

[Egg51] Harold G. Eggleston. Correction to "A property of Hausdorff measure.". *Duke Math. J.*, 18:593, 1951.

[Fal90] Kenneth Falconer. *Fractal geometry*. John Wiley & Sons Ltd., Chichester, 1990.

[GMW88] Siegfried Graf, R. Daniel Mauldin, and Stanley C. Williams. The exact Hausdorff dimension in random recursive constructions. *Mem. Amer. Math. Soc.*, 71(381):x+121, 1988.

[Hau18] Felix Hausdorff. Dimension und äußeres Maß. *Math. Ann.*, 79(1-2):157–179, 1918.

[Lut00] Jack H. Lutz. Gales and the constructive dimension of individual sequences. In Ugo Montanari, José D. P. Rolim, and Emo Welzl, editors, *Automata, languages and programming*, volume 1853 of *Lecture Notes in Comput. Sci.*, pages 902–913. Springer, Berlin, 2000.

[Lut03a] Jack H. Lutz. Dimension in complexity classes. *SIAM J. Comput.*, 32(5):1236–1259, 2003.

[Lut03b] Jack H. Lutz. The dimensions of individual strings and sequences. *Inform. and Comput.*, 187(1):49–79, 2003.

[LV93] Ming Li and Paul M. B. Vitányi. *An introduction to Kolmogorov complexity and its applications*. Texts and Monographs in Computer Science. Springer, 1993.

[May02] Elvira Mayordomo. A Kolmogorov complexity characterization of constructive Hausdorff dimension. *Inf. Process. Lett.*, 84(1):1–3, 2002.

[MGW87] R. Daniel Mauldin, Siegfried Graf, and Stanley C. Williams. Exact Hausdorff dimension in random recursive constructions. *Proc. Nat. Acad. Sci. U.S.A.*, 84(12):3959–3961, 1987.

[Mie08] Jöran Mielke. Refined bounds on Kolmogorov complexity for ω-languages. *Electr. Notes Theor. Comput. Sci.*, 221:181–189, 2008.

[Mie10] Jöran Mielke. *Verfeinerung der Hausdorff-Dimension und Komplexität von ω-Sprachen.* PhD thesis, Martin-Luther-Universität Halle-Wittenberg, 2010. [in German].

[MM09] R. Daniel Mauldin and Alexander P. McLinden. Random closed sets viewed as random recursions. *Arch. Math. Logic*, 48(3-4):257–263, 2009.

[Rei04] Jan Reimann. *Computability and Fractal Dimension.* PhD thesis, Ruprecht-Karls-Universität Heidelberg, 2004.

[Rog98] Claude A. Rogers. *Hausdorff measures.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1998.

[RS06] Jan Reimann and Frank Stephan. Hierarchies of randomness tests. In S. S. Goncharov, R. Downey, and H. Ono, editors, *Mathematical logic in Asia*, pages 215–232. World Scientific, Hackensack, NJ, 2006.

[Rya84] Boris Ya. Ryabko. Coding of combinatorial sources and Hausdorff dimension. *Dokl. Akad. Nauk SSSR*, 277(5):1066–1070, 1984.

[Rya86] Boris Ya. Ryabko. Noise-free coding of combinatorial sources, Hausdorff dimension and Kolmogorov complexity. *Problemy Peredachi Informatsii*, 22(3):16–26, 1986.

[Sch71] Claus-Peter Schnorr. *Zufälligkeit und Wahrscheinlichkeit. Eine algorithmische Begründung der Wahrscheinlichkeitstheorie.* Lecture Notes in Mathematics, Vol. 218. Springer-Verlag, Berlin, 1971.

[Sta93] Ludwig Staiger. Kolmogorov complexity and Hausdorff dimension. *Inform. and Comput.*, 103(2):159–194, 1993.

[Sta98] Ludwig Staiger. A tight upper bound on Kolmogorov complexity and uniformly optimal prediction. *Theory Comput. Syst.*, 31(3):215–229, 1998.

[Sta05] Ludwig Staiger. Constructive dimension equals Kolmogorov complexity. *Inform. Process. Lett.*, 93(3):149–153, 2005.

[Sta07] Ludwig Staiger. The Kolmogorov complexity of infinite words. *Theoret. Comput. Sci.*, 383(2-3):187–199, 2007.

[Sta08] Ludwig Staiger. On oscillation-free ε-random sequences. *Electr. Notes Theor. Comput. Sci.*, 221:287–297, 2008.

[Sta11] Ludwig Staiger. Constructive dimension and Hausdorff dimension: The case of exact dimension. In Olaf Owe, Martin Steffen, and Jan Arne Telle, editors, *Fundamentals of Computation Theory*, volume 6914 of *Lecture Notes in Computer Science*, pages 252–263, Heidelberg, 2011. Springer.

[Sta12] Ludwig Staiger. A correspondence principle for exact constructive dimension. In S. Barry Cooper, Anuj Dawar, and Benedikt Löwe, editors, *Computability in Europe*, volume 7318 of *Lecture Notes in Computer Science*, pages 686–695, Heidelberg, 2012. Springer.

[Sta16] Ludwig Staiger. Bounds on the Kolmogorov complexity function for infinite words. In Mark Burgin and Cristian S. Calude, editors, *Information and Complexity*, Singapore, 2016. World Scientific. to appear.

[Tad02] Kohtaro Tadaki. A generalization of Chaitin's halting probability Ω and halting self-similar sets. *Hokkaido Math. J.*, 31(1):219–253, 2002.

[Ter04] Sebastiaan A. Terwijn. Complexity and randomness. *Rend. Semin. Mat., Torino*, 62(1):1–37, 2004.

[US96] Vladimir A. Uspensky and Alexander Shen. Relations between varieties of Kolmogorov complexities. *Math. Systems Theory*, 29(3):271–292, 1996.

[Usp92] Vladimir A. Uspensky. Complexity and entropy: an introduction to the theory of Kolmogorov complexity. In Osamu Watanabe, editor, *Kolmogorov complexity and computational complexity*, EATCS Monogr. Theoret. Comput. Sci., pages 85–102. Springer, Berlin, 1992.

[USS90] V. A. Uspenskiĭ, A. L. Semenov, and A. Kh. Sheń. Can an (individual) sequence of zeros and ones be random? *Uspekhi Mat. Nauk*, 45(1(271)):105–162, 222, 1990.

[ZL70] Alexander K. Zvonkin and Leonid A. Levin. The complexity of finite objects and the basing of the concepts of information and randomness on the theory of algorithms. *Uspehi Mat. Nauk*, 25(6(156)):85–127, 1970.