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Non-separability and complete reducibility: $E_n$ examples
with an application to a question of Külshammer

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Abstract

Let $G$ be a simple algebraic group of type $E_n (n = 6, 7, 8)$ defined over an algebraically closed field $k$ of characteristic 2. We present examples of triples of closed reductive groups $H < M < G$ such that $H$ is $G$-completely reducible, but not $M$-completely reducible. As an application, we consider a question of Külshammer on representations of finite groups in reductive groups. We also consider a rationality problem for $G$-complete reducibility and a problem concerning conjugacy classes.

Keywords: algebraic groups, separable subgroups, complete reducibility, representations of finite groups

1 Introduction

Let $G$ be a connected reductive algebraic group defined over an algebraically closed field $k$ of characteristic $p$. In [17, Sec. 3], J.P. Serre defined the following:

Definition 1.1. A closed subgroup $H$ of $G$ is $G$-completely reducible ($G$-cr for short) if whenever $H$ is contained in a parabolic subgroup $P$ of $G$, $H$ is contained in a Levi subgroup $L$ of $P$.

This is a faithful generalization of the notion of semisimplicity in representation theory: if $G = GL_n (k)$, a subgroup $H$ of $G$ is G-cr if and only if $H$ acts semisimply on $k^n$ [17 Ex. 3.2.2(a)]. If $p = 0$, the notion of $G$-complete reducibility agrees with the notion of reductivity [17, Props. 4.1, 4.2]. In this paper, we assume $p > 0$. In that case, if a subgroup $H$ is $G$-cr, then $H$ is reductive [17 Prop. 4.1], but the other direction fails: take $H$ to be a unipotent subgroup of order $p$ of $G = SL_2$. See [20] for examples of connected non-$G$-cr subgroups. In this paper, by a subgroup of $G$, we always mean a closed subgroup.

Completely reducible subgroups have been much studied as important ingredients to understand the subgroup structure of connected reductive algebraic groups [12, 13, 21]. Recently, studies of complete reducibility via Geometric Invariant Theory (GIT for short) have been fruitful [3, 4, 2]. In this paper, we use a recent result from GIT (Proposition 2.4).

Here is the first problem we consider in this paper. Let $H < M < G$ be a triple of reductive algebraic groups. It is known to be hard to find such a triple with $H$ $G$-cr but not $M$-cr [3, 22]. The only known such examples are [3, Sec. 7] for $p = 2, G = G_2$ and [22] for $p = 2, G = E_7$. Recall that a pair of reductive groups $G$ and $M$ is called a reductive pair if Lie $M$ is an $M$-module direct summand of $g$. For more on reductive pairs, see [3]. Our main result is:
**Theorem 1.2.** Let $G$ be a simple algebraic group of type $E_6$ (respectively $E_7$, $E_8$) of any isogeny type defined over an algebraically closed field $k$ of characteristic $2$. Then there exist reductive subgroups $H < M$ of $G$ such that $H$ is finite, $M$ is semisimple of type $A_5, A_1$ (respectively $A_7$, $D_8$), $(G, M)$ is a reductive pair, and $H$ is $G$-cr but not $M$-cr.

In this paper, we present new examples with the properties of Theorem 1.2 giving an explicit description of the mechanism for generating such examples. We give 11 examples for $G = E_6$, 1 new example for $G = E_7$, and 2 examples for $G = E_8$. We use Magma [5] for our computations. Recall that $G$-complete reducibility is invariant under isogenies [2, Lem. 2.12]; in Sections 3, 4, and 5, we do computations for simply-connected $G$ only, but that is sufficient to prove Theorem 1.2 for $G$ of any isogeny type.

We recall a few relevant definitions and results from [3], [22], which motivated our work. We denote the Lie algebra of $G$ by $\mathfrak{g}$.

**Definition 1.3.** Let $H$ and $N$ be subgroups of $G$ where $H$ acts on $N$ by group automorphisms. The action of $H$ is called separable in $N$ if the global centralizer of $H$ in $N$ agrees with the infinitesimal centralizer of $H$ in Lie $N$, that is, $C_N(H) = \mathfrak{c}_{\text{Lie} N}(H)$. Note that the condition means that the set of fixed points of $H$ acting on $N$, taken with its natural scheme structure, is smooth.

This is a slight generalization of the notion of separable subgroups. Recall that

**Definition 1.4.** Let $H$ be a subgroup of $G$ acting on $G$ by inner automorphisms. Let $H$ act on $\mathfrak{g}$ by the corresponding adjoint action. Then $H$ is called separable if Lie $C_G(H) = \mathfrak{c}_{\mathfrak{g}}(H)$.

Note that we always have Lie $C_G(H) \subseteq \mathfrak{c}_{\mathfrak{g}}(H)$. In [3], Bate et al. investigated the relationship between $G$-complete reducibility and separability, and showed the following [3, Thm. 1.2, Thm. 1.4] (see [9] for more on separability).

**Proposition 1.5.** Suppose that $p$ is very good for $G$. Then any subgroup of $G$ is separable in $G$.

**Proposition 1.6.** Suppose that $(G, M)$ is a reductive pair. Let $H$ be a subgroup of $M$ such that $H$ is a separable subgroup of $G$. If $H$ is $G$-cr, then it is also $M$-cr.

Propositions 1.5 and 1.6 imply that the subgroup $H$ in Theorem 1.2 must be non-separable, which is possible for small $p$ only.

We recap our method from [22]. Fix a maximal torus $T$ of $G = E_6$ (respectively $E_7$, $E_8$). Fix a system of positive roots. Let $L$ be the $A_5$ (respectively $A_6$, $A_7$)-Levi subgroup of $G$ containing $T$. Let $P$ be the parabolic subgroup of $G$ containing $L$, and let $R_u(P)$ be the unipotent radical of $P$. Let $W_L$ be the Weyl group of $L$. Abusing the notation, we write $W_L$ for the group generated by canonical representatives $n_\zeta$ of reflections in $W_L$. (See Section 2 for the definition of $n_\zeta$.) Now $W_L$ is a subgroup of $L$.

1. Find a subgroup $K'$ of $W_L$ acting non-separably on $R_u(P)$.
2. If $K'$ is $G$-cr, set $K := K'$ and go to the next step. Otherwise, add an element $t$ from the maximal torus $T$ in such a way that $K := (K' \cup \{t\})$ is $G$-cr and $K$ still acts non-separably on $R_u(P)$.
3. Choose a suitable element $v \in R_u(P)$ in a 1-dimensional curve $C$ such that $T_1(C)$ is contained in $\mathfrak{c}_{\text{Lie}(R_u(P))}(K)$ but not contained in $\text{Lie}(C_{R_u(P)}(K))$. Set $H := vKv^{-1}$. Choose a connected reductive subgroup $M$ of $G$ containing $H$ such that $H$ is not $G$-cr. Show that $H$ is not $M$-cr using Proposition 2.4.
As the first application of our construction, we consider a rationality problem for $G$-complete reducibility. We need a definition first.

**Definition 1.7.** Let $k_0$ be a subfield of $k$. Let $H$ be a $k_0$-defined subgroup of a $k_0$-defined reductive algebraic group $G$. Then $H$ is $G$-completely reducible over $k_0$ ($G$-cr over $k_0$ for short) if whenever $H$ is contained in a $k_0$-defined parabolic subgroup $P$ of $G$, it is contained in some $k_0$-defined Levi subgroup of $P$.

Note that if $k_0$ is algebraically closed then $G$-cr over $k_0$ means $G$-cr in the usual sense. Here is the main result concerning rationality.

**Theorem 1.8.** Let $k_0$ be a nonperfect field of characteristic 2, and let $G$ be a $k_0$-defined split simple algebraic group of type $E_n (n = 6, 7, 8)$ of any isogeny type. Then there exists a $k_0$-defined subgroup $H$ of $G$ such that $H$ is $G$-cr but not $G$-cr over $k_0$.

**Proof.** Use the same $H = v(a)Kv(a)^{-1}$ as in the proof of Theorem 1.2 with $v := v(a)$ for $a \in k_0\backslash k_0^2$. Then a similar method to [22, Sec. 4] shows that subgroups $H$ have the desired properties. The crucial thing here is the existence of a 1-dimensional curve $C$ such that $T_1(C)$ is contained in $\mathfrak{c}_{\text{Lie}(R_6(P))}(K)$ but not contained in $\text{Lie}(C_{R_6(P)}(K))$ (see [22, Sec. 4] for details).

**Remark 1.9.** Let $k_0$ and $G = E_6$ be as in Theorem 1.8. Based on the construction of the $E_6$ examples in this paper, we found the first examples of nonabelian $k_0$-defined subgroups $H$ of $G$ such that $H$ is $G$-cr over $k_0$ but not $G$-cr; see [23]. Note that $G$-complete reducibility over $k_0$ is invariant under central isogenies [23, Sec. 2].

As the second application, we consider a problem concerning conjugacy classes. Given $n \in \mathbb{N}$, we let $G$ act on $G^n$ by simultaneous conjugation:

$$g \cdot (g_1, g_2, \ldots, g_n) = (gg_1g^{-1}, gg_2g^{-1}, \ldots, gg_ng^{-1}).$$

In [15], Slodowy proved the following result, applying Richardson’s tangent space argument [14, Sec. 3], [15, Lem. 3.1].

**Proposition 1.10.** Let $M$ be a reductive subgroup of a reductive algebraic group $G$ defined over an algebraically closed field $k$. Let $N \in \mathbb{N}$, let $(m_1, \ldots, m_N) \in M^N$ and let $H$ be the subgroup of $M$ generated by $m_1, \ldots, m_N$. Suppose that $(G, M)$ is a reductive pair and that $H$ is separable in $G$. Then the intersection $G \cdot (m_1, \ldots, m_N) \cap M^N$ is a finite union of $M$-conjugacy classes.

Proposition 1.10 has many consequences; see [2], [18], and [24, Sec. 3] for example. Here is our main result on conjugacy classes:

**Theorem 1.11.** Let $G$ be a simple algebraic group of type $E_6$ (respectively $E_7$, $E_8$) defined over an algebraically closed $k$ of characteristic $p = 2$. Let $M$ be the subsystem subgroup of type $A_5, A_1$ (respectively $A_7, D_8$). Then there exists $N \in \mathbb{N}$ and a tuple $m \in M^N$ such that $G \cdot m \cap M^N$ is an infinite union of $M$-conjugacy classes.

**Proof.** We give a sketch with one example for $G = E_6$ (see Section 3, case 4). Keep the same notation $P_\lambda, K$, $q_1, q_2$ therein. Define $K_0 := (K', Z(R_{q_2}(P_\lambda)))$. By a standard result, there exists a finite subset $F = \{z_1, \ldots, z_n\}$ of $Z(R_{q_2}(P_\lambda))$ such that $C_{P_\lambda}(K' \cup F) = C_{R_{q_2}(P_\lambda)}(K_0)$. Let $m := (q_1, q_2, z_1, \ldots, z_n)$. Set $N := n + 2$. Then, a similar computation to that of [22, Sec. 5] shows that the tuple $m \in M^N$ has the desired properties. The existence of a 1-dimensional curve $C$ such that $T_1(C)$ is contained in $\mathfrak{c}_{\text{Lie}(R_6(P))}(K')$ but not contained in $\text{Lie}(C_{R_6(P)}(K'))$ is crucial.

□
Now we discuss another application of our construction with a different flavor. Here, we consider a question of Kulshammer on representations of finite groups in reductive algebraic groups. Let $\Gamma$ be a finite group. By a representation of $\Gamma$ in a reductive algebraic group $G$, we mean a homomorphism from $\Gamma$ to $G$. We write $\text{Hom}(\Gamma, G)$ for the set of representations $\rho$ of $\Gamma$ in $G$. The group $G$ acts on $\text{Hom}(\Gamma, G)$ by conjugation. Let $\Gamma_p$ be a Sylow $p$-subgroup of $G$. In [1], Bate et al. presented an example where $p = 2, G = G_2$ and $G$ has a finite subgroup $\Gamma$ with Sylow 2-subgroup $\Gamma_2$ such that $\Gamma$ has an infinite family of pairwise non-conjugate representations $\rho_a$ such that $\rho_a \neq \rho_b$ for $a \neq b$ but the restrictions $\rho_a |_{\Gamma_2}$ are pairwise conjugate for all $a \in k$.

Note that the example of Theorem 1.13 is derived from Case 4 in the proof of Theorem 1.12.

We also present an example giving a negative answer to Question 1.12 for a non-connected reductive $G$ (this is much easier than the connected case):

**Theorem 1.13.** Let $G$ be a simply simply-connected algebraic group of type $E_6$ defined over an algebraically closed field $k$ of characteristic $p = 2$. Then there exist a finite group $\Gamma$ with a Sylow 2-subgroup $\Gamma_2$ and representations $\rho_a \in \text{Hom}(\Gamma, G)$ for $a \in k$ such that $\rho_a$ is not conjugate to $\rho_b$ for $a \neq b$ but the restrictions $\rho_a |_{\Gamma_2}$ are pairwise conjugate for all $a \in k$.

Here is the structure of this paper. In Section 2, we set out the notation and give a few preliminary results. Then in Section 3, 4, 5, we present a list of $G$-c.r but non $M$-c.r subgroups for $G = E_6, E_7, E_8$ respectively. This proves Theorem 1.12. Some details of our method will be explained in Section 3 using one of the examples for $G = E_6$. Finally in Section 6, we give proofs of Theorems 1.13 and 1.14.

## 2 Preliminaries

Throughout, we denote by $k$ an algebraically closed field of characteristic $p$. Let $G$ be an algebraic group defined over $k$. We write $R_u(G)$ for the unipotent radical of $G$, and $G$ is called (possibly non-connected) reductive if $R_u(G) = \{1\}$. In particular, $G$ is simple as an algebraic group if $G$ is connected and all proper normal subgroups of $G$ are finite. In this paper, when a subgroup $H$ of $G$ acts on $G$, we assume $H$ acts on $G$ by inner automorphisms. We write $CG(H)$ and $\epsilon_G(H)$ for the global and the infinitesimal centralizers of $H$ in $G$ and $g$ respectively. We write $X(G)$ and $Y(G)$ for the set of characters and cocharacters of $G$ respectively.

Let $G$ be a connected reductive algebraic group. Fix a maximal torus $T$ of $G$. Let $\Psi(G, T)$ denote the set of roots of $G$ with respect to $T$. We sometimes write $\Psi(G)$ for $\Psi(G, T)$. Let
\(\zeta \in \Psi(G)\). We write \(U_\zeta\) for the corresponding root subgroup of \(G\) and \(u_\zeta\) for the Lie algebra of \(U_\zeta\). We define \(G_\zeta := \langle U_\zeta, U_-\zeta \rangle\). Let \(\zeta, \xi \in \Psi(G)\). Let \(\xi'\) be the coroot corresponding to \(\xi\). Then \(\zeta \circ \xi' : k^* \to k^*\) is a homomorphism such that \((\zeta \circ \xi')(a) = a^n\) for some \(n \in \mathbb{Z}\). We define \((\zeta, \xi') := n\). Let \(s_\zeta\) denote the reflection corresponding to \(\zeta\) in the Weyl group of \(G\). Each \(s_\zeta\) acts on the set of roots \(\Psi(G)\) by the following formula [19 Lem. 7.1.8]: \(s_\zeta \cdot \zeta = \zeta - (\zeta, \xi')\xi\). By [6 Prop. 6.4.2, Lem. 7.2.1] we can choose homomorphisms \(\epsilon_\zeta : k \to U_\zeta\) so that \(n_\zeta \epsilon_\zeta(a)n_\zeta^{-1} = \epsilon_{s_\zeta \cdot \zeta}(\pm a)\) where \(n_\zeta = \epsilon_\zeta(1)\epsilon_{-\zeta}(-1)\epsilon_\zeta(1)\). We define \(\epsilon_\zeta := \epsilon_\zeta(0)\).

We recall [10 Sec. 2.1–2.3] for the characterization of a parabolic subgroup \(P\) of \(G\), a Levi subgroup \(L\) of \(P\), and the unipotent radical \(R_u(P)\) of \(P\) in terms of a cocharacter of \(G\) and state a result from GIT (Proposition 2.4).

**Definition 2.1.** Let \(X\) be an affine variety. Let \(\phi : k^* \to X\) be a morphism of algebraic varieties. We say that \(\lim_{a \to 0} \phi(a)\) exists if there exists a morphism \(\tilde{\phi} : k \to X\) (necessarily unique) whose restriction to \(k^*\) is \(\phi\). If this limit exists, we set \(\lim_{a \to 0} \phi(a) = \tilde{\phi}(0)\).

**Definition 2.2.** Let \(\lambda\) be a cocharacter of \(G\). Define \(P_\lambda := \{g \in G \mid \lim_{a \to 0} \lambda(a)g\lambda(a)^{-1}\} \exists\}\), \(L_\lambda := \{g \in G \mid \lim_{a \to 0} \lambda(a)g\lambda(a)^{-1} = 1\}\), \(R_u(P_\lambda) := \{g \in G \mid \lim_{a \to 0} \lambda(a)g\lambda(a)^{-1} = 1\}\).

Note that \(P_\lambda\) is a parabolic subgroup of \(G\), \(L_\lambda\) is a Levi subgroup of \(P_\lambda\), and \(R_u(P_\lambda)\) is the unipotent radical of \(P_\lambda\) [16 Sec. 2.1-2.3]. By [19 Prop. 8.4.5], any parabolic subgroup \(P\) of \(G\), any Levi subgroup \(L\) of \(P\), and any unipotent radical \(R_u(P)\) of \(P\) can be expressed in this form. It is well known that \(L_\lambda = C_G(\lambda(k^*))\).

Let \(M\) be a reductive subgroup of \(G\). There is a natural inclusion \(Y(M) \subseteq Y(G)\) of cocharacter groups. Let \(\lambda \in Y(M)\). We write \(P_\lambda(M)\) or just \(P_\lambda\) for the parabolic subgroup of \(G\) corresponding to \(\lambda\), and \(P_\lambda(M)\) for the parabolic subgroup of \(M\) corresponding to \(\lambda\). It is obvious that \(P_\lambda(M) = P_\lambda(G) \cap M\) and \(R_u(P_\lambda(M)) = R_u(P_\lambda(G)) \cap M\).

**Definition 2.3.** Let \(\lambda \in Y(G)\). Define a map \(c_\lambda : P_\lambda \to L_\lambda\) by \(c_\lambda(g) := \lim_{a \to 0} \lambda(a)g\lambda(a)^{-1}\).

Note that the map \(c_\lambda\) is the usual canonical projection from \(P_\lambda\) to \(L_\lambda \cong P_\lambda/R_u(P_\lambda)\). Now we state a result from GIT (see [2 Lem. 2.17, Thm. 3.1], [14 Thm. 3.3]).

**Proposition 2.4.** Let \(H\) be a subgroup of \(G\). Let \(\lambda\) be a cocharacter of \(G\) with \(H \subseteq P_\lambda\). If \(H\) is \(G\)-cr, there exists \(v \in R_u(P_\lambda)\) such that \(c_\lambda(h) = vhv^{-1}\) for every \(h \in H\).

## 3 The \(E_6\) examples

For the rest of the paper, we assume \(k\) is an algebraically closed field of characteristic 2. Let \(G\) be a simple algebraic group of type \(E_6\) defined over \(k\). Without loss, we assume that \(G\) is simply-connected. Fix a maximal torus \(T\) of \(G\). Pick a Borel subgroup \(B\) of \(G\) containing \(T\). Let \(\Sigma = \{\alpha, \beta, \gamma, \delta, \epsilon, \sigma\}\) be the set of simple roots of \(G\) corresponding to \(B\) and \(T\). The next figure defines how each simple root of \(G\) corresponds to each node in the Dynkin diagram of \(E_6\). We label the positive roots of \(G\) as shown in Table 4 in the Appendix [7 Appendix, Table B]. Define \(L := \langle T, G_{22}, \cdots, G_{36} \rangle\), \(P := \langle L, U_1, \cdots, U_{21} \rangle\), \(W_L := \langle n_\alpha, n_\beta, n_\gamma, n_\delta, n_\epsilon \rangle\). Then \(P\)
is a parabolic subgroup of $G$. Let $L$ be a Levi subgroup of $P$, and $\Psi(R_u(P)) = \{1, \cdots, 21\}$. Let $M = (L, G_21)$. Then $M$ is a subsystem subgroup of type $A_5 A_1$, $(G, M)$ is a reductive pair, and $\Psi(M) = \{\pm 21, \cdots, \pm 36\}$. Note that $L$ is generated by $T$ and all root subgroups with $\sigma$-weight $0$, and $M$ is generated by $L$ and all root subgroups with $\sigma$-weight $\pm 2$. Here, by the $\sigma$-weight of a root subgroup $\Upsilon_\zeta$, we mean the $\sigma$-coefficient of $\zeta$.

Using Magma, we found that there are 56 subgroups of $W_L$ up to conjugacy, and 11 of them act non-separably on $R_u(P)$. Table lists these 11 subgroups $K'$, and also gives the choice of $t$ we use to give $K := (K' \cup \{t\})$. Note that $[L, L] = SL_6$ since $G$ is simply-connected. We identify $n_\alpha, n_\beta, n_\gamma, n_\delta, n_e$ with $(12), (23), (34), (45), (56)$ in $S_6$. To illustrate our method, we look at Case 4 closely.

![Table 1: The $E_6$ examples](image)

**Case 4:**

Let $b \in k$ such that $b^3 = 1$ and $b \neq 1$. Define

$q_1 := n_\alpha n_\beta n_\gamma n_\delta n_e, q_2 := n_\alpha n_\beta n_\gamma n_\delta n_e, t := (\alpha' + \epsilon')(b), K' := (q_1, q_2, t)$.

It is easy to calculate how $W_L$ acts on $\Psi(R_u(P))$. The $n_\zeta$ with $[L, L] = SL_6$ since $G$ is simply-connected. We identify $n_\alpha, n_\beta, n_\gamma, n_\delta, n_e$ with $(12), (23), (34), (45), (56)$ in $S_6$. To illustrate our method, we look at Case 4 closely.

![Table 1: The $E_6$ examples](image)

**Proposition 3.1.** $e_7 + e_8 \in \xi_{\text{Aut}(R_u(P))}(K)$.

**Proposition 3.2.** Let $u \in C_{R_u(P)}(p_{e_7+e_8})(K)$. Then $u$ must have the form,

$$u = \prod_{i=1}^{6} e_i(a) \prod_{i=7}^{8} e_i(b) \prod_{i=9}^{14} e_i(c) \prod_{i=15}^{20} e_i(a + b + c) e_{21}(a_{21})$$

for some $a, b, c, a_{21} \in k$.

**Proof.** By [19] Prop. 8.2.1, $u$ can be expressed uniquely as $u = \prod_{i=1}^{21} e_i(a_i)$ for some $a_i \in k$. Since $p = 2$ we have $n_\zeta e_\zeta(a) n_\zeta^{-1} = e_\zeta^{-1} e_\zeta(a)$ for any $a \in k$ and $\zeta, \zeta \in \Psi(G)$. Then a calculation using the commutator relations ([10] Lem. 32.5, Lem. 33.3) shows that

$$q_2 u q_2^{-1} = e_1(a_2) e_2(a_1) e_3(a_4) e_4(a_3) e_5(a_5) e_6(a_6) e_7(a_7) e_8(a_8) e_9(a_{14}) e_{10}(a_{11}) e_{11}(a_{10}) e_{12}(a_{13}) e_{13}(a_{12})$$

$$e_{14}(a_9) e_{15}(a_{18}) e_{16}(a_{19}) e_{17}(a_{20}) e_{18}(a_{15}) e_{19}(a_{16}) e_{20}(a_{17}) e_{21}(a_{18} + a_{21}).$$

(3.1)
Since $q_1$ and $q_2$ centralize $u$, we have $a_1 = \cdots = a_6$, $a_7 = a_8$, $a_9 = \cdots = a_{14}$, $a_{15} = \cdots = a_{20}$. Set $a_1 = a$, $a_7 = b$, $a_9 = c$, $a_{15} = d$. Then (34) simplifies to

$$q_2 a q_2^{-1} = \prod_{i=1}^{6} \epsilon_i(a) \prod_{i=7}^{8} \epsilon_i(b) \prod_{i=9}^{14} \epsilon_i(c) \left( \prod_{i=15}^{20} \epsilon_i(d) \right) \epsilon_{21}(a^2 + b^2 + c^2 + d^2 + a_{21}).$$

Since $q_2$ centralizes $u$, comparing the arguments of the $\epsilon_{21}$ term on both sides, we must have

$$a_{21} = a^2 + b^2 + c^2 + d^2 + a_{21},$$

which is equivalent to $a + b + c + d = 0$. Then we obtain the desired result. □

**Proposition 3.3.** $K$ acts non-separably on $R_u(P)$.

**Proof.** Proposition 3.2 and a similar argument to that of the proof of [22, Prop. 3.3] show that $e_7 + e_8 \notin \text{Lie} C_{R_u(P)}(K)$. Then Proposition 3.1 gives the desired result. □

**Remark 3.4.** The following three facts are essential for the argument above:

1. The orbit $O_7$ contains a pair of roots corresponding to a non-commuting pair of root subgroups which get swapped by $q_2$: $q_2 \cdot (e_7(a) + e_8(a)) = e_8(a) + e_7(a) + e_7(a) + e_8(a) e_{21}(a^2)$.
2. The correction term $e_{21}(a^2)$ in the last equation is contained in $Z(R_u(P))$.
3. The root $21$ corresponding to the correction term is fixed by $\pi(q_2)$.

Now, let $C := \left\{ \prod_{i=7}^{8} \epsilon_i(a) \mid a \in k^* \right\}$, pick any $a \in k^*$, and let $v(a) := \prod_{i=7}^{8} \epsilon_i(a)$. Now set $H := v(a)Kv(a)^{-1} = (q_1, a, v(a) e_{21}(a^2), t)$. Note that $H \subset M, H \not\subset L$.

**Proposition 3.5.** $H$ is not $M$-cr.

**Proof.** Let $\lambda = \alpha^\vee + 2\beta^\vee + 3\gamma^\vee + 2\delta^\vee + 4\varepsilon^\vee + 2\sigma^\vee$. Then $L = L_\lambda, P = P_\lambda$. Let $c_\lambda : P_\lambda \to L_\lambda$ be the homomorphism from Definition 2.3. In order to prove that $H$ is not $M$-cr, by Proposition 2.3 it suffices to find a tuple $(h_1, h_2) \in H^2$ that is not $R_u(P_\lambda(M))$-conjugate to $c_\lambda((h_1, h_2))$. Set $h_1 := v(a) q_1 v(a)^{-1}, h_2 := v(a) q_2 v(a)^{-1}$. Then

$$c_\lambda((h_1, h_2)) = \lim_{x \to 0} (\lambda(x) q_1 v(x)^{-1}, \lambda(x) q_2 v(x)^{-1}, \lambda(x) q_2 v(x)^{-1}) = (q_1, q_2).$$

Now suppose that $(h_1, h_2)$ is $R_u(P_\lambda(M))$-conjugate to $c_\lambda((h_1, h_2))$. Then there exists $m \in R_u(P_\lambda(M))$ such that $mv(a) q_1 v(a)^{-1} = q_1, mv(a) q_2 v(a)^{-1} = q_2$. Thus we have $mv(a) \in C_{R_u(P_\lambda)}(K)$. Note that $\Psi(R_u(P_\lambda(M))) = \{21\}$. Let $m = e_{21}(a_{21})$ for some $a_{21} \in k$. Then we have $mv(a) = e_7(a) e_8(a) e_{21}(a_{21}) \in C_{R_u(P_\lambda)}(K)$. This contradicts Proposition 3.2. □

**Proposition 3.6.** $H$ is $G$-cr.

**Proof.** Since $H$ is $G$-conjugate to $K$, it is enough to show that $K$ is $G$-cr. Since $K$ is contained in $L$, by [17, Prop. 3.2] it suffices to show that $K$ is $L$-cr. Then by [2] Lem. 2.12, it is enough to show that $K$ is $[L, L]$-cr. Note that $[L, L] = SL_6$. An easy matrix computation shows that $K$ acts semisimply on $k^n$, so $K$ is $G$-cr by [17, Ex. 3.2.2(a)]. □

It is clear that similar arguments work for the other cases. We omit proofs.
4 The $E_7$ examples

Let $G$ be a simple simply-connected algebraic group of type $E_7$ defined over $k$. Fix a maximal torus $T$ of $G$, and a Borel subgroup of $G$ containing $T$. We define the set of simple roots $\Sigma = \{\alpha, \beta, \gamma, \delta, \epsilon, \eta, \sigma\}$ as in the following Dynkin diagram. The positive roots of $G$ are listed in [7, Appendix, Table B].

Let $L$ be the subgroup of $G$ generated by $T$ and all root subgroups of $G$ with $\sigma$-weight $0$. Let $P$ be the subgroup of $G$ generated by $L$ and all root subgroups of $G$ with $\sigma$-weight $1$ or $2$. Then $P$ is a parabolic subgroup of $G$ and $L$ is a Levi subgroup of $P$. Let $W_L := \langle n_\alpha, n_\beta, n_\gamma, n_\delta, n_\epsilon, n_\eta \rangle$.

Let $M$ be the subgroup of $G$ generated by $L$ and all root subgroups of $G$ with $\sigma$-weight $\pm 2$. Then $M$ is the subsystem subgroup of $G$ of type $A_7$, and $(G, M)$ is a reductive pair.

In the $E_7$ cases, we take $t = 1$ and $K' := K$; so each $K$ is a subgroup of $W_L$. We use the same method as the $E_6$ examples, so we just give a sketch.

Using Magma, we found 95 non-trivial subgroups $K$ of $W_L$ up to conjugacy, and 19 of them are $G$-cr. Only two of them act non-separably on $R_u(P)$ (see Table 2). We determined $G$-complete reducibility and non-separability of $K$ by a similar argument to that of the proof of Proposition 3.6. Note that $[L, L] = SL_7$. We identify $n_\alpha, \cdots, n_\eta$ with $(12), \cdots, (67)$ in $S_7$.

| Case | generators of $K$ | $|K|$ |
|------|------------------|-----|
| 1    | $(2 5)(3 7)(4 6), (1 4 3 2 5 7 6)$ | 14  |
| 2    | $(2 6 7)(3 5 4), (2 5)(3 7)(4 6), (1 6 7 5 2 3 4)$ | 42  |

Table 2: The $E_7$ examples

• Case 1 was in [22, Sec. 3].
• Case 2:

Let $q_1 = n_\alpha n_\gamma n_\delta, q_2 = n_\alpha n_\gamma n_\delta n_\epsilon n_\eta n_\gamma n_\delta n_\eta, K = \langle q_1, q_2 \rangle \cong \text{Frob}_{42}$ (Frobenius group of order 42). We label some roots of $G$ in Table 5 in Appendix. It can be calculated that $K$ has an orbit $\{1, \cdots, 14\}$ which contains only one non-commuting pair of roots $(2, 10)$ contributing to a correction term that lies in $U_{15}$. Also, $\pi(q_1)$ swaps 2 with 10, and fixes 15. Thus $K$ acts non-separably on $R_u(P)$ (see Remark 5.1). Now, set $v(a) = \prod_{i=1}^{14} \epsilon_i(a)$, and $H := v(a) \cdot K$. Then a similar argument to that of the proof of Proposition 5.4 show that $H$ is not $M$-cr.

5 The $E_8$ examples

Let $G$ be a simple simply-connected algebraic group of type $E_8$ defined over $k$. Fix a maximal torus $T$ and a Borel subgroup $B$ containing $T$. Define $\Sigma = \{\alpha, \beta, \gamma, \delta, \epsilon, \eta, \xi, \sigma\}$ by the next Dynkin diagram. All roots of $G$ are listed in [7, Appendix, Table B]. Let $L$ be

\begin{center}
\begin{tikzpicture}
  \foreach \x in {0,1,2,3,4,5,6,7}
  \draw (\x,0)node[anchor=north]{$\alpha$} -- (\x,1)node[anchor=south]{$\beta$};
  \draw (4,0)node[anchor=north]{$\sigma$} -- (4,1)node[anchor=south]{$\gamma$};
  \draw (4,1)node[anchor=south]{$\delta$} -- (6,1)node[anchor=south]{$\epsilon$};
  \draw (6,1)node[anchor=south]{$\eta$} -- (6,2)node[anchor=west]{$\xi$};
\end{tikzpicture}
\end{center}
the subgroup of \( G \) generated by \( T \) and all root subgroups of \( G \) with \( \sigma \)-weight 0. Let \( P \) be the subgroup of \( G \) generated by \( L \) and all root subgroups of \( G \) with \( \sigma \)-weight 1, 2, or 3. Let \( W_L := \langle n_\alpha, n_\beta, n_\gamma, n_\delta, n_e, n_\eta, n_\xi \rangle \). Then \( P \) is a parabolic subgroup of \( G \), and \( L \) is a Levi subgroup of \( P \). Let \( M \) be the subgroup of \( G \) generated by \( L \) and all root subgroups of \( G \) with \( \sigma \)-weight \( \pm 2 \). Then \( M \) is a subsystem subgroup of type \( D_8 \), and \( (G, M) \) is a reductive pair. In the \( E_8 \) cases, we take \( t = 1 \) and \( K' := K \); so each \( K \) is a subgroup of \( W_L \). We use the same method as in the \( E_6, E_7 \) examples, so we just give a sketch.

With Magma, we found 295 non-trivial subgroups \( K \) of \( W \) up to conjugacy, and 31 of them are \( G \)-cr. Only two of them act non-separably on \( R_u(P) \) (see Table 3). Note that \([L, L] \cong SL_8\).

We identify \( n_\alpha, \cdots, n_\xi \) with \((12), \cdots, (78)\) in \( S_8 \).

| case | generators of \( K \) | \(|K|\) |
|------|---------------------|-----|
| 1    | \((2 9)(4 6)(8 10), (1 4 2 8 7 6 5)\) | 14  |
| 2    | \((1 7 5)(2 6 8), (1 2)(5 8)(6 7), (1 2 7 5 4 8 6)\) | 42  |

Table 3: The \( E_8 \) examples

- **Case 1:**
  Let \( q_1 = n_\beta n_\gamma n_\delta n_\epsilon n_\xi \), \( q_2 = n_\alpha n_\beta n_\gamma n_\delta n_\epsilon n_\xi \), \( K = \langle q_1, q_2 \rangle \).

  We label some roots of \( G \) as in Table 8 in the Appendix. It can be calculated that \( K \) has an orbit \( O_1 = \{1, \cdots, 7\} \) which contains only one non-commuting pair of roots \{3, 4\}, contributing a correction term that lies in \( U_8 \). Also \( \pi(q_1) \) swaps 3 with 4, and fixes 8. So \( K \) acts nonseparably on \( R_u(P) \) (see Remark 3.4). Now let \( v(a) = \prod_{i=1}^{14} \epsilon_i(a) \), and define \( H = v(a) \cdot K \). Then it is clear that \( H \) is not \( M \)-cr by the same arguments as in the \( E_6 \) cases.

- **Case 2:**
  Let \( q_1 = n_\alpha n_\beta n_\gamma n_\delta n_\epsilon n_\xi \), \( q_2 = n_\alpha n_\beta n_\gamma n_\delta n_\epsilon n_\xi \), \( K = \langle q_1, q_2, q_3 \rangle \).

  We label some roots of \( G \) as in Table 7 in Appendix. It can be calculated that \( K \) has an orbit \( O_1 = \{1, \cdots, 14\} \) which contains only one non-commuting pair of roots \{4, 9\} contributing a correction term that lies in \( U_{15} \). Also \( \pi(q_1) \) swaps 4 with 9, and fixes 15. Let \( v(a) = \prod_{i=1}^{14} \epsilon_i(a) \) and define \( H := v(a) \cdot K \). It is clear that the same arguments work in the last case.

6 On a question of Külshammer for representations of finite groups in reductive groups

6.1 The \( E_6 \) example

*Proof of Theorem 1.13.* Let \( G \) be a simple simply-connected algebraic group of type \( E_6 \) defined over \( k \). We keep the notation from Sections 2 and 3. Pick \( c \in k \) such that \( c^3 = 1 \) and \( c \neq 1 \). Let

\[
t_1 := \alpha^\vee(c), \ t_2 := \beta^\vee(c), \ t_3 := \gamma^\vee(c), \ t_4 := \delta^\vee(c), \ t_5 := \epsilon^\vee(c),
\]

\[
q_1 := n_\alpha n_\beta n_\gamma n_\delta n_\epsilon n_\xi, \ q_2 := n_\alpha n_\beta n_\gamma n_\delta n_\epsilon n_\xi,
\]

\[
H' := \langle t_1, t_2, t_3, t_4, t_5, q_1, q_2 \rangle.
\]
Note that $q_1$ and $q_2$ here are the same as $q_1$ and $q_2$ in Case 4 of Section 3. Using Magma, we obtain the defining relations of $H'$:

$$t_1^3 = 1, q_1^3 = 1, q_2^2 = 1, q_1 \cdot t_1 = (t_1 t_2 t_3)^{-1}, q_1 \cdot t_2 = t_1 t_2 t_3 t_4 t_5, \quad q_1 \cdot t_3 = (t_2 t_3 t_4 t_5)^{-1},$$
$$q_1 \cdot t_4 = t_2, \quad q_1 \cdot t_5 = t_2 t_4, \quad q_2 \cdot t_1 = (t_3 t_4)^{-1}, \quad q_2 \cdot t_2 = t_2^{-1}, \quad q_2 \cdot t_3 = t_2 t_3 t_4 t_5,$$
$$q_2 \cdot t_4 = (t_1 t_2 t_3 t_4 t_5)^{-1}, \quad q_2 \cdot t_5 = t_1 t_2 t_3, \quad [t_1, t_2] = 1, \quad (q_1^2 q_2)^2 = 1.$$

Let

$$\Gamma := F \times C_2 = \langle r_1, r_2, r_3, r_4, r_5, s_1, s_2, z \mid r_i^3 = s_i^3 = 1, s_1 r_1 s_1 = (r_1 r_2 r_3)^{-1}, s_2 r_2 s_2 = (r_3 r_4)^{-1}, s_2 r_2 s_2 = (r_1 r_2 r_3 r_4)^{-1}, s_2 r_5 s_2 = (r_1 r_2 r_3)^{-1}, \rangle$$

Then $F \cong 3^{1+2} : 2^2 : S_3$ and $|F| = 1458 = 2 \times 3^6$. Let $\Gamma_2 := \langle s_2, z \rangle$ (a Sylow 2-subgroup of $\Gamma$).

It is clear that $F \cong H'$.

For any $a \in k$ define $\rho_a \in \text{Hom}(\Gamma, G)$ by

$$\rho_a(r_i) = t_i, \quad \rho_a(s_1) = q_1, \quad \rho_a(s_2) = q_2 \varepsilon_{21}(a), \quad \rho_a(z) = \varepsilon_{21}(1).$$

It is easily checked that this is well-defined.

**Lemma 6.1.** $\rho_a|_{\Gamma_2}$ is $G$-conjugate to $\rho_b|_{\Gamma_2}$ for any $a, b \in k$.

**Proof.** It is enough to prove that $\rho_0|_{\Gamma_2}$ is $G$-conjugate to $\rho_a|_{\Gamma_2}$ for any $a \in k$. Now let

$$u(\sqrt{a}) = \varepsilon_7(\sqrt{a}) \varepsilon_8(\sqrt{a}).$$

Then an easy computation shows that

$$u(\sqrt{a}) \cdot q_2 = q_2 \varepsilon_{21}(a), \quad u(\sqrt{a}) \cdot \varepsilon_{21}(1) = \varepsilon_{21}(1).$$

So we have

$$u(\sqrt{a}) \cdot (\rho_0|_{\Gamma_2}) = (\rho_a|_{\Gamma_2}). \quad \Box$$

**Lemma 6.2.** $\rho_a$ is not $G$-conjugate to $\rho_b$ for $a \neq b$.

**Proof.** Let $a, b \in k$. Suppose that there exists $g \in G$ such that $g \cdot \rho_a = \rho_b$. Since $\rho_a(r_i) = t_i$, we need $g \in C_G(t_1, t_2, t_3, t_4, t_5)$. A direct computation shows that $C_G(t_1, t_2, t_3, t_4, t_5) = TG_{21}$.

So let $g = tm$ for some $t \in T$ and $m \in G_{21}$. Note that $q_2$ centralizes $G_{21}$. So,

$$(tq_2 t^{-1})(tm \varepsilon_{21}(a)(m^{-1}t^{-1}) = (tm)q_2 \varepsilon_{21}(a)(m^{-1}t^{-1})$$

$$= g \cdot \rho_a(s_2)$$

$$= \rho_b(s_2)$$

$$= q_2 \varepsilon_{21}(b). \quad (6.1)$$

Note that $tq_2 t^{-1} \in G_{\alpha \beta \gamma \delta \epsilon}$ and $tm \varepsilon_{21}(a)(m^{-1}t^{-1}) \in G_{21}$. Since $[G_{\alpha \beta \gamma \delta \epsilon}, G_{21}] = 1$, it is clear that $G_{\alpha \beta \gamma \delta \epsilon} \cap G_{21} = 1$. Now (6.1) yields that $tq_2 t^{-1} = q_2$. We also have

$$q_1 = \rho_b(s_1) = g \cdot \rho_a(s_1) = tm \cdot q_1 = tq_1 t^{-1}.$$ 

So $t$ commutes with $q_1$ and $q_2$. Then a quick calculation shows that $t \in G_{21}$. So $g \in G_{21}$. But $G_{21}$ is a simple group of type $A_1$, so the pair $(q_2 \varepsilon_{21}(a), \varepsilon_{21}(1))$ is not $G_{21}$-conjugate to $(q_2 \varepsilon_{21}(b), \varepsilon_{21}(1))$ if $a \neq b$. Therefore $\rho_a$ is not $G$-conjugate to $\rho_b$ if $a \neq b. \quad \Box$
Now Theorem 1.13 follows from Lemmas 6.1 and 6.2.

Remark 6.3. One can obtain examples with the same properties as in Theorem 1.13 for \( G = E_7, E_8 \) using the \( E_7 \) and \( E_8 \) examples in Sections 4 and 5.

### 6.2 The non-connected \( A_2 \) example

**Proof of Theorem 1.14.** We have \( G^o = SL_3(k) \). Fix a maximal torus \( T \) of \( G^o \), and a Borel subgroup of \( G^o \) containing \( T \). Let \( \{ \alpha, \beta \} \) be the set of simple roots of \( G^o \). Let \( c \in k \) such that \(|c| = d \) is odd and \( c \neq 1 \). Define \( t := (\alpha - \beta)^c \). For each \( a \in k \), define \( \rho_a \in \text{Hom}(\Gamma, G) \) by

\[
\rho_a(r) = t, \quad \rho_a(s) = \sigma \epsilon_{\alpha+\beta}(a), \quad \rho_a(z) = \epsilon_{\alpha+\beta}(1).
\]

An easy computation shows that this is well-defined.

**Lemma 6.4.** \( \rho_a |_{\Gamma_2} \) is \( G \)-conjugate to \( \rho_b |_{\Gamma_2} \) for any \( a, b \in k \).

**Proof.** Let \( u(\sqrt{a}) := \epsilon_\alpha(\sqrt{a})\epsilon_\beta(\sqrt{a}) \). Then

\[
u(\sqrt{a}) \cdot \sigma = \sigma \epsilon_{\alpha+\beta}(a), \quad u(\sqrt{a}) \cdot \epsilon_{\alpha+\beta}(1) = \epsilon_{\alpha+\beta}(1).\]

This shows that \( u(\sqrt{a}) \cdot (\rho_0 |_{\Gamma_2}) = \rho_a |_{\Gamma_2} \).

**Lemma 6.5.** \( \rho_a \) is not \( G \)-conjugate to \( \rho_b \) if \( a \neq b \).

**Proof.** Let \( a, b \in k \). Suppose that there exists \( g \in G \) such that \( g \cdot \rho_a = \rho_b \). Since \( \rho_a(r) = t \), we have \( g \in C_G(t) = TG_{\alpha+\beta} \). So let \( g = hm \) for some \( h \in T \) and \( m \in G_{\alpha+\beta} \). We compute

\[
(hs^m)^{-1}(hm\epsilon_{\alpha+\beta}(a)m^{-1}h^{-1}) = (hm)\sigma\epsilon_{\alpha+\beta}(a)(m^{-1}h^{-1})
\]

\[
= g \cdot \rho_a(s)
\]

\[
= \rho_b(s)
\]

\[
= \sigma\epsilon_{\alpha+\beta}(b).
\]

(6.2)

Now (6.2) shows that \( h \) commutes with \( \sigma \). Then \( h \) is of the form \( h := (\alpha + \beta)^x \) for some \( x \in k^* \). So \( h \in G_{\alpha+\beta} \). Thus \( g \in G_{\alpha+\beta} \). But \( G_{\alpha+\beta} \) is a simple group of type \( A_1 \), so the pair \( (\sigma\epsilon_{\alpha+\beta}(a), \epsilon_{\alpha+\beta}(1)) \) is not \( G_{\alpha+\beta} \)-conjugate to \( (\sigma\epsilon_{\alpha+\beta}(b), \epsilon_{\alpha+\beta}(1)) \) unless \( a = b \). So \( \rho_a \) is not \( G \)-conjugate to \( \rho_b \) unless \( a = b \).

Theorem 1.14 follows from Lemmas 6.4 and 6.5.

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### Appendix
Table 4: The set of positive roots of $E_6$

Table 5: Case 2 ($E_7$)

Table 6: Case 1 ($E_8$)

Table 7: Case 2 ($E_8$)

References


