The Topology of Incidence Pseudographs

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Abstract

An incidence pseudograph models a (reflexive and symmetric) incidence relation between sets of various dimensions, contained in a countable family. Work by Klaus Voss in 1993 suggested that this general discrete model allows to introduce a topology, and some authors have done some studies into this direction in the past. This paper provides a comprehensive overview about the topology of incidence pseudographs. This topology has various applications, such as in modeling basic data in 2D or 3D digital picture analysis. This paper addresses especially also partially open sets which occur in common (non-binary) picture analysis.

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1 Introduction

An incidence pseudograph $[S, I, \dim]$ models a (reflexive and symmetric) incidence relation I between sets c of dimension $\dim(c) \geq 0$, contained in a countable family S. (Relation I represents the symmetric completion of the *subset-of*-relationship.) This very general discrete model allows to introduce a topology, and to derive combinatorial formulas assuming some kind of regularity for the underlying geometry of *cells* $c \in S$. Obviously, the generality of this model allows for applications in a wide range of situations.

For example, digital (2D or 3D) pictures may be considered to be substructures of a regular orthogonal grid in (2D or 3D) space, and S would be a set of m-cells c (i.e., $\dim(c) = m$ with $0 \leq m \leq 3$) in this case; a pixel is a 2-cell, a voxel is a 3-cell, two pixels are *vertex-adjacent* if they are both incident with the same 0-cell, two voxels are *face-connected* iff they are both incident with the same 2-cell, and so forth.

The book (3) decided for the model of incidence pseudographs for discussing the underlying digital topology of 2D or 3D digital pictures. (Equivalently, also some model of cell complexes could have been used; however, graphs might be seen as an even more abstract model compared to, for example, families of cells.)

Incidence pseudographs have been introduced in (6) for discussing combinatorial properties of sets of pixels or voxels, considered to be grid points (note: not cells!) in 2D or 3D regular orthogonal grids. [Applying the topological discussion of (3), finite incidence pseudographs as considered in (6) are *open sets*.] Historic origins of incidence pseudographs may be seen in work published in (1) or reported in (2).

For equivalent approaches of modeling the topology of digital pictures, using abstract complexes (5), see, for example, (4). Let $S, I, \dim]$ be an incidence pseudograph. We define that $c < c'$ iff

$$c' \in I(c),\ c \neq c',\ \text{and}\ \dim(c) < \dim(c')$$

Let $c \leq c'$ iff $c < c'$ or $c = c'$. It follows that $[S, \leq, \dim]$ is an abstract complex [see page 223 in (3)]. For example, see Figure 1, for 2-cells, 1-cells, and 0-cells. The sketch in this figure indicates a partition of the digital plane into those cells, and in case of more than two values in a digital picture, one of those defines non-open and non-closed regions, which will be studied as partially open in this paper. (For example, in the sketch on the right, black regions are open, there is one closed gray region, and one partially-open white region.

This paper recalls the discussion of topological subjects of incidence pseudographs as given in (3)] in a brief but concise form, and extends it then into a much more detailed analysis of topological properties of incidence pseudographs.

The paper is structured as follows: Section 2 introduces into incidence pseudographs. Sections 3 and 4 introduce the auxiliary notions of a descendence path

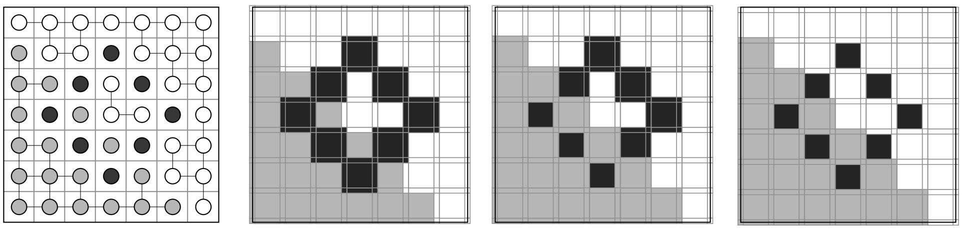

Fig. 1. Figure 5.21 in (3). The three-valued digital picture on the left is shown in three alternative topological interpretations. From left to right: black is 8-connected (forming a closed set), and gray is 4-connected (forming an open set), then gray is closed and white is open, and, finally, gray is again closed, but black is open.

and the rooted set, respectively. Section 5 introduces components and regions; subjects of major interest in this study. Section 6 then finally defines the topology by introducing open and closed sets. Section 7 shows that there is a unique topological closure for any finite set which has a connected nonempty core. Open, closed and complete sets are studied in Section 8. Section 9 shows that there is also a smallest open set containing a given set. Section 10 discusses a more technical concept (of 0-rooted sets), which is then applied in Section 11 for studying so-called 0-components and 0-regions. Section 12 concludes this paper.

2 Incidence Pseudographs

An *incidence structure* $[S, I, \dim]$ is defined by a countable set S of nodes, an *incidence relation* I on S that is reflexive and symmetric, and a function dim defined on S into a finite set $\{0, 1, \ldots, n\}$ of natural numbers.

Let $G = [S, I, \dim]$ be an incidence structure. If n is the maximum of the range of dim, then we call G an *n-incidence structure* and say that $ind(G) = n$. A node $c \in S$ is called an *i-cell* if $\dim(c) = i$ and if $i = n$ we also say c is a *principal node* otherwise we say c is a *marginal node* of G. The set of all principal nodes of G is called the *core of* G, written $core(G)$, or $core(M)$ for $M \subseteq G$.

Two nodes $p, q \in S$ are *connected wrt* $M \subseteq S$ iff there exists a finite sequence $\{p_0, \ldots, p_n\}$ where

$p = p_0$ and $q = p_n$,
$(\forall i \in \{0, \ldots, n\}\ p_i \in M) \vee (\forall i \in \{0, \ldots, n\}\ p_i \in \overline{M})$, and
$\forall i \in \{0, \ldots, n-1\}\ p_i \in I(p_{i+1})$.

The sequence $\{p_0, \ldots, p_n\}$ is called a *path from* p *to* q. If also $p_i \in M$, for all $i \in \{0, \ldots, n\}$, we say that p and q are *connected in* M. We say that p and q are *connected* if they are connected in S. For $p \in S$, we define the relation

$$\Gamma_M(p) = \{(p, q) : p \text{ and } q \text{ are connected wrt } M\}$$

If $\Gamma_M(p) \subseteq M$, it is called an *I-component* of M; otherwise it is called a *complementary I-component* of M. $A \subseteq M \subseteq S$ is *connected wrt* M iff all $p, q \in A$ are connected wrt M. We say that A is *connected* if A is connected wrt S.

Definition 1 An incidence structure $G = [S, I, \dim]$ is called an *incidence pseudograph* iff it has the following properties:

(1) For all $c \in S$, $I(c)$ is finite.
(2) The core of G is connected.

(3) Any finite set of principal nodes of G has at most one infinite complementary I-component of principal nodes.
(4) If $c' \in I(c), c' \neq c$, then $\dim(c) \neq \dim(c')$.
(5) Each marginal node of G is incident with at least one principal node of G.

G is said to be *monotonic* provided

(6) If $c' \in I(c), c'' \in I(c')$ and $\dim(c) \leq \dim(c') \leq \dim(c'')$ implies $c'' \in I(c)$.

Unless stated otherwise $G = [S, I, \dim]$ is assumed to be an incidence pseudograph. – For $i \in \mathbb{N}$ and $c \in S$, we define

$$I_i(c) = \{c' \in I(c) : \dim(c') = i\}$$
$$G_i(c) = \{c' \in I(c) : \dim(c') \geq i\}$$
$$G(c) = \{c' \in I(c) : \dim(c') > \dim(c)\}$$

We state four direct conclusions [with proofs].

(i) If $i > j$, then $G_i(c) \subseteq G_j(c)$. [Assume $i > j$ and $c' \in G_i(c)$. Thus $c' \in I(c) \wedge \dim(c') \geq i$ which implies $c' \in I(c) \wedge \dim(c') \geq j$. Hence $c' \in G_j(c)$ and therefore $G_i(c) \subseteq G_j(c)$.]

(ii) If $i \leq ind(G)$, then $G_i(c) \neq \emptyset$. [Assume $i \leq ind(G)$ and let $c \in S$. There exists $p \in core(S) \cap I(c)$. Since $\dim(p) = ind(G) \geq i,\ p \in G_i(c)$.]

(iii) If $\dim(c) < ind(G)$, then $G(c) \neq \emptyset$ [Assume $\dim(c) < ind(G)$ and let $i = \dim(c)$. From Property (ii), we have $G(c) = G_{i+1} \neq \emptyset$.]

(iv) If $i = \dim(c)$, then $G(c) = G_{i+1}(c)$. [Assume $i = \dim(c)$ and note that $G(c) = \{c' \in I(c) : \dim(c') > \dim(c)\} = \{c' \in I(c) : \dim(c') \geq i + 1\} = G_{i+1}(c)$.]

The following was not yet defined this way in (3), and will prove to be useful:

Definition 2 For $M \subseteq S, n = ind(G)$. For $0 \leq i \leq n$ we define M_i^+ recursively by

$$M_n^+ = M$$
$$M_{i-1}^+ = M_i^+ \cup \{c \in S : \dim(c) = i - 1 \wedge \emptyset \neq G(c) \subseteq M_i^+\}$$

We define $M^+ = M_0^+$; M is *complete* iff $M = M^+$

Lemma 3 *If $n = ind(G)$ and $M \subseteq S$, then*

(1) For $0 \leq i \leq j \leq n, M_i^+ \supseteq M_j^+$.
(2) $M_0^+ = \bigcup_{i=0}^n M_i^+$
(3) If $0 \leq i < n$, then $c \in M_i^+ \backslash M_{i+1}^+ \iff \dim(c) = i \wedge \emptyset \neq G(c) \subseteq M_{i+1}^+ \wedge c \notin M$.

(4) If $i = \dim(c) \wedge c \in M^+\backslash M$, *then* $c \in M_i^+ \wedge \emptyset \neq G(c) \subseteq M_{i+1}^+$.

PROOF. Property (1) follows immediately from the definition: $M_{i-1}^+ = M_i^+ \cup \{c \in S : \dim(c) = i-1\}$ and $G_i(c) \neq \emptyset$. Property (2) follows from $0 \leq i \leq j \leq n, M_i^+ \supseteq M_j^+$. Property (3) follows immediately from the definition.

To prove Property (4), let $0 \leq i < n$ and assume $i = \dim(c)$ and $c \in M_0\backslash M$ and let k be largest such that $c \in M_k^+$. Since $c \notin M = M_n^+$, we have $k < n$ and $c \in M_k^+\backslash M_{k+1}^+$ and thus $k = i \wedge \emptyset \neq G_i(c) \subseteq M_{i+1}^+$. □

Theorem 4 *For* $M \subseteq S$, M^+ *is the smallest subset of* S *satisfying:*
(1) $M \subseteq M^+$.
(2) If $\emptyset \neq G(c) \subseteq M^+$, *then* $c \in M^+$.

PROOF. Let $n = ind(G)$. Property (1) follows from the fact that $M = M_n^+ \subseteq M_n^+ = M^+$. To prove (2), assume $\emptyset \neq G(c) \subseteq M^+$. If $c \in M = M_0^+$, then $c \in M^+$ so assume $c \notin M$. Let $i = \dim(c)$. Thus, by Lemma 3, $\emptyset \neq G(c) \subseteq M_{i+1}^+ \wedge c \in M_i^+\backslash M_{i+1}^+$. Hence $c \in M^+$. Therefore M^+ satisfies Properties (1) and (2).

Suppose C satisfies Properties (1) and (2). Let $c \in M^+$. If $c \in M$, then, by Property (2), $c \in C$. Assume $c \in M^+\backslash M$. Thus, by definition and Lemma 3, $c \in M_i^+\backslash M_{i+1}$ where $i = \dim(c)$. Thus $c \in M_i^+\backslash M$ where $i = \dim(c)$. We claim this is sufficient to insure $c \in C$.

Let $\mathbb{P}(i)$ be the statement "If $\dim(c) = i \wedge c \in M_i^+\backslash M$, then $c \in C$". Let $n = ind(G)$. Since $M_n^+ = M$ and C satisfies Property (1), $\mathbb{P}(n)$ is true.

Assume $\mathbb{P}(j)$ is true for all j such that $i \leq j \leq n$ for some i such that $0 < i \leq n$ and let $\dim(c) = i-1$ and $c \in M_{i-1}^+\backslash M$. Thus $\emptyset \neq G(c) \subseteq M_i^+$. Let $c' \in G(c)$ and $k = \dim(c')$. Thus $c' \in M^+$ and $k \geq \dim(c) + 1 = i$. If $c' \in M$ then $c' \in C$ so assume $c' \notin M$. It follows that $c' \in M_k^+\backslash M$. Since $i \leq k \leq n$, by assumption, $\mathbb{P}(k)$ is true and hence $c' \in C$. Thus $\emptyset \neq G(c) \subseteq C$. Since C satisfies Property (2), $c \in C$. Therefore $M^+ \subseteq C$. □

3 Descendence Paths

For $M \subseteq S$ the *complement* of M is defined as $\overline{M} = M\backslash S$. A sequence of nodes $\{p_0, \ldots, p_k\}$ is called a *descendence path (from* p_0 *to* p_k*)* iff, for all $i \in \{0, \ldots, k-1\}$, $\dim(p_{i+1}) > \dim(p_i)$ and $p_{i+1} \in I(p_i)$. A descendence path $\{p_0, \ldots, p_k\}$ is called a *descendence path wrt* M *(from* p_0 *to* p_k*)* iff for $0 \leq i < k$, $p_i \notin M$ and $p_k \in S$. For $M \subseteq S, i \in \mathbb{N}$ define

$$\mathbb{C}(M, i) = \{c \in S : \exists \text{ descendence path } \{p_0, \ldots, p_i\} \text{ with } c = p_0 \wedge p_i \in M\}$$
$$\mathbb{C}(M) = \cup_{i=0}^{\infty} \mathbb{C}(M, i)$$

We say c' is a *descendent of* c iff there exists a path $\{ p_0, \dots, p_k\}$, $k \geq 0$, $c = p_0$, $c' = p_k$, and for all $i \in \{0, \dots, k-1\}$, $\dim(p_{i+1}) > \dim(p_i)$ and $p_{i+1} \in I(p_i)$. Let

$$D(c) = \{c' : c' \text{ is a descendent of } c\}$$

For $c \in S$, we define $D^M(c) = \{c' : \exists \text{ descendence path wrt } M \text{ from } c \text{ to } c' \}$.

Proposition 5 *If* $c' \in I(c) \wedge \dim(c') > \dim(c)$, *then* $D(c') \subseteq D(c)$.

PROOF. Let $b \in D(c')$. Thus there exists a descendence path $\{p_0, \dots, p_k\}$ from c' to b. Define $s_0 = c$ and $s_i = p_{i-1}$ for $1 \leq i \leq k+1$. Then $\{s_0, \dots, s_{k+1}\}$ is a descendence path from c to b. Thus $b \in D(c)$. Therefore $D(c') \subseteq D(c)$. □

Proposition 6 $D(c) = \bigcup\{D(c') : c' \in I(c) \wedge \dim(c') > \dim(c)\}$.

PROOF. Let $b \in D(c)$. Thus there exists a descendence path $\{p_0, \dots, p_k\}$ from c to b. Note that $b \in D(p_1), p_1 \in I(c), \dim(p_1) > \dim(c)$ since $c = p_0$. Therefore $D(c) \subseteq \bigcup\{D(c') : c' \in I(c) \wedge \dim(c') > \dim(c)\}$.

Let $b \in \bigcup\{D(c') : c' \in I(c) \wedge \dim(c') > \dim(c)\}$. There exists $c' \in I(c), \dim(c') > \dim(c)$, and $b \in D(c')$. Thus there is a descendence path $\{p_0, \dots, p_k\}$ from c' to b. Define $s_0 = c$ and $s_i = p_{i-1}$ for $1 \leq i \leq k+1$. Then $\{s_0, \dots, s_{k+1}\}$ is a descendence path from c to b and hence $b \in D(c)$. □

For $0 \leq i \leq ind(G)$, we define $D_i(c) = \{c' \in D(c) : \dim(c) = i\}$ and $D_i^M(c) = \{c' \in D^M(c) : \dim(c) = i\}$.

Proposition 7 *If* $A = M \cup \{c \in S \backslash M : D_n(c) \subseteq M\}$, $n = ind(G)$, *and if* $\{p_0, \dots, p_k\}$ *is a descendence path for which there exists a* $p_i \in A \backslash M$, *then* $p_j \in A$, *for all* j, $i \leq j \leq k$.

PROOF. Assume $\{p_0, \dots, p_k\}$ is a descendence path such that there exists a $p_i \in A \backslash M$. Let $i \leq j \leq k$. We need to show $D_n(p_j) \subseteq M$. If $j = i$ then we have $p_j \in A \backslash M$ and so $D_n(p_j) \subseteq M$. Assume $i < j$. Let $s \in D_n(p_j)$ Thus there exists a descendence path $\{s_0, \dots, s_m\}$ from p_j to s, with $\dim(s) = n$. For $0 \leq q \leq m - i + j$, define

$$t_q = \begin{cases} p_{i+q} & 0 \leq q \leq j - i \\ s_{q+i-j} & j - i < q \leq m - i + j \end{cases}$$

Then $\{t_0, \dots, t_{m-i+j}\}$ is a descendence path from p_i to s and hence $s \in D_n(p_i) \subseteq M$. Therefore $D_n(p_j) \subseteq M$ and thus $p_j \in A$. □

Proposition 8 *If* $c \notin core(S) \wedge D^M(c) \subseteq M$, *then* $c \in M^+$.

PROOF. Assume $c \notin core(S) \wedge D^M(c) \subseteq M$ and suppose $c \notin M^+$. Let $i = \dim(c)$. Since $c \notin M^+, c \notin M_i$. Thus $\emptyset = G(c)$ or $G(c) \not\subseteq M_{i+1}$. Since $c \notin core(S)$ we have $G(c) \neq \emptyset$. Thus there exists $c' \in G(c) \backslash M_{i+1}$. So $c' \in I(c), \dim(c') \geq i + 1 > \dim(c)$. $c' \notin M_{i+1}$ implies $c' \notin M$. Let $p_0 = c, p_1 = c'$, then $\{p_0, p_1\}$ is a descendence path from c to c', $p_i \notin M$ for $0 \leq i < 1$ Therefore $c' \in D^M(c) \backslash M$. This implies $D^M(c) \nsubseteq M$. Therefore $c \in M^+$. $\square$

Note that, if $p_i \notin M$ for all i satisfying $0 \leq i \leq k$, then $p_k \in D^M(c)$. Thus we have:

Corollary 9 *If $c \in M^+$ and $\{p_0, \ldots, p_k\}$ is a descendence path with $c = p_0$ and $p_k \notin M^+$, then there exists i, $0 \leq i \leq k$, with $p_i \in M$.*

Proposition 10 *If $core(M) = \emptyset$, then $M^+ = M$.*

PROOF. Suppose there exists a $c \in M^+ \backslash M$. Let $i = \dim(c)$ thus $c \in M_i^+ \backslash M_{i+1}^+$ and $\emptyset \neq G(c) \subseteq M_{i+1}$. There exists a principal node $p \in I(c)$. Note that $\dim(p) > i = \dim(c)$ and $p \in I(c)$ Therefore $p \in G(c) \subseteq M_{i+1}$ which implies that $p \in M^+$. Thus $p \in core(M)$ and hence $core(M) \neq \emptyset$. $\square$

4 Rooted Sets

A node $c \in M$ is said to be *rooted in M* iff c is incident to a principal node of M, otherwise c is said to be *unrooted in M*. $Rooted(c)$ is the set of rooted nodes in M. $Unrooted(c)$ is the set of unrooted nodes in M. If $M = Rooted(M)$, M is said to be *rooted*. – Immediately from the definitions, we have the following:

$$Rooted(M) = \{c \in M : core(M) \cap I(c) \neq \emptyset\}$$
$$Unrooted(M) = M \backslash Rooted(M) = \{c \in M : core(M) \cap I(c) = \emptyset\}$$

Proposition 11 *If $c \in M^+ \backslash M$ and $p \in core(S) \cap I(c)$, then $p \in M$.*

PROOF. Let $c \in M^+ \backslash M$ and $p \in core(S) \cap I(c)$. Let $i = \dim(c)$. Thus $c \in M_i^+ \backslash M_{i+1}^+$ and $\emptyset \neq G(c) \subseteq M_{i+1}$. We have $p \in I(c) \cap \dim(p) > \dim(c) = i$ so $p \in G(c)$ and therefore $p \in M_{i+1} \subseteq M^+$. Thus $p \in core(M^+) = core(M)$. Therefore $p \in M$. $\square$

Proposition 12 *If $c \in M^+ \backslash M$, then $core(M) \cap I(c) \neq \emptyset$*

PROOF. Let $c \in M^+ \backslash M$. Since G is an incidence pseudograph, $\exists p \in core(S) \cap I(c)$. By Proposition 11 this implies $p \in M$ and since p is a principal node, that $p \in core(M)$. Hence $core(M) \cap I(c) \neq \emptyset$. $\square$

Proposition 13 *If M is rooted, then M^+ is rooted.*

PROOF. Assume M is rooted and let $c \in M^+$. If $c \in M$, then $core(M) \cap I(c) \neq \emptyset$. Otherwise, by Proposition 11, $core(M) \cap I(c) \neq \emptyset$. Therefore $M+$ is rooted. $\square$

Proposition 14 *If $M \neq \emptyset$ and M is rooted, then $core(M) \neq \emptyset$.*

PROOF. Assume $M \neq \emptyset$ and M is rooted and let $c \in M$. Since M is rooted there exists $p \in core(M) \cap I(c)$. $\square$

Proposition 15 *If M is complete, then $Rooted(M)$ is complete and rooted.*

PROOF. Let $R = Rooted(M)$. Note that $core(R) = core(M)$. Suppose $c \in R$. Thus $c \in M$ and $core(M) \cap I(c) \neq \emptyset$ and so $core(R) \cap I(c) \neq \emptyset$. Therefore R is rooted.

To show that R is complete, suppose there exists a $c \in R^+ \backslash R$. Since $R \subseteq M, R^+ \subseteq M^+$. Since M is complete we have $R^+ \subseteq M^+ = M$ and hence $c \in M \backslash R$. Thus $core(M) \cap I(c) = \emptyset$. Let $i = \dim(c)$. Since $c \in R^+ \backslash R$ we have $G(c) \subseteq R^+_{i+1}$. Since G is an incidence pseudograph, there exists a $p \in core(S) \cap I(c)$. Since M is complete and $c \in M$ we have $p \in M$. Hence $core(M) \cap I(c) \neq \emptyset$ which implies $c \in R$ but $c \notin R$. Therefore $R^+ = R$ and hence R is complete. $\square$

5 Components and Regions

If M is complete and $C \subseteq M$, then C is called a *component* of M iff

(1) The principal nodes of C form a non-empty maximal connected (wrt M) subset of the principal nodes of M.
(2) If p is a principal node of $C, c \in M,$ and $c \in I(p)$, then $c \in C$.
(3) C is complete wrt G.

$M \subseteq S$ is said to be a *component* iff M is a component of M. A *region (of M)* is a finite component (of M).

Proposition 16 *If M is complete, rooted, and $core(M)$ connected and if C is a component of M, then $C = M$.*

PROOF. We have $C \subseteq M$. Let $c \in M$. We need to show $c \in C$. Since $core(M)$ is connected and $core(C)$ is a maximal connected subset of the $core(M)$ and since $core(M)$ is connected, we must have $core(C) = core(M)$. M is rooted so there exists a $p \in core(M) \cap I(c)$. Since $p \in core(C) \cap p \in I(c)$ and since C is a component, we have $c \in C$. $\square$

Proposition 17 *If C is a component of M, then $Rooted(M)$ is a rooted component of M.*

PROOF. Let $R = Rooted(M)$. Note $R \subseteq C \subseteq M$. The $core(R) = core(C)$ is a nonempty maximal connected subset of $core(M)$ since C is a component of M.

If $c \in M, core(R) \cap I(c) \neq \emptyset$, then $core(C) \cap I(c) \neq \emptyset$ and $c \in M$. Therefore $c \in C$ since C is a component of M. Thus $c \in R$ since $c \in C$ and $core(C) \cap I(c) \neq \emptyset$.

To show R is complete, assume that there exists a $c \in R^+ \backslash R$. Since $R^+ \subseteq C^+ = C, c \in C$, and $c \notin R$ we have $core(C) \cap I(c) = \emptyset$. Since G is an incidence pseudograph, there exists a $p \in core(S) \cap I(c)$. By Proposition 11, $p \in C$. Therefore $p \in core(R) = core(C)$ and $p \in I(c)$ which contradicts $core(C) \cap I(c) = \emptyset$, since $c \notin C$. Therefore R is complete.

Let $c \in R$ which implies $c \in C$ and $core(C) \cap I(c) \neq \emptyset$. Therefore $core(R) \cap I(c) \neq \emptyset$ and so R is rooted. $\square$

Proposition 18 *If M is complete and $p \in core(M)$, then M has a unique rooted component C containing p. Furthermore $C = core(C) \cup \{c \in M : I(c) \cap core(C) \neq \emptyset\}$.*

PROOF. Let $p \in core(M)$ for M complete. Let $A = \{c \in core(M) : c$ and p are connected wrt $M\}$. A is a nonempty maximal connected subset of $core(M)$.

Let $C = A \cup \{c \in M : A \cap I(c) \neq \emptyset\}$. $C \subseteq M$. Note that $core(C) = A$. Thus $core(C)$ is a non-empty maximal connected subset of M.

To show C is complete suppose there exists a $c \in C^+ \backslash C$. Since $c \notin C$, we have $A \cap I(c) = \emptyset$. There exists a principal node $q \in I(c)$. By theorem 11, $q \in C$ which implies $core(M) \cap I(c) \neq \emptyset$. This implies that $c \in C$ which contradicts the assumption that $c \notin C$. Therefore $C = C^+$ and hence C is complete.

Let $q \in core(C), c \in M$, and $c \in I(q)$. Thus $c \in M$ and $A \cap I(c) \neq \emptyset$ which implies $c \in C$. Therefore C is a component of M containing p.

To show C is rooted let c be a marginal node of C. By the definition of C we have $core(C) \cap I(c) \neq \emptyset$ and hence C is rooted.

To show C is unique, assume R is a rooted component of M containing p. Since $core(R)$ and $core(C)$ are both maximal connected subsets of the $core(M)$ each containing p, we must have $core(C) = core(R) = A$. Let $c \in C$ and hence $c \in M$. If $c \in A$ then $c \in R$ so assume $c \notin A$ which implies $A \cap I(c) \neq \emptyset$ and hence $core(R) \cap I(c) \neq \emptyset$ which, since R is a component of M implies $c \in R$. So $C \subseteq R$. Let $c \in R$ which implies $c \in M$. Since R is rooted there exists a principal node $p \in I(c)$. Thus $A \cap I(c) \leq \emptyset$ which implies $c \in C$. Therefore $C = R$. $\square$

Theorem 19 *If M is complete and rooted, then the rooted components of M form a partition of M.*

PROOF. For each $p \in core(M)$ let C_p be the unique rooted component of M containing p. Recall that $C_p = core(C_p) \cup \{c \in M : core(C_p) \cap I(c) \neq \emptyset\}$. Let $\mathbb{A} = \{C_p : p \in core(M)\}$. Let $p, q \in core(M)$ and assume $C_p \cap C_q \neq \emptyset$. Let $c \in C_p \cap C_q$. Since C_p and C_q are rooted, there exists a $p' \in core(C_p) \cap I(c)$ and there exists a $q' \in core(C_q) \cap I(c)$. We have p, p', c, q', q is a sequence of nodes in M each connected to the next and thus p and q are connected wrt M and thus $p \in C_q$ which implies $C_p = C_q$ since the rooted components of M containing p are unique. Thus $\mathbb{A}$ consists of disjoint subsets of M.

Let $c \in M$. Since M is rooted there exists a $p \in \mathit{core}(M) \cap I(c)$ and so $c \in C_p$ which implies $c \in \bigcup \mathbb{A}$. Since $\bigcup \mathbb{A} \subseteq M$ we conclude that $M = \bigcup \mathbb{A}$. □

Theorem 20 *If C is a component of $\mathit{Rooted}(M)$, then C is a rooted component of M.*

PROOF. Let $K = \mathit{Rooted}(M)$ and let C be a component of K. Note that $\mathit{core}(K) = \mathit{core}(M)$ and thus $\mathit{core}(C)$ is a maximal connected subset of $\mathit{core}(M)$.

Assume $p \in \mathit{core}(K)$, $c \in K$, and $c \in I(p)$. Thus $\mathit{core}(M) \cap I(c) \neq \emptyset$ and hence $c \in K$. Since C is a component of K we have $c \in C$ and since C being a component of K is complete, we have C is a component of M. Since $C \subseteq \mathit{Rooted}(M)$, we have $\mathit{core}(M) \cap I(c) \neq \emptyset$, for all $c \in C$. Therefore C is a rooted component of M. □

Theorem 21 *If M is complete and not rooted, then the set consisting of $\mathit{Unrooted}(M)$ along with the rooted components of M is a partition of M.*

PROOF. Let $K = M \backslash \mathit{Unrooted}(M) = M \cap \mathit{Rooted}(M)$. By Theorem' 15, K is complete and rooted. Thus, by Theorem 19, K is partitioned by the rooted components of K. Let $\mathbb{P}$ be the collection of the rooted components of K along with the set $\mathit{Unrooted}(M)$. From Theorem 20, we have that the components of K are rooted components of M. Clearly $M = \bigcup \mathbb{P}$. We have the rooted components of M are disjoint and disjoint from $\mathit{Unrooted}(M)$. Therefore $\mathbb{P}$ partitions M. □

6 Definition of Topology; Closed and Open Sets

A node c of a set M is called an *inner node of* M iff $I(c) \subseteq M$, otherwise it is called a *border node of* M. M^{∇} is the set of inner nodes of M. δM is the set of border nodes of M and is called the *border of* M.

Definition 22 $M \subseteq S$ is said to be *closed* iff, for all $c \in M$ and for all $c' \in I(c)$ with $\dim(c') < \dim(c)$, it follows that $c' \in M$.

M is said to be *open* iff $\overline{M} = S \backslash M$ is closed.

Figure 2 shows that there exists a finite, closed (and hence complete), non-empty M which has a non-rooted component (left), and also (right) that there exists an M which is closed (and hence complete) with $M \neq \mathit{core}(M)^+$; for this, let G be defined by the diagram; let M={a,c,d}. M is closed but $\mathit{core}(M)^+ = \{a, c\} \neq M$

Figure 3 shows on the left that there exists an M which is closed (and hence complete) and M^{∇} not open; for this let S and M be defined by the diagram;

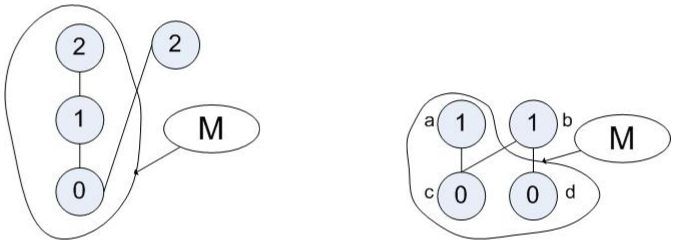

Fig. 2. Left: A finite, closed (and hence complete), non-empty M which has a non-rooted component. Right: An M which is closed (and hence complete) with $M \neq core(M)^+$.

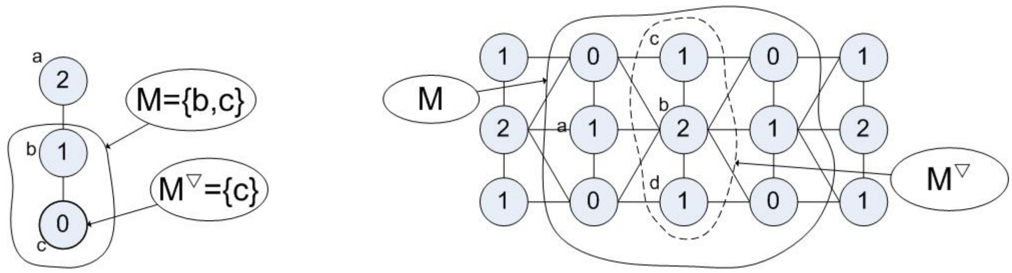

Fig. 3. Left: An M which is closed (and hence complete) and M^∇ not open. Right: An M which is closed, not open, and M^∇ not closed.

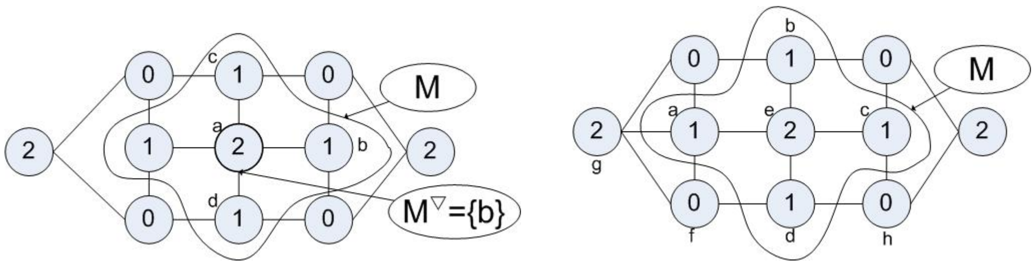

Fig. 4. Left: An M which is complete, open, not closed, and M^∇ is not complete. Right: An M which is complete and δM is not complete, not closed, and not open.

$M = \{b, c\} = M^+$, $M^\nabla = \{c\}$, and $\overline{M^\nabla} = \{a, b\}$ which is not closed since it is missing c. Therefore M^∇ is not open. The figure shows on the right that there exists an M which is closed, not open, and M^∇ not closed. For this, let S and M be defined by the diagram; $M^\nabla = \{b, c, d\}$, $b \in M^\nabla$, $\dim(a) < \dim(b)$, and $a \notin M^\nabla$. Therefore M^∇ is not closed.

Figure 4 shows on the left that there exists an M which is complete, open, not closed, and M^∇ is not complete. For this, let S and M be defined by the diagram; $M^\nabla = \{a\}$. Note $\emptyset \neq \{c' \in I(b) : \dim(c') > \dim(b)\} = a \subseteq M^\nabla$, but $b \notin M^\nabla$. – The figure shows on the right that there exists an M which is complete and δM is not complete, not closed, and not open. For this, let S and M be defined by the diagram; $\delta M = \{a, b, c, d\}$. Note $\emptyset \neq \{c' \in I(h) : \dim(c') > \dim(h)\} = \{c, d\} \subseteq \delta M$. But $h \notin \delta M$. Therefore δM is not complete and hence not closed. Furthermore, $g \in \overline{\delta M}$, $\dim(g) > \dim(a)$, and $a \notin \overline{\delta M}$. Therefore δM is not open.

Figure 5 shows on the left that there exists an M which is open but not

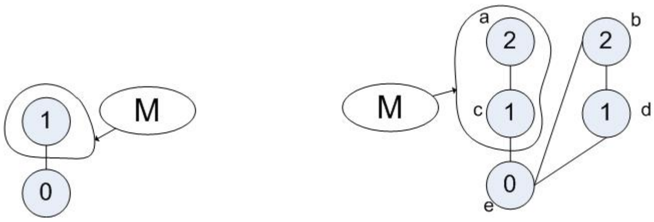

Fig. 5. Left: An M which is open but not complete. Right: A rooted, complete, finite $M \backslash \emptyset \neq core(M)$ which is connected wrt M, but $\mathbb{C}(M)$ is not a rooted component of M.

complete. For this, let S and M be defined by the diagram. Let G be such that $ind(G) \geq 1 \wedge \exists c \in S \backslash \dim(c) = 0$. Let $M = S\{c\}$. $\overline{M} = \{c\}$ is closed and thus M is open but M is not complete since for all $c' \in I(c), \dim(c') < \dim(c) \Rightarrow c' \in M$(vacuously) but $c \notin M$. – The figure shows on the right that there exists a rooted, complete, finite $M \backslash \emptyset \neq core(M)$ which is connected wrt M, but $\mathbb{C}(M)$ is not a rooted component of M. For this, let G be defined by the diagram and let $M = \{a, c\}$. M is rooted, finite, and complete. Note $M^+ = M$ and $\mathbb{C}(M) = \{a, c, e\}$. We have $e \in \mathbb{C}(M)$ but $core(\mathbb{C}(M) = \{a\}$. Thus $\mathbb{C}(M)$ is not rooted since $e \in \mathbb{C}(M)$ and $core(\mathbb{C}(M) \cap I(c) = \emptyset$.

Proposition 23 *M is open iff, for all $c \in M$ and $c' \in I(c)$ with $\dim(c') > \dim(c)$, it follows that $c' \in M$.*

PROOF. Assume M is open, $c \in M$, $c' \in I(c)$ and $\dim(c') > \dim(c)$. If $c' \in \overline{M}$ which is closed since M is open, we would have $c \in \overline{M}$. Therefore $c' \in M$.

Assume that for all $c \in M$ and $c' \in I(c)$ with $\dim(c') > \dim(c)$ it follows that $c' \in M$, and suppose $c \in \overline{M}$, $c' \in I(c)$ and $\dim(c') < \dim(c)$. If $c' \in M$ this would imply that $c \in M$. Thus $c' \in \overline{M}$. Therefore $\overline{M}$ is closed and hence M is open. □

Proposition 24 *M is closed iff $\overline{M}$ is open.*

PROOF. $\overline{\overline{M}} = M$ □

Proposition 25 *M is complete iff $\emptyset \neq G(c) \subseteq M \Rightarrow c \in M$.*

PROOF. Assume M is complete. Thus $M = M^+$. Thus, by Theorem 4,

$$\emptyset \neq \; G(c) \subseteq M^+ = M \Rightarrow c \in M.$$

Assume

$$\emptyset \neq G(c) \subseteq M \Rightarrow c \in M$$

Then M satisfies Properties (1) and (2) of Theorem 4. It follows that $M = M^+$. □

Theorem 26 *If M is closed, then M is complete.*

PROOF. Suppose M is closed. We claim $M_i^+ \subseteq M$ for $0 \leq i \leq n$ where $n = ind(G)$. Recall $M_n^+ = M$ so the claim is true for $i = n$.

Assume $M_i^+ \subseteq M$ for some $0 < i \leq n$ and let $c \in M_{i-1}^+$. Note that $\dim(c) = i-1 < n$, so $G(c) \neq \emptyset$ and thus $\emptyset \neq G(c) \subseteq M_i^+$. Hence there exists a $c' \in G(c)$. Thus $c' \in I(c)$ and $\dim(c') > \dim(c)$. By assumption $M_i^+ \subseteq M$. Hence $c' \in M$. Since M is closed, this implies $c \in M$. Therefore $M_i^+ \subseteq M$ for all i satisfying $0 < i \leq n$. This implies $M^+ = M$ and therefore M is complete. □

Theorem 27 *If M is closed, then M^∇ is complete.*

PROOF. Assume M is closed and $\emptyset \neq G(c) \subseteq M^\nabla$. To show $c \in M^\nabla$ let $b \in I(c)$. Note that there exists a $c' \in I(c)$ satisfying $\dim(c') > \dim(c)$ and $c' \in M^\nabla$. Thus $I(c') \subseteq M$. We have $c' \in I(c)$, $c \in I(c')$ and hence $c \in M$.

If $\dim(b) > \dim(c)$, then by assumption $b \in M^\nabla \subseteq M$. If $\dim(b) = \dim(c)$, then $b = c$ which implies $b \in M$. If $\dim(b) < \dim(c)$, then $b \in M$ since M is closed. □

Proposition 28 *M is open and closed iff $\delta M = \emptyset$ iff $M = M^\nabla$.*

PROOF. $\delta M = M \backslash M^\nabla$. Thus $\delta M = \emptyset$ iff $M = M^\nabla$.

Assume M is open and closed and let $c \in M$. To show $c \in M^\nabla$, let $c' \in I(c)$. If $\dim(c') > \dim(c)$, then $c' \in M$ since M is open. If $\dim(c') = \dim(c)$, then $c' = c$ and hence $c \in M$. If $\dim(c') < \dim(c)$, then $c' = c$ and hence $c \in M$ since M is closed. Therefore $M = M^\nabla$.

Assume $M = M^\nabla$ and suppose $c \in M$ satisfies $c' \in I(c)$. It follows that $I(c) \subseteq M$ since $M = M^\nabla$ and thus $c' \in M$ □

7 Topological Closure

In this section we show that any finite set $M \subseteq S$, where $\emptyset \neq core(M)$ is connected wrt M, does have a unique topologic closure.

Proposition 29 *If D is closed and $M \subseteq D$, then*

(1) for all $i \in \mathbb{N}$, $\mathbb{C}(M, i) \subseteq D$
(2) $\mathbb{C}(M) \subseteq D$

PROOF. Property (2) follows from Property (1) since $\mathbb{C}(M) = \bigcup_{i=0}^{\infty} \mathbb{C}(M, i)$.

Let $n = ind(G)$. Note that $\mathbb{C}(M, i) = \emptyset$, for all $i > n$. We prove Property (1) by induction. Since $\mathbb{C}(M, 0) = M$ we have $\mathbb{C}(M, 0) \subseteq D$.

Assume $\mathbb{C}(M, i) \subseteq D$ and let $c \in \mathbb{C}(M, i+1)$ for some $0 \leq i < n$. By definition there exists a descendence path $\{p_0, \ldots, p_{i+1}\}$ with $c = p_0$ and $p_{i+1} \in M$. For

$0 \leq j \leq i$ let $s_j = p_{j+1}$. We have $s_i = P_{i+1} \in M$ so $s_0 \in \mathbb{C}(M, i)$. By the assumption we have $s_0 \in D$. Hence $p_1 \in D$ and $p_0 \in I(p_1)$. Since D is closed, $c = p_0 \in D$. $\square$

Corollary 30 *If D is closed, $M \subseteq D$, and if $\{p_0, \ldots, p_k\}$ is a descendence path with $p_k \in M$, then $p_0 \in D$.*

Proposition 31 $\mathbb{C}(M, i+1) = \{c \in S : \exists c' \in \mathbb{C}(M, i) \cap I(c)$ *with* $\dim(c) < \dim(c')\}$

PROOF. Let $A = \{c \in S : \exists c' \in \mathbb{C}(M, i) \cap I(c) \text{ with } \dim(c) < \dim(c')\}$ and let $c \in \mathbb{C}(M, i+1)$. By definition, there exists a descendence path $\{p_0, \ldots, p_{i+1}\}$ with $c = p_0$ and $p_{i+1} \in M$.

Let $c' = p_1$. Thus, since $c = p_0 \in I(c')$ and $\dim(c) < \dim(c')$. Let $s_j = p_{j+1}$ for $0 \leq j \leq i+1$. Then, $\{s_0, \ldots, s_i\}$ is a descendence path with $c' = s_0$ and $s_i = p_{i+1} \in M$. Hence $c' \in \mathbb{C}(M, i)$ and thus $c \in A$.

Let $c \in A$. Then there exists a $c' \in \mathbb{C}(M, i)$ such that $c' \in I(c)$ and $\dim(c) < \dim(c')$. There exists a descendence path $\{p_0, \ldots, p_i\}$ such that $c' = p_0$ and $p_i \in M$. Let $s_0 = c$ and for $0 \leq j \leq i$, let $s_j = p_{j-1}$. We note that $\{s_0, \ldots, s_{i+1}\}$ is a descendence path from C to $p_i \in M$ and hence $c \in \mathbb{C}(M, i+1)$. Therefore $\mathbb{C}(M, i+1) = A$. $\square$

Proposition 32 *If M is finite, then $\mathbb{C}(M)$ is finite.*

PROOF. Assume M is finite. Thus $\mathbb{C}(M, 0) = M$ is finite. Let $n = ind(G)$ and assume $\mathbb{C}(M, i)$ is finite for $0 \leq i < n$. By Proposition 31

$$\mathbb{C}(M, i+1) = \{c \in S : \exists c' \in \mathbb{C}(M, i) \cap I(c) \wedge \dim(c) < \dim(c')\}$$

Note that if $c \in \mathbb{C}(M, i+1)$, then there exists a $c' \in \mathbb{C}(M, i)$ such that $c \in I(c')$. And thus

$$\mathbb{C}(M, i+1) \subseteq \bigcup\{I(c') : c' \in \mathbb{C}(M, i)\}$$

which is a finite union of finite sets and therefore finite. $\square$

Corollary 33 *If M is finite, then M^+ is finite.*

PROOF. $M^+ \subseteq \mathbb{C}(M)$. $\square$

Definition 34 $c \in S$ is said to be *invalid wrt M* iff $c \notin M \wedge c' \in M \wedge I(c) \neq \emptyset$. The set of all nodes invalid wrt M is called the *boundary of M*.

Proposition 35 *If G is monotonic and $M \subseteq S$, then $\mathbb{C}(M) = M \cup \mathbb{C}(M, 1)$.*

PROOF. Let $A = M \cup \mathbb{C}(M, 1) = \mathbb{C}(M, 0) \cup \mathbb{C}(M, 1)$. Let $n = ind(G)$. We will show $\mathbb{C}(M, i) \subseteq \mathbb{C}(M, 1)$ for all i, $1 \leq i \leq n$, by induction. Clearly it is true for $i = 1$.

Assume $\mathbb{C}(M, i) \subseteq \mathbb{C}(M, 1)$ and let $c \in \mathbb{C}(M, i+1)$. Thus there exists a descendence path $\{p_0, \ldots, p_k\}$ such that $c = p_0$ and $p_{i+1} \in M$. For $0 \leq$

$j \leq i$ define $s_j = p_{j+1}$. So $p_1 = s_0 \in \mathbb{C}(M, i)$. By assumption, $p_1 = c_0 \in \mathbb{C}(M, 1)$. Thus there exists a descendence path $\{t_0, t_1\}$ such that $p_1 = t_0$ and $t_1 \in M$. We have $c = p_0 \in I(p_1)$, $\dim(c) < \dim(p_1)$, $t_0 = p_0 \in I(t_1)$, and $\dim(p_1) < \dim(t_1)$. Since G is monotonic this implies $c \in I(t_1)$. Thus, for $r_0 = c, r_1 = t_1$, $\{r_0, r_1\}$ is a descendence path with $c = r_0$ and $r_1 \in M$. Therefore $c \in \mathbb{C}(M, 1)$. □

Theorem 36 $\mathbb{C}(M)$ *is the smallest closed set containing* M

PROOF. $M = \mathbb{C}(M, 0) \subseteq \mathbb{C}(M)$. To show $\mathbb{C}(M)$ is closed, let $c \in \mathbb{C}(M)$ and $c' \in I(c)$ such that $\dim(c') < \dim(c)$. Then there exists a descendence path $\{p_0, \ldots, p_k\}$ such that $c = p_0$ and $p_k \in M$. Define $s_0 = c'$ and for $1 \leq j \leq k+1$, define $s_j = p_{j-1}$. Thus $\{s_0, \ldots, s_{k+1}\}$ is a descendence path with $c' = s_0 \wedge s_{k+1} \in M$. Hence $c' \in \mathbb{C}(M)$ and therefore $\mathbb{C}(M)$ is closed.

Suppose $M \subseteq D$ and D is closed. By Proposition 29, $\mathbb{C}(M) \subseteq D$. □

Proposition 37 $\mathit{core}(M^+) = \mathit{core}(M)$

PROOF. Let $n = \mathit{ind}(G)$ and let $p \in \mathit{core}(M^+)$. Thus $\dim(p) = n$ which implies $p \in M^+_n = M$ which implies $p \in \mathit{core}(M)$. Therefore $\mathit{core}(M^+) \subseteq \mathit{core}(M)$. Since $m \subseteq M^+$, we have $\mathit{core}(M) \subseteq \mathit{core}(M^+)$. Therefore $\mathit{core}(M^+) = \mathit{core}(M)$. □

Proposition 38 $\mathit{core}(\mathbb{C}(M)) = \mathit{core}(M)$

PROOF. Let $n = \mathit{ind}(G)$. Since $M \subseteq \mathbb{C}(M)$, we have $\mathit{core}(M) \subseteq \mathit{core}(\mathbb{C}(M))$. Let $p \in \mathit{core}(\mathbb{C}(M))$. Thus there exists a descendence path $\{p_0, \ldots, p_k\}$ such that $p = p_0$ and $p_k \in m$. Since $\dim(p) = n, k = 0$. Thus $p = p_k \in M$. Therefore $\mathit{core}(\mathbb{C}(M)) = \mathit{core}(M)$. □

Proposition 39 $M^+ \subseteq \mathbb{C}(M)$

PROOF. $\mathbb{C}(M)$ is closed and therefore complete and contains M. Since M^+ is the smallest complete set containing M, $M^+ \subseteq \mathbb{C}(M)$. □

Proposition 40 $\mathbb{C}(M) = \mathbb{C}(M^+)$

PROOF. $M^+ \subseteq \mathbb{C}(M)$ and $\mathbb{C}(M)$ is closed. It follows from Proposition 36 that $\mathbb{C}(M^+) \subseteq \mathbb{C}(M)$. Since $M \subseteq M^+$, it follows that $\mathbb{C}(M) \subseteq \mathbb{C}(M^+)$ Therefore $\mathbb{C}(M) = \mathbb{C}(M^+)$. □

Proposition 41 *If* M *is finite,* $\mathit{core}(M)$ *is non-empty and connected wrt* M*, then* $\mathbb{C}(M)$ *is a closed region.*

PROOF. $\mathbb{C}(M)$ is finite since M is finite. $\mathbb{C}(M)$ is closed and therefore complete. $\mathit{core}(\mathbb{C}(M)) = \mathit{core}(M)$ is a non-empty maximal connected subset of $\mathit{core}(M)$. Suppose $p \in \mathit{core}(\mathbb{C}(M)) \cap I(c)$. If $c = p$ then $c \in \mathbb{C}(M)$ so assume $c \neq p$. We have $p \in \mathit{core}(\mathbb{C}(M))$, $c \in I(p)$ and $\dim(c) < \dim(p)$. Since $\mathbb{C}(M)$ is closed, $c \in \mathbb{C}(M)$. Therefore $\mathbb{C}(M)$ is a component. Hence $\mathbb{C}(M)$ is a closed component. □

Proposition 42 *If $G = [S, I, \dim]$ is monotonic and if M is a rooted subset of S, then $\mathbb{C}(M)$ is rooted.*

PROOF. Let c be a marginal node of $\mathbb{C}(M)$. If $c \in M$, then $core(M) \cap I(c) \neq \emptyset$ since M is rooted. So assume $c \notin M$. Since G is monotonic we have from Proposition 35 that $c \in \mathbb{C}(M, 1)$. So there exists a descendence path $\{p_0, p_1\}$ such that $c = p_0$ and $p_1 \in M$. If $p_1 \in core(M) = core(\mathbb{C}(M))$, then $p_1 \in core(\mathbb{C}(M)) \cap I(c)$. Assume $p_1 \notin core(M)$. Thus p_1 is marginal in M. Since M is rooted there exists a $p \in core(M) \cap I(p_1)$. We have $c \in I(p_1)$, $\dim(c) < \dim(p_1)$, $p_1 \in I(p)$ and $\dim(p_1) < \dim(p)$. Since G is monotonic this implies $c \in I(p)$. Thus $p \in core(M) = core(\mathbb{C}(M))$. Therefore $\mathbb{C}(M)$ is rooted. □

Proposition 43 *If $M \subseteq S$, $\emptyset \neq core(M)$ is connected wrt M, then $\mathbb{C}(M)$ is a closed component containing M.*

PROOF. Note that $M \subseteq \mathbb{C}(M)$ and $core(\mathbb{C}(M)) = core(M)$ is a non-empty maximal connected subset of $core(M)$. Assume $p \in core(\mathbb{C}(M))$, $c \in \mathbb{C}(M)$, and $p \in I(c)$. If $p = c$, then $c \in \mathbb{C}(M)$, otherwise $\dim(c) < \dim(p)$. Since $p \in \mathbb{C}(M)$ and $\mathbb{C}(M)$ we have $c \in \mathbb{C}(M)$. Thus $\mathbb{C}(M)$ is complete. Since $\mathbb{C}(M)$ is also closed, we have $\mathbb{C}(M)$ is a closed component containing M. □

Theorem 44 *If $M \subseteq S$ is finite and if $\emptyset \neq core(M)$ is connected wrt M, then $\mathbb{C}(M)$ is the unique closure of M.*

PROOF. From Propositions 43 and 32, $\mathbb{C}(M)$ is a finite closed component. Since $M \subseteq \mathbb{C}(M)$, $\mathbb{C}(M)$ is a closed region containing M.

Suppose R is a closed region containing M. Let $n = ind(G)$. To show $\mathbb{C}(M) \subseteq R$, we show $\mathbb{C}(M, i) \subseteq R$ for $0 \leq i \leq n$. Note that $\mathbb{C}(M, 0) = M \subseteq R$. Assume $\mathbb{C}(M, i) \subseteq R$ for some i, $0 \leq i < n$ and let $c \in \mathbb{C}(M, i+1)$. By Proposition 31, there exists a $c' \in \mathbb{C}(M, i)$ such that $c' \in I(c)$ and $\dim(c) < \dim(c')$. Hence $c' \in R$, $c' \in I(c)$, and $\dim(c) < \dim(c')$. Since R is closed this implies $c' \in R$. Therefore, $\mathbb{C}(M, i) \subseteq R$, for all i, $0 \leq i \leq n$, and hence $\mathbb{C}(M) = \bigcup_{i=0}^{n} \mathbb{C}(M, i) \subseteq R$. Therefore $\mathbb{C}(M)$ is the smallest closed region containing M. □

Definition 45 Suppose $M \subseteq S$ is finite and if $\emptyset \neq core(M)$ is connected wrt M, then we denote the unique (topological) *closure* of M by $M^{\bullet}$.

8 Open and Closed Regions; Complete Sets

Proposition 46 *If $M \subseteq S$ is finite then M is a closed region iff*

(1) $\emptyset \neq core(M)$ is a non-empty, maximal connected subset of $core(M)$.
(2) For all $c \in M$, $c' \in I(c)$ and $\dim(c') < \dim(c)$ it follows that $c' \in M$.

PROOF. Assume M is a closed region. Then Property (1) follows from the fact that M is a component. Property (2) follows from the fact that M is closed.

Assume M satisfies Properties (1) and (2). From Property (2), M is closed and hence complete.

Assume p is a principal node of M, $c \in I(p)$, and $c \in I(p)$. If $c = p$, then $c \in M$. So assume $s \neq p$. We have $p \in M, c \in I(p)$, and $\dim(c) < \dim(p)$. Since M is closed, this implies $c \in M$. It follows that M is a closed region. □

Proposition 47 *Let M be a finite subset of S where $G = [S, I, \dim]$ is an n-incidence pseudograph. Then*

(1) M is an open region of G iff
(1.1) $core(M)$ is non-empty and connected,
(1.2) $core(M) \cap I(c) \neq \emptyset \Rightarrow c \in M$, and
(1.3) if $\dim(c) < n$, then $c \in M \Leftrightarrow G(c) \subseteq M$.
(2) If G is monotonic then, M is an open region of G iff
(2.1) $core(M)$ is non-empty and connected, and
(2.2) if $\dim(c) < n$, then $c \in M \Leftrightarrow core(S) \cap I(c) \subseteq M$.

PROOF. Assume M is an open region of G. Properties (1.1) and (1.2) follow directly from the fact that M is a component. To prove Property (1.3), first assume $\dim(c) < n$ and $c \in M$ and $b \in G(c)$. Thus $b \in I(c)$ and $\dim(c) < \dim(b)$. Since M is open by Proposition 23, $b \in M$. Thus $G(c) \subseteq M$.

Next assume $\dim(c) < n$ and $G(c) \subseteq M$. There exists a $p \in core(S) \cap I(c)$. Thus $c \in I(p)$ and $\dim(c) < \dim(p)$ which implies $p \in G(c) \subseteq M$. Hence $p \in M$. Since M is closed this implies $c \in M$. Therefore Property (1.3) is satisfied by M.

Assume M is a region satisfying Properties (1.1), (1.2), and (1.3). To show M is open assume $c \in M$, $c' \in I(c)$, and $\dim(c') > \dim(c)$. By Property (1.3), $c' \in M$. Thus, by Proposition 23, M is open.

To show that M is complete suppose $c \in M^+ \backslash M$. Thus $G(c) \subseteq M_{i+1}$ where $i = \dim(c)$. It follows, by Property (1.3), that $c \in M$. This contradiction establishes $M^+ = M$ and hence M is complete. Thus we have M is an open region. □

Figure 6 shows that there exists an M which is finite, $core(M) \neq \emptyset$, connected, open, complete, and satisfies

$$\dim(c) < ind(G) \Rightarrow (c \in M \Leftrightarrow D(c) \subseteq M)$$

but is not a component as it fails to satisfy $core(M) \cap I(c) \neq \emptyset \Rightarrow c \in M$.

Theorem 48 *If M is closed, then $bd(M) = \emptyset$.*

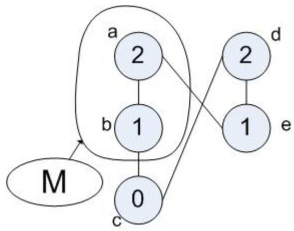

Fig. 6. A finite set M which is not a component.

PROOF. Assume M is closed and $c \in bd(M)$. Thus $c \notin M$ and there exists a $p \in core(M) \cap I(c)$. Hence $p \in I(c)$, $p \in M$, and $\dim(c) < \dim(p)$. Since M is closed this implies $c \in M$ but $c \notin M$. Therefore $bd(M) = \emptyset$. $\square$

Theorem 49 *If $bd(M) = \emptyset$, then M is complete.*

PROOF. Suppose M is not complete. Thus there exists a $c \in M^+ \backslash M$. Since there exists a principal node $p \in I(c)$, we have by Proposition 11, $p \in core(M) \cap I(c)$. Since $c \notin M$ this implies $c \in bd(M)$. But $bd(M) = \emptyset$. Therefore M is complete. $\square$

The following are some technical specifications, needed in the following auxiliary considerations.

A node c is an *upward rooted point* of a set M iff $c \in M$ and there exists a descendence path $\{p_0, \ldots, p_k\}$ with

$$c = p_0 \wedge p_k \in core(M) \wedge \forall i\,(0 \leq i \leq k \rightarrow p_i \in M)$$

The set of all upward rooted points of M is denoted by $URP(M)$.

A node c is a *downward exit point* of M iff $c \notin M$ and there exists a $c' \in M \cap I(c)$ with $\dim(c) < \dim(c')$. The set of all downward exit points of M is denoted by $DXP(M)$.

A node c is an *upward exit point* of M iff $c \notin M$ and there exists a $c' \in M \cap I(c)$ with $\dim(c) > \dim(c')$. The set of all upward exit points of M is denoted by $UXP(M)$.

Proposition 50 *M is closed iff $DXP(M) = \emptyset$.*

PROOF. Assume M is closed and suppose there exists a $c \in DXP(M)$. Thus $c \notin M$ and there exists a $c' \in M \cap I(c)$ such that $\dim(c) < \dim(c')$. Since M is closed this implies $c \in M$. This contradiction establishes $DXP(M) = \emptyset$.

Assume $DXP(M) = \emptyset$ and let $c \in M, c' \in I(c)$ such that $\dim(c') < \dim(c)$. Since $DXP(M) = \emptyset$, this implies $c' \in M$. Therefore M is closed. $\square$

Proposition 51 *M is open iff $UXP(M) = \emptyset$.*

PROOF. Assume M is open and $c \in UXP(M)$. This implies $c \notin M$ and there exists a $c' \in M \cap I(c)$ such that $\dim(c') > \dim(c)$. But M is open which implies $c \in M$. Therefore $UXP(M) = \emptyset$.

Assume $UXP(M) = \emptyset$. Let $c \in M \wedge c' \in I(c)$ such that $\dim(c') > \dim(c)$. Since $UXP(M) = \emptyset$, this implies $c' \in M$. Therefore M is open. $\square$

Theorem 52 *M is complete iff $DXP(M) \subseteq URP(\overline{M})$.*

PROOF. Assume M is complete and let $c \in DXP(M)$. Thus $c \notin M$ and there exists a vertex $b \in M \cap I(c)$ such that $\dim(c) < \dim(b)$. Thus $b \in G(c)$ and, since M is complete, $G(c) \neq \emptyset$, and $c \notin M = M^+$, we must have $G(c) \not\subseteq M$. Hence there exists a vertex $c' \in I(c) \cap \overline{M}$ such that $\dim(c') > \dim(c)$. Choose $p_0 = c$ and $p_1 = c'$.

Assume $p_0, \ldots, p_i$ have been chosen for some $i \geq 1$ satisfying for all j, $i \leq j \leq i$, $p_j \in I(p_{j-1}) \cap \overline{M}$ and $\dim(p_j) > \dim(p_{j-1})$. If $\dim(p_i) = ind(G)$, then we stop. Otherwise, since $p_i \in \overline{M}$ and M is complete we have $p_i \notin M^+$. Since $\dim(p_i) < ind(G)$, $G(p_i) \neq \emptyset$. Thus we must have $G(p_i) \not\subseteq M$. Hence there exists a $p_{i+1} \in G(p_i) \backslash M$. Then we have $p_{i+1} \in I(p_i) \cap \overline{M}$ and $\dim(p_{i+1}) > \dim(p_i)$. This process will eventually end. We choose k to be the final i and we end up with a descendence path $\{p_0, \ldots, p_k\}$ with $p_0 = c \wedge p_k \in core(\overline{M})$. Thus $c \in URP(\overline{M})$.

Assume $DXP(M) \subseteq URP(\overline{M})$. Let $n = ind(G)$ and for $0 \leq i \leq n$, consider the statement that $\mathbb{P}(i) \equiv \nexists c \in M^+ \backslash M$ with $\dim(c) = i$.

Suppose $c \in M^+ \backslash M \wedge \dim(c) = n$. Thus $c \in core(M^+) = core(M) \subseteq M$. Therefore $\mathbb{P}(n)$ is true.

Assume $\mathbb{P}(j)$ is true for all j such that $i \leq j \leq n$ for some i for $0 < i \leq n$ and suppose $c \in M^+ \backslash M$ such that $\dim(c) = i - 1$. Thus $\emptyset \neq G(c) \subseteq M_i^+$. Thus there exists a c' $\in G(c)$ such that $c' \in I(c)$, $\dim(c') > \dim(c) = i - 1$, and $c' \in M_i^+ \subseteq M^+$. By assumption, since $i \leq \dim(c') \leq n$, $c' \notin M^+ \backslash M$. Since $c' \in M^+$, we must have $c' \in M$. Hence $c \in DXP(M) \subseteq URP(\overline{M})$. Thus there exists a descendence path $\{p_0, \ldots, p_k\}$ such that $c = p_0$, $\dim(p_k) = n$, and $p_j \notin M$ for all j satisfying $0 \leq j \leq k$. Since $\dim(c) < n$, $k > 0$. Thus $p_1 \in I(c)$, $\dim(p_1) > \dim(c)$, and $p_1 \notin M$. However, $p_1 \in G(c) \subseteq M^+$ which implies $p_1 \in M^+ \backslash M$ and $i \leq \dim(p_1) \leq n$. This contradiction establishes $\mathbb{P}(i-1)$. Therefore $M^+ \backslash M = \emptyset$ and therefore M is complete. $\square$

Corollary 53 *M is open and complete iff $UXP(M) = \emptyset$ and $DXP(M) \subseteq URP(\overline{M})$*

9 Partially Open Sets

For $i \in \mathbb{N}$ and $c \in S$, we define that

$L_i(c) = \{c' \in I(c) : \dim(c') \leq i\}$
$L(c) = \{c' \in I(c) : \dim(c') < \dim(c)\}$

From these definitions it follows that $I(c) = L(c) \cup cG(c)$, and, if $\dim(c) = i$, then $L(c) = L_{i-1}(c)$. [For the latter, note that $L(c) = \{c' \in I(c) : \dim(c' < \dim(c)\} = \{c' \in (c) : \dim(c' <= i-1\} = L_{i-1}(c)$.]

Definition 54 Let $M \subseteq S$ and $n = ind(G)$. For $0 \leq i \leq n$ we define M^- recursively by

$M_0^- = M$
$M_{i+1}^- = M_i^- \cup \{c \in S : \dim(c) = i+1 \wedge \emptyset \neq L(c) \subseteq M_i^-\}$

We define $M^- = \bigcup_{i=0}^{n} M_i^-$; M is *partially open* iff $M = M^-$

Proposition 55 *If M is open, then M is partially open.*

PROOF. Let $n = ind(G)$ and suppose M is open. We claim $M_i^- \subseteq M$ for $0 \leq i \leq n$.

Since $M_0^- = M$, the claim is true for $i = 0$. Assume $M_i^- \subseteq M$ for some i, $0 \leq i < n$ and let $c \in M_{i+1}^-$. If $c \in M$ we are done so assume $c \notin M$. Thus we have $\dim(c) = i+1$ and $\emptyset \neq L(c) \subseteq M_i^-$. Thus there exists a $c' \in L(c)$ such that $c' \in M_i^-$, $c' \in I(c)$, and $\dim(c') < \dim(c)$. By assumption, $M_i^- \subseteq M$ and so $c' \in M$. Since M is open this implies $c \in M$. Therefore M is partially open. □

Lemma 56 *If $n = ind(G)$ and $M \subseteq S$, then*

(1) $M_i^- \subseteq M_j$, *for* $0 \leq i \leq j \leq n$
(2) $M_n^- = \bigcup_{i=0}^{n} M_i^-$
(3) if $0 < i \leq n$ *then* $c \in M_i^- \backslash M_{i-1}^- \Leftrightarrow \dim(c) = i \wedge \emptyset \neq L(c) \subseteq M_{i-1}^- \wedge c \notin M$
(4) if $\dim(c) = i \wedge c \in M^- \backslash M$ *then* $i > 0 \wedge c \in M_i^- \wedge \emptyset \neq L(c) \subseteq M_{i-1}^-$

PROOF. Properties (1), (2), and (3) follow immediately from the definitions. To prove Property (4) let $c \in M^- \backslash M$. Let k be the smallest natural number such that $c \in M_k^-$. Since $c \notin M = M_0^-$, $k > 0$ and $c \in M_k^- \backslash M_{k-1}^-$. Thus $i = k > 0$ and $\emptyset \neq L(c) \subseteq M_{i-1}^-$. □

Theorem 57 *For $M \subseteq S$, M^- is the smallest subset of S satisfying:*

(1) $M \subseteq M^-$
(2) if $\emptyset \neq L(c) \subseteq M^-$ *then* $c \in M^-$.

PROOF. Property (1) follows from $M = M_0 \subseteq M^-$. To prove Property (2), assume $\emptyset \neq L(c) \subseteq M^-$. If $c \in M$, then $c \in M^-$ so assume $c \notin M$. Thus, by Lemma 56, $\emptyset \neq L(c) \subseteq M_{i-1}^-$ and $c \in M_i^- \backslash M_{i-1}^-$. Hence $c \in M^-$. Therefore M satisfies Properties (1) and (2).

Suppose A satisfies Properties (1) and (2). Let $c \in M^-$. If $c \in M$, then $c \in A$ since A satisfies Property (1). Assume $c \notin M$. Thus, by Lemma 56,

$c \in M_i^- \backslash M_{i-1}^-$ and $\emptyset \neq L(c) \subseteq M_{i-1}^-$ where $i = \dim(c)$. Hence $c \in M^- \backslash M$. We claim that this is sufficient to show $c \in A$.

Let $\mathbb{P}(i)$ be the statement "If $c \in M_i^- \backslash M \wedge \dim(c) = i$, then $c \in A$". Since $M_0^- = M$, $\mathbb{P}(0)$ is true (vacuously.)

Let $n = ind(G)$ and assume $\mathbb{P}(j)$ is true for all $0 \leq j \leq i$ for some $i, 0 \leq i < n$. Let $c \in M_{i+1}^- \backslash M$ such that $\dim(c) = i+1$. It follows from Lemma 56 that $\emptyset \neq L(c) \subseteq M_i^-$. Let $c' \in L(c)$ and $k = \dim(c')$. Thus $c' \in M_i^-$ and $k \leq \dim(c) - 1 = i$. If $c' \in M$ then $c' \in A$ so assume $c' \notin M$. It follows that $c' \in M_k^- \backslash M$. Since $0 \leq k \leq i$, and the assumption $\mathbb{P}(k)$ is true, we have that $c' \in A$. Thus $\emptyset \neq L(c) \subseteq A$. Since A satisfies Property (2), $c \in A$. Hence $\mathbb{P}(i)$ is true for all $0 \leq i \leq n$. Therefore $M^- \subseteq A$. $\square$

For $M \subseteq S$, and $i \in \mathbb{N}$ we define

$$\mathbb{O}(M, i) = \{c \in S : \exists \text{ descendence path } \{p_0, \ldots, p_i\} \text{ with } c = p_0 \wedge p_0 \in M\}$$
$$\mathbb{O}(M) = \bigcup_{i=0}^{\infty} \mathbb{O}(M, i)$$

Proposition 58 *If A is open and $M \subseteq A$, then*

(1) $\mathbb{O}(M, i) \subseteq A$, for all $i \in \mathbb{N}$
(2) $\mathbb{O}(M) \subseteq A$

PROOF. Property (2) follows from Property (1) since $\mathbb{O}(M) = \bigcup_{i=0}^{\infty} \mathbb{O}(M, i)$. Let $n = ind(G)$. Note that $\mathbb{O}(M, i) = \emptyset$, $\forall i > n$. We will prove Property (1) by induction. Since $\mathbb{O}(M, 0) = M$ and $M \subset A$, we have $\mathbb{O}(M, 0) \subseteq A$.

Assume $\mathbb{O}(M, i) \subseteq A$ and let $c \in \mathbb{O}(M, i+1)$ for some i such that $0 \leq i < n$. By definition there exists a descendence path $\{p_0, \ldots, p_{i+1}\}$ with $c = p_{i+1}$ and $p_0 \in M$. Note $\{p_0, \ldots, p_i\}$ is a descendence path with $p_0 \in M$ and thus $p_i \in \mathbb{O}(M, i)$. By assumption this implies $p_i \in A$. We have $p_i \in A$, $p_i \in I(c)$, and $\dim(p_i) < \dim(c)$. Since A is open this implies $c \in A$. $\square$

Corollary 59 *If A is open, $M \subseteq A$, and if $\{p_0, \ldots, p_k\}$ is a descendence path with $p_0 \in M$, then $p_k \in A$*

Proposition 60 $\mathbb{O}(M, i+1) = \{c \in S : \exists c' \in \mathbb{O}(M, i)$ *with* $c' \in I(c) \wedge \dim(c') < \dim(c)\}$

PROOF. Let $A = \{c \in S : \exists c' \in \mathbb{O}(M, i)$ with $c' \in I(c) \wedge \dim(c') < \dim(c)\}$ and let $c \in \mathbb{O}(M, i+1)$. By definition there exists a descendence path $\{p_0, \ldots, p_{i+1}\}$ such that $p_0 \in M$ and $c = p_{i+1}$. Let $c' = p_i$. Note $\{p_0, \ldots, p_i\}$ is a descendence path with $p_0 \in M \wedge c' = p_i$. Thus $c' \in \mathbb{O}(M, i)$, $c' \in I(c)$, and $\dim(c') < \dim(c)$. Thus $c \in A$. Therefore $\mathbb{O}(M) \subseteq A$.

Let $c \in A$. Thus there exists a $c' \in \mathbb{O}(M, i)$ with $c' \in I(c)$ and $\dim(c') < \dim(c)$. Let $\{p_0, \ldots, p_i\}$ be a descendence path with $p_0 \in M \wedge c' = p_i$. Let $s_{i+1} = c$ and let $s_j = p_j$ for $0 \leq j \leq i$. Note that $\{s_0, \ldots, s_{i+1}\}$ is a descendence path with $s_0 \in M$ and $c = s_{i+1}$. Hence $c \in \mathbb{O}(M, i+1)$ and therefore $\mathbb{O}(M, i+1) = A$. $\square$

Proposition 61 *If M is finite, then $\mathbb{O}(M)$ is finite*

PROOF. Assume M is finite. Thus $\mathbb{O}(M, 0) = M$ is finite. Let $n = ind(G)$ and assume $\mathbb{O}(M, i)$ is finite for some i such that $0 \le i < n$. By Proposition 60

$$\mathbb{O}(M, i+1) = \{c \in S : \exists c' \in \mathbb{O}(M, i) \text{ with } c' \in I(c) \wedge \dim(c') < \dim(c)\}$$

Note if $c \in \mathbb{O}(M, i+1)$, then there exists a $c' \in \mathbb{O}(M, i)$ such that $c \in I(c')$. Thus $\mathbb{O}(M, i+1) \subseteq \bigcup\{I(c') : c' \in \mathbb{O}(M, i)\}$, which is a finite union of finite sets. $\square$

It follows that M^- is finite if M is finite. [Note that $M^- \subseteq \mathbb{O}(M)$.]

Proposition 62 *If G is monotonic, then $\mathbb{O}(M) = M \cup \mathbb{O}(M, 1)$.*

PROOF. Assume G is monotonic and let $A = M \cup \mathbb{O}(M, 1)$ and $n = ind(G)$. We claim $\mathbb{O}(M, i) \subseteq \mathbb{O}(M, 1)$ for $1 \le i \le n$. Clearly it is true for $i = 1$. Assume $\mathbb{O}(M, i) \subseteq \mathbb{O}(M, 1)$ for some i such that $1 \le i < n$ and let $c \in \mathbb{O}(M, i+1)$. Thus there exists a descendence path $\{p_0, \ldots, p_{i+1}\}$ such that $p_0 \in M$ and $c = p_{i+1}$. Since $\{p_0, \ldots, p_i\}$ is a descendence path with $p_0 \in M$, we have $p_i \in \mathbb{O}(M, i)$ which, by assumption implies $p_i \in \mathbb{O}(M, 1)$. Thus there exists a descendence path $\{a, p_i\}$ with $a \in M$. We have $a \in I(p_i)$, $\dim(a) < \dim(p_i)$, $p_i \in I(p_{i+1})$, and $\dim(p_i) < \dim(p_{i+1})$. Since G is monotonic, $a \in I(p_{i+1})$ and thus $\{a, p_{i+1}\}$ is a descendence path with $a \in M$ which implies $c = p_{i+1} \in \mathbb{O}(M, 1)$. Therefore $M \cup \mathbb{O}(M, 1) = \bigcup_{i=0}^{n} \mathbb{O}(M, i) = \mathbb{O}(M)$. $\square$

Theorem 63 *$\mathbb{O}(M)$ is the smallest open set containing M.*

PROOF. $M = \mathbb{O}(M, 0) \subseteq \mathbb{O}(M)$. To show $\mathbb{O}(M)$ is open, let $c \in \mathbb{O}(M)$ and $c' \in I(c)$ such that $\dim(c') > \dim(c)$. Thus there exists a descendence path $\{p_0, \ldots, p_k\}$ such that $p_0 \in M$ and $c = p_k$. Define $p_{k+1} = c'$ then $\{p_0, \ldots, p_{k+1}\}$ is a descendence path with $p_0 \in M$ and $c' = p_{k+1}$. Thus $c' \in \mathbb{O}(M)$. Therefore $\mathbb{O}(M)$ is open.

Suppose $M \subseteq A$ for some open set A. By theorem 58, $\mathbb{O}(M) \subseteq A$. $\square$

10 0-Rooted Sets

For $M \subseteq S$ we define $leaves(M) = \{c \in M : \dim(c) = 0\}$. A node $c \in M$ is said to be *0-rooted* in M iff $I(c) \cap leaves(M) \neq \emptyset$. The set of all 0-rooted nodes in M is denoted by *0-Rooted(M)*.

Let *0-Unrooted(M)* $= M \setminus$ 0-Rooted(M). If $M =$ 0-Rooted(M), then we say M is *0-rooted*. If S is 0-rooted, then we also say that G is *0-rooted*.

Proposition 64 $leaves(M^-) = leaves(M)$

PROOF. Let $c \in leaves(M^-)$. Thus $\dim(c) = 0 \wedge c \in M^-$. Suppose $c \notin M$. Then, by Lemma 56, $\dim(c) > 0$. Thus $c \in M$ which implies $c \in leaves(M)$. Therefore $leaves(M^-) \subseteq leaves(M)$.

Let $c \in leaves(M)$. Thus $c \in M \wedge \dim(c) = 0$ which implies $c \in M^- \wedge \dim(c) = 0$. Thus $c \in leaves(M^-)$. Therefore $leaves(M^-) = leaves(M)$. $\square$

Proposition 65 $leaves(\mathbb{O}(M)) = leaves(M)$

PROOF. Since $m \subseteq \mathbb{O}(M)$ it follows that $leaves(M) \subseteq leaves(\mathbb{O}(M))$. Let $c \in leaves(\mathbb{O}(M))$. Thus $c \in \mathbb{O}(M)$ and $\dim(c) = 0$. This implies there exists a descendence path $\{p_0, \ldots, p_k\}$ such that $p_0 \in M$ and $c = p_k$. Note that if $k > 0$, $p_{k-1} \in I(c)$ and $\dim(p_{k-1}) < \dim(c)$. But $\dim(c) = 0$ and thus $k = 0$ and $c = p_0 \in M$. Hence $c \in leaves(M)$. Therefore $leaves(\mathbb{O}(M)) = leaves(M)$. $\square$

Proposition 66 $M^- \subseteq \mathbb{O}(M)$

PROOF. $\mathbb{O}(M)$ contains M and is open and therefore partially open. Since M^- is the smallest partially open set containing M, $M^- \subseteq \mathbb{O}(M)$. $\square$

Proposition 67 *If $c \in M^- \backslash M$ and $b \in leaves(S) \cap I(c)$, then $b \in M$.*

PROOF. Let $c \in M^- \backslash M$ and $b \in leaves(S) \cap I(c)$. Let $i = \dim(c)$. Thus, by Lemma 56, $i > 0 \wedge c \in M_i^- \backslash M$ and $\emptyset \neq L(c) \subseteq M_{i-1}^-$. We have $b \in I(c)$ and $\dim(b) < \dim(c)$. Thus $b \in L(c)$ and hence $b \in M_{i-1}^-$. This implies $b \in M^-$. Since $\dim(b) = 0$, $b \in leaves(M^-) = leaves(M)$. Therefore $b \in M$. $\square$

Proposition 68 *If G is 0-rooted and $c \in M^- \backslash M$, then $leaves(M) \cap I(c) \neq \emptyset$.*

PROOF. Let $c \in M^- \backslash M$. Since G is 0-rooted, there exists a $b \in leaves(S) \cap I(c)$. By theorem 67, this implies $b \in M$ and therefore $leaves(M) \cap I(c) \neq \emptyset$. $\square$

Proposition 69 *If M is 0-rooted, then M^- is 0-rooted.*

PROOF. Assume M is 0-rooted and let $c \in M^-$. If $c \in M$, then $leaves(M) \cap I(c) \neq \emptyset$. Since $leaves(M) = leaves(M^-)$, this implies $leaves(M^-) \cap I(c) \neq \emptyset$. $\square$

Proposition 70 *If $M \neq \emptyset$ and M is 0-rooted, then $leaves(M) \neq \emptyset$*

PROOF. Since $M \neq \emptyset$ there exists a $c \in M$. Since M is 0-rooted, $leaves(M) \cap I(c) \neq \emptyset$. Therefore $leaves(M) \neq \emptyset$. $\square$

Proposition 71 *If G is 0-rooted and M is partially open, then 0-Rooted(M) is partially open and 0-rooted.*

PROOF. Let $A = 0\text{-Rooted}(M)$. Note that $leaves(A) = leaves(M)$. Suppose $c \in A$. Thus $c \in M$ and $leaves(M) \cap I(c) \neq \emptyset$ and so $leaves(A) \cap I(c) \neq \emptyset$. Therefore A is 0-rooted.

To show that A is partially open, suppose there exists a $c \in A^{-} \backslash A$. Since $A \subseteq M$, we have $A^{-} \subseteq M^{-}$. Since M is partially open this implies $A^{-} \subseteq M$ and hence $c \in M \backslash A$ which implies $leaves(M) \cap I(c) = \emptyset$. Let $i = \dim(c)$. Since G is 0-rooted, there exists a $b \in leaves(S) \cap I(c)$ and since $c \in A^{-} \backslash A$, we have, by Proposition 67, $b \in A$ and hence $leaves(A) \cap I(c) \neq \emptyset$. But, since $leaves(A) = leaves(B)$, this implies $leaves(M) \cap I(c) \neq \emptyset$. This contradiction established that A is partially open. □

11 0-Components and 0-Regions

$C \subseteq M$ is a *0-component of M* iff

(1) $leaves(A)$ form a non-empty maximal connected (wrt M) subset of $leaves(M)$,
(2) if $b \in leaves(C) \wedge c \in M \wedge c \in I(b)$, then $c \in C$, and
(3) C is partially open.

A finite 0-component of M is called a *0-region of M*. If M is a 0-component of M, then we call M a *0-region*

Proposition 72 *Let M be partially open, 0-rooted, and* $leaves(M)$ *is connected. If C is a 0-component of M, then* $C = M$.

PROOF. Since C is a 0-component of M, $C \subseteq M$. Since $leaves(M)$ is connected and $leaves(C)$ is a maximal connected subset of $leaves(M)$, we must have $leaves(C) = leaves(M)$. Let $c \in M$. Since M is 0-rooted, $I(c) \cap leaves(M) \neq \emptyset$ and thus $I(c) \cap leaves(C) \neq \emptyset$. Since C is a 0-component of M this implies $c \in C$. Therefore $M = C$. □

Proposition 73 *If G is 0-rooted and C is a 0-component of M, then 0-Rooted(C) is a 0-rooted component of M.*

PROOF. Let $R = 0\text{-Rooted}(C)$. Note that $leaves(R) = leaves(C)$ is a non-empty maximal connected subset of $leaves(M)$ since C is a 0-component of M. Clearly R is 0-rooted.

Let $c \in M$ such that $leaves(R) \cap I(c) \neq \emptyset$. Thus $c \in M$ and $leaves(C) \cap I(c) \neq \emptyset$. Since C is a 0-component this implies $c \in C$. Since $leaves(C) \cap I(c) \neq \emptyset$ we have $c \in R$.

To show R is 0-complete assume there exists a $c \in R^{-} \backslash R$. Since G is 0-rooted there exists a $b \in leaves(S) \cap I(c)$. Thus, by Proposition 67, $b \in R$ and thus $leaves(R) \cap I(c) \neq \emptyset$. Since $leaves(R) = leaves(C)$, $leaves(C) \cap I(c) \neq \emptyset$. Since $R \subseteq C, R^{-} \subseteq C^{-} = C, c \in C$, and $c \notin R$, we have $leaves(C) \cap I(c) \neq \emptyset$.

This contradiction establishes that R is 0-complete. Therefore R is a 0-rooted 0-component of M. $\square$

Proposition 74 *If G is 0-rooted, M is partially open, and $b \in leaves(M)$, then M has a unique 0-rooted 0-component C containing b. Furthermore $C = leaves(C) \cup \{c \in M : leaves(C) \cap I(c) \neq \emptyset\}$*

PROOF. Let $b \in leaves(M)$ for M partially open. Let $A = \{c \in leaves(M) : c$ and b are connected $wrtM\}$. Note A is a non-empty maximal connected subset of $leaves(M)$. Let $C = A \cup \{c \in M : A \cap I(c) \neq \emptyset\}$. Note $C \subseteq M$ and $leaves(C) = A$. Thus $leaves(C)$ is a non-empty maximal connected subset of $leaves(M)$.

To show C is 0-complete suppose there exists a $c \in C^{-} \backslash C$ which implies $A \cap I(c) = \emptyset$. Since G is 0-rooted there exists a $p \in leaves(S) \cap I(c)$ which, by theorem 67,implies $p \in leaves(C) \cap I(c)$. This contradiction establishes that C is 0-complete.

Let $p \in leaves(C)$ and $c \in M$ such that and $c \in I(p)$. Thus $c \in M$ and $A \cap I(c) \neq \emptyset$ which implies $c \in C$. Therefore C is a 0-component of M. Clearly C is 0-rooted. Therefore C is a 0-rooted 0-component of M containing b.

To show that C is unique, assume R is a 0-rooted 0-component of M containing B. Since both C and R are 0-rooted and contain b, there exists a $c \in leaves(C)$ and an $r \in leaves(R)$ such that $c \in I(b)$ and $r \in I(c)$. Thus $leaves(R)$ and $leaves(C)$ are connected in M by b and since $leaves(R)$ and $leaves(C)$ are both maximal connected subsets of $leaves(M)$, we must have $leaves(R) = leaves(C) = A$.

Let $c \in C$ and hence $c \in M$. If $c \in A$, then $c \in R$ so assume $c \notin A$ which implies $A \cap I(c) \neq \emptyset$ and hence $leaves(R) \cap I(c) \neq \emptyset$. Since R is a 0-component of M this implies $c \in R$. Thus $C \subseteq R$.

Let $c \in R$ which implies $c \in M$. Since R is 0-rooted there exists a $p \in leaves(R) \cap I(c)$. Thus $c \in M$ and $A \cap I(c) \neq \emptyset$ which implies $c \in C$. Therefore $C = R$. $\square$

Theorem 75 *If G is 0-rooted and M is partially open and 0-rooted, then the 0-rooted 0-components of M partition M*

PROOF. For each $b \in leaves(M)$, let C_b be the unique 0-rooted 0-component of M containing b. Recall $C_b = leaves(C_b) \cup \{c \in M : leaves(C_p) \cap I(c) \neq \emptyset\}$. Let $\mathbb{P} = \{C_b : b \in leaves(M)\}$. Suppose $a, b \in leaves(M)$ such that $C_a \cap C_b \neq \emptyset$. Let $c \in C_a \cap C_b$. Since C_a and C_b are 0-rooted there exists an $a' \in leaves(C_a) \cap I(c)$ and there exists a vertex $b' \in leaves(C_b) \cap I(c)$. We have that $[a, a', c, b', b]$ is a sequence of nodes in M each connected to the next and thus a and b are connected *wrt* M and hence $a \in C_b$ which implies $C_a = C_b$.

Fig. 7. Left: a finite set M which is complete, and $\overline{M}$ is not partially open. Right: a finite M which is partially open with $\overline{M}$ not complete.

Let $c \in M$. Since M is 0-rooted, there exists $b \in leaves(M) \cap I(c)$. Thus $c \in C_b$ which implies $c \in \bigcup \mathbb{P}$. Since $\bigcup \mathbb{P} \subseteq M$ we have $M = \bigcup \mathbb{P}$. Therefore $\mathbb{P}$ partitions M. □

Proposition 76 *If C is a 0-component of 0-Rooted(M), then C is a 0-rooted 0-component of M.*

PROOF. Let K = 0-Rooted(M) and let C be a 0-component of K. Note that $leaves(K) = leaves(M)$ and thus $leaves(C)$ is a non-empty maximal connected subset of $leaves(M)$.

Assume $p \in leaves(K)$, $c \in K$, and $c \in I(p)$. Thus $leaves(M) \cap I(c) \neq \emptyset$ and hence $c \in K$. Since C is a 0-component of K we have $c \in C$. Since C is partially open, C is a 0-component of M. Since $C \subseteq$ 0-Rooted(M), we have $leaves(M) \cap I(c) \neq \emptyset$, for all $c \in C$. Therefore C is a 0-rooted 0-component of M. □

Proposition 77 *If G is 0-rooted and M is partially open and not 0-rooted, then the set consisting of $0\!-\!Unrooted(M)$ along with the 0-rooted 0-components of M forms a partition of M.*

PROOF. Let $K = M \setminus$ 0-Unrooted(M) = Rooted(M). By Proposition 73, K is 0-complete and 0-rooted. Let $\mathbb{P}$ be the collection of the 0-rooted 0-components of K along with $Unrooted(M)$. By Proposition 76, the 0-rooted 0-components of K are 0-rooted 0-components of M. By Proposition 75, K is the union of the 0-rooted 0-components of K (and hence of M.) Since M = 0-Unrooted(M) $\cup K$, we have $M = \bigcup \mathbb{P}$. Since the 0-rooted 0-components of M are disjoint and distinct from each other and from 0-Unrooted(M), $\mathbb{P}$ partitions M. □

We demonstrate the existence of some particular kinds of set by means of examples. There exists a finite M which is complete with $\overline{M}$ not partially open. For this, see $M = \{1, c\} = M^+$ and $\overline{M} \neq \overline{M}^- = \{a, b, c, d\}$ on the left in Figure 7.

There exists a finite M which is partially open with $\overline{M}$ not complete; see right of Figure 7: $M = \{b, c\} = M^-$, and $\overline{M} = \{a, d, e\} \neq \overline{M}^- = \{a, b, c, d, e\}$.

There exists a finite M which is open (and hence partially open) with $M \neq leaves(M)^-$. See $M = \{a, b, d\} = M^-$ and $leaves(M)^- = \{d\} \neq M$ in Figure 8, left.

Fig. 8. Left: a finite set M which is open, with $M \neq leaves(M)^-$. Right: a finite set M which is closed, partially open, and for which M^∇ is not partially open.

Proposition 78 *If M is open, then M^∇ is partially open.*

PROOF. Assume M is open and $\emptyset \neq L(c) \subseteq M^\nabla$. To show $c \in M^\nabla$, let $b \in I(c)$. Note that there exists a $c' \in L(c)$ with $c' \in I(c)$ and $\dim(c')$. By assumption $c' \in M^\nabla$. Thus $I(c') \subseteq M$. Since $c' \in I(c)$ implies $c \in I(c')$, we have $c \in M$.

If $\dim(b) > \dim(c)$, then $b \in M$ since M is open. If $\dim(b) = \dim(c)$, then $b = c \in M$. If $\dim(b) < \dim(c)$, then $b \in L(c) \subseteq M^\nabla \subseteq M$. In all cases we have that $b \in M$. Hence $c \in M^\nabla$ and therefore M^∇ is partially open. □

There exists a closed, partially open M for which M^∇ is not partially open. Figure 8 shows such a set on the right, with $M = \{a, b, c\}$, $M^\nabla = \{c\}$ and $L(a) = M^\nabla$ but $a \notin M^\nabla$. Therefore M^∇ is not partially open.

Proposition 79 *If $M \neq \emptyset$ and $M^\nabla = \emptyset$ (i.e., $\delta M = M$), then M is not open or M is not closed.*

PROOF. Since $M \neq \emptyset$, there exists a $c \in M$. Since $c \notin M^\nabla$, there exists a $c' \in I(c) \backslash M$. If $\dim(c') > \dim(c)$, then M is not open. We cannot have $\dim(c') = \dim(c)$ which would imply $c' = c$ since $c \in M$ and $c' \notin M$. If $\dim(c') < \dim(c)$, then M is not closed. □

There exists an M which is partially open and closed (and therefore complete) with $M^\nabla = \emptyset$. See Figure 9, left.

There exists an M which is complete and open (and therefore partially open) with $M^\nabla = \emptyset$. See Figure 9, middle.

There exists an M which is partially open and complete, $core(M) \neq \emptyset$, $leaves(M) \neq \emptyset$, rooted and 0-rooted, with $M^\nabla = \emptyset$. See Figure 9, right.

A node c is a *downward 0-rooted point* of M iff $c \in M$ and there exists a descendence path $\{p_0, \ldots, p_k\}$ such that

$$p_0 \in leaves(M) \wedge c = p_k \wedge \forall i \, (0 \leq i \leq k \rightarrow p_i \in M)$$

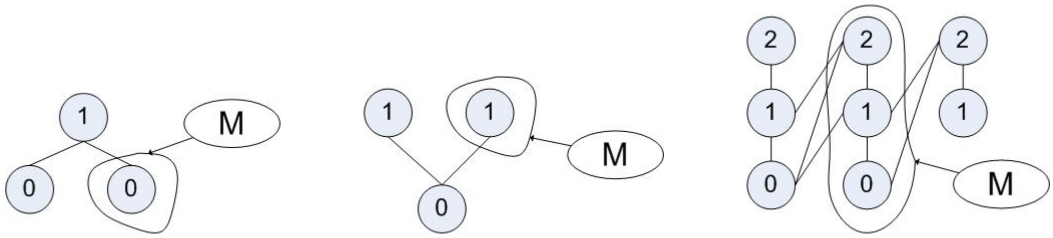

Fig. 9. Left: set M which is partially open and closed with $M^{\nabla} = \emptyset$. Middle: set M which is complete and open with $M^{\nabla} = \emptyset$. Right: set M which is partially open and complete, $core(M) \neq \emptyset$, $leaves(M) \neq \emptyset$, rooted and 0-rooted, with $M^{\nabla} = \emptyset$.

The set of all downward 0-rooted points of M is denoted by $DRP(M)$.

Theorem 80 *If G is 0-rooted, then M is partially open iff $UXP(M) \subseteq DRO(\overline{M})$.*

PROOF. Assume G is 0-rooted.

Assume M is partially open and let $c \in UXP(M)$. Thus $c \notin M$ and there exists a $b \in M$ such that $\dim(b) < \dim(c)$. Hence $L(c) \neq \emptyset$ and, since M is partially open, $L(c) \neq \emptyset$, and $c \notin M = M^{-}$, we must have $L(c) \neq M$. Thus there exists $c' \in I(c) \cap \overline{M}$ such that $\dim(c') < \dim(c)$. Chose $p_0 = c$ and $p_1 = c'$.

Assume $p_0, \ldots, p_i$ have been chosen for some $i \geq 1$, such that for all j such that $1 \leq j \leq i, p_j \in I(p_{j-1})$ and $\dim(p_j) < \dim(p_{j-1})$. If $\dim(p_i) = 0$ we set $k = i$ and stop. Otherwise, since G is 0-rooted and $\dim(p_i) > 0$, we have $L(p_i) \neq \emptyset$. Since $p_i \in \overline{M},\ L(p_i) \neq \emptyset$,and M is partially open, we have $p_i \notin M^{-}$. Hence there exists a $p_{i+1} \in L(p_i) \backslash M$. Thus we have $p_{i+1} \in I(p_i) \cap \overline{M}$ and $\dim(p_{i+1}) < \dim(p_i)$. This process will eventually end.

Define $s_j = p_{k-j}$, for each j such that $0 \leq j \leq k$. Then $\{s_0, \ldots, p_k\}$ is a descendence path with $s_0 \in leaves(\overline{M})$ and $s_k = c$. Thus $c \in DRP(\overline{M})$ and therefore $UXP(M) \subseteq DRP(\overline{M})$.

Assume $UXP(M) \subseteq DRP(\overline{M})$. Let $n = ind(G)$ and for $0 \leq i \leq n$ consider the statement

$$\mathbb{P}(i) \ \equiv\ \nexists c \in \overline{M} \backslash M \text{ with } \dim(c) = i$$

Suppose $c \in M^{-} \backslash M$ and $\dim(c) = 0$. Thus $c \in leaves(M^{-}) = leaves(M) \subseteq M$. Therefore $\mathbb{P}(0)$ is true.

Assume $\mathbb{P}(j)$ is true for all j such that $0 \leq j \leq i$ for some i with $0 \leq i < n$ and suppose $c \in M^{-} \backslash M$ with $\dim(c) = i + 1$. Thus $\emptyset \neq L(c) \subseteq M^{-}$. Hence there exists a $c' \in L(c)$ such that $c' \in I(c)$, $\dim(c') < \dim(c) = i + 1$, and $c' \in M_i^{-} \subseteq M^{-}$. By assumption, since $\dim(c') \leq i$, $c' \notin M^{-} \backslash M$. Since $c') \in M^{-}$, we must have $c' \in M$. Hence $c \in DXP(M) \subseteq URP(\overline{M})$. Thus there exists a descendence path $\{p_0, \ldots, p_k\}$ such that $p_0 \in leaves(\overline{M})$, $p_k = c$, and $p_j \in \overline{M}$, for all j satisfying $0 \leq j \leq k$. Since $\dim(c') < \dim(c)$, $\dim(c) > 0$ and so $k > 0$. Thus $p_{k-1} \in I(c)$, $\dim(p_{k-1}) < \dim(c)$, and $p_{k-1} \notin M$. However, $p_{k-1} \in L(c) \subseteq M^-$, which implies $p_{k-1} \in M^- \backslash M$ and $\dim(p_{k-1}) = i$. Thus, by assumption $\mathbb{P}(\dim(p_{k-1}))$ is true which implies $p_{k-1} \notin M^- \backslash M$. This contradiction establishes that $\mathbb{P}(i+1)$ is true and therefore $M^- \backslash M = \emptyset$. Therefore M is partially open. □

Corollary 81 *A set M is closed and partially open iff $DXP(M) = \emptyset$ and $UXP(M) \subseteq DRP(\overline{M})$.*

12 Concluding Remarks

This paper provided a comprehensive discussion of a topology on incidence pseudographs, as introduced by Klaus Voss in 1993, and further discussed by others in more recent years. (The references below only give a very limited account of such work; for an extensive bibliography see, for example, (3).) The paper also discussed (for the first time) especially partially open sets, as occurring in common (non-binary) digital picture analysis.

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