VERTEX-PRIMITIVE DIGRAPHS HAVING VERTICES WITH ALMOST EQUAL NEIGHBOURHOODS

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Abstract. We consider vertex-primitive digraphs having two vertices with almost equal neighbourhoods (that is, the set of vertices that are neighbours of one but not the other is small). We prove a structural result about such digraphs and then apply it to answer a question of Araújo and Cameron about synchronising groups.

1. Introduction

All sets, digraphs and groups considered in this paper are finite. For basic definitions, see Section 2. Let Γ be a digraph on a set Ω and suppose that Γ is *vertex-primitive*, that is, its automorphism group acts primitively on Ω. It is well known and easy to see that, in this case, Γ cannot have two distinct vertices with equal neighbourhoods, unless $\Gamma = \emptyset$ or $\Gamma = \Omega \times \Omega$ (see Lemma 2.3 for example).

We consider the situation when Γ has two vertices with "almost" equal neighbourhoods. Since Γ is vertex-primitive, it is regular, of valency d, say. Let Γ_i be the graph on Ω with two vertices being adjacent if the intersection of their neighbourhoods in Γ has size $d - i$. Our main result is the following.

Theorem 1.1. *Let Γ be a vertex-primitive digraph on a set Ω with $\Gamma \neq \emptyset$ and $\Gamma \neq \Omega \times \Omega$. Let n be the order of Γ and d its valency. If κ is the smallest positive i such that $\Gamma_i \neq \emptyset$, then either*

(1) $\Gamma_0 \cup \Gamma_\kappa = \Omega \times \Omega$ *and* $(n-1)(d-\kappa) = d(d-1)$, *or*
(2) *there exists* $i \in \{\kappa, \ldots, d-1\}$ *such that* Γ_i *has valency at least* 1 *and at most* $\kappa^2 + \kappa$.

Theorem 1.1 is most powerful when κ is small. To illustrate this, we completely determine the digraphs which occur when $\kappa = 1$ (see Corollary 4.2). We then apply this result to answer a question of Araújo and Cameron [2, Problem 2(a)] (see Theorem 4.3) concerning synchronising groups. Finally, in Section 5, we say a few words about the case $\Gamma_0 \cup \Gamma_\kappa = \Omega \times \Omega$.

2. Preliminaries

2.1. **Digraphs.** Let Ω be a finite set. A *digraph* Γ on Ω is a binary relation on Ω, in other words, a subset of $\Omega \times \Omega$. The elements of Ω are called the *vertices* of Γ while the cardinality of Ω is the *order* of Γ. The digraph Γ^{-1} is $\{(\alpha, \beta) \in \Omega \times \Omega : (\beta, \alpha) \in \Gamma\}$. Given two digraphs Γ and Λ on Ω, we define the digraph

$$\Gamma \circ \Lambda := \{(\alpha, \beta) \in \Omega \times \Omega : \text{there exists } \gamma \in \Omega \text{ with } (\alpha, \gamma) \in \Gamma, (\gamma, \beta) \in \Lambda\}.$$

Let v be a vertex of Γ. The *neighbourhood* of v is the set $\{u \in \Omega : (v, u) \in \Gamma\}$ and is denoted $\Gamma(v)$. Its cardinality is the *valency* of v. If every vertex of Γ has the same valency, say d, then we say that Γ is *regular of valency d*.

If Γ is a symmetric binary relation, then it is sometimes called a *graph*. If Ψ is a subset of Ω, then the *subgraph of Γ induced by Ψ* is $\Gamma \cap (\Psi \times \Psi)$ viewed as a graph on Ψ. We denote by Ω^* the set $\{(v, v) : v \in \Omega\}$. The graph $(\Omega \times \Omega) \setminus \Omega^*$ is called the *complete graph* on Ω.

2.2. **Groups.** The *automorphism group* of Γ, denoted $\mathrm{Aut}(\Gamma)$, is the group of permutations of Ω that preserve Γ. A permutation group G on Ω is *transitive* if for every $x, y \in \Omega$ there exists $g \in G$ with $x^g = y$, that is, mapping x to y. A permutation group on Ω is *primitive* if it is transitive and preserves no nontrivial partition of Ω. We say that Γ is *vertex-transitive* if $\mathrm{Aut}(\Gamma)$ is transitive.

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2.3. **A few basic results.**

Lemma 2.1. *Let* Γ *be a vertex-transitive digraph on* Ω*. If* $\Gamma_0 = \Omega \times \Omega$*, then* $\Gamma = \emptyset$ *or* $\Gamma = \Omega \times \Omega$*.*

Proof. Suppose that $\Gamma \neq \emptyset$ and thus there exists $(\alpha, \beta) \in \Gamma$. As $\Gamma_0 = \Omega \times \Omega$, all vertices of Γ have the same neighbourhood and thus $\beta \in \Gamma(\omega)$ for every $\omega \in \Omega$ but then vertex-transitivity implies that $\Gamma = \Omega \times \Omega$. □

Lemma 2.2. *Let* Γ *be a vertex-primitive graph on* Ω*. If* Γ *is a transitive relation on* Ω *and* $\Gamma \not\subseteq \Omega^*$*, then* $\Gamma = \Omega \times \Omega$*.*

Lemma 2.3. *Let* Γ *be a vertex-primitive digraph on* Ω*. If* $\emptyset \neq \Gamma \neq \Omega \times \Omega$*, then* $\Gamma_0 = \Omega^*$*.*

Proof. Clearly, $\Omega^* \subseteq \Gamma_0$. If $\Gamma_0 \not\subseteq \Omega^*$, then Lemma 2.2 implies that $\Gamma_0 = \Omega \times \Omega$ but this contradicts Lemma 2.1. □

3. Proof of Theorem 1.1

Since $\emptyset \neq \Gamma \neq \Omega \times \Omega$, we have $n \geq 2$ and Lemma 2.1 implies that $\Gamma_0 \neq \Omega \times \Omega$ and thus κ is well-defined. Since Γ is vertex-primitive, it is regular, of valency d, say. By minimality of κ, $\Gamma_i = \emptyset$ for $i \in \{1, \ldots, \kappa - 1\}$ but $\Gamma_\kappa \neq \emptyset$. Note that, for every integer i, $\mathrm{Aut}(\Gamma) \leq \mathrm{Aut}(\Gamma_i)$ hence Γ_i is also vertex-primitive and regular, of valency d_i, say. By Lemma 2.3, we have $\Gamma_0 = \Omega^*$ and thus

$$d_0 = 1. \tag{1}$$

Moreover, it is easy to see that

$$\bigcup_{i \in \{0, \ldots, d-1\}} \Gamma_i = \Gamma \circ \Gamma^{-1}. \tag{2}$$

We now show that, for every $i \geq 0$, we have

$$\Gamma_i \circ \Gamma_\kappa \subseteq \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_{i+\kappa}. \tag{3}$$

Let $(u, v) \in \Gamma_i \circ \Gamma_\kappa$. By definition, there exists $w \in \Omega$ such that $(u, w) \in \Gamma_i$ and $(w, v) \in \Gamma_\kappa$. In particular, $|\Gamma(u) \cap \Gamma(w)| = d - i$ while $|\Gamma(v) \cap \Gamma(w)| = d - \kappa$ and thus $\Gamma(u) \setminus \Gamma(w) = i$ and $\Gamma(w) \setminus \Gamma(v) = \kappa$ but $\Gamma(u) \setminus \Gamma(v) \subseteq (\Gamma(u) \setminus \Gamma(w)) \cup (\Gamma(w) \setminus \Gamma(v))$. This implies that $|\Gamma(u) \setminus \Gamma(v)| \leq i + \kappa$ hence $(u, v) \in \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_{i+\kappa}$, which concludes the proof of (3). Let

$$\ell := \min\{i \geq \kappa : \Gamma_{i+1} = \Gamma_{i+2} = \cdots = \Gamma_{i+\kappa} = \emptyset\}. \tag{4}$$

By definition, we have $\ell \geq \kappa$. Recall that $\Gamma_i = \emptyset$ for every $i \geq d + 1$ hence

$$\kappa \leq \ell \leq d. \tag{5}$$

Let $\overline{\Gamma_\kappa}$ be the transitive closure of the relation Γ_κ. (That is, the minimal transitive relation containing Γ_κ.) By Lemma 2.2, we have $\overline{\Gamma_\kappa} = \Omega \times \Omega$. Let

$$\Lambda := \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_\ell.$$

Note that

$$\begin{aligned} \Lambda \circ \Gamma_\kappa &= (\Gamma_0 \circ \Gamma_\kappa) \cup (\Gamma_1 \circ \Gamma_\kappa) \cup \cdots \cup (\Gamma_\ell \circ \Gamma_\kappa) \\ &\overset{(3)}{\subseteq} \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_{\kappa+\ell} \\ &\overset{(4)}{=} \Gamma_0 \cup \Gamma_1 \cup \cdots \cup \Gamma_\ell = \Lambda. \end{aligned}$$

As $\Gamma_\kappa \subseteq \Lambda$, it follows by induction that $\overline{\Gamma_\kappa} \subseteq \Lambda$ and thus

$$\Lambda = \Omega \times \Omega.$$

This implies that

$$n = \sum_{i=0}^{\ell} d_i = 1 + d_1 + \cdots + d_\ell. \tag{6}$$

We now consider two cases, according to whether $\ell = \kappa$ or $\ell \geq \kappa + 1$.

3.1. $\boldsymbol{\ell = \kappa}$. If $\ell = \kappa$, then the minimality of κ implies $\Lambda = \Gamma_0 \cup \Gamma_\kappa$ and hence $\Gamma_0 \cup \Gamma_\kappa = \Omega \times \Omega$. Let $\mathcal{B} := \{\Gamma(\alpha) \mid \alpha \in \Omega\}$ and let

$$\mathcal{S} := \{(\alpha, b, b') \mid \alpha \in \Omega, b, b' \in \mathcal{B}, \alpha \in b \cap b', b \neq b'\}.$$

Now,

$$|\mathcal{S}| = \sum_{\alpha \in \Omega} |\{(b, b') \mid \alpha \in b \cap b', b \neq b'\}| = \sum_{\alpha \in \Omega} d(d-1) = nd(d-1).$$

On the other hand, observe that $|b \cap b'| = d - \kappa$ for every $b, b' \in \mathcal{B}$ with $b \neq b'$ and thus

$$|\mathcal{S}| = \sum_{\substack{b, b' \in \mathcal{B} \\ b \neq b'}} |b \cap b'| = \sum_{\substack{b, b' \in \mathcal{B} \\ b \neq b'}} (d - \kappa) = n(n-1)(d-\kappa).$$

Therefore $(n-1)(d-\kappa) = d(d-1)$ and the theorem follows.

3.2. $\boldsymbol{\ell \geq \kappa + 1}$. We assume that $\ell \geq \kappa + 1$. By minimality of ℓ, we have $\Gamma_\ell \neq \emptyset$ and thus there exist two vertices of Γ whose neighbourhoods intersect in $d - \ell$ vertices hence, considering the union of their neighbourhoods, we obtain

$$n \geq d + \ell. \tag{7}$$

Let $\alpha \in \Omega$ and let

$$\mathcal{S}(\alpha) := \{(\beta, \gamma) \in \Omega \times \Omega : \beta \in \Gamma(\alpha) \cap \Gamma(\gamma)\}.$$

Clearly,

$$|\mathcal{S}(\alpha)| = \sum_{\beta \in \Gamma(\alpha)} |\{\gamma \in \Omega : \gamma \in \Gamma^{-1}(\beta)\}| = \sum_{\beta \in \Gamma(\alpha)} d = d^2. \tag{8}$$

On the other hand,

$$\begin{aligned}
|\mathcal{S}(\alpha)| &= \sum_{\gamma \in (\Gamma \circ \Gamma^{-1})(\alpha)} |\Gamma(\alpha) \cap \Gamma(\gamma)| \\
&\overset{(2)}{=} \sum_{i=0}^{d-1} \sum_{\gamma \in \Gamma_i(\alpha)} |\Gamma(\alpha) \cap \Gamma(\gamma)| = \sum_{i=0}^{d-1} \sum_{\gamma \in \Gamma_i(\alpha)} (d - i) = \sum_{i=0}^{d} d_i(d-i) \\
&\overset{(5)}{\geq} \sum_{i=0}^{\ell} d_i(d-i) \\
&\overset{(1)}{=} d + \sum_{i=1}^{\ell} d_i(d-i) \\
&\overset{(6)}{=} d + \left(n - 1 - \sum_{i=1}^{\ell-1} d_i\right)(d - \ell) + \sum_{i=1}^{\ell-1} d_i(d-i) \\
&= d + (n-1)(d-\ell) + \sum_{i=1}^{\ell-1} d_i(\ell - i) \\
&\overset{(7)}{\geq} d^2 - \ell(\ell-1) + \sum_{i=1}^{\ell-1} d_i(\ell-i).
\end{aligned}$$

Combining this with (8), we have

$$\ell(\ell-1) \geq \sum_{i=1}^{\ell-1} d_i(\ell-i).$$

Let $\mathcal{I} := \{i \in \{1, \ldots, \ell-1\} : d_i \neq 0\}$ and $d^* := \min\{d_i : i \in \mathcal{I}\}$. We have

$$\ell(\ell-1) \geq d^* \sum_{i \in \mathcal{I}} (\ell - i). \tag{9}$$

Since $\ell \geq \kappa+1$, κ is the minimum element of $\mathcal{I}$. Note also that $(\ell - i)$ is decreasing with respect to i and, by definition of ℓ, any two elements of $\mathcal{I}$ are at most κ apart. Let

$$\sigma := \left\lfloor \frac{\ell-1}{\kappa} \right\rfloor.$$

Then

$$\begin{aligned}\sum_{i\in\mathcal{I}}(\ell-i) &\geq (\ell-\kappa)+(\ell-2\kappa)+\cdots+(\ell-\sigma\kappa)\\ &= \sigma\ell - \frac{\kappa\sigma(\sigma+1)}{2}.\end{aligned}$$

Combining this with (9), we find

$$\ell(\ell-1) \geq d^*\left(\sigma\ell - \frac{\kappa\sigma(\sigma+1)}{2}\right). \tag{10}$$

Write $\ell := \sigma\kappa + r$, with $r \in \{1,\ldots,\kappa\}$. Now, (10) gives

$$\frac{2(\sigma\kappa+r)(\sigma\kappa+r-1)}{\sigma(\sigma\kappa+2r-\kappa)} \geq d^*. \tag{11}$$

Calculating the derivative of the left side with respect to σ, one finds

$$-\frac{2\left(\sigma^2\kappa^2(\kappa-1)+r(r-1)((2\sigma-1)\kappa+2r)\right)}{(\sigma(\sigma\kappa+2r-\kappa))^2},$$

which is clearly nonpositive since $r,\kappa,\sigma \geq 1$. It follows that the maximum of the left side of (11) is attained when $\sigma = 1$, hence

$$\frac{(\kappa+r)(\kappa+r-1)}{r} \geq d^*. \tag{12}$$

If $r = \kappa$, then the left side of (12) is $4\kappa-2$ and an easy computation shows that $4\kappa - 2 \leq \kappa^2+\kappa$. If $r \leq \kappa-1$, then another easy computation yields that the left side of (12) is a decreasing function of r, hence the minimum is attained when $r = 1$ and $\kappa^2+\kappa \geq d^*$. This completes the proof. □

Remark 3.1. The upper bound $\kappa^2+\kappa$ in Theorem 1.1 (2) is actually tight (see some of the examples in Section 4). However, the proof of Theorem 1.1 reveals that when more information about Γ is available, this upper bound can be drastically improved. For instance, following the argument in the last part of the proof, one finds that if $\sigma \geq 2$, then $d^* \leq (4\kappa^2+2\kappa)/(\kappa+2) \leq 4\kappa-2$ and hence there exists $i \in \{\kappa,\ldots,d-1\}$ such that Γ_i has nonzero valency bounded by a linear function of κ.

4. The case $\kappa = 1$ and an application to synchronising groups

We now completely determine the digraphs that arise when $\kappa = 1$ in Theorem 1.1. Let p be a prime, let $d \in \mathbb{Z}$ with $0 \leq d \leq p$ and let $x \in \mathbb{Z}_p$. We define $\Delta_{p,x,d}$ to be the Cayley digraph on $\mathbb{Z}_p$ with connection set $\{x+1, x+2, \ldots, x+d\}$. (That is, $(u,v) \in \Delta_{p,x,d}$ if and only if $v-u \in \{x+1,x+2,\ldots,x+d\}$.) We will need the following easy lemma.

Lemma 4.1. *Let Γ be a vertex-primitive digraph on Ω of valency 1. Then $\Gamma = \Omega^*$ or $\Gamma \cong \Delta_{p,0,1}$ for some prime p.*

Proof. If $\Gamma \cap \Omega^* \neq \emptyset$, then, by vertex-primitivity, $\Gamma = \Omega^*$. Otherwise, Γ must be a directed cycle and, again by vertex-primitivity, must have prime order p and hence $\Gamma \cong \Delta_{p,0,1}$. □

Corollary 4.2. *Let Γ be a vertex-primitive digraph on Ω. If $\Gamma_1 \neq \emptyset$, then one of the following occurs:*

(1) $\Gamma = \Omega^*$,
(2) Γ *is a complete graph, or*
(3) $\Gamma \cong \Delta_{p,x,d}$, *for some prime p and $d \geq 1$.*

Proof. Let n be the order of Γ and d its valency. Since $\Gamma_1 \neq \emptyset$, $d \geq 1$ and $\emptyset \neq \Gamma \neq \Omega \times \Omega$. If $d = 1$, then the result follows by Lemma 4.1. We thus assume that $d \geq 2$. Applying Theorem 1.1 with $\kappa = 1$, we get that either $(n-1)(d-1) = d(d-1)$ and thus $n = d+1$, or there exists $i \in \{1, \dots, d-1\}$ such that Γ_i is regular of valency at least 1 and at most 2.

Suppose first that $n = d+1$. This implies that $(\Omega \times \Omega) \setminus \Gamma$ has valency 1. By Lemma 4.1, $(\Omega \times \Omega) \setminus \Gamma = \Omega^*$ or $(\Omega \times \Omega) \setminus \Gamma \cong \Delta_{p,0,1}$ for some prime p. In the former case, Γ is a complete graph, while in the latter case, $\Gamma \cong \Delta_{p,1,p-1}$.

We may thus assume that there exists $i \in \{1, \dots, d-1\}$ such that Γ_i is regular of valency at least 1 and at most 2. This implies that Γ_i must have order 2 or be a vertex-primitive cycle and thus have prime order. It follows that Γ also has prime order and is thus a Cayley digraph on $\mathbb{Z}_p$ for some prime p. Up to isomorphism, we may assume that $(0,1) \in \Gamma_1$. Let y be the unique element of $\Gamma(1) \setminus \Gamma(0)$. Now, for every $s \in \Gamma(0) \setminus \{y-1\}$, we have that $s+1 \in \Gamma(1) \setminus \{y\}$ and thus $s+1 \in \Gamma(0)$. In other words, if $s \in \Gamma(0)$, then $s+1 \in \Gamma(0)$, unless $s = y-1$. It follows that $\Gamma(0)$ is of the form $\{x+1, x+2, \dots, y-1\}$ for some x and the result follows. □

We note that the second author asked for a proof of Corollary 4.2 on the popular MathOverflow website (see `http://mathoverflow.net/q/186682/`). The question generated some interest there but no answer.

We now use Corollary 4.2 to answer Problem 2(a) in [2]. In fact, we will prove a slightly more general result. First, we need some terminology regarding synchronising groups (see also [3]). Let G be a permutation group and let f be a map, both with domain Ω. The *kernel* of f is the partition of Ω into the inverse images of points in the image of f. The *kernel type* of f is the partition of $|\Omega|$ given by the sizes of the parts of its kernel. We say that G *synchronises* f if the semigroup $\langle G, f\rangle$ contains a constant map, while G is said to be *synchronising* if G synchronises every non-invertible map on Ω.

Theorem 4.3. *Let Ω be a set, let G be a primitive permutation group on Ω and let f be a map on Ω. If f has kernel type $(p, 2, 1, \dots, 1)$ with $p \geq 2$, then G synchronises f.*

Proof. By contradiction, we assume that G does not synchronise f. Let Γ be the graph on Ω such that $(v, w) \in \Gamma$ if and only if there is no element of $\langle G, f\rangle$ which maps v and w to the same point.

By [2, Theorem 5(a),(b)], we have $\Gamma \neq \emptyset$ and $G \leq \mathrm{Aut}(\Gamma)$ and thus Γ is vertex-primitive. Since f is not a permutation, Γ is not complete. Transitive groups of prime degree are synchronising (see for example [6, Corollary 2.3]) hence we may assume that $|\Omega|$ is not prime. It thus follows by Corollary 4.2 that $\Gamma_1 = \emptyset$.

Let d be the valency of Γ and let A and B be the parts of the kernel of f with sizes 2 and p, respectively. Let $a = f(A)$ and $b = f(B)$, let $K = A \cup B$ and let Y be the subgraph of Γ induced by K. By definition, Y is bipartite, with parts A of size 2 and B of size p. By [2, Lemma 10], every vertex of Y has degree at least one.

Suppose that there exist $b_1, b_2 \in B$ having valency one in Y. Then $\Gamma(b_1) \setminus A$ and $\Gamma(b_2) \setminus A$ are mapped injectively and hence bijectively into $\Gamma(b) \setminus \{a\}$ hence we have $\Gamma(b_1) \setminus A = \Gamma(b_2) \setminus A$. This implies that $(b_1, b_2) \in \Gamma_1$, a contradiction.

Now suppose that there exist $b_1, b_2 \in B$ having valency two in Y. Then $\Gamma(b_1) \cap K = A = \Gamma(b_2) \cap K$ and, as before, $\Gamma(b_1) \setminus A$ and $\Gamma(b_2) \setminus A$ are mapped injectively into $\Gamma(b) \setminus \{a\}$. Since $|\Gamma(b_1) \setminus A| = d-2 = |\Gamma(b_2) \setminus A|$ while $|\Gamma(b) \setminus \{a\}| = d-1$, it follows that $|(\Gamma(b_1) \setminus A) \cap (\Gamma(b_1) \setminus A)| \geq d-3$ and thus $|\Gamma(b_1) \cap \Gamma(b_2)| \geq d-1$, which is again a contradiction.

Since every vertex of B has valency either one or two in Y, we conclude that $|B| \leq 2$ thus $p = 2$ and the result follows by [2, Theorem 3(a)]. □

Note that Theorem 4.3 was later proved independently in [1].

5. The case $\Gamma_0 \cup \Gamma_\kappa = \Omega \times \Omega$

We now say a few words about part (1) of the conclusion of Theorem 1.1, that is, when $\Gamma_0 \cup \Gamma_\kappa = \Omega \times \Omega$ and

$$(n-1)(d-\kappa) = d(d-1). \tag{13}$$

Let $\mathcal{B} := \{\Gamma(\alpha) \mid \alpha \in \Omega\}$, as in Section 3.1. Note that $\mathcal{B}$ is a set of d-subsets of Ω. Moreover, any two distinct elements of $\mathcal{B}$ intersect in $d-\kappa$ elements. In particular, $(\Omega, \mathcal{B})$ is a symmetric 2-design with parameters $(n, d, d-\kappa)$ and with a point-primitive automorphism group. (For undefined terminology, see for example [4, Chapter 1].)

Given a specific value of κ, one can often push the analysis further and determine all the possibilities for Γ. Recall that $1 \leq \kappa \leq d$. If $\kappa = d$, then, since $d \geq 1$, Eq. (13) implies that $d = 1$ and we may apply Lemma 4.1. The case when $\kappa = 1$ was dealt with in Corollary 4.2. From now on, we assume that

$$2 \leq \kappa \leq d - 1.$$

Observe that now Eq. (13) yields

$$n = d + \kappa + \frac{\kappa(\kappa - 1)}{d - \kappa}. \tag{14}$$

In particular, $d - \kappa \leq \kappa(\kappa - 1)$, that is,

$$d \leq \kappa^2. \tag{15}$$

A computation using Eq. (14) also yields that, for fixed κ, n is a non-decreasing function of d. Therefore the maximum for n (as a function of κ) is achieved when $d = \kappa + 1$ and $n \leq \kappa^2 + \kappa + 1$.

In our opinion, the most interesting situation occurs when $d = \kappa^2$ or (dually) when $d = \kappa + 1$. By Eq. (13), we have $n = \kappa^2 + \kappa + 1$ and thus $(\Omega, \mathcal{B})$ is a symmetric 2-design with parameters $(\kappa^2 + \kappa + 1, \kappa + 1, 1)$, that is, a finite projective plane of order κ. Note that $\mathrm{Aut}(\Gamma)$ cannot be 2-transitive and thus this is a non-Desarguesian projective plane. By a remarkable theorem of Kantor [5, Theorem B (ii)] (which depends upon the classification of the finite simple groups), n is prime.

We conclude by showing how, given an explicit value of κ, one can often pin down the structure of Γ. We do this using $\kappa := 4$ as an example. By Eq. (14), we have $n = d + 4 + 12/(d - 4)$ and hence $d \in \{5, 6, 7, 8, 10, 16\}$. Moreover, replacing Γ by its complement $(\Omega \times \Omega) \setminus \Gamma$ we may assume that $2d \leq n$. Therefore $(d, n) \in \{(5, 21), (6, 16), (7, 15)\}$. The previous paragraph shows that the case $(d, n) = (5, 21)$ does not arise because 21 is not a prime. When $(d, n) = (7, 15)$, a careful analysis of the primitive groups of degree 15 reveals that Γ is isomorphic to the Kneser graph with parameters $(6, 2)$ with a loop attached at each vertex. Finally, if $(d, n) = (6, 16)$, then going through the primitive groups of degree 16, one finds that Γ is isomorphic to either the Clebsch graph with a loop added at each vertex or to the Cartesian product of two copies of a complete graph of order 4.

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